On $K_{s,t}$ Minors in $(s+t)$-Chromatic Graphs

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Received December 9, 2008; Revised November 11, 2009

Published online 14 January 2010 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.20485

Abstract: Let $K^*_{s,t}$ denote the graph obtained from the complete graph K_{s+t} by deleting the edges of some K_t-subgraph. We prove that for each fixed s and sufficiently large t, every graph with chromatic number $s+t$ has a $K^*_{s,t}$ minor. © 2010 Wiley Periodicals, Inc. J Graph Theory 65: 343–350, 2010

MSC 2000: *05 C 15; 05 C 83*

Keywords: *graph coloring; graph minors; complete bipartite graphs*

1. INTRODUCTION

A graph is *k-chromatic* if its chromatic number is exactly k. A *minor* of a graph G is a graph H that can be obtained from G by a sequence of vertex and edge deletions and edge contractions. If H is a minor of G, we will also say that G *has an H-minor*.

Contract grant sponsor: NSF; Contract grant number: DMS-06-50784; Contract grant sponsor: Russian Foundation for Basic Research; Contract grant numbers: 06-01-00694; 08-01-00673.

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In search of ways to attack Hadwiger's Conjecture, Woodall [7] and independently Seymour [6] suggested to prove the following weaker statement.

Conjecture 1. *Every $(s+t)$-chromatic graph has a $K_{s,t}$-minor.*

The conjecture is evident for $s=1$. The validity of the conjecture for $s=2$ and all t (and even of the list-coloring version of it) was proved by Woodall [7]. It also follows from an observation of Seymour (see Lemma 12 below) and the following result by Chudnovsky et al. [1].

Theorem 1. *Let G be a graph with $n \geq 3$ vertices such that $e(G) > \frac{1}{2}(t+1)(n-1)$. Then G has a $K_{2,t}$-minor.*

Note that this result for $t > 10^{29}$ was earlier proved by Myers [5]. Very recently, Kostochka and Prince [4] using Lemma 12 below (due to Seymour) proved the conjecture for $s=3$ and $t \geq 6500$. They proved that for $t \geq 6500$, every $(3+t)$-chromatic graph has a $K^*_{3,t}$-minor, where $K^*_{s,t}$ denotes the graph obtained from $K_{s,t}$ by adding all edges between the vertices of the partite set of size s. In other words, $K^*_{s,t} = K_{s+t} - E(K_t)$.

In this article, elaborating some ideas of [4, 3], we show that for every s, Conjecture 1 holds in a slightly stronger form if t is sufficiently large in comparison with s.

Theorem 2. *Let s and t be positive integers such that*

$$t > t_0(s) := \max\{4^{15s^2+s}, (240s \log_2 s)^{8s \log_2 s+1}\}. \tag{1}$$

Then every $(s+t)$-chromatic graph has a $K^*_{s,t}$-minor.

In the next section, we introduce notation and cite or prove auxiliary statements. In Section 3, we prove the key lemma on minors in dense subgraphs of moderate order. We deliver the main proof in Section 4 and conclude the article with some comments.

2. PRELIMINARIES

For a graph G, $V(G)$ is the set of its vertices, $E(G)$ is the set of its edges, $e(G) = |E(G)|$, and $\overline{G}$ is the complement of G. By $G[X]$ we denote the subgraph of G induced by the vertex set X. For $v \in V(G)$, we let $N_G(v)$ denote the set of neighbors of v in G, $d_G(v) = |N_G(v)|$, and $N_G[v] = N_G(v) \cup \{v\}$.

For a graph G, a set $T \subseteq V(G)$ is *totally dominating*, if every vertex of G has a neighbor in T. We say that a set $T \subseteq V(G)$ is *connected* if $G[T]$ is connected.

The following simple fact is proved in [4].

Lemma 3 (Kostochka and Prince [4])**.** *Let G be an n-vertex connected graph with minimum degree $k \geq 1$. Then*:

(a) *G contains a totally dominating set T with $|T| \leq \lfloor \log_{n/(n-k)} n \rfloor + 1$; and*
(b) *G contains a connected totally dominating set T' with $|T'| \leq 2 \log_{n/(n-k)} n$.*

Applying Lemma 3 s times, we obtain the following corollary.

Lemma 4. *Let s, k, and n be positive integers. Suppose $n>k>2$. Let H be a graph of order n with $\delta(H)\geq k+2(s-1)\log_{n/(n-k)} n$ and connectivity greater than $2(s-1)\log_{n/(n-k)} n$. Then $V(H)$ contains s pairwise-disjoint subsets $A_1,\ldots,A_s$ such that, for every $i=1,\ldots,s$,*

(i) *$H[A_i]$ is connected,*
(ii) *$|A_i|\leq 2\log_{n/(n-k)} n$,*
(iii) *A_i dominates $H-\bigcup_{j=1}^{i-1} A_j$.*

An important tool will be the following result in [3].

Theorem 5 (Kostochka and Prince [3]). *Let s and t be positive integers with*

$$t>(240s\log_2 s)^{8s\log_2 s+1}.$$

Let G be a graph such that $e(G)\geq((t+3s)/2)(n(G)-s+1)$. Then G has a $K^*_{s,t}$-minor. Furthermore, for n large, there exists a graph G of order n and size at least $(t+3s-5\sqrt{s}/2)(n-s+1)$ that has no $K_{s,t}$-minor.

As it was mentioned in the introduction, it is known that Conjecture 1 holds for $s\leq 2$ and all t. For $s=1$, graph $K^*_{s,t}$ equals $K_{s,t}$. To extend the base of induction a bit more, we use the following result for $s=2,3$ also mentioned above.

Theorem 6 (Kostochka and Prince [4]). *Let $t\geq 6500$. Then every $(3+t)$-chromatic graph has a $K^*_{3,t}$-minor and every $(2+t)$-chromatic graph has a $K^*_{2,t}$-minor.*

3. DENSE SUBGRAPHS OF MODERATE ORDER

Let $\mathcal{U}=\{U_1,U_2,\ldots,U_q\}$ be a family of pairwise-disjoint sets of vertices in a graph G. Then a path P is a *strict $\mathcal{U}$-path* if the ends of P are in distinct members of $\mathcal{U}$ and all internal vertices of P are disjoint from $\bigcup_{i=1}^{q} U_i$. Furthermore, a family $(P_1,\ldots,P_{q-1})$ of paths is *$\mathcal{U}$-connecting* if all paths in the family are strict $\mathcal{U}$-paths and the graph whose vertices are $U_1,U_2,\ldots,U_q$ and two vertices are adjacent if they are connected by a P_j is connected.

Lemma 7. *Let G be a graph and let $\mathcal{U}=\{U_0,U_1,\ldots,U_{q-1}\}$ be a family of pairwise-disjoint sets of vertices in G. If G contains $q-1$ pairwise-disjoint paths $Q_1,\ldots,Q_{q-1}$ such that Q_i connects U_0 with U_i for $i=1,\ldots,q-1$, then $Q_1\cup\cdots\cup Q_{q-1}$ contains a $\mathcal{U}$-connecting family of paths.*

Proof. We use induction on q. For $q=2$, the statement is obvious. Suppose that the statement holds for each family of $q-1$ sets and the corresponding family of $q-2$ paths. Consider a family $\mathcal{U}=\{U_0,U_1,\ldots,U_{q-1}\}$ with the corresponding family $\{Q_1,\ldots,Q_{q-1}\}$ of paths connecting U_0 with the remaining sets. Let $\mathcal{U}'=\mathcal{U}-U_{q-1}$. By the induction assumption, $Q_1\cup\cdots\cup Q_{q-2}$ contains a $\mathcal{U}'$-connecting family $\{P_1,\ldots,P_{q-2}\}$ of paths. Let P_{q-1} be the subpath of Q_{q-1} whose first vertex is the last vertex z of Q_{q-1} that belongs to $\bigcup_{i=0}^{q-2} U_i$ and whose last vertex is the first vertex in Q_{q-1} after z that belongs to U_{q-1}. Then the family $\{P_1,\ldots,P_{q-2},P_{q-1}\}$ is $\mathcal{U}$-connecting. ■

Lemma 8. *Let s and q be positive integers. Let G be an $s(q-1)$-connected graph and let $\mathcal{U}=\{U_1,U_2,\ldots,U_q\}$ be a family of pairwise-disjoint sets of vertices in G such that $|U_i|\geq s(q-1)$ for $i=1,\ldots,q$. Then G contains s vertex-disjoint $\mathcal{U}$-connecting families of paths.*

Proof. For $j=1,\ldots,q$, choose $F_j\subset U_j$ with $|F_1|=(q-1)s$ and $|F_2|=\cdots=|F_q|=s$. Since G is $(q-1)s$-connected, G contains $(q-1)s$ vertex-disjoint paths $Q_{1,2},Q_{1,3},\ldots,Q_{s,q}$ from F_1 to $F_2\cup\cdots\cup F_q$ such that for $i=1,\ldots,s$ and $j=2,\ldots,q$, $Q_{i,j}$ connects U_1 with U_j. Let $\mathcal{Q}_i=\{Q_{i,2},\ldots,Q_{i,q}\}$ for $i=1,\ldots,s$. By Lemma 7, for each such i, $Q_{i,2}\cup\cdots\cup Q_{i,q}$ contains a $\mathcal{U}$-connecting family of paths. ■

Lemma 9. *Let s and t be positive integers such that $t>t_0(s)$. Let G be a $15s^2$-connected graph. Suppose that G contains a vertex subset U with*

$$t+700s^3\ln t\leq |U|\leq 3t$$

such that $\delta(G[U])\geq t/(4s+1)$. Then G has a $K^*_{s,t}$-minor.

Proof. Let $u=|U|$. Perform the following procedure on $G[U]$. Let $i=1$ and $G_1=G[U]$.

Step i: If every component of G_i has connectivity greater than $50s^2\ln t$ and the number of components in G_i is exactly i, then stop. Otherwise, choose a set S_i with $|S_i|\leq\lfloor 50s^2\ln t\rfloor$ such that G_i-S_i has more than i components and let $G_{i+1}=G_i-S_i$.

Let ℓ be the step in which our procedure terminated. By construction, G_ℓ has exactly ℓ components. Let $H_1,H_2,\ldots,H_\ell$ be the components of G_ℓ and let $U_i=V(H_i)$ and $u_i=|U_i|$ for $i=1,\ldots,\ell$. We may assume that $u_1\geq\cdots\geq u_\ell$. First, we show that

$$\ell\leq 15s. \tag{2}$$

Suppose that (2) does not hold. Consider G_{15s}. By construction, G_{15s} has at least $15s$ components. Since by construction and by (1),

$$\delta(G_{15s})\geq\delta(G)-15s(50s^2\ln t)\geq\frac{t}{4s+1}-750s^3\ln t\geq\frac{t}{5s}+100s^2\ln t, \tag{3}$$

each component of G_{15s} has more than $t/(5s)$ vertices. Since $|U|\leq 3t$, this is impossible. This proves (2). In particular, similarly to (3) we get

$$\delta(G_\ell)\geq\frac{t}{5s}+100s^2\ln t. \tag{4}$$

Furthermore, since the procedure has stopped, each H_j is $50s^2\ln t$-connected.

Let $\mathcal{U}=\{U_1,U_2,\ldots,U_\ell\}$. Since G is $15s^2$-connected and $\ell\leq 15s$, by Lemma 8, G contains s vertex-disjoint $\mathcal{U}$-connecting families of paths $\mathcal{P}_1,\ldots,\mathcal{P}_s$. For $i=1,\ldots,s$, let $W_i=\bigcup_{P\in\mathcal{P}_i}V(P)$, and let $W=\bigcup_{i=1}^{s}W_i$. For $j=1,\ldots,\ell$, let $U'_j=U_j-W$ and $H'_j=H_j-W$.

Since each path in each of the families $\mathcal{P}_1,\ldots,\mathcal{P}_s$ is a strict $\mathcal{U}$-path, for every $1\leq j\leq\ell$, we have $|W\cap U_j|\leq s(\ell-1)$, and hence by (4), $\delta(H'_j)\geq t/(5s)+85s^2\ln t$. Let us check that each H'_j satisfies the conditions of Lemma 4 for $n=|V(H'_j)|\leq 3t$

and $k=\lceil t/(5s)\rceil$. For this, we need only to check that $85s^2\ln t\geq 2(s-1)\log_{n/(n-k)}n$ (to verify the restrictions on the minimum degree and the connectivity of H'_j). Indeed,

$$\frac{n}{n-k}\geq\frac{3t}{3t-t/5s}=1+\frac{1/5s}{3-1/5s}=1+\frac{1}{15s-1},$$

and hence

$$2(s-1)\log_{n/(n-k)}n\leq\frac{2(s-1)\ln n}{\ln\left(1+\dfrac{1}{15s-1}\right)}\leq\frac{2(s-1)\ln 3t}{1/15s}=30s(s-1)\ln 3t<40s^2\ln t.$$

Thus, by Lemma 4 for every $j\in\{1,\dots,\ell\}$, $V(H'_j)$ contains s pairwise-disjoint subsets $A_{1,j},\dots,A_{s,j}$ such that, for every $i=1,\dots,s$, (i) $H'_j[A_{i,j}]$ is connected, (ii) $|A_{i,j}|\leq 2\log_{1+1/15s}3t\leq 40s\ln t$, and (iii) $A_{i,j}$ dominates $H'_j-\bigcup_{q=1}^{i-1}A_{q,j}$.

For $j=1,\dots,\ell$, let $A_j=\bigcup_{i=1}^{s}A_{i,j}$. By (ii), $|A_j|\leq 40s^2\ln t$. Since each $w\in W\cap U_j$ has by (4) at least $t/(5s)+100s^2\ln t$ neighbors in U_j and $|W\cap U_j|\leq 15s^2$, we have $|(N(w)\cap U_j)-W-A_j|\geq t/(5s)+50s^2\ln t$, and so we can choose for each $w\in W\cap U_j$ a neighbor $y(w)\in(N(w)\cap U_j)-W-A_j$ so that all $y(w)$ are distinct for distinct w.

Now for $i=1,\dots,s$, we let

$$B_i:=W_i\cup\bigcup_{j=1}^{\ell}(A_{i,j}\cup\{y(w):w\in W_i\}).$$

By (iii), each $v\in\bigcup_{j=1}^{\ell}(U'_j-\bigcup_{q=1}^{i}A_{q,j})$ has a neighbor in B_i. In particular, if $w\in W_i\cap U_j$, then $y(w)$ has a neighbor in $A_{i,j}$. It follows that $G[B_i]$ is connected for every i. It also implies that each B_i dominates $U-\bigcup_{i=1}^{s}B_i$ and that between any sets B_i and $B_{i'}$ there is an edge. In other words, we have found a $K^*_{s,z}$-minor in G, where $z=|U|-2|U\cap W|-\sum_{i=1}^{s}\sum_{j=1}^{\ell}|A_{i,j}|$. By definition, $|U\cap W|=2s(\ell-1)\leq 30s^2$. By (ii),

$$\sum_{i=1}^{s}\sum_{j=1}^{\ell}|A_{i,j}|\leq s\ell(40s\ln t)\leq 600s^3\ln t.$$

Since $|U|\geq t+700s^3\ln t$,

$$z\geq t+700s^3\ln t-60s^2-600s^3\ln t>t.$$

This proves the lemma. ■

4. THE MAIN ARGUMENT

Suppose that the theorem is proved for all $s'<s$ and $t>t_0(s')$. By Theorem 6, it is enough to consider $s\geq 4$. Let G_0 be a counter-example for s and some $t>t_0(s)$ which is minimal with respect to $|V(G)|+|E(G)|$. Then G_0 is color-critical, namely $(s+t)$-critical. Let $n_0=|V(G_0)|$. We will need the following celebrated result of Gallai [2].

Theorem 10 (Gallai [2]). *Let $k\geq 3$ and G be a k-critical graph. If $|V(G)|\leq 2k-2$, then G has a spanning complete bipartite subgraph.*

Lemma 11. $n_0 \geq 2(s+t)-1$.

Proof. Suppose not. Then by Theorem 10, $V(G_0)$ can be partitioned into non-empty V_1 and V_2 so that each vertex in V_1 is adjacent to each vertex in V_2. Suppose that $\chi(G_0[V_1])=k_1$ and $\chi(G_0[V_2])=k_2$. By definition, $k_1+k_2=s+t$. We may assume that $k_1 \leq k_2$. Since the theorem holds for $s=1$ and any t, $G_0[V_1]$ has a K_{1,k_1-1}-minor. Since $t \geq t_0(s)$ and $t_0(s)>3t_0(s-1)$, $k_2-s+1>t_0(s-1)$. Thus, by the minimality of s, $G_0[V_2]$ has a K^*_{s-1,k_2-s+1}-minor. But then using the edges of the complete bipartite subgraph, we construct a K^*_{s,k_1+k_2-s}-minor of G_0 from these two minors. ■

By $\alpha(G)$ we denote the independence number of the graph G. The next lemma is due to Seymour.

Lemma 12 (Seymour [6]). *Let k be a non-negative integer. If $v \in V(G_0)$ and $d(v)=s+t-1+k$, then $\alpha(G_0[N(v)]) \leq k+1$.*

Proof. Suppose that $v \in V(G_0)$, $d(v)=s+t-1+k$, and $G[N(v)]$ has an independent set I with $|I|=k+2$. Let G' be obtained from G_0 by contracting all edges connecting v with vertices in I and let v^* be the new vertex which is the result of these contractions. Since G' is a minor of G_0, it does not have a $K^*_{s,t}$-minor. Therefore, by the minimality of G_0, G' is $(t+s-1)$-colorable. Let f' be a proper $(t+s-1)$-coloring of G'. Let $f'(v^*)=\alpha$. Coloring f' naturally yields a proper $(t+s-1)$-coloring f of G_0-v in which the color of each $w \in I$ is α. But then at most $d(v)-|I|+1=s+t-2$ colors are present on $N(v)$, and we have an admissible color for v, a contradiction to the definition of G_0. ■

Lemma 13. *Let k be a non-negative integer. If $v \in V(G_0)$ and $d(v)=s+t-1+k$, then there exists a subset $Y(v) \subseteq N(v)$ such that $\delta(G_0[Y(v) \cup \{v\}]) \geq t/(k+1)$*

Proof. Suppose that the lemma is not true for some $v \in V(G_0)$, and let $F_0=G_0[N(v)]$. Then F_0 is d-degenerate for some $d<t/(k+1)-1$. Therefore, F_0 is $(d+1)$-colorable and hence

$$\alpha(F_0) \geq \frac{s+t-1+k}{d+1} > \frac{t+3}{t/(k+1)} > k+1,$$

a contradiction to Lemma 12. ■

Lemma 14. *If $t \geq 4^{x+s}$, then the connectivity of G_0 is greater than x.*

Proof. Suppose that the connectivity of G_0 is x and that X is a separating set in G_0 of size x. Let V_1 be the vertex set of a component of G_0-X and $V_2=V(G_0)-X-V_1$. For $i=1,2$, let $G_i=G_0[V_i]$. Since neither of G_1 and G_2 has a $K^*_{s,t}$-minor, by Theorem 5, for $i=1,2$ we have $\sum_{v \in V_i} d_{G_i}(v)<(t+3s)|V_i|$ and hence

$$\sum_{v \in V_i} d_{G_0}(v) \leq \sum_{v \in V_i} (d_{G_i}(v)+x) < (t+3s+x)|V_i|. \tag{5}$$

Therefore, there is a vertex $v_i \in V_i$ such that $d_i := d_{G_0}(v_i) \leq t+3s+x-1$. By Lemma 12, $\alpha(G[N_{G_0}(v_i)]) \leq x+2s$. Since G_0 is $(s+t)$-critical, $d_i \geq s+t-1$. Since the Ramsey number, $r(x+2s,x)$, is at most $\binom{(x+2s)+x-1}{x-1} < 4^{x+s}$ and $s+t-1>4^{x+s}$, graph $G[N_{G_0}(v_i)]$ has a clique W_i of size x. Since G_0 is x-connected, there are x vertex-disjoint paths connecting X to W_i (if $W_i \cap X \neq \emptyset$, then some paths will have length 0). By contracting each of these paths into a vertex, we see that G_0 has the minor G'_{3-i}, which is obtained from $G_0 - V_i$ by adding all edges between the vertices in X.

Since G_0 has no $K^*_{s,t}$-minor, neither of G'_1 and G'_2 has a $K^*_{s,t}$-minor. The minimality of G_0 implies then that for $i=1,2$, G'_i has an $(s+t-1)$-coloring f_i. Since $G'_1[X]=G'_2[X]=K_x$, in both of these colorings, the colors of all vertices in X are distinct, and we can change the names of the colors in f_2 so that $f_1 \cup f_2$ is an $(s+t-1)$-coloring of G_0, a contradiction. ■

Now we are ready to prove Theorem 2. By Theorem 5,

$$\sum_{v \in V(G_0)} d(v) < (t+3s)(n_0 - s+1). \tag{6}$$

Since G_0 is color-critical, $\delta(G_0) \geq t+s-1$. Say that a vertex $v \in V(G_0)$ is *low* if $d(v) \leq t+5s-1$, and let L be the set of low vertices in G_0. By (6),

$$|L|(t+s-1)+(n_0-|L|)(t+5s) < (t+3s)(n_0-s+1).$$

It follows that

$$|L| \geq \frac{2s}{4s+1} n_0 + \frac{s-1}{4s+1} t. \tag{7}$$

Since $t > 4^{15s^2+s}$, by Lemma 14, G_0 is $15s^2$-connected. Recall that by Lemma 11, $n_0 \geq 2t+2s-1$. If $n_0 \leq 3t$, then G_0 with $U=V(G_0)$ satisfies the conditions of Lemma 9 and hence has a $K^*_{s,t}$-minor, a contradiction. So,

$$n_0 \geq 3t+1. \tag{8}$$

Thus for $s \geq 3$ by (7),

$$|L| \geq \tfrac{6}{13} 3t + \tfrac{2}{13} t = \tfrac{20}{13} t.$$

By Lemma 13, for every $v \in L$, there exists a subset $Y(v) \subseteq N(v)$ such that $\delta(G_0[Y(v) \cup \{v\}]) \geq t/(4s+1)$. Let $v_1, \ldots, v_{|L|}$ be the vertices of L, and for $j=1,\ldots,|L|$, let $Z_j = \bigcup_{i=1}^{j} (Y(v_i) \cup \{v_i\})$. By construction, for every $j \geq 2$, $\delta(G_0[Z_j]) \geq t/(4s+1)$ and $j \leq |Z_j| \leq |Z_{j-1}|+t+5s$. It follows that there exists j_0 such that

$$\tfrac{20}{13} t \leq |Z_{j_0}| \leq 3t. \tag{9}$$

Since $\frac{20}{13} t \geq t+700s^3 \ln t$, the graph G_0 with $U=Z_{j_0}$ satisfies the conditions of Lemma 9 and hence has a $K^*_{s,t}$-minor, a contradiction.

5. CONCLUDING REMARKS

1. It follows from the proof of the theorem that if $t > t_0(s)$ and an $(s+t)$-critical graph G is $15s^2$-connected and has at least $1.5t$ vertices, then G has a $K^*_{s,t'}$-minor for some $t' > t$. However, for every $s,t \geq 3$, there are infinitely many $(s+t)$-critical graphs that do not have $K_{s,t+1}$-minors. Some such graphs can be obtained by repetition of Hajos's construction starting from K_{s+t}.

2. One of the reasons that we demand $t_0(s)$ to be so large is that we directly use Theorem 5 from [3]. One can revise the proof of this theorem in order to somewhat weaken its restrictions. Another reason for large $t_0(s)$ is the direct use of Ramsey bound. Maybe one can get around it by using rooted minors, for example.

ACKNOWLEDGMENTS

I thank Noah Prince for the helpful comments and suggestions. My research is supported in part by NSF grant DMS-06-50784 and by grants 06-01-00694 and 08-01-00673 of the Russian Foundation for Basic Research.

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