M-Degrees of Quadrangle-Free Planar Graphs

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Abstract: The *M-degree* of an edge *xy* in a graph is the maximum of the degrees of *x* and *y*. The *M-degree* of a graph *G* is the minimum over *M*-degrees of its edges. In order to get upper bounds on the game chromatic number, He et al showed that every planar graph *G* without leaves and 4-cycles has *M*-degree at most 8 and gave an example of such a graph with *M*-degree 3. This yields upper bounds on the game chromatic number of C_4-free planar graphs. We determine the maximum possible *M*-degrees for

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planar, projective-planar and toroidal graphs without leaves and 4-cycles. In particular, for planar and projective-planar graphs this maximum is 7.

1. INTRODUCTION

For an edge xy in a graph G, the *maximum degree* (for short, *M*-degree) $M(xy)$ is the maximum of the degrees of x and y. The *minimax degree* (*M*-degree) of a graph G is $M^*(G) = \min\{M(xy) \,|\, xy \in E(G)\}$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of a graph G, respectively.

Kotzig [4] proved that $M^*(G) \leq 5$ for every triangle-free planar graph G with $\delta(G) \geq 3$. Borodin [1,2] showed that $M^*(G) \leq 10$ for every planar graph G with $\delta(G) \geq 3$, $M^*(G) \leq 7$ for every planar graph G with $\delta(G) \geq 4$, and $M^*(G) \leq 6$ for every planar graph G with $\delta(G) \geq 5$. All these bounds are tight.

The *M*-degree of planar graphs with $\delta(G) \geq 2$ is not bounded from above. For example, $M^*(K_{2,n}) = n$. Note that every cycle in $K_{2,n}$ has length 4. He et al. [3] found upper bounds on *M*-degrees of planar graphs with $\delta(G) \geq 2$ and restrictions on girth. They used these bounds to estimate from above the game chromatic number and the game coloring number of such graphs. In particular, one of the main results in [3], Theorem 2.2, says that $M^*(G) \leq 8$ for planar graphs G having no leaves and 4-cycles. This result yields that every C_4-free planar graph can be decomposed into a forest and a graph with maximum degree at most 7, which in turn implies that the game chromatic number and the game coloring number of every C_4-free planar graph is at most 11. It is also mentioned in [3] that it is not known whether 8 is the exact bound and that the *M*-degree of dodecahedron is 3.

In this note we determine the exact upper bounds on *M*-degrees for C_4-free graphs G with $\delta(G) \geq 2$ embeddable into the plane, the projective plane, the torus and the Klein bottle.

Theorem 1. *Let G be a graph without 4-cycles and leaves. If G is planar or projective-planar, then $M^*(G) \leq 7$. If G is embeddable into the torus or the Klein bottle, then $M^*(G) \leq 8$. Both bounds are sharp.*

As explained in [3], this implies

Corollary 1. *If G is a planar graph without 4-cycles, then*

(i) *G has an edge-partition into a forest and a subgraph H with $\Delta(H) \leq 6$;*
(ii) *the game chromatic number and the game coloring number of G is at most* 10.

Let $N(S)$ denote the Euler characteristic of a surface S. Recall that $N(S) = 2$ if S is the plane, $N(S) = 1$ if S is the projective plane, and $N(S) = 0$ if S is the torus or the Klein bottle; for the other surfaces $N(S) < 0$.

From the proof of Theorem 1, it is easy to deduce the following theorem.

Theorem 2. *Let G be a graph without 4-cycles and leaves embedded into a surface S, where $N(S) < 0$. If G has more than $-72N(S)$ edges, then $M^*(G) \le 8$.*

Thus, a large graph on a fixed surface S with $N(S) < 0$ behaves in terms of M^* as a graph embeddable into the torus or the Klein bottle.

2. CONSTRUCTIONS

For $k \ge 5$, a *k-belt* B_k is the graph obtained from two disjoint k-cycles $(v_1, \ldots, v_k)$ and $(u_1, \ldots, u_k)$ by adding, for every $i = 1, \ldots, k$, paths (v_i, x_i, u_i), (v_i, y_i, u_{i+1}), (v_i, w_i, v_{i+1}) and (u_i, z_i, u_{i+1}) (the summation of subscripts taken modulo k). Graph B_6 is depicted below.
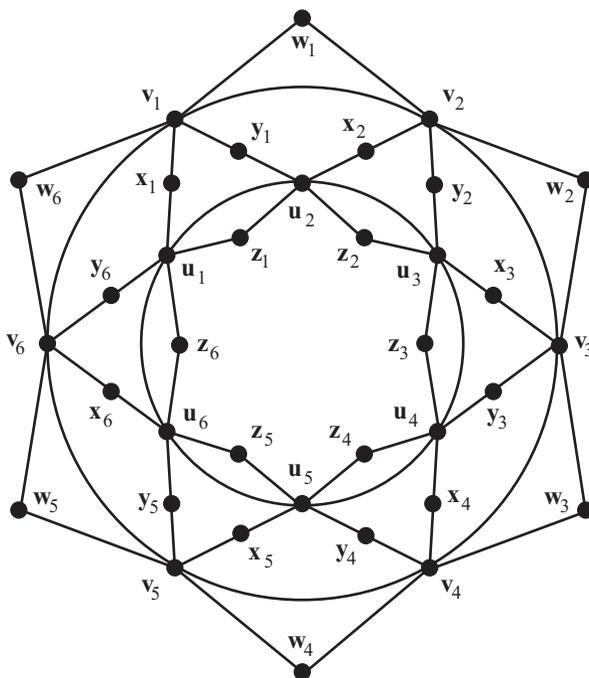
For $k \ge 5$ and $m \ge 2$, an *m-multiple k-belt* $B(k, m)$ is the graph obtained from m disjoint copies B_k^j of a k-belt B_k by identifying for every $j = 1, \ldots, m - 1$ and every $i = 1, \ldots, k$, vertex u_i^j in B_k^j with vertex v_i^{j+1} in B_k^{j+1} and vertex z_i^j with vertex w_i^{j+1}. Graph B_k can be also considered as $B(k, 1)$.

By the construction of $B(k, m)$,

(b1) the degree of v_i^1 and of u_i^m in $B(k, m)$ is 6, $(i = 1, \ldots k)$;

(b2) for $j = 2, \ldots, m$, the degree of v_i^j in $B(k, m)$ is 8, $(i = 1, \ldots k)$;

(b3) $B(k, m)$ has no 4-cycles (if $k \geq 5$).

Extremal examples for Theorem 1 can be easily obtained from $B(k, m)$. Indeed, let graph $X(k)$ be obtained from B_k by adding a vertex p connected by internally disjoint paths of length two with each v_i $(i = 1, \ldots, k)$ and a vertex q connected by internally disjoint paths of length two with each u_i $(i = 1, \ldots, k)$. For $k \geq 5$, by (b3), graph $X(k)$ has no 4-cycles. For $k \geq 7$, by (b1) and (b2), $M^*(X(k)) = 7$. Since $X(k)$ is planar, it shows that the bound of Theorem 1 for planar and projective-planar graphs is tight (if true).

Similarly, for any $k \geq 5$ and $m \geq 5$, let $Y(k, m)$ be obtained from $B(k, m)$ by identifying for $i = 1, \ldots, k$ vertex v_i^1 with vertex u_i^m and vertex w_i^1 with vertex z_i^m. Then $M^*(Y(k, m) = 8$ and again $Y(k, m)$ has no 4-cycles. Clearly, $Y(k, m)$ is embeddable into the torus. In order to get an analogous graph embeddable into the Klein bottle, we will identify v_i^1 not with u_i^m but with u_{k+1-i}^m and identify w_i^1 with z_{k-i}^m.

3.   UPPER BOUNDS

Let G be a counterexample to Theorem 1 with the fewest vertices. Let $k = 7$ if G is projective-planar and $k = 8$ if G is not projective-planar, but embeddable into the torus or the Klein bottle. Fix some such embedding of G and let $F(G)$ be the set of faces of this embedding. Let $d(y)$ denote the degree of a vertex y or of a face y. Call a vertex v *senior*, if $d(v) \geq k + 1$. By the choice of G, $M^*(G) \geq k + 1$, and hence

$$\text{Each closed walk } (w_1, \ldots, w_t) \text{ encounters senior vertices at least } \left\lceil \frac{t}{2} \right\rceil \text{ times.} \quad (1)$$

Recall that the same vertex can be encountered several times on a closed walk.

Claim 1.   *G is connected.*

Proof.   Suppose that G is the vertex-disjoint union of graphs G_1 and G_2, and there are no edges between $V(G_1)$ and $V(G_2)$. Let G' be obtained from G by identifying a vertex $v \in V(G_1)$ with a vertex $w \in V(G_2)$. Since G does not have leaves and 4-cycles, G' also does not. By the minimality of G, $M^*(G') \leq k$. But then $M^*(G) \leq k$, as well.   ■

Since G is connected, the boundary of every face is a closed walk and hence (1) applies to it.

Claim 2.   *For every vertex v, the total number of incident triangular faces and 2-vertices is at most d(v).*

Proof.   Suppose that the neighbors of v in the clockwise direction are $w_1, \ldots, w_t$. Let f_i be the face containing walk (w_i, v, w_{i+1}), $i = 1, \ldots, t$ (some

f_is and/or w_is can coincide). If f_i is a triangle, then since $M^*(G) \geq k + 1$, at least one of w_i and w_{i+1} is senior; call such a vertex x_i. Since G has no 4-cycles, all x_is are distinct. This yields the claim. ∎

We now use discharging to obtain a contradiction with the assumption $M^*(G) \geq k + 1$.

Let the *initial charge* of every $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$. By Euler's formula $|V(G)| - |E(G)| + |F(G)| \geq N(S)$, we have

$$\sum_{x \in V(G) \cup F(G)} (d(x) - 4) = \sum_{x \in V(G) \cup F(G)} \mu(x) \leq -4N(S). \tag{2}$$

The vertices and faces of G discharge their initial charge by the following rules:

Rule 1. Every non-triangular face gives 1/2 to each incident vertex of degree 2 or 3.
Rule 2. Every non-triangular face gives additional 1/2 to each incident vertex v of degree 2, such that the other face incident with v is a triangle.
Rule 3. Every senior vertex gives 1/2 to each adjacent 2-vertex and each incident triangular face.

It remains to show that the *final charge* $\mu^*(y)$ is nonnegative for each $y \in V(G) \cup F(G)$, and that the final charge of every vertex of degree at least 9 is strictly positive. This yields a contradiction to (2), since the total charge does not change, and hence should be strictly negative when G is projective-planar and non-positive when G is embeddable into the torus or the Klein bottle.

If y is a triangular face, then by (1) it has at least two senior vertices incident to it, and it gets from them $1/2 + 1/2$ by Rule 3; so $\mu^*(y) \geq 3 - 4 + 2 \times 1/2 = 0$.

If y is a face with $d(y) \geq 7$, then by (1), it encounters at most $\lfloor d(y)/2 \rfloor$ incident non-senior vertices, and hence by Rules 1 and 2, $\mu^*(y) \geq d(y) - 4 - \lfloor d(y)/2 \rfloor \geq 0$.

Suppose y is a 5-face $(v_1, \ldots, v_5)$. Again by (1), it has at most two incident non-senior vertices. Moreover, neither of these vertices is a 2-vertex incident with a triangular face, since if a vertex v_i would be such a 2-vertex, then $v_{i-1} v_{i+1} \in E(G)$, and G has a 4-cycle containing this edge. Thus, Rule 2 is not applicable to y, and $\mu^*(y) \geq 5 - 4 - 2(1/2) = 0$.

Similarly, if y is a 6-face, then it has at most three incident non-senior vertices, and at most one of these non-senior vertices is a 2-vertex incident with a triangular face. So, $\mu^*(y) \geq 6 - 4 - 3(1/2) - 1/2 = 0$.

Suppose now that y is a vertex.

If $d(y) = 2$, then by Rule 3 it gets $2(1/2) = 1$ from the adjacent vertices and by Rules 1 and 2, exactly 1 from the incident faces. Hence, $\mu^*(y) = 2 - 4 + 1 + 1 = 0$.

If $d(y) = 3$, then since G has no 4-cyles, by Rule 1, it gets at least $2(1/2) = 1$ from the incident non-triangular faces. Hence, $\mu^*(y) \geq 3 - 4 + 1 = 0$.

If $4 \leq d(y) \leq 7$, then v gives out nothing, and $\mu^*(y) = d(y) - 4 \geq 0$.

If $d(y) = 8$, then by Rule 3 and Claim 2, $\mu^*(y) \geq d(y) - 4 - d(y)/2 = 0$. Finally, if $d(y) \geq 9$, then for the same reasons, $\mu^*(y) \geq d(y) - 4 - d(y)/2 > 0$.

This completes the proof of Theorem 1.

4.   DEDUCING THEOREM 2

Let G be a counterexample to Theorem 2. In particular, this means that every edge is incident with a vertex of degree at least 9 (a *senior vertex*). Consider the discharging from the previous section. Then, as shown a few lines above, each senior vertex v in G has $\mu^*(v) \geq d(v) - 4 - d(v)/2 = (d(v) - 8)/2 \geq 1/2$. For each senior vertex v, we distribute $\mu^*(v)$ evenly among the edges incident with v.

Let $v(x)$ denote the new charge of an element $x \in V(G) \cup E(G) \cup F(G)$. Then $v(e) \geq (d(v) - 8)/2d(v) \geq 1/18$ for every edge $e \in E(G)$, $v(f) = \mu^*(f) \geq 0$ for every $f \in F(G)$, and $v(v) = 0$ for every $v \in V(G)$. Now (2) implies

$$|E(G)|/18 \leq \sum_{x \in E(G) \cup F(G)} v(x) = \sum_{x \in V(G) \cup F(G)} \mu^*(x) \leq -4N(S),$$

which contradicts the assumption that $|E(G)| > -72N(S)$.

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