# Constructions of Sparse Uniform Hypergraphs With High Chromatic Number*

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**ABSTRACT:** A random construction gives new examples of simple hypergraphs with high chromatic number that have few edges and/or low maximum degree. In particular, for every integers $k \geq 2$, $r \geq 2$, and $g \geq 3$, there exist $r$-uniform non-$k$-colorable hypergraphs of girth at least $g$ with maximum degree at most $\lceil r\,k^{r-1} \ln k \rceil$. This is only $4r^2 \ln k$ times greater than the lower bound by Erdős and Lovász (Colloquia Math Soc János Bolyai 10 (1973), 609–627) that holds for all hypergraphs (without restrictions on girth). © 2009 Wiley Periodicals, Inc. Random Struct. Alg., 36, 46–56, 2010

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## 1. INTRODUCTION

A number of articles discuss chromatic number of sparse (hyper)graphs. Measures of sparseness could be the girth or clique number, maximum degree, degeneracy, number of edges, etc. Recall that a *cycle* of *length* $k \geq 2$ in a hypergraph $\mathcal{H}$ is an alternating cyclic sequence $e_0, v_0, e_1, v_1, \dots, e_{k-1}, v_{k-1}, e_0$ of distinct edges and vertices in $\mathcal{H}$ such that $v_i \in e_i \cap e_{i+1}$ for all $i$ modulo $k$. In particular, if two distinct vertices $x$ and $y$ both belong to distinct edges $e_0$ and $e_1$, then $e_0, x, e_1, y, e_0$ is a cycle of length 2. The *girth* of a hypergraph is the length of its shortest cycle. In this article, we consider hypergraphs with restrictions on girth, in

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particular, *simple hypergraphs* (i.e., having girth at least 3). A variation of this is the notion of *b-simple hypergraphs*—hypergraphs in which no two distinct edges share more than $b$ vertices. We present a simple random construction that improves known upper bounds on the maximum degree and on the number of edges of $b$-simple $r$-uniform hypergraphs that are not $k$-colorable for $r$ large in comparison with $k$ and $b$. The case when $k$ is large in comparison with $r$ was studied in [1, 8].

Let $\Delta(G)$ denote the maximum degree of $G$. For $k, r, g \geq 2$, let $\Delta(k, r)$ (respectively, $\Delta(k, r, g)$) denote the minimum $D$ such that there exists an $r$-uniform non-$k$-colorable hypergraph $G$ (respectively, and with girth $g$) with maximum degree $D$.

In their seminal article [3], Erdős and Lovász proved the following bound.

**Theorem 1** ([3]). *If $k, r \geq 2$, then*

$$\Delta(k, r) > \frac{1}{4}k^r r^{-1}.$$

Radhakrishnan and Srinivasan [10] improved the bound for $k = 2$ and large $r$. They showed that for large $r$,

$$\Delta(2, r) > \frac{0.17}{\sqrt{r \ln r}} 2^r.$$

Szabó [12] showed that the bound of Theorem 1 for simple hypergraphs can be improved further.

**Theorem 2** ([12]). *If $k \geq 2$ and $\epsilon > 0$ are fixed and $r$ is sufficiently large, then*

$$\Delta(k, r, 3) > k^r r^{-\epsilon}.$$

On the other hand, it follows from Theorem 1′ in the article of Erdős-Lovász [3] that for every $k \geq 2$, $r \geq 2$, and $g \geq 3$,

$$\Delta(k, r, g) \leq 20r^2 k^{r+1}. \tag{1}$$

We refine bound (1) as follows.

**Theorem 3.** *Let $k \geq 2$ and $r \geq 2$ be integers. If $d$ is a positive integer such that*

$$\left(1 - \frac{1}{k^{r-1}}\right)^{d/r} < 1/k, \tag{2}$$

*then for every $g \geq 3$, $\Delta(k, r, g) \leq d$. In particular, $\Delta(k, r, g) \leq \lceil r\, k^{r-1} \ln k \rceil$.*

This bound is only $r^{1+\epsilon} \ln k$ times larger than the lower bound in Theorem 2. Although in this article we are concerned with $r$ that are much larger than $k$ and $g$, as a by-product, Theorem 3 gives also good bounds for $r = 2$, i.e., for graphs. It yields a simple proof for a couple of known upper bounds on $\Delta(k, 2, g)$. Recall that Kim [5] proved that $\Delta(k, 2, 5) > (k + o(k)) \ln k$ for sufficiently large $k$. On the other hand, random constructions by Kostochka and Mazurova [7] and Bollobás [2] show that $\Delta(k, 2, g) \leq \lceil 2\, k \ln k \rceil$ for any $g$. This bound is a partial case of Theorem 3. Tashkinov [13] proved that $\Delta(3, 2, g) \leq 6$

for every $g$, i.e., there exist non-3-colorable graphs of arbitrary girth with maximum degree at most 6. Observe that this result immediately follows when we plug $k = 3$ and $d = 6$ into (2). It is not known whether there are non-3-colorable graphs of large girth with maximum degree 5.

Let $f(r, k)$ denote the minimum number of edges in an $r$-uniform hypergraph that is not $k$-colorable. Let $f(r, k, b)$ denote the minimum number of edges in an $r$-uniform $b$-simple hypergraph that is not $k$-colorable. It is known (see, e.g., [3, 10]) that $f(k, r) \leq k^r r^2 \ln k$. Erdős and Lovász [3] proved that $f(k, r, 1)$ is much larger:

$$\frac{k^{2(r-2)}}{16r(r-1)^2} \leq f(r, k, 1) \leq 1600r^4 k^{2(r+1)}. \tag{3}$$

The lower bound in (3) is obtained from Theorem 1. The same idea with the help of Theorem 2 gives a better lower bound on $f(r, k, 1)$ for fixed $k$ and $\epsilon$ and large $r$: if $r \geq r(k, \epsilon)$, then

$$f(r, k, 1) \geq \frac{k^{2r}}{r^{1+\epsilon}}. \tag{4}$$

Recently, Kostochka and Kumbhat [6] improved Szabó's bound by a factor of $r$ and generalized the bound to $b$-simple graphs as follows.

**Theorem 4** ([6]). *Let $k \geq 2$, $b \geq 1$ and $\epsilon > 0$ be fixed. There exists $r_0 = r_0(k, b, \epsilon)$ such that for every $r \geq r_0$,*

$$f(r, k, b) \geq \frac{k^{r(1+1/b)}}{r^{\epsilon}}. \tag{5}$$

They also gave the following upper bound on $f(r, k, b)$: for large $r$,

$$f(r, k, b) \leq 40k^2(k^r r^2)^{1+1/b}. \tag{6}$$

Our second construction allows us to improve the upper bounds on $f(r, k, b)$ in (3) and (6) so that the ratio of the new bounds to the lower bounds (5) for fixed $k$ and $b$ is of order $r^{1+\epsilon+1/b}$.

**Theorem 5.** *Let $k \geq 2$ and $b \geq 1$ be integers. There is $c = c(k, b)$ such that for every sufficiently large $r$, there are $b$-simple non-$k$-colorable $r$-uniform hypergraphs with at most $c \cdot (r \ln k)^{1+1/b} k^{r+r/b}$ edges.*

In the next section, we describe the idea of our construction. In Section 3, we prove Theorem 5 modulo somewhat technical Lemma 6. In Section 4, we discuss constructions of hypergraphs with low maximum degrees; in particular, we prove Theorem 3. In the last section, we give a (rather standard) proof of Lemma 6 for Section 3.

## 2. TEMPLATE OF THE CONSTRUCTION

For a hypergraph $G$, let *S-property* be either the property "to have girth at least $g$" or the property "to be $b$-simple" for some $b < r$. We want to construct $r$-uniform non-$k$-colorable hypergraphs with a given $S$-property either with few edges or with low maximum degree.

Let $0 < \alpha < 1$. We will say that an $r$-uniform hypergraph $H_1$ *satisfies $\alpha$-Condition* if the following holds: Let $R$ be a set with $|R| := |V(H_1)|$ and let $f$ be any fixed $k$-coloring of $R$. Then

$$\text{the probability that for a random placement of } H_1 \text{ onto } R,$$
$$f \text{ is not a proper coloring of } H_1 \text{ is at least } \alpha. \tag{7}$$

The construction goes as follows.

- Consider an $r$-uniform hypergraph $H_1 = H_1(k, r, S)$ with $S$-property. Let $r_1 = |V(H_1)|$. For a fixed $r_1$, we want $H_1$ to satisfy $\alpha$-Condition with $\alpha$ as large as possible.
- Another part of the construction is an $r_1$-uniform hypergraph $H_2$ with the $S$-property that has as large average degree as possible.
- Now we let $\mathbf{G}$ be the random $r$-uniform hypergraph obtained from $H_2$ by replacing each edge with a randomly placed copy of $H_1$. For each such edge of $H_2$, every of the possible $r_1!$ placements of a copy of $H_1$ has the same probability $1/r_1!$, and for different edges of $H_2$ the placements are independent.

As both $H_1$ and $H_2$ possess the $S$-property, every value of the random variable $\mathbf{G}$ also has this property. Also, since by the definition of $S$-property any two edges of $H_2$ share less than $r$ vertices, $|E(\mathbf{G})| = |E(H_1)||E(H_2)|$. By the independence of the placements of distinct copies of $H_1$ and by Condition (7), for every $k$-coloring $f$ of $V(H_2)$, the probability $P(f)$ that $f$ is a proper coloring for $\mathbf{G}$ is at most $(1-\alpha)^{|E(H_2)|}$.

Let $n = |V(H_2)|$. Since there are only $k^n$ different $k$-colorings of $V(H_2)$, if

$$k^n(1-\alpha)^{|E(H_2)|} < 1, \tag{8}$$

then with positive probability, $\mathbf{G}$ is not $k$-colorable.

So, the proofs below provide that (A) $H_1$ and $H_2$ have the desired $S$-property, (B) $H_1$ satisfies (7) (the larger is $\alpha$, the better), (C) (8) holds, and (D) either the number of edges of $\mathbf{G}$ or its maximum degree is “small”.

## 3. HYPERGRAPHS WITH FEW EDGES

In this section, we prove Theorem 5.

Let $r$ be large in comparison with $k$ and $b$. Let $q = q(r)$ be the smallest prime that is larger than $r$. It is known that $q = r + o(r)$. Let $H_1 = H_1(r, b)$ be the $r$-uniform hypergraph with $rq$ vertices defined as follows. (We use a construction from Kuzjurin's paper [9].) The vertex set of $H_1$ is $S = S_1 \cup \ldots \cup S_r$ where all $S_i$ are disjoint copies of $GF(q) = \{0, 1, \ldots, q-1\}$. The edges of $H_1(r, b)$ are $r$-tuples $(x_1, \ldots, x_r) \in S_1 \times \cdots \times S_r$ that are solutions of the system of linear equations

$$\sum_{i=1}^{r} i^j x_i = 0, \qquad j = 0, 1, \ldots, r-b-2 \tag{9}$$

over $GF(q)$.

For any arbitrarily fixed $b+1$ variables in (9), we have a square system of linear equations with Vandermond's determinant which has a unique solution over $GF(q)$. This means:

(a1) $|E(H_1(r,b))| = q^{b+1}$; and
(b1) no two distinct edges can share more than $b$ vertices; i.e., $H_1(r,b)$ is $b$-simple.

The next lemma says that $H_1$ satisfies $\alpha$-Condition with $\alpha = 0.5^b e^{-k} q^{b+1}/k^r$.

**Lemma 6.** *Let $k \geq 2$ and $b \geq 1$ be integers and let $r$ be large in comparison with $k$ and $b$ and $q = q(r)$ be the smallest prime that is larger than $r$. Let $f$ be any $k$-coloring of a set $V$ with $|V| = rq$. If we randomly place a copy of $H_1$ on $V$, then the probability that $f$ is a proper coloring of $H_1$ is at most $1-\alpha$, where $\alpha = 0.5^b e^{-k} q^{b+1}/k^r$.*

The proof of the lemma is intuitively clear and is a standard application of the second moment method. So, we postpone the proof to the last section.

Let $p$ be a prime such that

$$2e^k k^{r/b}(r\ln k)^{1/b}/q < p \leq 4e^k k^{r/b}(r\ln k)^{1/b}/q. \tag{10}$$

Similarly to $H_1$, define an $rq$-uniform hypergraph $H_2 = H_2(r,b,p)$ as follows. The vertex set of $H_2$ is $T = T_1 \cup \ldots \cup T_{rq}$, where all $T_i$ are disjoint copies of $GF(p) = \{0,1,\ldots,p-1\}$. The edges of $H_2(r,b,p)$ are $rq$-tuples $(x_1,\ldots,x_{rq}) \in T_1 \times \cdots \times T_{rq}$ that are solutions of the system of linear equations

$$\sum_{i=1}^{rq} i^j x_i = 0, \qquad j = 0,1,\ldots,rq-b-2 \tag{11}$$

over $GF(p)$.

Similarly to $H_1$, we have

(a2) $|E(H_2(r,b,p))| = p^{b+1}$; and
(b2) no two distinct edges can share more than $b$ vertices; i.e., $H_2(r,b,p)$ is $b$-simple.

As in the previous section, let $\mathbf{G}$ be the random $r$-uniform hypergraph obtained from $H_2$ by replacing each edge with a random copy of $H_1$.

As $H_1$ and $H_2$ are $b$-simple, every value of $\mathbf{G}$ is a $b$-simple $r$-uniform hypergraph. Note that by (10) always

$$|E(\mathbf{G})| = q^{b+1}|E(H_2)| = (qp)^{b+1} \leq (4e^k k^{r/b}(r\ln k)^{1/b})^{b+1} \leq (4e^k)^{b+1}(r\ln k)^{\frac{1+b}{b}} k^{\frac{r(1+b)}{b}}.$$

Thus, $\mathbf{G}$ satisfies (A) and (D) stated at the end of the previous section. By Lemma 6, (B) is also satisfied for $\alpha = 0.5^b q^{b+1} e^{-k}/k^r$.

Therefore, we only need to verify (8). We prove the slightly stronger inequality $n\ln k < \alpha|E(H_2)|$ which in our case is

$$0.5^b (qp)^{b+1}/e^k k^r > |V(\mathbf{G})|\ln k = rqp\ \ln k.$$

This is equivalent to $qp > 2(re^k k^r \ln k)^{1/b}$ which holds by (10).

## 4. HYPERGRAPHS WITH LOW MAXIMUM DEGREE

Let $f$ be a $k$-coloring of a set $M$ with $|M| = mr$. Let $\phi(f, M)$ be the probability that a randomly chosen $r$-element subset of $M$ is not monochromatic and let $\phi(k, r, m) = \max\{\phi(f, M)\}$, where the maximum is taken over all $k$-colorings $f$ of $M$. Similarly, let $\psi(f, M)$ be the probability that, in a random placement of $m$ disjoint $r$-element hyperedges onto $M$, none of the edges is monochromatic, and let $\psi(k, r, m) = \max\{\psi(f, M)\}$, where the maximum is taken over all $k$-colorings $f$ of $M$.

Suppose that for $i = 1, \dots, k$, $x_i$ vertices of $M$ have color $i$ in $f$. Then by definition,

$$\phi(f, M) = 1 - \sum_{i=1}^{k} \frac{\binom{x_i}{r} r!((m-1)r)!}{(mr)!} = 1 - \sum_{i=1}^{k} \binom{x_i}{r} \binom{mr}{r}^{-1}. \tag{12}$$

As $x_1 + \cdots + x_k = mr$,

$$\phi(k, r, m) \le 1 - k\binom{mr/k}{r}\binom{mr}{r}^{-1} \le 1 - \begin{cases} \left(\frac{1-k/m}{k}\right)^{r-1}, & \text{if } k < m; \\ 0, & \text{if } k \ge m. \end{cases} \tag{13}$$

To give a bound for $\psi(k, r, m)$, we need the following lemma.

**Lemma 7.** *Let $k \ge 2$, $r \ge 2$ and $m$ be positive integers. Then*

$$\psi(k, r, m) \le \prod_{i=1}^{m} \phi(k, r, i). \tag{14}$$

*Proof.* We use induction on $m$ for fixed $k$ and $r$. For $m = 1$, the statement is evident. Suppose that (14) holds for all $m' < m$. Let $f$ be any $k$-coloring of an $mr$-element set $M$. Let $H_1$ be the $r$-uniform hypergraph on $mr$ vertices comprising a matching of $m$ edges. Let $A$ be an edge in $H_1$. Consider random placements of $V(H_1)$ onto $M$. Let $P_1(A,f)$ be the probability that $A$ will be monochromatic in $f$, and let $P_2(A,f)$ be the conditional probability that some edge of $H_1$ distinct from $A$ will be monochromatic in $f$ under condition that $A$ is not monochromatic. As $P_1(A,f) = 1 - \phi(f, M)$,

$$1 - \psi(f, M) = P_1(A,f) + (1 - P_1(A,f))P_2(A,f) = 1 - \phi(f, M) + \phi(f, M)P_2(A,f).$$

By definition, $P_2(A,f) \ge 1 - \psi(k, r, m-1)$ and $\phi(f, M) \le \phi(k, r, m)$. Hence,

$$1 - \psi(f, M) \ge 1 - \phi(k, r, m)\psi(k, r, m-1).$$

As $f$ is arbitrary, we get $\psi(k, r, m) \le \phi(k, r, m)\psi(k, r, m-1)$. Applying the induction hypothesis, we are done. ∎

By the lemma and (13), for every $m \ge k + 1$,

$$\psi(k, r, m) \le \prod_{i=k+1}^{m} \left(1 - \left(\frac{1 - k/i}{k}\right)^{r-1}\right). \tag{15}$$

**Corollary 8.** *Let $k \geq 2$ and $r \geq 2$ be integers. For every $\epsilon > 0$, there exists an $m_0 = m_0(k, r, \epsilon)$ such that for every $m \geq m_0$,*

$$\psi(k,r,m) \leq \left(1 - \frac{1-\epsilon/2}{k^{r-1}}\right)^{m(1-\epsilon/3)}. \tag{16}$$

*In particular, for such m,*

$$\psi(k,r,m) \leq \exp\{-m(1-\epsilon)/k^{r-1}\}. \tag{17}$$

*Proof.* We may assume that $\epsilon < 0.1$. Choose $m_0$ so that $(1 - k/3\epsilon m_0)^{r-1} > 1 - \epsilon/3$. For any $m \geq m_0$, let $i_0 = \lceil 3\epsilon m \rceil$. By definition, $i_0 > k$. Then by (15) and the choice of $m_0$ and $i_0$,

$$\psi(k,r,m) \leq \prod_{i=i_0}^{m} \left(1 - \left(\frac{1-k/i}{k}\right)^{r-1}\right) \leq \left(1 - \left(\frac{1-k/i_0}{k}\right)^{r-1}\right)^{m-i_0+1}$$

$$\leq \left(1 - \frac{1-\epsilon/2}{k^{r-1}}\right)^{m(1-\epsilon/3)} \leq \exp\left\{-\frac{1-\epsilon/2}{k^{r-1}} m(1-\epsilon/3)\right\} < \exp\{-m(1-\epsilon)/k^{r-1}\}.$$

■

Now we are ready to prove Theorem 3.

Let $d$ satisfy (2). As (2) holds, there exists an $\epsilon > 0$ so small that

$$\left(1 - \frac{1-\epsilon/2}{k^{r-1}}\right)^{d(1-\epsilon/3)/r} < 1/k. \tag{18}$$

Let $m_0$ be the number guaranteed by Corollary 8 for this $\epsilon$.

It is known that for every integers $d, g, R \geq 2$, there exists an $R$-uniform $d$-regular hypergraph with girth at least $g$. Some constructions of such hypergraphs are given by Sauer[1] [11] and (in the language of biregular bipartite graphs of girth $2g$) by Füredi et al.[2] [4]. Thus, there exists an $m_0 r$-uniform $d$-regular hypergraph $H_2 = H_2(m_0 r, d, g)$ of girth at least $g$. Let $V = V(H_2)$ and $n = |V|$. By construction, $|E(H_2)| = dn/(m_0 r)$. As suggested in Section 2, we form the random $r$-uniform $d$-regular hypergraph $\mathbf{G}$ by replacing each edge (of size $m_0 r$) of $H_2$ with a random matching of $m_0$ edges of size $r$. Replacement of each edge is independent of all other replacements. By construction, each value of the random variable $\mathbf{G}$ is $d$-regular and has no cycles shorter than $g$. It remains to verify (8). In our case, the left-hand side of (8) is at most

$$k^n \left(1 - \frac{1-\epsilon/2}{k^{r-1}}\right)^{m_0(1-\epsilon/3)dn/m_0 r} < k^n \left[\left(1 - \frac{1-\epsilon/2}{k^{r-1}}\right)^{d(1-\epsilon/3)/r}\right]^n.$$

By (18), the last expression is less than 1. This finishes the proof. ■

[1]Formally, Sauer does not consider cycles of length 2 as cycles. But his construction produces appropriate hypergraphs without cycles of length 2.

[2]They construct bipartite graph of girth at least $2g$ in which all vertices in one partite set have degree $R$, and in the other partite set have degree $d$. But such graphs are exactly the incidence graphs of $R$-uniform $d$-regular hypergraphs with girth at least $g$.

## 5. PROOF OF LEMMA 6

Suppose that for $i = 1, \dots, k$, $x_i$ vertices of $V$ have color $i$ in $f$. Our sample space consists of $(qr)!$ equiprobable placements of $V(H_1)$ onto $V$. For an edge $A$ of $H_1$, let $Y(A)$ be the event that $A$ is monochromatic. Then for any $A$,

$$\Pr[Y(A)] = \sum_{i=1}^{k} \frac{\binom{x_i}{r} r!(rq-r)!}{(rq)!} = \sum_{i=1}^{k} \binom{x_i}{r}\binom{rq}{r}^{-1}. \tag{19}$$

If two edges $A_1$ and $A_2$ share $y > 0$ vertices, then similarly

$$\Pr[Y(A_1) \cap Y(A_2)] = \sum_{i=1}^{k} \binom{x_i}{2r-y}\binom{rq}{2r-y}^{-1}. \tag{20}$$

If $A_1$ and $A_2$ are disjoint, then

$$\Pr[Y(A_1) \cap Y(A_2)] = \binom{rq}{2r}^{-1}\left[\sum_{i=1}^{k}\binom{x_i}{2r} + 2\binom{2r}{r}^{-1}\sum_{1\le i<j\le k}\binom{x_i}{r}\binom{x_j}{r}\right]. \tag{21}$$

*Case 1.* $x_1 \geq r + 1.5r^2/k$. Let $A$ be any edge of $H_1$. Since $r$ is large, by (19),

$$\Pr[Y(A)] \geq \binom{r+\frac{3r^2}{2k}}{r}\binom{rq}{r}^{-1} = \frac{\left(r+\frac{3r^2}{2k}\right)\left(r-1+\frac{3r^2}{2k}\right)\cdot\dots\cdot\left(1+\frac{3r^2}{2k}\right)}{(rq)(rq-1)\cdot\dots\cdot(rq-r+1)}$$
$$\geq \left(\frac{\frac{3r^2}{2k}}{rq}\right)^{r} \geq \left(\frac{3r}{2kq}\right)^{r} > \frac{(r+1)^{1+b}}{k^r}.$$

Since $q = r + o(r)$ and $r$ is large, the last expression is at most $(q/2)^{b+1}k^{-r}$. Let $Y$ be the event that at least one edge becomes monochromatic in $f$. Since $Y = \cup_A Y(A)$, we have $\Pr[Y] \geq \Pr[Y(A)]$. It follows that $1 - P[Y] \leq 1 - (q/2)^{b+1}k^{-r} < 1 - q^{b+1}2^{-b}e^{-k}k^{-r}$, i.e., the lemma holds.

*Case 2.* For every $1 \leq i \leq k$, $x_i \leq r - 1 + 1.5r^2/k$. We will use the simple second moment inequality:

$$\Pr[Y] \geq \sum_{A\in E(H_1)} \Pr[Y(A)] - \sum_{A_1\in E(H_1)}\ \sum_{A_2\in E(H_1), A_2\neq A_1} \Pr[Y(A_1)\cap Y(A_2)]. \tag{22}$$

To this end we will need to show that $\Pr[Y(A_1) \cap Y(A_2)]$ is small in comparison with $\Pr[Y(A_1)]$ for all possible $A_1$ and $A_2$.

In our case, the ratio of every summand in (20) to the corresponding summand in (19) is

$$\binom{x_i}{2r-y}\binom{rq}{2r-y}^{-1}\binom{x_i}{r}^{-1}\binom{rq}{r} = \frac{(x_i-r)\cdot\dots\cdot(x_i-2r+1+y)}{(rq-r)\cdot\dots\cdot(rq-2r+y+1)}$$
$$\leq \left(\frac{x_i-r}{rq-r}\right)^{r-y} \leq \left(\frac{1.5r^2/k}{rq-r}\right)^{r-y} \leq \left(\frac{3}{2k}\right)^{r-b}. \tag{23}$$

It follows that for any two edges $A_1$ and $A_2$ sharing at least one vertex,

$$\Pr[Y(A_1) \cap Y(A_2)] \le \left(\frac{3}{2k}\right)^{r-b} \Pr[Y(A_1)].$$

Similarly, we will show that

$$\Pr[Y(A_1) \cap Y(A_2)] \le 2e^{2k} \left(\frac{3}{2k}\right)^{r} \Pr[Y(A_1)] \tag{24}$$

for any disjoint $A_1$ and $A_2$. For this, we compare the ratios of every summand in (21) to the corresponding summand in (19) and show that

$$2e^{2k}\binom{x_i}{r}\binom{rq}{r}^{-1}\binom{rq}{2r} \ge \left(\frac{2k}{3}\right)^r \left(\binom{x_i}{2r} + k\binom{2r}{r}^{-1}\binom{x_i}{r}\binom{r-1+\frac{3r^2}{2k}}{r}\right), \tag{25}$$

which will yield (24). The proof of the fact that $\binom{x_i}{r}\binom{rq}{r}^{-1}\binom{rq}{2r} \ge (\frac{2k}{3})^r\binom{x_i}{2r}$ is the same as that of (23).

Suppose that

$$e^{2k}\binom{x_i}{r}\binom{rq}{r}^{-1}\binom{rq}{2r} < k\left(\frac{2k}{3}\right)^r\binom{2r}{r}^{-1}\binom{x_i}{r}\binom{r-1+\frac{3r^2}{2k}}{r}.$$

Cancelling several factors yields

$$e^{2k}\frac{(rq-r)!}{(rq-2r)!} < k\left(\frac{2k}{3}\right)^r \frac{\left(r-1+\frac{3r^2}{2k}\right)!}{\left(-1+\frac{3r^2}{2k}\right)!}.$$

As the left-hand side of this for fixed $r$ and $k$ grows with $q$ when $q \ge r$, we have

$$e^{2k}\frac{(r^2-r)!}{(r^2-2r)!} < k\left(\frac{2k}{3}\right)^r \frac{\left(r-1+\frac{3r^2}{2k}\right)!}{\left(-1+\frac{3r^2}{2k}\right)!}.$$

Dividing both sides by $\frac{(r^2-r)!}{(r^2-2r)!}$ and cancelling some factors from the right-hand side, we obtain

$$e^{2k} < k\left(\frac{2k}{3}\right)^r \prod_{i=1}^{r}\frac{i-1+\frac{3r^2}{2k}}{r^2-2r+i} \le k\left(\frac{2k}{3}\right)^r\left(\frac{r-1+\frac{3r^2}{2k}}{r^2-r}\right)^r < k\left(\frac{\frac{2k}{3r}+r}{r-1}\right)^r. \tag{26}$$

As under the conditions of the lemma $r >> k$,

$$\left(\frac{\frac{2k}{3r}+r}{r-1}\right)^r \le \left(\frac{1/2+r}{r-1}\right)^r = \left(1+\frac{3/2}{r-1}\right)^r < e^{1.5r/(r-1)} < 9.$$

Thus, in (26) we have $e^{2k} < 9k$ which is not true for $k \ge 2$. This proves (24).

So, for any distinct edges $A_1$ and $A_2$ in $H_1$,

$$\Pr[Y(A_1) \cap Y(A_2)] \leq \left(\frac{3}{2k}\right)^r \Pr[Y(A_1)] \max\left\{2e^{2k}, \left(\frac{2k}{3}\right)^b\right\}. \tag{27}$$

Let $\lambda(k,b) = \max\{2e^{2k}, (\frac{2k}{3})^b\}$. We plug it into (22).

By (22), (27) and the fact that $|E(H_1)| = q^{b+1}$, we have

$$\Pr[Y] \geq \sum_{A \in E(H_1)} \Pr[Y(A)] \left(1 - q^{b+1}\lambda(k,b)\left(\frac{3}{2k}\right)^r\right).$$

As $r$ is large in comparison with $k$ and $b$ and $r < q = r + o(r)$, we have $q^{b+1}\lambda(k,b)(\frac{3}{2k})^r < 1/2^b$. So,

$$\Pr[Y] \geq |E(H_1)| \Pr[Y(A)](1 - 2^{-b}) \geq 0.5q^{b+1} \Pr[Y(A)].$$

Recall that $\Pr[Y(A)] = \sum_{i=1}^k \binom{x_i}{r}\binom{rq}{r}^{-1}$. The minimum of $\sum_{i=1}^k \binom{x_i}{r}$ under condition $x_1 + \cdots + x_k = rq$ is attained when $x_1 = \cdots = x_k = rq/k$. Therefore,

$$\Pr[Y(A)] \geq k\binom{\frac{rq}{k}}{r}\binom{rq}{r}^{-1} \geq k\left(\frac{rq}{k} - r + 1\right)^r (rq)^{-r} \geq k^{1-r}\left(1 - \frac{k(r-1)}{rq}\right)^r > \frac{k^{1-r}}{e^k}.$$

It follows that $\Pr[Y] \geq 0.5q^{b+1}k^{1-r}e^{-k} \geq 0.5^b e^{-k} q^{b+1}/k^r$, which proves the lemma.

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## REFERENCES

[1] N. Alon, Hypergraphs with high chromatic number, Graphs Combin 1 (1985), 387–389.

[2] B. Bollobás, Random graphs, Academic Press, London, 1984.

[3] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, Infinite and finite sets, Colloquia Math Soc János Bolyai 10 (1973), 609–627.

[4] Z. Füredi, F. Lazebnik, Á Seress, V. A. Ustimenko, and A. J. Woldar, Graphs of prescribed girth and bi–degree, J Combin Theory Ser B 64 (1995), 228–239.

[5] J. H. Kim, On Brooks' theorem for sparse graphs, Comb Probab Comput 4 (1995), 97–132.

[6] A. Kostochka and M. Kumbhat, Coloring simple uniform hypergraphs with few edges, Random Struct Algorithms 35 (2009), 348–368.

[7] A. V. Kostochka and N. P. Mazurova, An inequality in the theory of graph coloring, Met Diskret Analiz 30 (1977), 23–29 (in Russian).

[8] A. Kostochka, D. Mubayi, V. Rödl, and P. Tetali, On the chromatic number of set systems, Random Struct Algorithms 19 (2001), 87–98.

[9] N. N. Kuzjurin, On the difference between asymptotically good packings and coverings, European J Combin 16 (1995), 35–40.

[10] J. Radhakrishnan and A. Srinivasan, Improved bounds and algorithms for hypergraph two-coloring, Random Struct Algorithms 16 (2000), 4–32.

[11] N. Sauer, On the existence of regular *n*-graphs with given girth, J Combin Theory 9 (1970), 144–147.

[12] Z. Szabó, An application of Lovász' local lemma — A new lower bound for the van der Waerden number, Random Struct Algorithms 1 (1990), 343–360.

[13] V. A. Tashkinov, A lower bound for the chromatic number of graphs with a given maximal degree and girth, Sib Elektron Mat Izv 1 (2004), 99–109 (in Russian).