S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 6, стр. 465–504 (2009) УДК 519.172.2
MSC 05C69, 05C40, 05C35

A NEW BOUND ON THE DOMINATION NUMBER OF CONNECTED CUBIC GRAPHS

A. V. KOSTOCHKA, C. STOCKER

Abstract. In 1996, Reed proved that the domination number, $\gamma(G)$, of every n-vertex graph G with minimum degree at least 3 is at most $3n/8$. This bound is sharp for cubic graphs if there is no restriction on connectivity. In this paper, improving an upper bound by Kostochka and Stodolsky we show that for $n > 8$ the domination number of every n-vertex cubic connected graph is at most $\lfloor 5n/14 \rfloor$. This bound is sharp for even $8 < n \leq 18$.

Keywords and phrases: cubic graphs, domination, connected graphs.

1. Introduction

A set A of vertices in a graph G *dominates* itself and the vertices at distance one from it. If a set A dominates all vertices of G, then it is called *dominating in* G. The *domination number*, $\gamma(G)$, of a graph G is the minimum size of a dominating set in G.

Naturally, graphs G with high minimum degree, $\delta(G)$, have small domination number. Ore [8] proved that $\gamma(G) \leq n/2$ for every n-vertex graph without isolated vertices (i.e., with $\delta(G) \geq 1$). Blank [1] proved that $\gamma(G) \leq 2n/5$ for every n-vertex graph with $\delta(G) \geq 2$ if $n \geq 8$. Reed [10] proved that $\gamma(G) \leq 3n/8$ for every n-vertex graph with $\delta(G) \geq 3$. All these bounds are sharp. Reed [10] conjectured that the domination number of each connected 3-regular (cubic) n-vertex graph is at most $\lceil n/3 \rceil$. Kostochka and Stodolsky [5] disproved this conjecture. They proved:

Theorem 1. [5] *There is a sequence* $\{G_k\}_{k=1}^{\infty}$ *of cubic connected graphs such that for every* k, $|V(G_k)| = 46k$ *and* $\gamma(G_k) \geq 16k$, *and thus* $\frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{8}{23} = \frac{1}{3} + \frac{1}{69}$.

A. V. Kostochka, C. Stocker, A new bound on the domination number of connected cubic graphs.

First author is supported in part by NSF grant DMS-0650784 and grants 05-01-00816 and 06-01-00694 of the Russian Foundation for Basic Research.

Received January, 6, 2009, published November, 24, 2009.

Kelmans [4] elaborated new examples giving stronger lower bounds on the domination number of cubic connected graphs:

Theorem 2. [4] *There is a sequence $\{G_k\}_{k=1}^{\infty}$ of cubic 2-connected graphs such that for every k, $|V(G_k)| = 60k$ and $\gamma(G_k) \geq 21k$, and thus $\frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{60}$.*

He also provided an example of a 54-vertex connected cubic graph L with $\gamma(L) = 19 = \left(\frac{1}{3} + \frac{1}{54}\right)|V(L)|$.

On the other hand, Kostochka and Stodolsky [6] improved Reed's upper bound of $3n/8$ for connected cubic graphs:

Theorem 3. [6] *Let $n > 8$. If G is a connected cubic n-vertex graph, then*

$$\gamma(G) \leq \frac{4n}{11} = \left(\frac{1}{3} + \frac{1}{33}\right) n.$$

The aim of this paper is to improve the bound of Theorem 3. Our main result is:

Theorem 4. *Let $n > 8$. If G is a connected cubic n-vertex graph, then*

$$\gamma(G) \leq \frac{5n}{14} = \left(\frac{1}{3} + \frac{1}{42}\right) n.$$

The bound $\left\lfloor \frac{5n}{14} \right\rfloor$ is sharp for $8 < n \leq 18$. For example, a 3-connected cubic 14-vertex hamiltonian graph G with $\gamma(G) = 5$ is presented in [2].

Our proofs exploit the ideas and techniques of Reed's seminal paper [10] and of [6]. We modify and elaborate the technique of [6] substantially. In the next section, we describe the setup of the Reed's paper [10] with some small changes and the procedure of constructing a dominating set. In the same section we state the basic lemmas that we will prove later. In Section 3, we describe a discharging that proves the bound modulo basic lemmas. In the next three sections we prove the basic lemmas.

2. The setup

We use standard notation. In particular, for a vertex v in a graph G, $N(v)$ denotes the set of neighbors of v.

We elaborate and extend the proof in [6]. A *vdp-cover* of a graph G is a covering of $V(G)$ by vertex-disjoint paths. The *order*, $|P|$, of a path P is the number of its vertices. For $i \in \{0, 1, 2\}$, a path P is an *i-path*, if $|P| \equiv i \pmod 3$. If P is a path, $x \in V(P)$ and $P - x$ consists of an i-path and a j-path, then x is called an *(i, j)-vertex of P*.

Let G be a connected cubic graph and S be a vdp-cover of G. An endpoint x of a path $P \in S$ is an *out-endpoint* if x has a neighbor outside of P. An endpoint x of a 2-path $P \in S$ is a *$(2,2)$-endpoint* if x is not an out-endpoint and is adjacent to a $(2,2)$-vertex of P. By S_i we denote the set of i-paths in S.

A vdp-cover S of G is *optimal* if

(R1) $2|S_1| + |S_2|$ is minimized;
(R2) Subject to (R1), $|S_2|$ is minimized;
(R3) Subject to (R1) and (R2), $\sum_{P \in S_0} |P|$ is minimized;
(R4) Subject to (R1)–(R3), $\sum_{P \in S_1} |P|$ is minimized;
(R5) Subject to (R1)–(R4), the total number of out-endpoints of all paths in S is maximized;
(R6) Subject to (R1)–(R5), the total number of $(2,2)$-endpoints of all 2-paths in S is maximized.

It turns out that optimal vdp-covers possess several useful properties. The next lemma is Lemma 1 in [6].

Lemma 1. *Suppose that an out-endpoint x of a 1-path or a 2-path P_i in an optimal vdp-cover S is adjacent to a vertex $y \in P_j$, where $j \neq i$. Let $P_j = P_j' y P_j''$. Then*
(B1) P_j is not a 1-path;
(B2) If P_j is a 0-path, then both P_j' and P_j'' are 1-paths;
(B3) If P_j is a 2-path, then both P_j' and P_j'' are 2-paths;
(B4) If P_j is a 2-path and z is the common endpoint of P_j and P_j', then each neighbor of z on P_j'' should be a $(2,2)$-vertex.

Properties (B3), (R1), (R2) and (R3) yield the following fact.

Lemma 2. *If a path $(v_1, \ldots, v_5)$ in an optimal vdp-cover S has chord v_1v_4 (see Fig. 1a), or chord v_1v_5, then none of its vertices is adjacent to an end vertex of another path in S.*

We also will use the following result.

Theorem 5. [2] *If G is a hamiltonian cubic $(3k+1)$-vertex graph, then $\gamma(G) \leq k$.*

A path P in a vdp-cover S is a *special path of type 1* (respectively, *of type 2*), if P has 35 vertices (respectively, 38 vertices) and none of the hamiltonian paths on $V(P)$ has an out-endpoint or a $(2,2)$-endpoint. A *special vertex* in a special path P is a vertex at distance 17 in P from some of its end. By definition, each special path of type 1 has exactly one special vertex (its center), and each special path of type 2 has two special vertices (at distance 3 from each other). A special path P in a vdp-cover S will be called *very special* if there exists a path P_1 in S whose end-vertex is adjacent to the special vertices of at least two special paths one of which is P. The other special paths in the definition of a very special path are, by definition, also very special.

Now we essentially repeat construction in [6] of a dominating set with some modifications. Let S be an optimal vdp-cover.

(C1) If a 1-path $P \in S$ has no dominating set of size at most $(|P|-1)/3$, but has an out-endpoint, choose a vertex $y \notin V(P)$ which is a neighbor of an out-endpoint $x(P)$ of P. Call this $y \notin V(P)$ *an acceptor for P*. If $x(P)$ or the other endvertex of P has an outneighbor that is not a special vertex of a special path, then let the acceptor of P be not the special vertex of a special path. Furthermore, if there is a choice between special paths of type 1 and type 2, then we choose the acceptor in a path of type 2. In particular, if $|P| = 4$ and $G[V(P)]$ is a 4-cycle, then we choose, if possible, an outneighbor of $V(P)$ that is not a special vertex of a special path.

(C2) Say that a path $P \in S$ with $|P| = 5$ forms a *δ-subgraph*, if for some hamiltonian path on P, the center vertex, x' is adjacent to an endpoint of the path (see Fig 1.b) and the other end of P has an outneighbor. For each P forming a δ-subgraph, choose an outneighbor of x and call it an *acceptor for P*. If $G[V(P)]$ is the 5-cycle, then choose as *acceptors* the outneighbors of two adjacent vertices of $G[V(P)]$. If $G[V(P)]$ is $K_{2,3}$, then choose as *acceptors* the outneighbors of two vertices of degree two in $G[V(P)]$. In all cases, if there is a choice, we try to minimize the number of acceptors that are special vertices of special paths.

(C3) Let $P \in S$ be a 2-path not described in (C2). If either P has two out-endpoints, or $|P| \leq 11$ and P has one out-endpoint, then for each of the out-endpoints of P, choose a neighbor outside P and designate it as an acceptor corresponding to that endpoint. If possible, choose the acceptors that are not special vertices of special paths.

Call a path *accepting* if at least one of its vertices was designated as an acceptor.

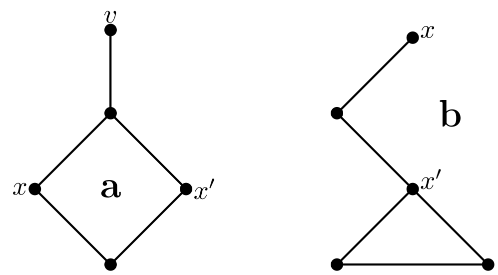

FIGURE 1

(C4) Construct a family $A \subseteq S$ of 2-paths as follows. Initially, let A be the set of accepting 2-paths in S. While there is any out-endpoint x of a path in A for which we have not already chosen an acceptor (because the path has only one out-endpoint), choose a neighbor y of x in $G - P$ and designate it as an acceptor for x. Moreover, if we can choose an acceptor that is not a special vertex of a special path, we do not choose a special vertex. If we have choice between special vertices of special paths of type 1 and type 2, then we choose the vertex in a special path of type 2. If y is on a previously non-accepting 2-path P', then add P' to A. Continue this process until there is an acceptor for every out-endpoint in A. In addition, for each $(2,2)$-endpoint x of each path P in A, designate a $(2,2)$-vertex y adjacent to x as an in-acceptor for x.

(C5) When we finish the procedure above, we look at special paths again. If a special vertex y of a special path $P \in S$ was designated as the acceptor for a path P_1 with an endvertex x_1 adjacent to y and some other vertex of P also is an acceptor, then we leave the situation as it is. If y is the only acceptor in P and x_1 has an outneighbor y' in a path that has other acceptors, then we redesignate the y' as the acceptor for x_1 (and P_1). Moreover, if P_1 is a path with 4 vertices, and $G[P_1]$ is a 4-cycle, then we choose y as an acceptor only if each other outneighbor of this 4-cycle also is a special vertex of a special path and no other vertices on all these paths are acceptors. If P_1 is a path with 5 vertices, and $G[P_1]$ is a 5-cycle or $K_{2,3}$, then we also, if possible switch to an acceptor in a path that contains another acceptor.

Each accepting 2-path $P \in S$ can be written in the form $P_1P_2P_3$, where P_1 and P_3 are both 1-paths containing no acceptors (including in-acceptors) and are maximal with this property. By (B3), the second and the penultimate vertices of P_2 are acceptors. The paths P_1 and P_3 are called *tips of* P, and P_2 is the *central path of* P. Now a dominating set D is defined as follows.

(C6) For each 0-path $P \in S$, every $(1,1)$-vertex of P is included in D.

(C7) For each accepting 2-path $P \in S$, every $(2,2)$-vertex of P that is in the central path of P is included in D.

(C8) Let $P \in S$ be a 1-path. If $G[P]$ has a dominating set D' with $|D'| \le \lfloor |P|/3 \rfloor$, then we include D' into D. If no such set exists and P has an out-endpoint, then P has an out-endpoint, say $x(P)$, adjacent to the acceptor of P. In this case, choose some $\lfloor |P|/3 \rfloor$ vertices that dominate all vertices of P except for $x(P)$, and include these $\lfloor |P|/3 \rfloor$ vertices in D.

(C9) For each non-accepting 2-path in S on 5 vertices that forms a δ-subgraph, include vertex x' from the definition of δ-subgraphs into D. If $G[V(P)]$ is $K_{2,3}$, then include into D the vertex of degree two in

$G[V(P)]$ that is not adjacent to the acceptors of P. If $G[V(P)] = C_5$, then include into D the vertex not adjacent to the two vertices adjacent with the acceptors of P.

(C10) For each other non-accepting 2-path $P \in S$ in which each of the ends is either an out-endpoint or a $(2, 2)$-endpoint, include in D all $(2, 2)$-vertices of P. Note that there are $\lfloor |P|/3 \rfloor$ of them and these $(2, 2)$-vertices dominate all vertices of P except possibly for the out-endpoints of P. If a non-accepting 2-path $P \in S$ has exactly one out-endpoint x and $|P| \leq 11$, then include into D a smallest subset of $V(P)$ that dominates $V(P) - x$.

(C11) Let $P \in S$ be a 1-path, or a non-accepting 2-path with no out-endpoints, or a non-accepting 2-path with exactly one out-endpoint and $|P| \geq 14$. Choose a smallest dominating set in $G[V(P)]$ and include it in D. Note that in any case, this set has at most $\lceil |P|/3 \rceil$ vertices.

(C12) Let P_1 be a tip of an accepting 2-path $P \in S$ and x be the common end of P and P_1. If x is an out-endpoint or a $(2, 2)$-endpoint, then include in D all $(2, 2)$-vertices of P that are in P_1. There are $\lfloor |P_1|/3 \rfloor$ of them and these $(2, 2)$-vertices dominate all vertices of P_1 except for x (which is dominated by a vertex already included in D by (C6) or (C7)). If x is neither an out-endpoint nor a $(2, 2)$-endpoint, then include in D a smallest dominating set in the subgraph of G induced by P_1. Similarly to (C11), this set has at most $\lceil |P_1|/3 \rceil$ vertices.

(C13) An *exceptional path* is a non-accepting 2-path $P \in S$ such that
(i) both ends of P are out-endpoints and P does not form a δ-subgraph,
(ii) the acceptors of both ends are vertices of 2-paths $P' = P'_1 P'_2 P'_3$ and $P'' = P''_1 P''_2 P''_3$ with no outneighbors,
(iii) $|P'_1| \geq 16$, $|P'_3| \geq 16$, $|P''_1| \geq 16$, and $|P''_3| \geq 16$,
(iv) paths P' and P'' do not contain other acceptors, $|P'_2| = |P''_2| = 3$, and
(v) according to (C12), $|D \cap V(P')| = (|P'| + 4)/3$ and $|D \cap V(P'')| = (|P''| + 4)/3$.

The paths P' and P'' in the definition of an exceptional path P are called *dependants of* P.

For every exceptional path, we replace the $\lfloor |P|/3 \rfloor$ vertices of D in P (they dominated P apart from the endpoints) by a set of size $1 + \lfloor |P|/3 \rfloor$ dominating all vertices of P, but replace the $(|P'|+4)/3 + (|P''|+4)/3$ vertices of D in $P' \cup P''$ by $(|P'| + 1)/3 + (|P''| + 1)/3$ vertices dominating $V(P' \cup P'')$.

This finishes the definition of D.

By construction (see [10, P. 283]), the set D is dominating. We will prove that $|D| \leq 5|V(G)|/14$ if $|V(G)| > 8$ and G is connected. Note that a path P (or P_1) can contribute to D more than $|P|/3$ (or $|P_1|/3$) vertices only in cases (C11), (C12) or (C13). Thus the following lemmas will be helpful (and are extensions of Lemmas 2, 3, and 4 in [6]).

Lemma 3. *If a* 1*-path* P *in an optimal vdp-cover is such that each of the hamiltonian paths in* $G[V(P)]$ *has no out-endpoints, then either some* $(|P| - 1)/3$ *vertices dominate all vertices of* P *or* P *has at least* 28 *vertices.*

Lemma 4. *If a* 2*-path* P *in an optimal vdp-cover is such that each of the hamiltonian paths in* $G[V(P)]$ *has at most one out-endpoint, then either some* $(|P| - 2)/3$ *vertices dominate all vertices of* P *apart from an out-endpoint or* P *has at least* 14 *vertices.*

Lemma 5. *Let* $P_1 = (x_1, \ldots, x_k)$ *be a tip of an accepting* 2*-path* P *in an optimal vdp-cover. Let* $X(P_1)$ *be the set of the hamiltonian paths in* $G[V(P_1)]$ *one of whose ends is* x_k*. If none of the other ends of any path* *in $X(P_1)$ is an out-endpoint of P or a $(2,2)$-endpoint, then either some $(k-1)/3$ vertices dominate $V(P_1)$, or $k \geq 16$.*

In the next section, we will use discharging in order to prove our upper bound on $|D|$ provided that Lemmas 3, 4 and 5 hold. In the subsequent sections we prove these lemmas.

3. Discharging

Consider the following discharging. Initially, every vertex in D has charge 1 and every other vertex of G has charge 0, so the total sum of charges is $|D|$. We will change the charges of vertices in such a way that
(a) the sum of charges does not decrease, and
(b) the charge of every vertex becomes at most 5/14.

The properties (a) and (b) together imply that $|D| \leq 5|V(G)|/14$. We do the discharging in several steps and at every step check that the charge of each so far involved vertex is not greater than 5/14.

Step 1: For each 0-path P, every $(1,1)$-vertex of P gives 1/3 of its charge to either of the two neighbors on P. After this step, each vertex of each 0-path P has charge 1/3.

Step 2: For each accepting 2-path P, every $(2,2)$-vertex of P that is in the central path of P gives 1/3 of its charge to either of the two neighbors on P. After this step, each vertex in the central path of each accepting 2-path P has charge 1/3.

Step 3: If P is a 1-path and $D \cap V(P)$ dominates all vertices in P, then we distribute the charges of vertices in $D \cap V(P)$ evenly among vertices in P. If $|D \cap V(P)| \leq \lfloor |P|/3 \rfloor$, then each vertex of P will have charge less than 1/3. If $|D \cap V(P)| > \lfloor |P|/3 \rfloor$, then, by (C8) and (C11), P has no out-endpoints and $|D \cap V(P)| = (|P|+2)/3$. Furthermore, by Lemma 3, $|P| \geq 28$ and hence the charge of each vertex will be at most $\frac{1}{3} + \frac{2}{3|P|} \leq \frac{1}{3} + \frac{2}{3 \cdot 28} = \frac{5}{14}$.

Step 4: If P is a 1-path and $D \cap V(P)$ does not dominate all vertices in P, then by (C8) and (C11), P has an out-endpoint, say $x(P)$, adjacent to the acceptor of P. Distribute the charges of the $\lfloor |P|/3 \rfloor$ vertices of D in $V(P)$ evenly among the vertices in $V(P) - \{x(P)\}$. After this step, the vertex $x(P)$ has charge 0 and every other vertex of P has charge 1/3.

Step 5: Let P be a non-accepting and non-exceptional 2-path that does not form a δ-subgraph and in which each of the ends is either an out-endpoint or a $(2,2)$-endpoint. Distribute the charges of the $\lfloor |P|/3 \rfloor$ vertices of D in $V(P)$ evenly among the internal vertices of P. After this step, either of the ends of P has charge 0 and every other vertex of P has charge 1/3.

Step 6: For each 2-path P on 5 vertices forming a δ-subgraph, the only vertex x' of P in D gives 1/4 to each of its neighbors. After this step, the out-endpoint x of P has charge 0 and every other vertex of P has charge 1/4.

Step 7: Let P be a non-accepting 2-path with at most one out-endpoint that does not form a δ-subgraph. Since P has at most one out-endpoint, it is not exceptional. If $|V(P)| \geq 14$ or P has no out-endpoints, then similarly to Step 3, distribute the charges of the vertices in $D \cap V(P)$ evenly among the vertices of P. In this case, if $|V(P)| < 14$, then by Lemma 4, $|D \cap V(P)| < |V(P)|/3$, and each vertex of P will have charge less than 1/3. If $|V(P)| \geq 14$, then

$$|D \cap V(P)| \leq (|V(P)|+1)/3 = (1 + 1/|V(P)|)|V(P)|/3 \leq (1 + 1/14)|V(P)|/3 = 5|V(P)|/14,$$

and, hence, each vertex of P has charge at most 5/14. Suppose now that $|V(P)| \leq 11$ and P has exactly one out-endpoint $x(P)$. Distribute the charges of the vertices in $D \cap V(P)$ evenly among the vertices of $P - x(P)$.

By (C10) and Lemma 4, $|D \cap V(P)| \leq (|V(P)| - 2)/3$, and so each vertex of $P - x(P)$ has the charge less than $1/3$, and $x(P)$ has charge 0.

Step 8: Let P be an exceptional path and P' and P'' be its dependants. By the definition of exceptional paths, P is non-accepting, and P' and P'' contain acceptors only for P. Distribute the charges of the vertices in $D \cap (V(P) \cup V(P') \cup V(P''))$ evenly among vertices in $V(P) \cup V(P') \cup V(P'')$. Recall that $|V(P) \cup V(P') \cup V(P'')| \geq 2 + 35 + 35 = 72$. By (C13),

$$|D \cap (V(P) \cup V(P') \cup V(P''))| = \frac{|V(P)| + |V(P')| + |V(P'')|}{3} + 1.$$

Hence, the charge of each vertex in $V(P) \cup V(P') \cup V(P'')$ is at most $1/3 + 1/72 = 25/72 < 5/14$.

Step 9: Let P_1 be a tip of an accepting 2-path P such that the common end, $x(P_1)$, of P and P_1 is either an out-endpoint or a $(2,2)$-endpoint of P. Distribute the charges of the $\lfloor |P_1|/3 \rfloor$ vertices of D in $V(P_1)$ evenly among the vertices of P_1 apart from $x(P_1)$. After this step, $x(P_1)$ has charge 0 and each other vertex of P_1 has charge $1/3$.

Step 10: Let P_1 be a tip of an accepting 2-path P such that the common end, $x(P_1)$, of P and P_1 is neither an out-endpoint nor a $(2,2)$-endpoint of P, and the central path of P has more than 3 vertices. Since the central path of P has more than 3 vertices, P is not a dependant of an exceptional path. Suppose that $P_1 = (x_1 \dots x_k)$, $P_2 = (y_1 \dots y_m)$, and $P_3 = (z_1 \dots z_l)$, so that $P = (x_1 \dots x_k y_1 \dots y_m z_1 \dots z_l)$. Recall that, by definition, y_2 is an acceptor for an out-endpoint y' of a path or for $y' = z_l$ if z_l is a $(2,2)$-endpoint. Recall also that so far all out-endpoints and $(2,2)$-endpoints of non-exceptional paths had charges equal to 0. If $|V(P_1)| \geq 16$, then we distribute the charges of at most $(|V(P_1)| + 2)/3$ vertices of $D \cap V(P_1)$ as follows: each vertex of P_1 gets $5/14$, then we add $1/42$ to the charge of each of y_1, y_2 and y_3 and give $3/14$ to the vertex y' whose acceptor is y_2. The total charge that the vertices of $P_1 \cup \{y_1, y_2, y_3, y'\}$ get at this step is $5|P_1|/14 + 3/42 + 3/14$ which is at least $(|V(P_1)| + 2)/3$ when $|P_1| \geq 16$. Each of y_1, y_2 and y_3 had charge $1/3$ after Step 2 and for each of them the charge changed to $5/14$. Note that, since $m > 3$, the vertices y_1, y_2, y_3, and y' will not get any charge from the tip P_3.

If $|V(P_1)| < 16$, then since $x(P_1)$ is not an out-endpoint, by Lemma 5, $|D \cap V(P_1)| < |V(P_1)|/3$, and after distributing the charges of vertices of $D \cap V(P_1)$ evenly among vertices of P_1, each vertex of P_1 will have charge less than $1/3$.

Step 11: Let P be an accepting 2-path such that exactly one endpoint of P is an out-endpoint or a $(2,2)$-endpoint, and the central path of P has exactly 3 vertices. By definition, P is not a dependant of an exceptional path. Suppose that $P_1 = (x_1 \dots x_k)$, $P_2 = (y_1 y_2 y_3)$, and $P_3 = (z_1 \dots z_l)$, so that $P = (x_1 \dots x_k y_1 y_2 y_3 z_1 \dots z_l)$. We may assume that x_1 is neither an out-endpoint nor a $(2,2)$-endpoint of P. By definition, y_2 is an acceptor for an out-endpoint y' of a path P' or for $y' = z_l$ if z_l is a $(2,2)$-endpoint. Since z_l is either a $(2,2)$-endpoint or an out-endpoint of P, the charges of vertices in P_3 were defined at Step 9 (if the acceptor of z_l is on a 2-path, then the charge of z_l could be changed at Step 10 or Step 11). We define the charges of vertices in P_1, P_2 and the charge of y' exactly as at Step 10.

Step 12: Let P be an accepting 2-path such that each of the endpoints of P is neither an out-endpoint nor a $(2,2)$-endpoint, the central path of P has exactly 3 vertices, and $|D \cap V(P)| \leq (|V(P)| + 1)/3$. By Lemma 5, $|P| \geq 16$. Hence, after distributing the charges of vertices of $D \cap V(P)$ evenly among all vertices of P, each vertex of P will have charge at most

$$\frac{|V(P)| + 1}{3|V(P)|} = \frac{1}{3} + \frac{1}{3|V(P)|} \leq \frac{1}{3} + \frac{1}{48} < \frac{5}{14}.$$

Step 13: Let P be an accepting 2-path such that each of the endpoints of P is neither an out-endpoint nor a $(2,2)$-endpoint, the central path of P has exactly 3 vertices, and $|D \cap V(P)| > (|V(P)|+1)/3$. If P is a dependant of an exceptional path, then we are done at Step 8. Suppose not. Let P_1, P_2, and P_3 be defined as at Step 11. Then $|D \cap V(P)| = (|V(P)|+4)/3$ and this may happen only if $|D \cap V(P_1)| = (|P_1|+2)/3$ and $|D \cap V(P_3)| = (|P_3|+2)/3$. In this case, by Lemma 5, $k \geq 16$ and $l \geq 16$. If $k+3+l > 38$, then $k+3+l \geq 41$ and $|D \cap V(P)| \leq \lceil k/3 \rceil + 1 + \lceil l/3 \rceil = (|V(P)|+4)/3$. Distributing the charge evenly among the vertices of $V(P) \cup \{y'\}$, where y' is the out-endpoint of another path P' whose acceptor is y_2, we obtain that the charge of each vertex in $V(P) \cup \{y'\}$ is at most

$$\frac{|V(P)|+4}{3(|V(P)|+1)} = \frac{1}{3} + \frac{3}{3(|V(P)|+1)} \leq \frac{1}{3} + \frac{1}{42} = \frac{5}{14}.$$

This is the only case so far that the end-vertex of a tip of a non-exceptional path gets charge greater than 3/14. Note that it happens only when each of the tips of P has at least 16 vertices, P has no out-endpoints or $(2,2)$-endpoints, $|D \cap V(P)| = (|V(P)|+4)/3$, and P accepts only one vertex. Recall that the other possibility for an end-vertex $y*$ of a 1-path or of a tip of a 2-path to get a positive charge occurs only at Step 10 or 11. In such a case, the following conditions hold:

(r1) $y*$ receives at most 3/14 of charge;

(r2) the accepting vertex y is either the second or the penultimate vertex in the central path, say P_2^*, of some 2-path P^*;

(r3) if P_2^* has more than 3 vertices (Case 10), then the closest to y tip of P^* has at least 16 vertices and no out-endpoints;

(r4) if P_2^* has exactly 3 vertices (Case 11), then one of the tips of P^* has at least 16 vertices and no out-endpoints and the other tip has either an out-endpoint or a (2,2)-endpoint.

The only case we have not yet considered is that $|P_2| = 3$, $k, l \geq 16$ and $k+l+3 \leq 38$. In particular, this means that P is a special path. In this case, $|D \cap V(P)| = 13$, when P has type 1 and $|D \cap V(P)| = 14$, when P has type 2. In both cases, the only accepting vertex is a special vertex. In both cases, y' has the current charge 0. We give to y' and to every vertex of P charge 5/14, but $(35+1)\cdot 5/14 = 13 - 1/7$ and $(38+1)\cdot 5/14 = 14 - 1/14$; so we need to distribute either 1/7 (when P has type 1) or 1/14 (when P has type 2) among some other vertices. Consider the following cases for distributing this charge.

Case 1: Vertex y' is the out-endpoint of a 1-path P' of length at least 4. In this case, we add 1/42 to the charge of each of the vertices of $P' - y'$. At Step 3 or Step 4, each of these vertices got charge 1/3, so now each of them has charge 5/14. If P is a special path of type 2 or P' has at least 7 vertices, then we are done; so suppose that P has type 1 and $|P'| = 4$. Let $P' = (w_1w_2w_3w_4)$, where $y' = w_1$. If w_1 has another outneighbor v apart from its acceptor, then by (C1) and (C5), v is a special vertex of a special path P'' of type 1, and this path is non-accepting. In this case, every vertex of P'' has charge 12/35, and after distributing evenly our surplus charge of 1/14 among vertices of P'', each of these vertices will have charge $12/35 + 1/(14\cdot 35) < 5/14$. So, w_1 has no other outneighbors. By (C1), no vertex in P' dominates all the others. If w_1 has two neighbors in P' and no vertex in P' dominates all the others, then $G(P')$ is the 4-cycle (w_1, w_2, w_3, w_4). By (C5), the outneighbors of w_2, w_3 and w_4 are special vertices of special paths which are not accepting. So, we can distribute our surplus 1/14 among these vertices, as above.

Case 2: Vertex y' is the out-endpoint of a tip of an accepting 2-path P'. Then P' can be written as $P'_1P'_2P'_3$, where P'_1 and P'_3 are the tips, and P'_2 is the center. Suppose that $P'_2 = (v_1v_2\ldots v_t)$. Note that by the definition of a center, v_2 is the acceptor for a vertex v' and the charge of v' (maybe received because of P_3' at Step 10 or 11) is at most $3/14$. We give $1/7$ to v'.

Case 3: Vertex y' is the out-endpoint of a 2-path P' that forms a δ-subgraph. From (B3) we get that the center vertex is the only possible accepting vertex, but it has degree 3 in P'. Hence P' is non-accepting. We give $1/28$ to each of the remaining vertices of F. Since each of them got the charge $1/4$ at Step 6, now it will have $1/4 + 1/28 = 2/7$.

Case 4: Vertex y' is the out-endpoint of a non-accepting 2-path P' that does not form a δ-subgraph. Let $P' = (w_1 \dots w_s)$, where $y' = w_1$. Since P' is not accepting and we chose an acceptor for w_1, according Rules (C2)-(C5), either w_s also is an out-endpoint or $s \leq 11$. Suppose first that $s \leq 11$ and w_s has no outneighbors. Then $s \in \{5, 8, 11\}$ and on Step 7 each of the vertices of $P' - w_1$ got the charge $\frac{s-2}{3(s-1)}$. We distribute $1/7$ evenly among these vertices so that each of them will now have charge

$$\frac{s-2}{3(s-1)} + \frac{1}{7(s-1)} = \frac{7s-11}{21(s-1)} < \frac{1}{3}.$$

Suppose now that w_s is an out-endpoint. Since P' is not an exceptional path, the path P'' accepting w_s does not give charge to any vertex apart from w_s and by (r1) w_s has charge at most $3/14$. Adding the surplus to this vertex leaves it with charge $3/14 + 1/7 = 5/14$.

Case 5: The path P' containing y' has no other vertices. Since P is special, by (C1) this might happen only if P is very special and y' is adjacent to special vertices in special paths P_1 and P_2 that are non-accepting. We add $1/(14 \cdot 35)$ to the charge of each vertex in P_1 and P_2. This finishes the discharging.

Thus, what is left to prove Theorem 3 is to prove Lemmas 3, 4 and 5. We will do it in the next sections. In Section 4 we describe the approach we use and prove a number of auxiliary statements. Applying these statements, we prove Lemmas 4 and 5 in Section 5. Lemma 3 has the longest proof. It will be proved in Section 6.

4. Structure of proofs and technical statements

We will need some notation. Let G' be a subgraph of a graph G and $u, v \in V(G')$, $u \neq v$. Say that u is (G', v)-*distant* if G' contains a hamiltonian v, u-path. Sometimes, if it is clear which G' we have in mind, we will simply say that u is v-*distant*.

A v-*lasso* is a graph consisting of a cycle, say C, and a path connecting v with C. In this case, C is the *loop* of this v-lasso, and H is the remaining path which we will call the *handle*. If $v \in V(C)$, then C itself is a v-lasso. A v-lasso with k vertices, l of whose belong to the loop, will be sometimes called a (v, k, l)-*lasso*.

A typical structure used in the proofs of Lemmas 3, 4, and 5 will be as follows. We will consider a path $P_1 = (v_1 \dots v_k)$ and let $G_1 = G[V(P_1)]$. We will know that k is not large, for example, $k \leq 11$. For some reasons, we will know that v_1 has no neighbors outside of P_1 and, moreover, that no (G_1, v_k)-distant vertex has a neighbor outside of P_1. If k is 2 (mod 3), then we will want to prove that some $(k-2)/3$ vertices dominate $V(P_1) - v_k$. If k is 1 (mod 3), then we will want to prove that some $(k-1)/3$ vertices dominate $V(P_1)$. We will show that we do not need to consider the case of $k = 0$ (mod 3). Thus, we need that some $\lfloor k/3 \rfloor$ vertices dominate the first $3\lfloor k/3 \rfloor + 1$ vertices of P_1. For example, if $P_1 = P = (v_1 \dots v_8)$ and v_8 is the only out-endpoint of P, then we will prove that some two vertices dominate $V(P_1) - v_8$. We will do this as follows.

Since v_1 has no neighbors outside of P_1, it has two neighbors, v_i and v_j distinct from v_2 on P_1. Path P_1 together with edge v_1v_i forms a v_k-lasso. Among all v_k-lassos on $V(P_1)$ choose a lasso L with the largest loop C. By renumbering vertices, we may assume that L consists of the cycle $C = (v_1 \dots v_r)$ and the path $(v_r \dots v_k)$. If r is divisible by 3, then the set $D = \{v_3, v_6, \dots, v_{3\lfloor k/3 \rfloor}\}$ dominates what we need. So, we will need to consider only $r \not\equiv 0 \pmod 3$. The problem of finding $\lfloor k/3 \rfloor$ vertices that dominate the first $3\lfloor k/3 \rfloor + 1$ vertices of P_1 reduces to the problem of finding $\lfloor r/3 \rfloor$ vertices that dominate $\{v_1, \dots, v_{3\lfloor r/3 \rfloor + 1}\}$, since the remaining $3(\lfloor k/3 \rfloor - \lfloor r/3 \rfloor)$ vertices of P_1 that we need to dominate are easily dominated by the vertices $v_{3(\lfloor r/3 \rfloor + 1)}, v_{3(\lfloor r/3 \rfloor + 2)}, \dots, v_{3(\lfloor k/3 \rfloor)}$.

Let $G' = G[V(C)]$. By the above condition on P_1, no (G', v_r)-distant vertex has a neighbor outside of P_1. By the maximality of $|C|$, no (G', v_r)-distant vertex has a neighbor in $V(P_1) - V(C)$. Thus, no (G', v_r)-distant vertex has a neighbor outside of C. In the rest of this section we will prove that under these conditions, some $\lfloor r/3 \rfloor$ vertices dominate $\{v_1, \dots, v_{3\lfloor r/3 \rfloor + 1}\}$ for $r = 4, 5, 7, 8, 10, 11, 13$ and 14. This will be heavily used later.

Lemma 6. *Let G' be the subgraph of a cubic graph G induced by vertices v_1, v_2, v_3, and v_4. If v_1 has no neighbor outside of G', then v_1 dominates $V(G')$.*

Proof. This is because the only possible neighbors of v_1 are v_2, v_3, and v_4. □

Lemma 7. *Let G' be the subgraph of a cubic graph G induced by the vertices of a path $(v_1v_2v_3v_4v_5)$. If no (G', v_5)- distant vertex has a neighbor outside of $V(G')$, then some vertex dominates $V(G') - v_5$.*

Proof. If $v_1v_3 \in E(G)$, then v_3 dominates $V(G') - v_5$. Suppose that $v_1v_3 \notin E(G)$. Then $v_1v_4, v_1v_5 \in E(G)$. The paths $(v_3v_2v_1v_4v_5)$ and $(v_2v_3v_4v_1v_5)$ show that each of v_2 and v_3 can play the role of v_1 and thus by the above argument should be adjacent to v_5 if no vertex dominates $V(G') - v_5$. But v_5 cannot be adjacent to all of v_1, v_2, v_3, v_4. □

Lemma 8. *If a graph G' on $3k+1$ vertices has a hamiltonian path $P = (v_1 \dots v_{3k+1})$ and an edge v_iv_{i+3j-1}, where i is not divisible by 3, then G' has a dominating set of size k.*

Proof. If $i = 3m+1$, then we let $D = \{v_2, v_5, \dots, v_{3m-1}, v_{3m+3}, v_{3m+6}, \dots, v_{3k}\}$. Note that then $v_{i+3j-1} \in D$. Thus every $v \in D$ dominates its neighbors on P, and v_{i+3j-1} also dominates v_i.

If $i = 3m + 2$, then we let $D = \{v_2, v_5, \dots, v_{3m+3j-1}, v_{3m+3j+3}, v_{3m+3j+6}, \dots, v_{3k}\}$. In this case $v_i \in D$, every $v \in D$ dominates its neighbors on P, and $v_i = v_{3m+2}$ also dominates $v_{i+3j-1} = v_{3m+3j+1}$. □

An immediate corollary of this lemma is the following fact.

Lemma 9. *If a graph G' on $3k + 1$ vertices has a hamiltonian cycle $(v_1 \dots v_{3k+1})$ and an edge v_iv_j with $j - i + 1$ divisible by 3, then G' has a dominating set of size k.*

Lemma 10. *Let graph G' on $3k + 1$ vertices form a subdivision of K_4 with the set R of the 4 branching vertices. Then either G' has a dominating set of size k or the lengths (mod 3) of the paths between the vertices in R in this subdivision of G' are equivalent to those in one of the three graphs in Figure 2 (graphs D, E, and F).*

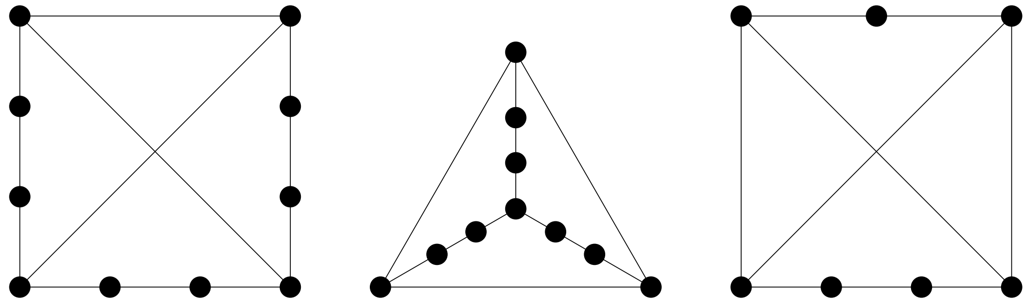

FIGURE 2. Graphs D, E, and F

Proof. A *thread* in a graph is a path connecting two vertices of degree at least 3 whose all internal vertices have degree 2. Say that two subdivisions of K_4 are *equivalent* if the lengths of their threads are the same (mod 3). Since every vertex of degree 2 dominates exactly three consecutive vertices in a thread, it is enough to prove the lemma for subdivisions of K_4 in which the length of each thread is in $\{1, 2, 3\}$. Since every edge subdivision in a graph adds one vertex and one edge, each K_4 subdivision with $3k+1$ vertices has $3k+3$ edges.

Case 1: Two threads of length 2 share an endvertex v. Then v dominates all but a path with $3k-3$ vertices. Taking the natural dominating set in this path yields a dominating set of G' with size k.

Case 2: G' contains two vertex disjoint threads of length 2, but Case 1 does not hold. Since G' has $3k+3$ edges, the other threads necessarily have the lengths $1, 1, 3$, and 3. This yields two possible graphs. The graphs and their dominating sets are shown as graphs G and H in Figure 3.

Case 3: Exactly one thread has length 2. The possible lengths of the remaining threads are $1, 1, 1, 1, 3$ or $1, 3, 3, 3, 3$. This yields four possible graphs, the bad case shown as F, and the three graphs shown with their dominating sets of size k are shown as graphs I, J, and K.

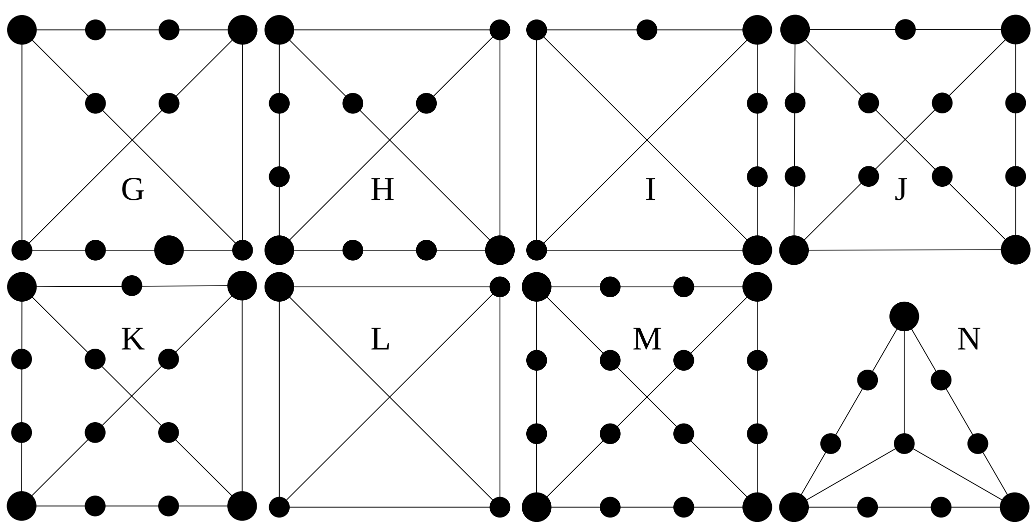

FIGURE 3. Graphs G, H, I, J, K, L, M, and N along with their dominating sets

Case 4: All threads have the same length (mod 3). This yields the graphs L and M each of which has a dominating set of size k.

Case 5: The lengths of the threads in our subdivision are $1, 1, 1, 3, 3, 3$. The three possible graphs with these thread lengths are graphs D, E in Figure 2, and graph N in Figure 3. □

Sometimes, it will be simpler to check that Case 1 of Lemma 10 holds. We state this case as a separate claim:

Lemma 11. *Suppose that a graph G' on $3k+1$ vertices has a spanning subgraph G'' consisting of 3 internally disjoint paths P_1, P_2 and P_3 connecting some vertices x and y. Suppose that the distances on P_1 from an internal vertex z of P_1 to x and to y are $2 \pmod 3$. Then either G' has a dominating set of size k, or z has no third neighbor in G', or the third neighbor of z belongs to P_1.*

Lemma 12. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_7$. If G' contains a hamiltonian cycle $(v_1 v_2 \ldots v_7)$ and v_7 has an outneighbor, then either some two vertices dominate $V(G')$, or there are two (G', v_7)-distant vertices such that each of them has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. For each $i = 1, \ldots, 7$, the *third neighbor* of v_i is the in-neighbor different from v_{i-1} and v_{i+1} (if it exists). Since both v_1 and v_6 are (G', v_7)-distant, under conditions of the lemma, at least one of them has no outneighbors. By symmetry, we may assume that v_1 has no outneighbors. By Lemma 9, the only possible third neighbors of v_1 are v_4 and v_5.

Case 1: $v_1 v_5 \in E(G')$. By Lemma 9, v_4 has no third neighbors in G'. Thus it has an outneighbor. But the path $(v_4 v_3 v_2 v_1 v_5 v_6 v_7)$ is hamiltonian in G'. So if the lemma does not hold, then v_6 has no outneighbors. Symmetrically to v_1, the possible third neighbors of v_6 are v_2 and v_3. If $v_6 v_3 \in E(G')$, then $\{v_1, v_3\}$ dominates $V(G')$. If $v_6 v_2 \in E(G')$, then symmetrically to v_4, v_3 must have an outneighbor, a contradiction to our assumptions.

Case 2: $v_1 v_4 \in E(G')$. If $\{v_1, v_6\}$ dominates $V(G')$, then we are done. Suppose not. Then $v_6 v_3 \notin E(G)$. Thus by Lemma 9, v_3 has an outneighbor. Since the path $(v_3 v_2 v_1 v_4 v_5 v_6 v_7)$ is hamiltonian in G', v_3 is v_7-distant. Hence if the lemma does not hold, then v_6 has the third neighbor in G'. By the symmetry with v_1, it should be v_2 or v_3. But we assumed that $v_6 v_3 \notin E(G)$. Hence, $v_6 v_2 \in E(G)$ and we have Case 1 again. □

Lemma 13. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_8$. If G' contains a hamiltonian cycle $(v_1 v_2 \ldots v_8)$ and v_8 has an outneighbor, then either some two vertices v_i and v_j dominate $V(G') - v_8$, or some (G', v_8)-distant vertex has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. In particular, this implies that v_1 and v_7 have third neighbors in G'. If $v_1 v_7 \in E(G')$, then Lemma 12 yields our lemma. Let $v_1 v_7 \notin E(G')$. By Lemma 8, $v_1 v_6 \notin E(G')$ and $v_1 v_3 \notin E(G')$. Hence, the only possible third neighbors for v_1 are v_4 and v_5, and by symmetry, the only possible third neighbors for v_7 are v_4 and v_3. If v_4 is not a neighbor of $\{v_1, v_7\}$, then $v_7 v_3, v_1 v_5 \in E(G')$ and hence $\{v_3, v_5\}$ dominates $V(G') - v_8$. Thus, (by symmetry) we may assume that $v_1 v_4 \in E(G')$ and hence $v_7 v_3 \in E(G')$.

The existence of the path $(v_6 v_5 v_4 v_1 v_2 v_3 v_7 v_8)$ yields that v_6 has no outneighbors. The only possible third in-neighbor for v_6 is v_2. Then v_5 must have an outneighbor, but this contradicts the existence of the hamiltonian path $(v_5v_4v_3v_7v_6v_2v_1v_8)$. □

Lemma 14. [6] *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{10}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{10})$, and that v_{10} has an outneighbor. Then either some three vertices dominate $V(G')$, or some (G', v_{10})-distant vertex has an outneighbor.*

Lemma 15. [6] *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{10}, v_{11}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{11})$, and that v_{11} has an outneighbor. Then either some three vertices dominate $V(G') - v_{11}$, or some (G', v_{11})-distant vertex has an outneighbor.*

Lemma 16. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{13}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{13})$ and v_{13} has an outneighbor. Then either some four vertices dominate $V(G')$, or some (G', v_{13})-distant vertex has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. By Lemma 9,

(1) no edge of the form v_iv_{i+3j-1} is present in G'.

Further, if the hamiltonian cycle is drawn as a planar graph, then any two crossing edges along with the hamiltonian cycle determine a K_4 subdivision on 13 vertices. Hence Lemma 10 may be applied whenever a potential edge crosses an edge already forced. Since v_1 is v_{13}-distant, it has a third neighbor in G'.

Case 1: $v_1v_4 \in E(G')$. The path $(v_{13}v_{12}...v_4v_1v_2v_3)$ forces v_3 to have its third neighbor in G'. Since any such neighbor forces an edge crossing v_1v_4, Lemma 10 restricts this neighbor to one of v_7 and v_{10}.

Case 1.1: $v_3v_7 \in E(G')$. Then Lemmas 9 and 10 forbid edges $v_{12}v_2$, $v_{12}v_5$, and $v_{12}v_{10}$. So, the third neighbor of v_{12} is one of v_6, v_8, and v_9. If $v_{12}v_6 \in E(G')$, then the set $\{v_1,v_6,v_7,v_{10}\}$ dominates G'. Suppose that $v_{12}v_8 \in E(G')$. Then the path $(v_{13}v_1v_2...v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (1) and by Lemma 10 with $R = \{v_1, v_2, v_4, v_9\}$ and $R = \{v_3, v_5, v_7, v_9\}$, we have $v_9v_6 \in E(G')$. So, the set $\{v_1, v_6, v_7, v_{11}\}$ dominates G'. Thus, $v_{12}v_9 \in E(G')$, and by symmetry, $v_{10}v_6 \in E(G')$. Then $\{v_1, v_6, v_7, v_{12}\}$ dominates G'.

Case 1.2: $v_3v_{10} \in E(G')$. Lemma 10 applied successively with $R \supset \{v_1, v_4, v_{12}\}$ and $R \supset \{v_3, v_{10}, v_{12}\}$ restricts the third neighbor of v_{12} to one of v_6 and v_9. In either case, the set $\{v_1, v_6, v_9, v_{10}\}$ dominates G'.

Hence $v_1v_4 \notin G'$, and symmetry gives $v_{12}v_9 \notin G'$.

Case 2: $v_1v_5 \in E(G')$. The path $(v_{13}v_{12}...v_5v_1v_2v_3v_4)$ forces v_4 to have its third neighbor in G'. By (1), this neighbor is one of v_7,v_8,v_{10}, or v_{11}.

Case 2.1: $v_4v_7 \in E(G')$. The path $(v_{13}v_{12} \ldots v_7v_4v_3v_2v_1v_5v_6)$ forces v_6 to have its third neighbor in G'. By Lemma 10 with $R \supset \{v_6, v_4, v_7\}$ and $R \supset \{v_6, v_1, v_5\}$, this neighbor must be v_{10}. Then by Lemma 10 with $R \supset \{v_{12}, v_6, v_{10}\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} must be v_2 and hence the set $\{v_1, v_2, v_7, v_{10}\}$ dominates G'.

Case 2.2: $v_4v_8 \in E(G')$. By Lemma 10 with $R \supset \{v_{12}, v_4, v_8\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} must be v_2. Then the path $(v_{13}v_1v_2v_{12}v_{11} \ldots v_3)$ forces v_3 to have its third neighbor in G', but Lemma 10 with $R \supset \{v_3, v_1, v_5\}$ eliminates all possible neighbors of v_3.

Case 2.3: $v_4v_{10} \in E(G')$. By Lemma 10 with $R \supset \{v_{12}, v_4, v_{10}\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} must be v_2. But then the set $\{v_1, v_2, v_7, v_{10}\}$ dominates G'.

Case 2.4: $v_4v_{11} \in E(G')$. By Lemma 10 with $R \supset \{v_{12}, v_4, v_{11}\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} is either v_2 or v_8. If $v_{12}v_2 \in E(G')$, then the path $(v_{13}v_{12}v_2v_1v_5v_6 \ldots v_{11}v_4v_3)$ forces v_3 to have its third neighbor in G', but Lemma 10 with $R \supset \{v_3, v_1, v_5\}$ eliminates all possible neighbors of v_3. So, $v_{12}v_8 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6 \dots v_{10})$ forces v_{10} to have its third neighbor in G', but Lemma 10 with $R \supset \{v_{12}, v_8, v_{10}\}$ eliminates all possible neighbors of v_{10}.

Hence $v_1v_5 \notin G'$, and symmetry gives $v_{12}v_8 \notin G'$.

Case 3: $v_1v_7 \in E(G')$. Each allowable edge from v_{12} crosses v_1v_7, and Lemma 10 gives a dominating set of size 4.

Hence $v_1v_7 \notin G'$, and symmetry gives $v_{12}v_6 \notin G'$.

Case 4: $v_1v_{10} \in E(G')$. Each allowable edge from v_{12} crosses v_1v_{10}, and Lemma 10 gives a dominating set of size 4.

Hence $v_1v_{10} \notin G'$, and symmetry gives $v_{12}v_3 \notin G'$.

Case 5: $v_1v_8 \in E(G')$. The two possible third neighbors of v_{12} are v_2, and v_5.

Case 5.1: $v_{12}v_2 \in E(G')$. The path $(v_{13}v_1v_8v_7 \dots v_2v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (1), this third neighbor is not in $\{v_4, v_7, v_{11}\}$. Then Lemma 10 with $R = \{v_1, v_9, v_8, v_i\}$ for $i \in \{4, 6\}$ forces $v_9v_5 \in E(G')$. Now the path $(v_{13}v_{12} \dots v_8v_1v_2 \dots v_7)$ forces v_7 to have its third neighbor in G'. This contradicts Lemma 11 with $x = v_5$, $y = v_9$ and $z = v_7$.

Case 5.2: $v_{12}v_5 \in E(G')$. The path $(v_{13}v_1v_2 \dots v_5v_{12}v_{11} \dots v_6)$ forces v_6 to have its third neighbor in G'. Lemma 11 with $x = v_1$, $y = v_8$ and $z = v_6$ forces this neighbor to be one of v_2 and v_3. If $v_6v_3 \in E(G')$, then the set $\{v_1, v_5, v_6, v_{10}\}$ dominates G'. So, $v_6v_2 \in E(G')$. By the symmetry between v_6 and v_7, $v_7v_{11} \in E(G)$. The path $(v_{13}v_1v_2v_6v_7 \dots v_{12}v_5v_4v_3)$ forces v_3 to have its third neighbor in G', a contradiction to Lemma 11 with $x = v_1$, $y = v_8$ and $z = v_3$.

Hence $v_1v_8 \notin G'$, and symmetry gives $v_{12}v_5 \notin G'$.

Case 6: $v_1v_{11} \in E(G')$, and $v_{12}v_2 \in E(G')$. The path $(v_{13}v_{12}v_2v_1v_{11}v_{10} \dots v_3)$ forces v_3 to have its third neighbor in G'. By (1), this neighbor is one of v_6,v_7,v_9, and v_{10}. Note that v_{10} is symmetric with v_3.

Case 6.1: $v_3v_6 \in E(G')$. The path $(v_{13}v_{12}v_2v_1v_{11}v_{10} \dots v_6v_3v_4v_5)$ forces v_5 to have its third neighbor in G'. Lemma 10 with $R \supset \{v_6, v_3, v_5\}$ restricts this neighbor to v_9. By (1), $v_{10}v_8 \notin E(G)$. By Lemma 11 with $x = v_5$, $y = v_9$ and $z = v_7$, $v_{10}v_7 \notin E(G)$. So, $v_{10}v_4 \in E(G)$. So, the path $(v_{13}v_{12}v_2v_1v_{11}v_{10}v_4v_3v_6v_5v_9v_8v_7)$ forces v_7 to have its third neighbor in G', but no possible third neighbor remains.

Case 6.2: $v_3v_7 \in E(G')$. By symmetry, we may assume that v_{10} is adjacent to either v_4 or v_6. If $v_{10}v_4 \in E(G')$, then the path $(v_{13}v_1v_{11}v_{12}v_2v_3v_4v_{10}v_9 \dots v_5)$ forces v_5 to have its third neighbor in G', a contradiction to Lemma 11 with $x = v_3$, $y = v_7$ and $z = v_5$. So, $v_{10}v_6 \in E(G')$. The path $(v_{13}v_1v_{11}v_{12}v_2v_3v_7v_8v_9v_{10}v_6v_5v_4)$ forces v_4 to have its third neighbor in G'. By (1), this neighbor must be v_8, but Lemma 10 with $R = \{v_4, v_8, v_6, v_{10}\}$ gives a dominating set of size 4.

Case 6.3: $v_3v_9 \in E(G')$, and necessarily $v_{10}v_4 \in E(G')$. The path $(v_{13}v_{12}v_2v_1v_{11}v_{10}v_4v_3v_9v_8 \dots v_5)$ forces v_5 to have its third neighbor in G', and (1) forces it to be v_8. Finally the path $(v_{13}v_{12}v_2v_1v_{11}v_{10}v_4v_3v_9v_8v_5v_6v_7)$ forces v_7 to have its third neighbor in G' which is impossible.

Case 6.4: $v_3v_{10} \in E(G')$. The path $(v_{13}v_1v_{11}v_{12}v_2v_3v_{10}v_9 \dots v_4)$ forces v_4 to have its third neighbor in G'. By (1), this neighbor is one of v_7 and v_8. Note that v_9 is symmetric with v_4. If $v_4v_7 \in E(G')$, then Lemma 10 with $R \supset \{v_4, v_7, v_9\}$ eliminates v_5 and v_6 as possible third neighbors of v_9. So, $v_4v_8 \in E(G')$, and by symmetry, $v_9v_5 \in E(G')$. The path $(v_{13}v_1v_{11}v_{12}v_2v_3v_{10}v_9v_5v_4v_8v_7v_6)$ forces v_6 to have its third neighbor in G' which is impossible. This proves the lemma. □

Lemma 17. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \dots, v_{14}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2\dots v_{14})$ and v_{14} has an outneighbor. Then either some four vertices dominate $V(G') - v_{14}$, or some (G', v_{14})-distant vertex has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. Then by Lemma 8, for every hamiltonian path $(u_1 \dots u_{13})$ in $G' - v_{14}$,

$$\text{if } u_iu_{i+3j-1} \in E(G'), \text{ then } i = 0 \pmod 3. \tag{2}$$

By (2) for the path P=$(v_1v_2\dots v_{13})$, the only possible third neighbors of v_1 are v_4,v_5,v_7,v_8,v_{10},v_{11}, and v_{13}. Note that v_{13} is symmetric with v_1.

Case 1: $v_1v_4 \in E(G')$. The path $(v_{13}v_{12}\dots v_4v_1v_2v_3)$ forces v_3 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_5,v_7,v_8,v_{10},v_{11}, and v_{13}.

Case 1.1: $v_3v_5 \in E(G')$. The path $(v_{13}v_{12}\dots v_5v_3v_4v_1v_2)$ forces v_2 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is either v_8, or v_{11}. In either case, the set $\{v_5,v_8,v_{11},v_{14}\}$ dominates G'.

Case 1.2: $v_3v_7 \in E(G')$. The third neighbor of v_{13} is amongst v_6,v_9, and v_{10}.

Case 1.2.1: $v_{13}v_6 \in E(G')$. The path $(v_1v_4v_5v_6v_{13}v_{12}\dots v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. By (2) for P, this neighbor is among v_5,v_8,v_9,v_{11}, and v_{12}. If $v_2v_8 \in E(G')$, then the set $\{v_4, v_8, v_{10}, v_{13}\}$ dominates G'. If $v_2v_9 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{11}\}$ dominates $G' - v_{14}$. If $v_2v_{11} \in E(G')$, then the set $\{v_4,v_6,v_8,v_{11}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_4,v_6,v_9,v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_2v_5 \in E(G')$. The path $(v_{13}v_6v_5v_2v_1v_4v_3v_7v_8\dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for P, this neighbor is either v_8, or v_9. If $v_{12}v_8 \in E(G')$, then the path $(v_{13}v_6v_5v_2v_1v_4v_3v_7v_8v_{12}v_{11}\dots v_9)$ forces v_9 to have its third neighbor in G'. Then (2) for this path disallows all possible third neighbors. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_6v_5v_2v_1v_4v_3v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G'. Thus $v_{10}v_8 \in E(G')$, and the set $\{v_2, v_3, v_{10}, v_{13}\}$ dominates G'.

Case 1.2.2: $v_{13}v_9 \in E(G')$. The path $P' = (v_1v_2\dots v_9v_{13}v_{12}\dots v_{10})$ forces v_{10} to have its third neighbor in G', and (2) for P' forces $v_{10}v_6 \in E(G')$. The path $(v_1v_4v_5v_6v_{10}v_{11}v_{12}v_{13}v_9v_8v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. If $v_2v_5 \in E(G')$, then the path $(v_{13}v_9v_8v_7v_3v_2v_1v_4v_5v_6v_{10}v_{11}v_{12})$ forces v_{12} to have its third neighbor in G', and hence $v_{12}v_8 \in E(G')$. Then the set $\{v_2, v_3, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_8 \in E(G')$ or $v_2v_{12} \in E(G')$, then the set $\{v_4, v_6, v_8, v_{12}\}$ dominates $G' - v_{14}$. Finally, if $v_2v_{11} \in E(G')$, then the set $\{v_4, v_6, v_9, v_{11}\}$ dominates $G' - v_{14}$.

Case 1.2.3: $v_{13}v_{10} \in E(G')$. The path $(v_1v_2\dots v_{10}v_{13}v_{12}v_{11})$ forces v_{11} to have its third neighbor in G'. By (2) for P and the symmetry between v_{11} and v_3, $v_{11}v_6 \in E(G')$. The path $(v_1v_4v_5v_6v_{11}v_{12}v_{13}v_{10}v_9v_8v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. If $v_2v_5 \in E(G')$, then the path $(v_1v_2v_5v_4v_3v_7v_6v_{11}v_{12}v_{13}v_{10}v_9v_8)$ forces v_8 to have the third neighbor in G', hence $v_8v_{12} \in E(G')$. Then the set $\{v_2, v_5, v_8, v_{10}\}$ dominates $G' - v_{14}$. If $v_2v_8 \in E(G')$, then the set $\{v_4, v_8, v_{11}, v_{14}\}$ dominates G'. If $v_2v_9 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$.

Case 1.3: $v_3v_8 \in E(G')$. The path $(v_{13}v_{12}\dots v_8v_3v_2v_1v_4v_5\dots v_7)$ forces v_7 to have its third neighbor in G'. By (2) for this path this neighbor must be amongst v_{10}, v_{11}, and v_{13}.

Case 1.3.1: $v_7v_{10} \in E(G')$. Then (2) with the path $(v_{13}v_{12}v_{11}v_{10}v_7v_6v_5v_4v_1v_2v_3v_8v_9)$ forces $v_{13}v_6 \notin G'$. Hence by 2) for P $v_{13}v_9 \in E(G')$. Then the path $(v_{13}v_9v_8v_3v_2v_1v_4v_5v_6v_7v_{10}v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Using (2) on this path forces $v_{12}v_6 \in E(G')$. Then the set $\{v_3, v_6, v_{10}, v_{14}\}$ dominates G'.

Case 1.3.2: $v_7v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_7v_6v_5v_4v_1v_2v_3v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. Then (2) for this path and P forces $v_{10}v_{13} \in E(G')$. This is then symmetric with Case 1.2.

Case 1.3.3: $v_7v_{13} \in E(G')$. The path $(v_{13}v_7v_6v_5v_4v_1v_2v_3v_8v_9 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_2, v_5, v_6, and v_9. If $v_{12}v_2 \in E(G')$, then the set $\{v_2, v_4, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_5 \in E(G')$, then the set $\{v_2, v_5, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_6 \in E(G')$, then the set $\{v_3, v_6, v_{10}, v_{14}\}$ dominates G'. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_7v_6v_5v_4v_1v_2v_3v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (2) for P forces $v_{10}v_6 \in E(G')$. Then the set $\{v_3, v_6, v_{12}, v_{14}\}$ dominates G'.

Case 1.4: $v_3v_{10} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_3v_2v_1v_4v_5 \dots v_9)$ forces v_9 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is amongst v_2, v_5, v_6, v_{11}, and v_{13}.

Case 1.4.1: $v_9v_2 \in E(G')$. The set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$.

Case 1.4.2: $v_9v_5 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_3v_2v_1v_4v_5v_9v_8v_7v_6)$ forces v_6 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_2, v_{11}, and v_{13}. If $v_6v_2 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G'-v_{14}$. If $v_6v_{11} \in E(G')$, then by (2) for P $v_{13}v_8 \notin E(G')$ and hence $v_{13}v_7 \in E(G')$. So, in this case $\{v_2, v_5, v_7, v_{11}\}$ dominates $G' - v_{14}$. Thus, $v_6v_{13} \in E(G')$. The path $(v_1v_2 \dots v_6v_{13}v_{12} \dots v_7)$ forces v_7 to have its third neighbor in G'. By (2) for P, $v_7v_{11} \in E(G')$. Then the path $(v_1v_4v_5v_9v_8v_7v_6v_{13}v_{12}v_{11}v_{10}v_3v_2)$ forces v_2 to have the third neighbor in G', and (2) for this path yields $v_2v_{12} \in E(G')$. Thus the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$.

Case 1.4.3: $v_9v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_3v_2v_1v_4v_5v_6v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (2) for P, this neighbor is one of v_{11}, and v_{13}. If $v_7v_{11} \in E(G')$, then by (2) for P, no vertex in G' can be adjacent to v_{13}. If $v_7v_{13} \in E(G')$, then the path $(v_1v_2 \dots v_6v_9v_{10} \dots v_{13}v_7v_8)$ forces v_8 to have its third neighbor in G'. By (2) for this path, this neighbor is one of v_{11}, or v_{12}. If $v_8v_{11} \in E(G')$, then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'. If $v_8v_{12} \in E(G')$, then the set $\{v_3, v_6, v_{12}, v_{14}\}$ dominates G'.

Case 1.4.4: $v_9v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_9v_{10}v_3v_2v_1v_4v_5 \dots v_8)$ forces v_8 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is one of v_2, and v_5. If $v_8v_2 \in E(G')$, then the path $(v_{13}v_{12}v_{11}v_9v_{10}v_3v_4v_1v_2v_8v_7v_6v_5)$ forces v_5 to have its third neighbor in G'. Then by (2) for this path (2) for P eliminates the remaining possible neighbors of v_5. So, $v_8v_5 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_9v_{10}v_3v_2v_1v_4v_5v_8v_7v_6)$ forces v_6 to have its third neighbor in G'. By (2) for this path this neighbor is one of v_2 and v_{13}. If $v_6v_2 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_6v_{13} \in E(G')$, then the set $\{v_1, v_8, v_{10}, v_{13}\}$ dominates G'.

Case 1.4.5: $v_9v_{13} \in E(G')$. The path $(v_{13}v_9v_8 \dots v_4v_1v_2v_3v_{10}v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for this path and for P, this neighbor is amongst v_2, v_5, and v_8. If $v_{12}v_2 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_{12}v_5 \in E(G')$, then the set $\{v_1, v_7, v_{10}, v_{12}\}$ dominates G'. If $v_{12}v_8 \in E(G')$, then the set $\{v_1, v_6, v_{10}, v_{12}\}$ dominates G'.

Case 1.5: $v_3v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_3v_2v_1v_4v_5 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) for P, this neighbor is amongst v_6, v_7, and v_{13}.

Case 1.5.1: $v_{10}v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_3v_2v_1v_4v_5v_6v_{10}v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (2) for P, this neighbor is v_{13}. Now the path $(v_1v_2 \dots v_7v_{13}v_{12} \dots v_8)$ forces v_8 to have its third neighbor in G'. By (2) for this path, $v_8v_{12} \in E(G')$. Then the set $\{v_3, v_6, v_8, v_{14}\}$ dominates G'.

Case 1.5.2: $v_{10}v_7 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_3v_2v_1v_4v_5v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. By (2) for this path and for P, $v_8v_6 \in E(G')$. Then by (2) for P $v_{13}v_9 \in E(G')$, and the set $\{v_1, v_6, v_9, v_{11}\}$ dominates G'.

Case 1.5.3: $v_{10}v_{13} \in E(G')$. The path $(v_1v_4v_5 \dots v_{10}v_{13}v_{12}v_{11}v_3v_2)$ forces v_2 to have its third neighbor in G'. By (2) for P, this neighbor is amongst v_5, v_6, v_8, v_9, and v_{12}. If $v_2v_5 \in E(G')$ or $v_2v_8 \in E(G')$, then the set $\{v_5, v_8, v_{11}, v_{14}\}$ dominates G'. If $v_2v_6 \in E(G')$ or $v_2v_9 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_2, v_4, v_7, v_{10}\}$ dominates $G' - v_{14}$.

Case 1.6: $v_3v_{13} \in E(G')$. The path $(v_1v_4v_5 \dots v_{13}v_3v_2)$ forces v_2 to have its third neighbor in G'. By (2) for this path and for the path $(v_2v_1v_4v_3v_{13}v_{12} \dots v_5)$ this neighbor is amongst v_5, v_8, and v_{11}.

Case 1.6.1: $v_2v_5 \in E(G')$. Identifying the vertices $v_{13}, v_{14}, v_1, v_2, v_3$, and v_4 as one vertex v gives a new graph G'' on 8 vertices. A hamiltonian path in G'' starting at v has a corresponding hamiltonian path in G' which starts at v_{14} by using either the path $(v_{14}v_{13}v_3v_2v_1v_5)$ or the path $(v_{14}v_1v_4v_5v_2v_3v_{13})$. A dominating set of $G'' - v$ not using v can be extended to a dominating set of $G' - v_{14}$ with size 2 greater by including the vertices v_3 and v_4. A dominating set of G'' which contains v can be extended to a dominating set of G' with size 2 greater by replacing v by the vertices v_2, v_5, and v_{13}. Hence Lemma 13 gives the desired result for G'' which extends to G'.

Case 1.6.2: $v_2v_8 \in E(G')$. The path $(v_{13}v_3v_2v_1v_4v_5 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for this path and P, this neighbor is one of v_5 or v_9. Also the path $(v_1v_4v_5 \dots v_8v_2v_3v_{13}v_{12} \dots v_9)$ forces v_9 to have its third neighbor in G'. By (2) for this path and P, this neighbor is one of v_5 or v_{12}. This then forces the edge $v_9v_{12} \in E(G')$. Next the paths $(v_1v_4v_5 \dots v_8v_2v_3v_{13}v_{12}v_9v_{10}v_{11})$ and $(v_1v_2v_8v_7 \dots v_3v_{13}v_{12}v_9v_{10}v_{11})$ force v_{11} to have its third neighbor in G', and these paths along with (2) force this edge to be to v_5. Finally the path $(v_{13}v_3v_2v_1v_4v_5 \dots v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (2) on this path forces this edge to be to v_6. Then the set $\{v_2, v_4, v_6, v_{12}\}$ dominates $G' - v_{14}$.

Case 1.6.3: $v_2v_{11} \in E(G')$. The path $(v_1v_4v_5 \dots v_{11}v_2v_3v_{13}v_{12})$ forces v_{12} to have its third neighbor in G'. The set $\{v_2, v_3, v_6, v_9\}$ is a dominating set if v_{12} is adjacent to either of v_6 or v_9. By (2) on the path P the third neighbor of v_{12} must be either v_5 or v_8.

Case 1.6.3.1: $v_{12}v_5 \in E(G')$. The path $(v_1v_2v_{11}v_{12}v_{13}v_3v_4 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) on P this vertex must be either v_6 or v_7. The inclusion of the edge $v_{10}v_6$ gives the amended path $(v_1v_2v_{11}v_{12}v_{13}v_3 \dots v_6v_{10} \dots v_7)$ forcing $v_7v_9 \notin E(G')$. Then the path $(v_1v_2v_{11}v_{12}v_{13}v_3 \dots v_6v_{10}v_9v_7v_8)$ gives the (G', v_{14})-distant vertex v_8 with an outneighbor. Hence the edge $v_{10}v_7 \in E(G')$. Then the path $(v_1v_2v_{11}v_{12}v_{13}v_3v_4 \in v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. Hence $v_8v_6 \in E(G')$. Then the set $D = \{v_1, v_6, v_{10}, v_{13}\}$ dominates G'.

Case 1.6.3.2: $v_{12}v_8 \in E(G')$. The path $(v_1v_2v_{11}v_{12}v_{13}v_3v_4 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. Since $G' - \{v_{14}, v_9, v_{10}, v_{11}\}$ has the hamiltonian cycle $(v_1v_2v_3v_{13}v_{12}v_8v_7 \dots v_4)$, v_{10} dominates all but a P_9 in $G' - v_{14}$ so $G' - v_{14}$ has a dominating set of size 4.

Case 2: $v_1v_5 \in E(G')$. The path $(v_{13}v_{12} \dots v_5v_1v_2v_3v_4)$ forces v_4 to have its third neighbor in G'. By (2) for P, this neighbor is amongst v_7, v_8, v_{10}, v_{11}, and v_{13}.

Case 2.1: $v_4v_7 \in E(G')$. Then by the symmetry with v_1, the third neighbor of v_{13} is in $\{v_3, v_6, v_9\}$.

Case 2.1.1: $v_{13}v_3 \in E(G')$. The set $\{v_1, v_7, v_{10}, v_{13}\}$ dominates G'.

Case 2.1.2: $v_{13}v_6 \in E(G')$. The path $(v_{13}v_6v_5v_1v_2v_3v_4v_7v_8 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for this path and for P, this neighbor is amongst v_3, v_8, and v_9. If $v_{12}v_3 \in E(G')$, then the set $\{v_1, v_7, v_9, v_{12}\}$ dominates G'. If $v_{12}v_8 \in E(G')$, then the path $(v_{13}v_6v_5v_1v_2v_3v_4v_7v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G', and (2) for this path, forces $v_9v_3 \in E(G')$. Then the set $\{v_1, v_7, v_9, v_{12}\}$ dominates G'. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_6v_5v_1v_2v_3v_4v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (2) for P forces $v_{10}v_3 \in E(G')$. Then the set $\{v_1, v_3, v_7, v_{12}\}$ dominates G'.

Case 2.1.3: $v_{13}v_9 \in E(G')$. The path $(v_1v_2 \dots v_9v_{13}v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G'. If $v_{10}v_3 \in E(G')$, then the set $\{v_1, v_7, v_{10}, v_{12}\}$ dominates G'. So, $v_{10}v_6 \in E(G')$. The path $(v_1v_5v_6v_{10}v_{11}v_{12}v_{13}v_9v_8v_7v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. By (2) for this path, this neighbor is one of v_8 and v_{12}. If $v_2v_8 \in E(G')$, then the set $\{v_2, v_4, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_2, v_4, v_9, v_{10}\}$ dominates $G' - v_{14}$.

Case 2.2: $v_4v_8 \in E(G')$. The path $(v_{13}v_{12} \dots v_8v_4v_3v_2v_1v_5v_6v_7)$ forces v_7 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is amongst v_{10}, v_{11}, and v_{12}.

Case 2.2.1: $v_7v_{10} \in E(G')$. The path $(v_{13}v_{12} \dots v_{10}v_7v_6v_5v_1v_2v_3v_4v_8v_9)$ forces v_9 to have its third neighbor in G'. By (2) for P, $v_9v_{11} \notin E(G)$. If $v_9v_{12} \in E(G)$ or for some $i \in \{2,3\}$, $v_9v_i \in E(G)$, then the set $\{v_i, v_5, v_7, v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_9v_6 \in E(G)$. Now by (2) for P, only v_3 can be the third neighbor of v_{13}. Then the set $\{v_3, v_5, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 2.2.2: $v_7v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_7v_6v_5v_1v_2v_3v_4v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) for this path, this neighbor is one of v_3, or v_{13}. Symmetry with Case 1 forces $v_{10}v_3 \in E(G')$. The set $\{v_3, v_5, v_8, v_{12}\}$ dominates $G' - v_{14}$.

Case 2.2.3: $v_7v_{13} \in E(G')$. The path $(v_{13}v_7v_6v_5v_1v_2v_3v_4v_8v_9 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_2, v_3, v_6, and v_9. If $v_{12}v_2 \in E(G')$, then the set $\{v_2, v_4, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_3 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{10}\}$ dominates G'. So, either $v_{12}v_6 \in E(G')$ or $v_{12}v_9 \in E(G')$.

Case 2.2.3.1: $v_{12}v_6 \in E(G')$. The path $(v_1v_5v_6v_7v_{13}v_{12} \dots v_8v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. By (2) for the path P, this neighbor is one of v_9 and v_{11}. If $v_2v_9 \in E(G')$, then the set $\{v_2, v_4, v_7, v_{11}\}$ dominates $G' - v_{14}$. If $v_2v_{11} \in E(G')$, then the set $\{v_2, v_5, v_9, v_{13}\}$ dominates G'.

Case 2.2.3.2: $v_{12}v_9 \in E(G')$. The path $(v_{13}v_7v_6v_5v_1v_2v_3v_4v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have the third neighbor in G'. By (2) for P, this neighbor is one of v_3, or v_6. If $v_{10}v_3 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{12}\}$ dominates G'. If $v_{10}v_6 \in E(G')$, then the set $\{v_1, v_4, v_6, v_{12}\}$ dominates G'.

Case 2.3: $v_4v_{10} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_4v_3v_2v_1v_5v_6 \dots v_9)$ forces v_9 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_2, v_3, v_6, v_{11}, and v_{13}.

Case 2.3.1: $v_9v_2 \in E(G')$. The set $\{v_2, v_4, v_7, v_{12}\}$ dominates $G' - v_{14}$.

Case 2.3.2: $v_9v_3 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_4v_5v_3v_9v_8 \dots v_5v_1v_2)$ forces v_2 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is one of v_6, or v_{11}. If $v_2v_6 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$. So, $v_2v_{11} \in E(G')$. The third neighbor of v_{13} is one of v_6, or v_7. If $v_{13}v_6 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{13}\}$ dominates G'. If $v_{13}v_7 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{11}\}$ dominates G'.

Case 2.3.3: $v_9v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_4v_3v_2v_1v_5v_6v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (2) for P, this neighbor is amongst v_3, v_{11}, and v_{13}. If $v_7v_3 \in E(G')$, then the set $\{v_1, v_3, v_9, v_{12}\}$ dominates G'. If $v_7v_{11} \in E(G')$, then the path $(v_{13}v_{12}v_{11}v_7v_6v_5v_1v_2v_3v_4v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. So (2) for this path and for P forces $v_8v_3 \in E(G')$. Then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'. Thus, $v_7v_{13} \in E(G')$. The path $(v_1v_2 \dots v_7v_{13}v_{12} \dots v_8)$ forces v_8 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_3, v_{11}, and v_{12}. If $v_8v_3 \in E(G')$, or $v_8v_{11} \in E(G')$, then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'. If $v_8v_{12} \in E(G')$, then the set $\{v_1, v_4, v_6, v_{12}\}$ dominates G'.

Case 2.3.4: $v_9v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_9v_{10}v_4v_3v_2v_1v_5v_6v_7v_8)$ forces v_8 to have its third neighbor in G', and (2) for this path and for P, forces $v_8v_3 \in E(G')$. Then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'.

Case 2.3.5: $v_9v_{13} \in E(G')$. The path $(v_1v_5v_6 \dots v_9v_{13}v_{12}v_{11}v_{10}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is one of v_6, or v_{11}. If $v_2v_6 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{11} \in E(G')$, then the set $\{v_2, v_4, v_7, v_{13}\}$ dominates G'.

Case 2.4: $v_4v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. Then (2) for this path and for P limits this neighbor to one of v_6, v_7, and v_{13}. By the symmetry between v_1 and v_{13}, $v_{10}v_{13} \notin E(G)$.

Case 2.4.1: $v_{10}v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6v_{10}v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (2) for P and this path, this neighbor is v_{13}. The path $(v_1v_2 \dots v_6v_{10}v_{11}v_{12}v_{13}v_7v_8v_9)$ forces v_9 to have its third neighbor in G'. By (2) for this path, this neighbor is one of v_3 and v_{12}. If $v_9v_3 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{11}\}$ dominates G'. If $v_9v_{12} \in E(G')$, then the set $\{v_1, v_4, v_7, v_9\}$ dominates G'.

Case 2.4.2: $v_{10}v_7 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G', and (2) for this path and for P, forces $v_8v_2 \in E(G')$. Then the set $\{v_2, v_5, v_{10}, v_{13}\}$ dominates G'.

Case 2.5: $v_4v_{13} \in E(G')$. The path $(v_1v_5v_6 \dots v_{13}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3 and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a P_9 (i.e., a path with 9 vertices) undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 3: $v_1v_7 \in E(G')$. The path $(v_{13}v_{12} \dots v_7v_1v_2 \dots v_6)$ forces v_6 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst $v_2, v_3, v_8, v_{10}, v_{11}$, and v_{13}.

Case 3.1: $v_6v_2 \in E(G')$. By the symmetry between v_1 and v_{13} and by (2) for P, the third neighbor of v_{13} is either v_4 or v_3. If $v_{13}v_4 \in E(G)$, then as in Case 2.5, the set $\{v_2, v_4, v_8, v_{11}\}$ dominates $G' - v_{14}$. So, $v_{13}v_3 \in E(G)$. The path $(v_{13}v_3v_4v_5v_6v_2v_1v_7v_8 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for P, this neighbor is amongst v_5, v_8, and v_9. If $v_{12}v_5 \in E(G')$, then the set $\{v_3, v_5, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_8 \in E(G')$, then the path $(v_{13}v_3v_4v_5v_6v_2v_1v_7v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G', and (2) for this path and P forces $v_9v_5 \in E(G')$. In this case, the set $\{v_3, v_5, v_7, v_{11}\}$ dominates $G' - v_{14}$. Thus, $v_{12}v_9 \in E(G')$. The path $(v_{13}v_3v_4v_5v_6v_2v_1v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have the third neighbor in G', and (2) for P forces $v_{10}v_4 \in E(G')$. Then $\{v_1, v_4, v_7, v_{12}\}$ dominates G'.

Case 3.2: $v_6v_3 \in E(G')$. By the symmetry between v_1 and v_{13} and by (2) for P, $v_{13}v_4 \in E(G')$. The path $(v_1v_7v_8 \dots v_{13}v_4v_5v_6v_3v_2)$ forces v_2 to have its third neighbor in G'. As in Case 2.5, v_2 dominates v_1, v_2, v_3 and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a path with 9 vertices undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 3.3: $v_6v_8 \in E(G')$. The path $(v_{13}v_{12} \dots v_8v_6v_7v_1v_2 \dots v_5)$ forces v_5 to have its third neighbor in G'. By (2) for this path and for P this neighbor is one of v_2 and v_{11}. If $v_5v_{11} \in E(G')$, then the set $\{v_3, v_8, v_{11}, v_{14}\}$ dominates G'. Thus, $v_5v_2 \in E(G')$. By the symmetry between v_1 and v_{13}, the third neighbor of v_{13} is either v_4 or v_3. If $v_{13}v_4 \in E(G)$, then as in Case 2.5, the set $\{v_2, v_8, v_{11}, v_{13}\}$ dominates G'. If $v_3v_{13} \in E(G')$, then the set $\{v_2, v_3, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 3.4: $v_6v_{10} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_6v_5 \dots v_1v_7v_8v_9)$ forces v_9 to have its third neighbor in G'. By (2) for this path and for P, and by the symmetry with Case 2, this neighbor is in $\{v_2, v_5, v_{11}\}$. If $v_9v_2 \in E(G')$, then the set $\{v_4, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_9v_5 \in E(G')$, then the set $\{v_3, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_9v_{11} \in E(G')$. Then v_{13} is adjacent to one of v_3 and v_4. If $v_{13}v_3 \in E(G')$, then the set $\{v_1, v_5, v_9, v_{13}\}$ dominates G'. If $v_{13}v_4 \in E(G')$, then the set $\{v_1, v_4, v_7, v_{11}\}$ dominates G'.

Case 3.5: $v_6v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_6v_5 \dots v_1v_7v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) for P and the symmetry with Case 2, this neighbor is one of v_3 and v_4. If $v_{10}v_3 \in E(G')$, then $v_{13}v_4 \in E(G')$, and the set $\{v_1, v_4, v_8, v_{11}\}$ dominates G'. So, $v_{10}v_4 \in E(G')$. Then $v_{13}v_3 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12}v_{11}v_{10}v_4v_5\dots v_9)$ forces v_9 to have its third neighbor in G', and (2) for this path forces $v_9v_5 \in E(G')$. Then the set $\{v_1, v_3, v_9, v_{11}\}$ dominates G'.

Case 3.6: $v_6v_{13} \in E(G')$. The path $(v_{13}v_6v_5\dots v_1v_7v_8\dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (2) for this path and for P, this neighbor is in $\{v_2, v_5, v_8, v_9\}$. If $v_{12}v_2 \in E(G')$, then the set $\{v_4, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_{12}v_5 \in E(G')$, then the set $\{v_3, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_6v_5\dots v_1v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (2) for this path and for P forces $v_{10}v_4 \in E(G')$. In this case, the set $\{v_1, v_4, v_7, v_{12}\}$ dominates G'. Thus, $v_{12}v_8 \in E(G')$. The path $(v_{13}v_6v_5\dots v_1v_7v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (2) for this path and for P, this neighbor is either v_2 or v_5. If $v_9v_2 \in E(G')$, then the set $\{v_4, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_9v_5 \in E(G')$, then the set $\{v_3, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$.

Case 4: $v_1v_8 \in E(G')$. The third neighbor of v_{13} is amongst v_3, v_4, and v_6.

Case 4.1: $v_{13}v_3 \in E(G')$. The set $\{v_3, v_5, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 4.2: $v_{13}v_4 \in E(G')$. The path $(v_{13}v_{12}\dots v_8v_1v_2\dots v_7)$ forces v_7 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_3, v_{10}, and v_{11}.

Case 4.2.1: $v_7v_3 \in E(G')$. The path $(v_1v_8v_9\dots v_{13}v_4v_5v_6v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3 and one vertex of the cycle $(v_4v_5\dots v_{13})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 4.2.2: $v_7v_{10} \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12}\dots v_5)$ forces v_5 to have its third neighbor in G'. By (2) for this path, this neighbor is in $\{v_3, v_9, v_{11}, v_{12}\}$. If $v_5v_3 \in E(G')$, then the set $\{v_1, v_5, v_{10}, v_{12}\}$ dominates G'. If $v_5v_9 \in E(G')$, then the set $\{v_2, v_5, v_7, v_{12}\}$ dominates $G' - v_{14}$. If $v_5v_{12} \in E(G')$, then the set $\{v_1, v_4, v_5, v_{10}\}$ dominates G'. Thus, $v_5v_{11} \in E(G')$. The path $(v_1v_8v_9v_{10}v_7v_6v_5v_{11}v_{12}v_{13}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3, and one vertex of the cycle $(v_4v_5\dots v_{13})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 4.2.3: $v_7v_{11} \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12}\dots v_5)$ forces v_5 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_3, v_9, and v_{12}.

Case 4.2.3.1: $v_5v_3 \in E(G')$. The path $(v_1v_2v_3v_5v_4v_{13}v_{12}\dots v_6)$ forces v_6 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_9, v_{10}, and v_{12}. If $v_6v_9 \in E(G')$, then the set $\{v_3, v_9, v_{11}, v_{14}\}$ dominates G'. If $v_6v_{10} \in E(G')$, then the set $\{v_3, v_8, v_{10}, v_{13}\}$ dominates G'. If $v_6v_{12} \in E(G')$, then the set $\{v_1, v_4, v_6, v_{10}\}$ dominates G'.

Case 4.2.3.2: $v_5v_9 \in E(G')$. The set $\{v_1, v_4, v_5, v_{11}\}$ dominates G'.

Case 4.2.3.3: $v_5v_{12} \in E(G')$. The path $(v_1v_8v_9v_{10}v_{11}v_7v_6v_5v_{12}v_{13}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3, and one vertex of the cycle $(v_4v_5\dots v_{13})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 4.3: $v_6v_{13} \in E(G')$. The set $\{v_3, v_6, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 5: $v_1v_{10} \in E(G')$. Then v_{13} is adjacent to one of v_3 and v_4.

Case 5.1: $v_{13}v_3 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12}\dots v_4)$ forces v_4 to have its third neighbor in G'. By (2) for this path, this neighbor is amongst v_7, v_8, and v_{11}.

Case 5.1.1: $v_4v_7 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12}\dots v_7v_4v_5v_6)$ forces v_6 to have its third neighbor in G'. By (2) for this path, this neighbor is one of v_8 and v_{11}. If $v_6v_8 \in E(G')$, then the set $\{v_1, v_4, v_8, v_{12}\}$ dominates G'. So, $v_6v_{11} \in E(G')$. The path $(v_{13}v_3v_2v_1v_{10}v_9v_8v_7v_4v_5v_6v_{11}v_{12})$ forces v_{12} to have the third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13} and one vertex of the cycle $(v_1v_2\dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.1.2: $v_4v_8 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12}\dots v_8v_4v_5v_6v_7)$ forces v_7 to have its third neighbor in G', and (2) for the path P forces $v_7v_{11} \in E(G')$. Now the path $(v_{13}v_3v_2v_1v_{10}v_9v_8v_4v_5v_6v_7v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. So v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2\dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.1.3: $v_4v_{11} \in E(G')$. The path $(v_{13}v_3v_2v_1v_{10}v_9\dots v_4v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2\dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.2: $v_{13}v_4 \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12}\dots v_5)$ forces v_5 to have its third neighbor in G'. By (2) for this path and for the path $(v_5v_6\dots v_{10}v_1v_2v_3v_4v_{13}v_{12}v_{11})$, this neighbor is amongst v_8, v_9, and v_{11}. Note that a similar argument works for v_9.

Case 5.2.1: $v_5v_8 \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12}\dots v_8v_5v_6v_7)$ forces v_7 to have its third neighbor in G'. By (2) for the path P, this neighbor is one of v_3 and v_{11}. If $v_7v_3 \in E(G')$, then the set $\{v_3, v_5, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. So, $v_7v_{11} \in E(G')$. The path $(v_{13}v_4v_3v_2v_1v_{10}v_9v_8v_5v_6v_7v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2\dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.2.2: $v_5v_9 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_1v_2\dots v_5v_9v_8v_7v_6)$ forces v_6 to have its third neighbor in G', and (2) for this path and for the path $(v_6v_7v_8v_9v_5v_4v_{13}v_{12}v_{11}v_{10}v_1v_2v_3)$ forces $v_6v_3 \in E(G')$. Similarly, $v_8v_{11} \in E(G')$, and then the set $\{v_1, v_6, v_8, v_{13}\}$ dominates G'.

Case 5.2.3: $v_5v_{11} \in E(G')$. The path $(v_{13}v_4v_3v_2v_1v_{10}v_9\dots v_5v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2\dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 6: $v_1v_{11} \in E(G')$. By the symmetry between v_1 and v_{13}, $v_{13}v_3 \in E(G')$, and the set $\{v_3, v_5, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 7: $v_1v_{13} \in E(G')$. As in the proof of Lemma 16, for the cycle $C = (v_1v_2\dots v_{13})$ (1) holds. The path $(v_1v_{13}v_{12}\dots v_2)$ forces v_2 to have its third neighbor in G'. By (1), this neighbor is amongst $v_5, v_6, v_8, v_9, v_{11}$, and v_{12}. Note that a similar argument works for v_{12}.

Case 7.1: $v_2v_5 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_5v_2v_3v_4)$ forces v_4 to have the third neighbor in G'. By (2) for this path and (1) for C, this neighbor is one of v_8 and v_{11}.

Case 7.1.1: $v_4v_8 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_8v_4v_3v_2v_5v_6v_7)$ forces v_7 to have its third neighbor in G'. By (2) for this path and (1) for C, this neighbor is one of v_3 and v_{11}. If $v_7v_3 \in E(G')$, then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'. So, $v_7v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_7v_6v_5v_2v_3v_4v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) for this path and (1) for C, this neighbor is one of v_3 and v_6. If $v_{10}v_3 \in E(G')$, then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'. If $v_{10}v_6 \in E(G')$, then the set $\{v_1, v_4, v_{10}, v_{11}\}$ dominates G'.

Case 7.1.2: $v_4v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_4v_3v_2v_5v_6\dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) for this path and (1) for C, this neighbor is amongst v_3, v_6, and v_7. If $v_{10}v_3 \in E(G')$, then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'. If $v_{10}v_7 \in E(G')$, then the path $(v_1v_{13}v_{12}v_{11}v_4v_3v_2v_5v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. So (2) for this path and (1) for C force $v_8v_{12} \in E(G')$. Then the set $\{v_2, v_5, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_{10}v_6 \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_4v_3v_2v_5v_6v_{10}v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. So (2) for this path and (1) for C force $v_7v_3 \in E(G')$. Then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'.

Case 7.2: $v_2v_6 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_6v_2v_3v_4v_5)$ forces v_5 to have its third neighbor in G'. By (1) for C, this neighbor is amongst v_8, v_9, v_{11}, and v_{12}.
Case 7.2.1: $v_5v_8 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_8v_5v_4v_3v_2v_6v_7)$ forces v_7 to have its third neighbor in G'. By (2) for this path, and the path $(v_7v_6v_2v_1v_{13}v_{12}\dots v_8v_5v_4v_3)$, this neighbor is one of v_3 and v_{11}.
If $v_7v_3 \in E(G')$, then the set $\{v_3, v_5, v_{10}, v_{13}\}$ dominates G'. Thus, $v_7v_{11} \in E(G')$. Then the path $(v_1v_{13}v_{12}v_{11}v_7v_6v_2v_3v_4v_5v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. So, (2) for this path and (1) for C force $v_{10}v_4 \in E(G')$. Then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$.
Case 7.2.2: $v_5v_9 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_9v_5v_4v_3v_2v_6v_7v_8)$ forces v_8 to have its third neighbor in G'. By (2) for this path and (1) for C, this neighbor is either v_{11} or v_{12}. If $v_8v_{12} \in E(G')$, then the path $(v_{13}v_1v_2v_6v_7v_8v_{12}v_{11}v_{10}v_9v_5v_4v_3)$ forces v_3 to have its third neighbor in G', and (2) for this path forces $v_3v_{10} \in E(G')$. In this case, the set $\{v_1, v_5, v_8, v_{10}\}$ dominates G'. So, $v_8v_{11} \in E(G')$. By (1) for C, we need $v_{12}v_3 \in E(G')$. Then the path $(v_1v_{13}v_{12}v_{11}v_8v_7v_6v_2v_3v_4v_5v_9v_{10})$ forces v_{10} to have its third neighbor in G'. If $v_{10}v_4 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$. If $v_{10}v_7 \in E(G')$, then the set $\{v_1, v_5, v_7, v_{12}\}$ dominates G'.
Case 7.2.3: $v_5v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_5v_4v_3v_2v_6v_7\dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (2) for the path $(v_{10}v_9\dots v_6v_2v_1v_{13}v_{12}v_{11}v_5v_4v_3)$ this neighbor is either v_3 or v_7. If $v_{10}v_3 \in E(G')$, then the set $\{v_3, v_5, v_8, v_{13}\}$ dominates G'. So, $v_{10}v_7 \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_5v_4v_3v_2v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. Then (1) for C and (2) for the path $(v_8v_9v_{10}v_7v_6v_2v_1v_{13}v_{12}v_{11}v_5v_4v_3)$ eliminate all possible third neighbors of v_8.
Case 7.2.4: $v_5v_{12} \in E(G')$. The path $P' = (v_{13}v_1v_2v_6v_7\dots v_{12}v_5v_4v_3)$ forces v_3 to have its third neighbor in G', and (2) for P' and (1) for C force $v_3v_9 \in E(G')$. Now path $(v_1v_{13}v_{12}v_5v_4v_3v_2v_6v_7\dots v_{11})$ forces v_{11} to have its third neighbor in G'. By (2) for the path $(v_{10}v_{11}v_{12}v_{13}v_1v_2v_3v_9v_8\dots v_4)$, $v_{11}v_8 \in E(G')$. Hence, the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'.
Case 7.3: $v_2v_8 \in E(G')$. By the symmetry between v_2 and v_{12} and by (1) for C, v_{12} is adjacent to one of v_3, v_5 and v_6.
Case 7.3.1: $v_{12}v_5 \in E(G')$. The path $(v_{13}v_1v_2\dots v_5v_{12}v_{11}\dots v_6)$ forces v_6 to have its third neighbor in G'. By (1) for C, this neighbor is amongst v_3, v_9, and v_{10}. The case $v_6v_{10} \in E(G)$ contradicts Lemma 11 with $x = v_6$, $y = v_{10}$ and $z = v_8$.
Case 7.3.1.1: $v_6v_3 \in E(G')$. The path $(v_{13}v_1v_2v_8v_9\dots v_{12}v_5v_4v_3v_6v_7)$ forces v_7 to have its third neighbor in G'. By (1) for C, this neighbor is amongst v_4, v_{10}, and v_{11}. If $v_7v_4 \in E(G')$, then the set $\{v_2, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_7v_{10} \in E(G')$, then the set $\{v_2, v_5, v_{10}, v_{14}\}$ dominates G'. Thus, $v_7v_{11} \in E(G')$. The path $(v_{13}v_1v_2v_8v_7v_6v_3v_4v_5v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G', and (1) for C eliminates all possible third neighbors of v_9.
Case 7.3.1.2: $v_6v_9 \in E(G')$. The path $(v_1v_{13}v_{12}v_5v_4v_3v_2v_8v_7v_6v_9v_{10}v_{11})$ forces v_{11} to have its third neighbor in G', and (2) for this path and (1) for C force $v_{11}v_7 \in E(G')$. Then the set $\{v_1, v_4, v_9, v_{11}\}$ dominates G'.
Case 7.3.2: $v_{12}v_6 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_8v_2v_3\dots v_7)$ forces v_7 to have its third neighbor in G'. By (2) for this path and the symmetric path, this neighbor is one of v_3 and v_{11}. W.l.o.g. assume that $v_7v_3 \in E(G')$. Then the path $(v_1v_{13}v_{12}\dots v_8v_2v_3v_7v_6v_5v_4)$ forces v_4 to have its third neighbor in G', and (2) for this path and (1) for C force $v_4v_{11} \in E(G')$. So, the set $\{v_1, v_4, v_6, v_9\}$ dominates G'.
Case 7.3.3: $v_{12}v_3 \in E(G')$. The path $(v_{13}v_1v_2v_8v_7\dots v_3v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (1) for C, this neighbor is in $\{v_5, v_6\}$. The path $(v_{13}v_1v_2v_8v_9\dots v_{12}v_5v_4v_3v_6v_7)$ forces v_7 to have its third neighbor in G'. If $v_5v_9 \in E(G)$, this contradicts Lemma 11 with $x = v_5$, $y = v_9$ and $z = v_7$. Thus

$v_6v_9 \in E(G)$. Then the path $(v_1v_{13}v_{12}\dots v_9v_6v_7v_8v_2v_3v_4v_5)$ forces v_5 to have its third neighbor in G'. By (1) for C, it is v_{11}. Now the path $(v_1v_{13}v_{12}v_{11}v_5v_4v_3v_2v_8v_7v_6v_9v_{10})$ forces v_{10} to have its third neighbor in G'. This contradicts Lemma 11 with $x = v_8$, $y = v_2$ and $z = v_{10}$.

Case 7.4: $v_2v_9 \in E(G')$. The path $(v_1v_{13}v_{12}\dots v_9v_2v_3\dots v_8)$ forces v_8 to have its third neighbor in G'; so by Lemma 11 with $x = v_6$, $y = v_{10}$ and $z = v_8$,

$$v_6v_{10} \notin E(G). \tag{3}$$

By the symmetry between v_2 and v_{12} and by (1) for C, v_{12} is adjacent to either v_3 or v_5.

Case 7.4.1: $v_{12}v_3 \in E(G')$. The path $(v_{13}v_1v_2v_9v_8\dots v_3v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G'. By (1) for C, this neighbor is in $\{v_4, v_6, v_7\}$. By (3), it is in $\{v_4, v_7\}$, and the set $\{v_2, v_4, v_7, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.4.2: $v_{12}v_5 \in E(G')$. The path $(v_{13}v_1v_2\dots v_5v_{12}v_{11}\dots v_6)$ forces v_6 to have its third neighbor in G'. By (1) for C and (3), this neighbor is v_3. Symmetrically, $v_8v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_8v_7\dots v_2v_9v_{10})$ forces v_{10} to have its third neighbor in G', and similarly v_4 has its third neighbor in G'. Thus $v_4v_{10} \in E(G')$, and the set $\{v_2, v_4, v_7, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.5: $v_2v_{11} \in E(G')$ and symmetrically $v_{12}v_3 \in E(G')$. Let G'' be obtained from $G' - \{v_1, v_2, v_{13}, v_{12}\}$ by identifying v_3 and v_{11} into a new vertex v^*. Graph G'' with 8-cycle $C'' = (v_4v_5\dots v_{10}v^*)$ satisfies the conditions of Lemma 13. So by this lemma, either (a) some v^*-distant vertex $x \in G''$ has an outneighbor in G, or (b) a set $\{y, z\}$ of two vertices dominates $G'' - v^*$. Suppose (a) holds. By symmetry, we may assume that a hamiltonian path P in G'' from v^* to x starts from the edge v^*v_4. Then adding to $P - v^*$ the path $v_{14}v_{13}v_1v_2v_{11}v_{12}v_3v_4$ we produce a hamiltonian in G' path from v_{14} to the vertex x having an outneighbor, a contradiction. Thus (b) holds. Since v^* has only two neighbors in $G'' - v^*$, $v^* \notin \{y, z\}$. Hence the set $\{y, z, v_2, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.6: $v_2v_{12} \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11}\dots v_3)$ forces v_3 to have its third neighbor in G'. By (1) for C, this neighbor is amongst v_6, v_7, v_9, and v_{10}.

Case 7.6.1: $v_3v_6 \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11}\dots v_6v_3v_4v_5)$ forces v_5 to have its third neighbor in G', and (2) for this path and (1) for C force $v_5v_9 \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11}v_{10}v_9v_5v_4v_3v_6v_7v_8)$ forces v_8 to have its third neighbor in G', and (2) for this path and (1) for C force $v_8v_4 \in E(G')$, a contradiction to Lemma 11 with $x = v_4$, $y = v_8$ and $z = v_6$.

Case 7.6.2: $v_3v_7 \in E(G')$. Symmetry forces v_{11} to be adjacent to one of v_4 and v_5. By Lemma 11 with $x = v_3$, $y = v_7$ and $z = v_5$, $v_{11}v_5 \notin E(G')$. Thus, $v_{11}v_4 \in E(G')$. The path $(v_1v_{13}v_{12}v_2v_3v_7v_6v_5v_4v_{11}v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G', and (1) for C forces $v_8v_5 \in E(G')$. Then the set $\{v_1, v_7, v_8, v_{11}\}$ dominates G'.

Case 7.6.3: $v_3v_9 \in E(G')$. Then v_{11} is adjacent to one of v_4 and v_5. Both cases are forbidden by Lemma 11 with $x = v_9$, $y = v_3$ and $z = v_{11}$.

Case 7.6.4: $v_3v_{10} \in E(G')$. This forces $v_{11}v_4 \in E(G')$. Then the path $(v_{13}v_1v_2v_{12}v_{11}v_4v_3v_{10}v_9\dots v_5)$ forces v_5 to have its third neighbor in G'. By (1) for C, this neighbor is one of v_8 and v_9. If $v_5v_8 \in E(G')$, then $v_9v_6 \in E(G')$. Now the path $(v_{13}v_1v_2v_{12}v_{11}v_4v_3v_{10}v_9v_6v_5v_8v_7)$ forces v_7 to have its third neighbor in G', but no possible neighbor exists. Thus, $v_5v_9 \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11}v_4v_3v_{10}v_9v_5v_6v_7v_8)$ forces v_8 to have its third neighbor in G', but (1) for C eliminates all possible neighbors of v_8. □

5. Proofs of Lemmas 4 and 5

For convenience, we restate Lemma 4 here.

Lemma 4 *If a 2-path P in an optimal vdp-cover is such that each of the hamiltonian paths in $G[V(P)]$ has at most one out-endpoint, then either some $(|P|-2)/3$ vertices dominate all vertices of P apart from an out-endpoint or P has at least 14 vertices.*

Proof. If a 2-path $P=(v_1v_2\dots v_k)$ has at most 11 vertices, then $k\in\{2,5,8,11\}$. If $k=2$, then clearly both vertices of P are out-endpoints. The case $k=5$ was considered in Reed's paper [10], and the case $k=8$ is proved in [6]. Hence we may assume that $k=11$. If one of v_1 and v_{11} is an out-endpoint, then we may assume that it is v_{11}. Consider a v_{11}-lasso on $V(P)$ with a largest loop. As described in Section 4, we may assume that this loop is the cycle $C=(v_1\dots v_r)$. Let $G'=G[V(P)]$ and $G''=G[V(C)]$.

Case 1: Vertex v_{11} is an out-endpoint of P. By Lemma 8, if $r\in\{3,6,9\}$, then there exists a dominating set of $G'-v_{11}$ of size 3. If $r=11$, then by Lemma 15, some three vertices dominate $V(P)-v_{11}$. Consider the remaining cases.

Case 1.1: $r=10$. Since v_{11} is an out-endpoint of P, it has at most two neighbors in $V(G'')$ (one of which is v_{10}), and we are done by Lemma 14.

Case 1.2: $r=8$. By Lemma 13 either there exists a dominating set of $G''-v_8$ of size two, and this set together with v_9 dominates $V(P)-v_{11}$, or a (G'',v_8)-distant vertex is adjacent to a vertex in $\{v_9,v_{10},v_{11}\}$, a contradiction to the maximality of r.

Case 1.3: $r=7$. By Lemma 12, either there exists a dominating set of G'' with size two, or a (G'',v_7)-distant vertex is adjacent to a vertex in $\{v_8,v_9,v_{10},v_{11}\}$, a contradiction to the maximality of r.

Case 1.4: $r\le 5$. Since $d_{G''}(v_1)=3$ and by Lemma 8 $v_1v_3\notin E(G)$, $r=5$ and $v_1v_4,v_1v_5\in E(G'')$. Then the path $P_1=(v_2v_3v_4v_1v_5v_6...v_{11})$ shows that v_2 is (G',v_{11})-distant. Hence, v_2 has a neighbor in G' distinct from v_1 and v_3. This neighbor is not in $\{v_4,v_5\}$, since $v_1v_4,v_1v_5\in E(G)$. This contradicts the maximality of r.

Case 2: P has no out-endpoints. We consider a lasso on G' with the largest loop. Since a cubic graph must have an even number of vertices, some vertex of G' must have an outneighbor. In particular, some vertex in G' is not the end of a hamiltonian path in G'. This then gives that $r\ne 11$. Consider the remaining cases.

Case 2.1: $r=10$. Since G' has no out-endpoints, v_{11} has all three of its neighbors in G'. Viewing G' as the 10-cycle C together with the extra vertex v_{11}, we conclude that each vertex v_i adjacent along C to a neighbor of v_{11} is the end of a hamiltonian path on G' connecting v_i with v_{11}. It follows that

(4) each v_i adjacent along C to a neighbor of v_{11} has no outneighbors.

If two neighbors of v_{11} are adjacent along C, then G' is hamiltonian contradicting the maximality of r. If the shortest distance along C between two neighbors of v_{11} is at least 3, then we may assume that $v_{11}v_3\in E(G')$ and $v_{11}v_7\in E(G')$. Then by (4), only v_5 has an outneighbor. Then any choice of neighbors for v_4 gives a hamiltonian path starting at v_5. Hence every vertex of G' is the end of a hamiltonian path in G' which is a contradiction. Thus, the shortest distance along C between two neighbors of v_{11} is 2. We may assume that $v_{11}v_2\in E(G')$.

Case 2.1.1: $v_8v_{11}\in E(G')$. By (4), v_1 has its third neighbor in G'. Each of the edges v_1v_3, v_1v_7, or v_1v_9 then forces a hamiltonian cycle in G'. Hence this third neighbor is amongst v_4,v_5, and v_6. If $v_1v_5\in E(G')$, then every vertex of G' is the end of some hamiltonian path, a contradiction. Hence v_1 is adjacent to one of

v_4 or v_6. Symmetry forces v_9 to be adjacent to the other of these vertices, and again every vertex in G' is the end of some hamiltonian path. Hence $v_8v_{11} \notin E(G')$ and $v_4v_{11} \notin E(G')$ by symmetry.

Case 2.1.2: $v_7v_{11} \in E(G')$. Then adding an edge from v_1 to v_3, v_6, v_8, or v_9 gives the hamiltonian cycles $(v_1v_3v_4 \dots v_{11}v_2)$, $(v_1v_6v_5 \dots v_2v_{11}v_7v_8v_9v_{10})$, $(v_8v_9v_{10}v_{11}v_7v_6 \dots v_1)$, and $(v_1v_{10}v_{11}v_2v_3 \dots v_9)$ respectively. Thus v_1 must be adjacent to one of v_4 or v_5. However, if v_1 is adjacent to either v_4, or v_5, the other is the start of a hamiltonian path in G', so G' has an out-endpoint for some hamiltonian path which contradicts the assumption of Case 2. Hence v_{11} is not adjacent to v_7 or v_5.

Case 2.1.3: $v_6v_{11} \in E(G')$. Then by the symmetry between v_{11} and v_1, in order to avoid Cases 2.1.1 and 2.1.2, we need $v_6v_1 \in E(G')$. But v_6 cannot have 4 neighbors.

Case 2.2: $r = 9$. The maximality of r restricts the neighbors of v_{11} to v_3, v_4, v_5, or v_6. If v_{11} is adjacent to either of v_3, or v_6, then the set $\{v_3, v_6, v_9\}$ dominates G'. Hence v_{11} is adjacent to both of v_4, and v_5. This then gives the lasso having the loop $(v_1 \dots v_4v_{11}v_5 \dots v_9)$ which contradicts the maximality of r.

Case 2.3: $r = 8$. The maximality of r restricts the neighbors of v_{11} to v_4 and v_9. Then v_{10} has a third neighbor in G', but any possible neighbor contradicts the maximality of r.

Case 2.4: $r = 7$. The only possible neighbors of v_{11} not contradicting the maximality of r are v_8 and v_9. Then the path $(v_1 \dots v_9v_{11}v_{10})$ is also hamiltonian in G', and similarly we have $v_{10}v_8 \in E(G)$. Then $d(v_8) > 3$, a contradiction.

Case 2.5: $r = 6$. Since G' has maximum degree 3, the lowest indexed neighbor of v_{11} is at least v_7. So, by Lemmas 6 and 7, a single vertex dominates $\{v_{11}, v_{10}, v_9, v_8\}$, and this vertex along with v_3 and v_6 gives a dominating set of G' with size 3. □

Case 2.6: $r \leq 5$. The highest indexed neighbor of v_1 is smaller than the lowest indexed neighbor of v_{11}. So, by Lemmas 6 and 7, a vertex dominates $\{v_1, v_2, v_3, v_4\}$, a vertex dominates $\{v_{11}, v_{10}, v_9, v_8\}$, and a v_6 dominates v_5 and v_7. □

For convenience, we also restate Lemma 5.

Lemma 5 *Let $P_1 = (x_1, \dots, x_k)$ be a tip of an accepting 2-path P in an optimal vdp-cover. Let $X(P_1)$ be the set of the hamiltonian paths in $G[V(P_1)]$ one of whose ends is x_k. If none of the other ends of any path in $X(P_1)$ is an out-endpoint of P or a $(2,2)$-endpoint, then either some $(k-1)/3$ vertices dominate $V(P_1)$, or $k \geq 16$.*

Proof. For $k \leq 7$, it was proved in [10][Fact 11], for $k = 10$ it was proved in [6][Lemma 14]. Both cases will also be clear from the proof for $k = 13$ below. So, suppose that a tip $P_1 = (v_1v_2 \dots v_{13})$ of an accepting 2-path P has no out-endpoint and no $(2,2)$-endpoint. Let v_{14} be the second (i.e. the other than v_{12}) neighbor of v_{13} in the path P. Let G' be the subgraph of G induced by $V(P_1) + v_{14}$. Since our system of paths was chosen to maximize the number of out-endpoints and $(2,2)$-endpoints and taking into account (B4) of Lemma 1,

(5) no (G', v_{14})-distant vertex in G' has an outneighbor (with respect to $V(G')$).

We choose a (G', v_{14})-distant vertex in G' and an edge incident to this vertex so that to maximize the length of the loop of a v_{14}-lasso in G'. We renumber the vertices in G' so that this vertex is v_1 and this loop is $(v_1v_2 \dots v_r)$. Then let G'' be the graph induced by the set $\{v_1, v_2, \dots, v_r\}$. By the maximality of r and (5),

(6) no (G'', v_r)-distant vertex in G'' has an outneighbor with respect to G''.

If $r = 14$, then we are done by Lemma 17.

Let $r < 14$. Then v_1 has two neighbors in $G'' - v_2$. By Lemma 8,

$$v_1 v_{3j} \notin E(G) \text{ for } j = 1, 2, 3, 4, \tag{7}$$

and hence $r \notin \{3, 6, 9, 12\}$.

Case 1: $r = 13$. By Lemma 16, either some 4 vertices dominate $V(P_1)$ (in which case we are done), or some (G'', v_{13})-distant vertex v_j has an outneighbor with respect to G'', a contradiction to (6).

Case 2: $r \in \{10, 11\}$. By Lemma 15 (if $r = 11$) or Lemma 14 (if $r = 10$), either some 3 vertices dominate $v_1, v_2, \dots, v_{10}$ (then this set along with v_{12} dominates $G' - v_{14}$), or some (G'', v_r)-distant vertex v_j has an outneighbor, a contradiction to (6).

Case 3: $r \in \{7, 8\}$. By Lemma 13 (if $r = 8$) or Lemma 12 (if $r = 7$), either some 2 vertices dominate $v_1, v_2, \dots, v_7$ (then this set along with v_9 and v_12 dominates G'), or some (G'', v_r)-distant vertex v_j has an outneighbor, a contradiction to (6).

Case 4: $r \leq 5$. By (7) $r = 5$ and the three neighbors of v_1 are v_2, v_4, and v_5. Since there is the path $(v_3 v_2 v_1 v_4 v_5 \dots v_{13})$, by (6), v_3 has no neighbors outside of G''. So by (7), $v_3 v_5 \in E(G)$, but v_5 already has 3 other neighbors. □

6. Proof of Lemma 3

Recall that Lemma 3 states that each 1-path P in an optimal vdp-cover S that does not have an out-endpoint and does not contain a dominating set of size at most $(|P| - 1)/3$, has at least 28 vertices. Fact 9 in [10] states that such a path must have at least 16 vertices. Lemma 2 in [6] extends this by proving that such a path has at least 22 vertices. Hence we need to prove that such path cannot have 25 vertices and cannot have 22 vertices. We will prove this in two big lemmas. But first we introduce the notion of (H, v)-distant vertices for $v \notin V(H)$. If H is a subgraph of G and $x \in V(G) - V(H)$, then a vertex $y \in V(H)$ is (H, x)-*distant*, if H contains a hamiltonian path connecting y with a neighbor of x.

Lemma 18. *If a* 1*-path* P *in an optimal vdp-cover* S *does not have an out-endpoint and does not contain a dominating set of size at most* $(|P| - 1)/3$*, then* P *cannot have* 22 *vertices.*

Proof. Let $P = (v_1 v_2 \dots v_{22})$ be a counter-example to the lemma, and let $G' = G[V(P)]$. Consider a v_{22}-lasso on $V(P)$ with a largest loop $C = (v_1 \dots v_r)$. Let $H = G' - C$. If $r = 22$, then by the definition of P, no vertex of P has an outneighbor. So in this case G' is a cubic hamiltonian graph and by Theorem 5 is dominated by 7 vertices. Thus $r \leq 21$. Also by Lemma 8, r is not divisible by 3. If $r \leq 14$, then since each end of every hamiltonian path in G' has no outneighbors, Lemmas 6, 7 and 12–17 imply that for some i, some set D of i vertices dominates the set $\{v_1, \dots, v_{3i+1}\}$. Then the set $D \cup \{v_{3(i+1)}, v_{3(i+2)}, \dots, v_{21}\}$ dominates G' and has 7 vertices. Thus $r \in \{16, 17, 19, 20\}$.

Case 1: $r = 16$. By the maximality of r, for each (H, v_{16})-distant vertex of H, only v_7, v_8, and v_9 are possible neighbors on C. By Lemma 8, $v_{22} v_8 \notin E(G)$. So $N(v_{22}) - v_{21} \subset \{v_7, v_9, v_{18}, v_{19}\}$.

Case 1.1: $|N(v_{22}) \cap \{v_7, v_9\}| = 1$. By symmetry, we may assume that $v_{22} v_7 \in E(G')$.

Case 1.1.1: $v_{22} v_{18} \in E(G')$. Because of the path $(v_{16} v_{17} v_{18} v_{22} v_{21} v_{20} v_{19})$, vertex v_{19} is (H, v_{16})-distant. By Lemma 8 for P, v_{19} has only two neighbors in H. Since v_7 already has 3 neighbors, v_{19} is adjacent to v_9. If v_{17} has two neighbors in C, then since it is (H, v_7)-distant, the second (apart from v_{16}) neighbor in C

should be v_{14}. On the other hand, since v_{17} is (H, v_9)-distant, this neighbor should be v_2, a contradiction. So v_{17} has two neighbors in H. If $v_{17}v_{20} \in E(G)$, then the path $(v_{16}v_{17}v_{20}v_{19}v_{18}v_{22}v_{21})$ shows that v_{21} is (H, v_{16})-distant. Hence the third neighbor of v_{21} is in $H \cup \{v_7, v_9\}$. But all vertices in this set already have degree 3. Thus $v_{17}v_{21} \in E(G)$. Then the path $(v_{16}v_{17}v_{21}v_{22}v_{18}v_{19}v_{20})$ shows that v_{20} is (H, v_{16})-distant, and all possible neighbors of v_{20} already have degree 3.

Case 1.1.2: $v_{22}v_{19} \in E(G')$. If $v_{17}v_{20} \in E(G)$, then the set $\{v_{17}, v_{22}, v_2, v_5, v_8, v_{11}, v_{14}\}$ dominates G'. If $v_{17}v_{21} \in E(G)$, then we have Case 1.1.1 with v_{17} in place of v_{22}. So $v_{17}v_{14} \in E(G)$. The path $(v_{17}v_{18}v_{19}v_{22}v_{21}v_{20})$ shows that v_{20} is (H, v_{16})-distant and (H, v_{14})-distant. So if its third neighbor is in C, then it should be v_9 because of v_{16} and v_5 because of v_{14}, a contradiction. So $v_{20}v_{18} \in E(G')$, a contradiction to Lemma 8 for the path $(v_{17}v_{18} \dots v_{22}v_7v_8 \dots v_{16}v_1 \dots v_6)$.

Case 1.2: $v_{22}v_{18} \in E(G')$ and $v_{22}v_{19} \in E(G')$. Because of the path $H' = (v_{16}v_{17}v_{18}v_{22}v_{19}v_{20}v_{21})$, vertex v_{21} can play the role of v_{22}. By Lemma 8, $v_{21}v_{17} \notin E(G')$. So we have Case 1.1 for C and H'.

Case 1.3: $v_{22}v_9 \in E(G')$ and $v_{22}v_7 \in E(G')$. Consider G' as a lasso with the cycle $C' = (v_7v_8 \dots v_{22})$ and handle $H' = (v_{16}v_1v_2 \dots v_6)$. As above, only v_7 and v_9 can be the neighbors of v_6 on C'. Since v_9 already has 3 neighbors, we are in Case 1.1 for C' and H', which is proved.

Case 2: $r = 17$. By the maximality of r and Lemma 8, only v_7, and v_{10} can be the neighbors on C of any (H, v_{17})-distant vertex. So as in Case 1, by Lemma 8, $N(v_{22}) - v_{21} \subset \{v_7, v_{10}, v_{18}, v_{19}\}$.

Case 2.1: Exactly one of v_7 and v_{10} is a neighbor of v_{22}. Again, we may assume that $v_{22}v_7 \in E(G')$. If $v_{22}v_{18} \in E(G)$, then v_{21} is (H, v_7)-distant and v_{19} is (H, v_{17})-distant. They are not adjacent by Lemma 8 for P, and so $v_{21}v_{14} \in E(G')$ and $v_{19}v_{10} \in E(G')$. Now the set $\{v_{19}, v_{21}, v_2, v_5, v_8, v_{12}, v_{16}\}$ dominates G'. Thus $v_{22}v_{19} \in E(G)$. Then v_{20} is (H, v_{17})-distant. If $v_{20}v_{18} \in E(G')$, then the set $\{v_{18}, v_{22}, v_2, v_5, v_9, v_{12}, v_{15}\}$ dominates G'. So, $v_{20}v_{10} \in E(G')$. If $v_{18}v_{21} \in E(G')$, then the set $\{v_{18}, v_2, v_5, v_7, v_{10}, v_{12}, v_{15}\}$ dominates G'. Otherwise, since v_{18} is (H, v_7)-distant it is adjacent to v_{14}, but since it also is (H, v_{10})-distant it is adjacent to v_3, a contradiction.

Case 2.2: $v_{22}v_{18} \in E(G')$ and $v_{22}v_{19} \in E(G')$. We just repeat the proof of Case 1.2.

Case 2.3: $v_{22}v_7 \in E(G')$ and $v_{22}v_{10} \in E(G')$. Then by symmetry v_{18} has its third neighbor, say v_i. Since Case 2.1 is proved, $i < 17$, and v_i is at distance 7 along C from both v_7, and v_{10}, an impossibility.

Case 3: $r = 19$. First note that if v_{21} has its third neighbor in G', then v_{21} dominates all but a P_{18}, which can be dominated by 6 vertices. Thus v_{21}'s third neighbor is outside of G'. Also if $v_{20}v_{22} \in E(G')$, then v_{20} dominates all but a P_{18}. Thus we may assume that each of v_{20} and v_{22} has two neighbors on C. Furthermore, each vertex in G' that is adjacent to a neighbor of v_{20} or v_{22} is an endpoint of a hamiltonian path in G', and hence has its third neighbor in G'.

Case 3.1: The neighbors of v_{20} and v_{22} on C do not alternate. Let d be the maximum of the distance between the neighbors of v_{20} on C and the distance between the neighbors of v_{22} on C. We can assume that v_{20} is adjacent to v_{19} and v_d on C. We can further assume that the neighbors of v_{22} on C are v_{d+a} and v_{d+a+c}. Let $b = 19 - d - a - c$ (See the left graph in Figure 4). By symmetry, we may assume that $a \le b$. Maximality of r forces the neighbors of v_{20} to be at least distance 4 apart on C from the neighbors of v_{22}, in particular, $b \ge a \ge 4$. It also forces $c, d \ge 2$. Lemma 8 for P and symmetry eliminate all cases where neighbors of v_{20} are distance $5, 8, 11$, or 14 apart on C from the neighbors of v_{22}. Summarizing, we have

$$b \ge a \ge 4, \quad a, b, a+c, b+c, a+d, b+d \notin \{5, 8, 11, 14\}, \text{ and } 2 \le c \le d. \tag{8}$$

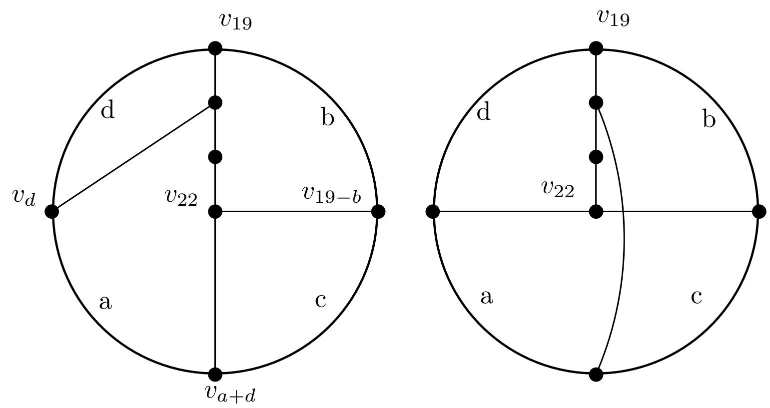

Figure 4

Case 3.1.1: $d > 9$. Then $c = 19 - (a + b + d) \leq 19 - (4 + 4 + 10) = 1$, a contradiction to (8).

Case 3.1.2: $d = 9$. Then $a + b = 19 - c - 9 = 10 - c \leq 8$. So by (8), $a = b = 4$. By Lemma 9, the third neighbor of v_{14} is one of $v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{17}$, and v_{18}. The path $(v_{14}v_{15} \ldots v_{22}v_{13}v_{12} \ldots v_1)$ along with Lemma 8 restricts this set of possible neighbors to $\{v_1, v_4, v_7, v_{10}, v_{17}, v_{18}\}$. Then Lemma 8 with the path $(v_{14}v_{13} \ldots v_9v_{20}v_{21}v_{22}v_{15}v_{16} \ldots v_{19}v_1v_2 \ldots v_8)$ restricts the set of possible neighbors of v_{14} to $\{v_{10}, v_{18}\}$. Since either of these edges forms a 4-arc in G' and since v_{16}, and v_{12} both have third neighbors in G', by Lemma 11, no good third neighbor exists for v_{14}.

Case 3.1.3: $d = 8$. If $a = 6$, then $a + d = 14$, a contradiction to (8). Similarly, $b \neq 6$. Hence $a = b = 4$. By Lemma 9, the third neighbor of v_{14} is one of $v_1, v_2, v_4, v_5, v_7, v_{10}, v_{11}, v_{17}$, and v_{18}. The path $(v_{13}v_{14} \ldots v_{22}v_{12}v_{11} \ldots v_1)$ along with Lemma 8 restricts this set of possible neighbors to $\{v_2, v_5, v_{11}, v_{17}, v_{18}\}$. The symmetry of the role of H with the role of the set $\{v_{18}, v_{17}, v_{16}\}$ eliminates v_{17} as a possible neighbor of v_{14}. If $v_{14}v_{18} \in E(G')$, then Lemma 11 with $x = v_{14}$, $y = v_{18}$ and $z = v_{16}$ yields that v_{16} has no third neighbor in G', a contradiction to the fact it is adjacent to a neighbor of v_{22}. Thus $v_{14}v_{18} \notin E(G')$. Now Lemma 11 with $x = v_{12}$, $y = v_{14}$ and $z = v_{19}$ eliminates all remaining potential neighbors of v_{14}.

Case 3.1.4: $d = 7$. If $a = 4$, then $a + d = 11$, a contradiction to (8). Similarly, $b \neq 4$. Then $a, b \geq 6$, and $a + b + c + d \geq 6 + 6 + 2 + 7 = 21$, a contradiction.

Case 3.1.5: $d = 6$. Then $a + b \leq 19 - 2 - 6 = 11$.

Case 3.1.5.1: $a = b = 4$. Let v_i be the third neighbor of v_{14}. By Lemma 9, $i \in \{1, 2, 4, 5, 7, 8, 11, 17, 18\}$. The path $(v_{14}v_{13} \ldots v_6v_{20}v_{21}v_{22}v_{15}v_{16} \ldots v_{19}v_1v_2 \ldots v_5)$ with Lemma 8 restricts this set to $\{2, 5, 7, 8, 11, 18\}$. The path $(v_{11}v_{12} \ldots v_{22}v_{10}v_9 \ldots v_1)$ with Lemma 8 shrinks this set to $\{7, 11, 18\}$. If $i = 18$, then by Lemma 11 with $x = v_{14}$, $y = v_{16}$ and $z = v_{18}$, graph $G'' = G' - \{v_{20}, v_{21}, v_{22}\}$ has a dominating set of size 6. If $i = 7$, then the path $(v_8v_9 \ldots v_{14}v_7v_6 \ldots v_1v_{19}v_{18} \ldots v_{15}v_{22}v_{21}v_{20})$ forces the third neighbor of v_8 to be in G', which contradicts the fact that the role of H can be switched with $\{v_7, v_8, v_9\}$. Thus $i = 11$.

Then the path $(v_{12}v_{13}v_{14}v_{11}v_{10}\dots v_1v_{19}v_{18}\dots v_{15}v_{22}v_{21}v_{20})$ forces v_{12} to have its third neighbor in G'. By Lemma 8 for this path and Lemma 9 for C, this neighbor is in $\{v_2, v_5, v_8, v_{18}\}$. For $j \in \{2,5,8\}$, Lemma 10 with $R = \{v_j, v_{10}, v_{12}, v_{19}\}$ eliminates v_j from the list. Thus $v_{12}v_{18} \in E(G')$. Then the hamiltonian cycle $(v_{11}v_{10}\dots v_1v_{19}v_{20}v_{21}v_{22}v_{15}v_{16}v_{17}v_{18}v_{12}v_{13}v_{14})$ contradicts the maximality of r.

Case 3.1.5.2: $a = 4, b = 6$. Let v_i be the third neighbor of v_{11}. By Lemma 9, $i \in \{1,2,4,5,7,8,14,15,17,18\}$. The path $(v_{11}v_{12}\dots v_{22}v_{10}v_9\dots v_1)$ and Lemma 8 further reduces this set to $\{1,4,7,14,15,17,18\}$. The path $(v_{12}v_{11}\dots v_6v_{20}v_{21}v_{22}v_{13}v_{14}\dots v_{19}v_1v_2\dots v_5)$ and Lemma 8 eliminate 15 and 18 from this list. If $i \in \{1,4,14,17\}$, then Lemma 10 with $R = \{v_6, v_{11}, v_{13}, v_i\}$ gives a dominating set of size 7. Thus $i = 7$. Then Lemma 11 with $x = v_7$, $y = v_{11}$ and $z = v_9$ gives a dominating set of size 7 in G'.

Case 3.1.5.3: $a = 4, b = 7$. Lemma 9 limits the third neighbor of v_{11} to one of $v_1, v_2, v_4, v_5, v_7, v_8, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_{11}v_{10}\dots v_6v_{20}v_{21}v_{22}v_{12}v_{13}\dots v_{19}v_1v_2\dots v_{15})$ and Lemma 8 limit this neighbor to one of $v_2, v_5, v_7, v_8, v_{15}$ and v_{18}. The path $(v_{11}v_{12}\dots v_{22}v_{10}v_9\dots v_1)$ and Lemma 8 further limit this neighbor to one of v_7, v_{15} and v_{18}. Now Lemma 11 with $x = v_7$, $y = v_9$ and $z = v_{11}$ (respectively, with $x = v_{11}$, $y = v_{13}$ and $z = v_{15}$) yields a dominating set of size 7 if $v_{11}v_7 \in E(G)$ (respectively, if $v_{11}v_{15} \in E(G)$). So $v_{11}v_{18} \in E(G)$. Then the hamiltonian cycle $(v_{12}v_{13}\dots v_{18}v_{11}v_{10}\dots v_1v_{19}v_{20}v_{21}v_{22})$ contradicts the maximality of r.

Thus $a, b \geq 6$, and hence $a + b + c + d \geq 20$, a contradiction.

Case 3.1.6: $d = 5$. By (8), $a, b \neq 11 - d = 6$. If $a, b \geq 7$, then $a + b + c + 5 > 19$, a contradiction. So one of a and b, say a, is 4. Then by the maximality of d, $b \geq 5$, and so $b \in \{5,6,7,8\}$. Hence by (8), $b = 7$. Then Lemma 9 limits the third neighbor of v_{11} to one of $v_1, v_2, v_4, v_7, v_8, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_{11}v_{10}, \dots v_1v_{19}v_{20}v_{21}v_{22}v_{12}v_{13}\dots v_{18})$ and Lemma 8 eliminate v_{14} and v_{17} from the list of possible neighbors. The path $(v_{10}v_{11}\dots v_{22}v_9v_8\dots v_1)$ and Lemma 8 limit this neighbor to one of v_2, v_8, v_{15}, and v_{18}. By Lemma 11 with $x = v_{11}$, $y = v_{15}$, and $z = v_{13}$, $v_{15}v_{11} \notin E(G)$. Lemma 10 with $R = \{v_9, v_{11}, v_{19}, v_2\}$ and $R = \{v_9, v_{11}, v_{19}, v_8\}$ eliminates v_2, and v_8 as neighbors of v_{11}. Thus $v_{11}v_{18} \in E(G')$. Then the hamiltonian cycle $(v_{18}v_{17}\dots v_{12}v_{22}v_{21}v_{20}v_{19}v_1v_2\dots v_{11})$ contradicts the maximality of r.

Case 3.1.7: $d = 4$. By (8), $a, b \notin \{4,5,7,8\}$. Then if $\max\{a,b\} \geq 9$, then $a + b + c + d \geq 6 + 9 + 2 + 4 = 21$, hence $a, b = 6$. By Lemma 9, the third neighbor of v_{11} is in $\{v_1, v_2, v_5, v_7, v_8, v_{14}, v_{15}, v_{17}, v_{18}\}$. The path $(v_{11}v_{12}\dots v_{22}v_{10}v_9\dots v_1)$ and Lemma 9 shrink this set to $\{v_1, v_7, v_{14}, v_{15}, v_{17}, v_{18}\}$. The path $(v_{12}v_{11}\dots v_1v_{19}v_{20}v_{21}v_{22}v_{13}v_{14}\dots v_{18})$ and Lemma 9 yield that this neighbor is in $\{v_1, v_7, v_{14}, v_{17}\}$. By Lemma 11 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_7v_{11} \notin E(G)$. If $v_{11}v_1 \in E(G')$, then the hamiltonian cycle $(v_1v_2\dots v_{10}v_{22}v_{21}\dots v_{11})$ contradicts the maximality of r. If $v_{11}v_{17} \in E(G')$, then the set $\{v_2, v_5, v_8, v_{11}, v_{15}, v_{19}, v_{22}\}$ dominates G'. Finally, if $v_{11}v_{14} \in E(G')$, then symmetry gives $v_{12}v_9 \in E(G')$, and hence the set $\{v_1, v_3, v_6, v_9, v_{14}, v_{17}, v_{21})\}$ dominates G'.

Case 3.1.8: $d = 3$. In this case, $2 \leq c \leq 3$. If $a = 4$, then $b \in \{19-4-3-3, 19-4-2-3\} = \{9, 10\}$. If $a = 6$, then similarly, $7 \leq b \leq 8$, but by (8), $b \neq 8$. Finally, if $a \geq 7$, then $b \leq 19 - 7 - 2 - 4 = 7$, and hence in this case $a = b = 7$.

Case 3.1.8.1: $a = 4, b = 9$. Lemma 9 limits the third neighbor of v_8 to one of $v_1, v_2, v_4, v_5, v_{11}, v_{12}, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_8v_9\dots v_{22}v_7v_6\dots v_1)$ and Lemma 8 limit this neighbor to one of $v_1, v_4, v_{11}, v_{12}, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_9v_8\dots v_3v_{20}v_{21}v_{22}v_{10}v_{11}\dots v_{19}v_1v_2)$ and Lemma 8 limit this neighbor to one of v_1, v_4, v_{11}, v_{14} or v_{17}. By Lemma 11 with $x = v_4$, $y = v_8$, and $z = v_6$, $v_8v_4 \notin E(G)$. For $i \in \{1, 11, 14, 17\}$, Lemma 11 with $x = v_8$, $y = v_i$, and $z = v_3$ eliminates v_i as a neighbor of v_8.

Case 3.1.8.2: $a = 4, b = 10$. By Lemma 10 with $R \supset \{v_{19}, v_2, v_7\}$, the third neighbor of v_2 is in $\{v_4, v_5, v_6\}$. Lemma 8 for P yields $v_2v_4 \notin E(G)$. By Lemma 11 with $x = v_2$, $y = v_6$, and $z = v_4$, $v_2v_6 \notin E(G)$. Thus $v_2v_5 \in E(G')$. Then the 21-cycle $(v_3v_4v_5v_2v_1v_{19}v_{18} \dots v_7v_{22}v_{21}v_{20})$ contradicts the maximality of r.
Case 3.1.8.3: $a = 6, b = 7$. Lemma 9 for C limits the third neighbor of v_{11} to one of $v_1, v_2, v_4, v_5, v_7, v_8, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_{11}v_{10} \dots v_3v_{20}v_{21}v_{22}v_{12}v_{13} \dots v_{19}v_1v_2)$ limits this neighbor to one of $v_2, v_4, v_5, v_7, v_8, v_{15}$, and v_{18}. The path $(v_{10}v_{11} \dots v_{22}v_9v_8 \dots v_1)$ limits the neighbor to one of v_2, v_5, v_8, v_{15}, and v_{18}. For $i \in \{2, 5, 8\}$, Lemma 11 with $x = v_i$, $y = v_{11}$, and $z = v_{19}$ eliminates v_i as a neighbor of v_8. By Lemma 11 with $x = v_{11}$, $y = v_{15}$, and $z = v_{13}$, $v_{11}v_{15} \notin E(G)$. Thus $v_{11}v_{18} \in E(G')$. Then the Hamiltionian cycle $(v_{19}v_{20}v_{21}v_{22}v_{12}v_{13} \dots v_{18}v_{11}v_{10} \dots v_1)$ contradicts the maximality of r.
Case 3.1.8.4: $a = 7, b = 7$. Lemma 11 with $x = v_{12}$, $y = v_3$, and $z = v_1$ yields that the third neighbor of v_1 is in $\{v_{13}, v_{14}, \dots, v_{18}\}$. Lemma 9 for C shortens the list to $\{v_{13}, v_{14}, v_{16}, v_{17}\}$. For $i = 14, 17$, Lemma 10 with $R = \{v_1, v_3, v_{19}, v_i\}$ gives $v_1v_i \notin E(G')$. If $v_1v_{13} \in E(G)$, then the cycle $(v_1v_2 \dots v_{12}v_{22}v_{21}v_{20}v_{19}v_{18} \dots v_{13})$ contradicts the maximality of r. So $v_1v_{16} \in E(G)$. By symmetry, $v_2v_6 \in E(G)$. Again by symmetry and by Lemma 9 for C, we may assume that the third neighbor of v_{11} is in $\{v_7, v_8\}$. Edge v_7v_{11} contradicts Lemma 11 with $x = v_{11}$, $y = v_7$, and $z = v_9$. So $v_8v_{11} \in E(G)$. Then the set $\{v_1, v_3, v_5, v_8, v_{14}, v_{18}, v_{22})\}$ dominates G'.
Case 3.1.9: $d = 2$. By (8), $a, b \notin \{5, 6, 8, 9, 11, 12\}$. Since $c = 2$, we have $d + c = 4$ and hence $a, b \notin \{4, 7, 10\}$. This proves the case.

Case 3.2: The neighbors of v_{20} and v_{22} alternate on C. Let $v_{22}v_d, v_{20}v_{d+a}, v_{22}v_{19-b} \in E(G')$. Define $c = 19 - a - b - d$ (see the graph on the right in Figure 4). By symmetry, we may assume that $d = \max\{a, b, c, d\}$. By Lemma 8, $5 \notin \{a, b, c, d\}$. By the maximality of r, $\min\{a, b, c, d\} \geq 4$, and so $d \leq 7$. Furthermore, if $d = 7$, then $a, b, c = 4$, and the set $\{v_2, v_5, v_9, v_{13}, v_{17}, v_{20}, v_{22}\}$ dominates G'. If $d = 6$, $a + b + c = 13$, a contradiction to $5 \notin \{a, b, c, d\}$. Finally, if $d = 4$, then $a + b + c + d = 16 < 19$.

Case 4: $r = 20$. Lemma 8 for the paths P and $\{v_{22}v_{21}v_{20}v_1v_2 \dots v_{19}\}$ gives the possible neighbors of v_{22} as v_i, $i \in \{1, 4, 7, 10, 13, 16, 19\}$. By the maximality of r, $i \in \{4, 7, 10, 13, 16\}$. It follows that the distance on C from a neighbor of v_{22} to both neighbors of v_{21} is the same modulo 3 (and vice versa). Thus the distance on C between the two neighbors of $v \in \{v_{21}, v_{22}\}$ is 0 (mod 3). Since $r \equiv 2 \pmod 3$ and the neighbors of v_{21} and v_{22} are distance 1 (mod 3) apart on C, these neighbors cannot alternate. Let $v_{22}v_d, v_{22}v_{d+a}, v_{21}v_{20-b} \in E(G')$. Define $c = 20 - a - b - d$. We may assume that $d \leq c$ and $a \leq b$. Therefore,

$$d \leq c,\ a \leq b,\ d + a \leq 10, \quad c, d \equiv 1 \pmod 3, \text{ and } \quad a, b \equiv 0 \pmod 3. \tag{9}$$

Case 4.1: $d = 4$. By (9), $d + a \in \{7, 10\}$. Then v_5 has its third neighbor. By Lemma 8 for the paths $(v_5v_6 \dots v_{20}v_1v_2v_3v_4v_{22}v_{21})$ and $(v_{d+a-1}v_{d+a-2} \dots v_1v_{20}v_{19} \dots v_{d+a}v_{22}v_{21})$, for either choice of $d + a$, the possible neighbors of v_5 are in $\{v_1, v_8, v_9, v_{12}, v_{15}, v_{18}\}$. If $v_5v_1 \in E(G')$, then the hamiltonian cycle $(v_5v_6 \dots v_{22}v_4v_3v_2v_1)$ contradicts the maximality of r. By Lemma 11 with $x = v_{20}$, $y = v_{d+a}$, and $z = v_5$, the third neighbor of v_5 must be in the set $\{v_1, v_2, \dots, v_{d+a-1}\}$. This contradicts the above statement when $d + a = 7$ and leaves v_8 and and v_9 as possible neighbors of v_5 when $d + a = 10$. In this case by (9), $c = 4$. Note that now v_5 is symmetric with v_9, v_{15}, and v_{19}.
Case 4.1.1: $v_5v_9 \in E(G)$. The path $(v_8v_7v_6v_5v_9v_{10} \dots v_4v_{22}v_{21})$ yields that v_8 has its third neighbor in G'. By Lemma 11 with $x = v_{20}$, $y = v_{10}$, and $z = v_8$, this neighbor is in $\{v_1, v_2, v_3\}$. By Lemma 8 for paths

$(v_9v_8\dots v_1v_{20}v_{19}\dots v_{10}v_{22}v_{21})$ and $(v_8v_7v_6v_5v_9v_{10}\dots v_{20}\dots v_4v_{22}v_{21})$ eliminates v_3 and v_2 from this list. So $v_8v_1 \in E(G)$. Then the cycle $(v_8v_7v_6v_5v_9v_{10}\dots v_{20}v_{21}v_{22}v_4v_3v_2v_1)$ contradicts the minimality of r.

Case 4.1.2: $v_5v_9 \notin E(G)$. Then $v_5v_8 \in E(G)$. Since v_5 is symmetric with v_9, v_{15}, and v_{19}, we conclude that $v_6v_9, v_{15}v_{18}, v_{19}v_{16} \in E(G)$. The path $(v_7v_6v_9v_5v_4\dots v_1v_{20}\dots v_{10}v_{22}v_{21})$ yields that v_7 has its third neighbor, say v_i, in G'. If $i \in \{3, 12, 13, 17\}$, then for $j \in \{12, 13\}$ the set $\{v_3, v_5, v_{10}, v_j, v_{14}, v_{17}, v_{20}\}$ dominates G'. Since vertices v_1, v_2 and v_{11} with respect to v_8 are symmetric to v_{13}, v_{12} and v_3, respectively, no possible neighbors for v_8 left.

Case 4.2: $d = 7$. By (9), $d + a = 10$ and so $c = 7$. By Lemma 11 with $x = v_{17}$, $y = v_7$, and $z = v_{19}$, the third neighbor of v_{19} is in $\{v_1, v_2, \dots, v_6\}$. Lemma 8 for P and the 16-vertex path $(v_{19}v_{18}v_{17}v_{21}v_{22}v_{10}v_9\dots v_1v_{20})$ reduces this list to $\{v_3, v_6\}$. If $v_{19}v_6 \in E(G)$, then the cycle $(v_{19}v_{18}\dots v_7v_{22}v_{21}v_{20}v_1\dots v_6)$ contradicts the maximality of r. So $v_{19}v_3 \in E(G)$. Symmetrically, $v_8v_4 \in E(G)$. Now the hamiltonian cycle $(v_{19}v_{18}\dots v_8v_4v_5v_6v_7v_{22}v_{21}v_{20}v_1v_2v_3)$ contradicts the maximality of r. □

Lemma 19. *If a* 1*-path P in an optimal vdp-cover S does not have an out-endpoint and does not contain a dominating set of size at most* $(|P| - 1)/3$*, then* $|P| \neq 25$.

Proof. Let $P = (v_1v_2\dots v_{25})$ be a counter-example to the lemma, and let $G' = G[V(P)]$. Consider a v_{25}-lasso on $V(P)$ with the largest loop. Call the loop C, and the remaining handle H. We may assume that it is a $(v_{25}, 25, r)$-lasso. If $r = 25$, then no vertex of G' has an outneighbor, and hence $G = G'$. But a cubic graph cannot have 25 vertices. Thus $r \leq 24$. Also r is not divisible by 3 by Lemma 8. If $r \leq 17$, then by the maximality of r each neighbor of an (H, v_r)-distant vertex must lie in H. Thus again by the maximality of r, considering the largest lasso L in H with v_{r+1} as the endpoint of the handle, we know that the loop in L has at most 12 vertices. So, we may apply one of the Lemmas 6, 7, 12, 13, 14, and 15 to L. This then gives a contradiction to the maximality of the loop in L or a dominating set extendable to a dominating set of size 8 of G'. Thus $r \in \{19, 20, 22, 23\}$.

Case 1: $r = 19$. By the maximality of r, and Lemma 8, each (H, v_{19})-distant vertex of H has only v_7, v_9, v_{10}, and v_{12} as possible neighbors in C. Also by the maximality of r, if a vertex z of C is adjacent to the end of a handle H, then a vertex adjacent to z along C cannot have neighbors in H.

Case 1.1: Vertex v_{25} has two neighbors on C. By symmetry and the maximality of r, we have the following three cases:

Case 1.1.1: $v_{25}v_7 \in E(G')$ and $v_{25}v_9 \in E(G')$. By Lemma 8 for the paths $(v_{20}v_{21}\dots v_{25}v_7v_8\dots v_{19}v_1v_2\dots v_6)$ and $(v_{20}v_{21}\dots v_{25}v_9v_{10}\dots v_{19}v_1v_2\dots v_8)$, the third neighbor of v_{20} is in $\{v_{16}, v_{23}, v_{24}\}$.

Case 1.1.1.1: $v_{20}v_{16} \in E(G')$. In this case, v_8 has its third neighbor in G', and by Lemma 8 for the paths $(v_8v_9\dots v_{19}v_1v_2\dots v_7v_{25}v_{24}\dots v_{20})$, $(v_8v_9\dots v_{16}v_{20}v_{21}\dots v_{25}v_7v_6\dots v_1v_{19}v_{18}v_{17})$, and $(v_8v_7\dots v_1v_{19}v_{20}\dots v_{25}v_9v_{10}\dots v_{18})$, this neighbor is in $\{v_1, v_4, v_{12}, v_{15}\}$. If $v_8v_1 \in E(G')$ (which is symmetric with the case $v_8v_{15} \in E(G')$), then the cycle $(v_1v_2\dots v_7v_{25}v_{24}\dots v_8)$ contradicts the maximality of r. If $v_8v_4 \in E(G')$ (which is symmetric with the case $v_8v_{12} \in E(G')$), then the cycle $(v_8v_9\dots v_{25}v_7v_6v_5v_4)$ gives $r \geq 22$ contradicting the maximality of r.

Case 1.1.1.2: $v_{20}v_{23} \in E(G')$. The path $(v_8v_9\dots v_{19}v_1v_2\dots v_7v_{25}v_{24}v_{23}v_{20}v_{21}v_{22})$ forces v_{22} to have its third neighbor in G'. By Lemma 8 for this path, $(v_8v_7\dots v_1v_{19}v_{18}\dots v_9v_{25}v_{24}v_{23}v_{20}v_{21}v_{22})$, and P, and by the maximality of r, $v_{22}v_{16} \in E(G')$. Then by Lemma 8 for the paths $(v_8v_9\dots v_{19}v_1v_2\dots v_7v_{25}v_{24}\dots v_{20})$, $(v_8v_9\dots v_{16}v_{22}v_{21}v_{20}v_{23}v_{24}v_{25}v_7v_6\dots v_1v_{19}v_{18}v_{17})$, and

$(v_8v_7\dots v_1v_{19}v_{20}\dots v_{25}v_9v_{10}\dots v_{18})$, the third neighbor of v_8 is in $\{v_1, v_4, v_{12}, v_{15}\}$. Just as in Case 1.1.1.1, each of these possibilities forces $r > 19$.

Case 1.1.1.3: $v_{20}v_{24} \in E(G')$. The path $(v_8v_7\dots v_1v_{19}v_{18}\dots v_9v_{25}v_{24}v_{20}v_{21}v_{22}v_{23})$ forces v_{23} to have its third neighbor in G'. Since the path $(v_{25}v_{24}v_{20}v_{21}v_{22}v_{23})$ covers H, Lemma 8 forces $v_{23}v_{16} \in E(G')$. Then just as in Case 1.1.1.1, we eliminate all neighbors of v_8.

Case 1.1.2: $v_{25}v_7 \in E(G')$ and $v_{25}v_{10} \in E(G')$. The maximality of r and Lemma 8 for the path $(v_{20}v_{21}\dots v_{25}v_7v_6\dots v_1v_{19}v_{18}\dots v_8)$ force the third neighbor of v_{20} to be in $\{v_{17}, v_{23}, v_{24}\}$. Note that equivalent paths restrict the third neighbor of each (H, v_{25})-distant vertex to be in H or to be v_{17}. If an (H, v_{25})-distant vertex v_i is adjacent to v_{17}, then v_{18} has its third neighbor in G', and by Lemma 8 for the paths $(v_{18}v_{17}\dots v_{10}v_{25}v_{24}\dots v_{19}v_1v_2\dots v_9)$ and $(v_{18}v_{19}v_1......v_7v_{25}\dots v_iv_{17}v_{16}\dots v_8)$, this neighbor is either in H or in $\{v_3, v_6, v_{11}, v_{14}\}$. In any case, as in Case 1.1.1, any such neighbor contradicts the maximality of r. If $v_{20}v_{23} \in E(G')$, then v_{22} is (H, v_{25})-distant and hence its third neighbor is in H. In this case, $v_{22}v_{24} \in E(G)$, a contradiction to Lemma 8 for P. Thus, $v_{20}v_{24} \in E(G')$. Similarly v_{23} is an (H, v_{25})-distant vertex; hence its third neighbor is in H, but Lemma 8 for the path $(v_{23}v_{22}v_{21}v_{20}v_{24}v_{25}v_{10}v_9\dots v_1v_{19}v_{18}\dots v_{11})$ eliminates all possible neighbors.

Case 1.1.3: $v_{25}v_7 \in E(G')$, and $v_{25}v_{12} \in E(G')$. In this case, no (H, v_{25})-distant vertex can have a neighbor in C other than v_{19}. Hence the third neighbor of v_{20} lies in H. By Lemma 8 for the path $(v_{20}v_{21}\dots v_{25}v_7v_6\dots v_1v_{19}v_{18}\dots v_8)$, this neighbor is one of v_{23} and v_{24}. If $v_{20}v_{23} \in E(G')$, then the path $(v_{22}v_{21}v_{20}v_{23}v_{24}v_{25}v_7v_6\dots v_1v_{19}v_{18}\dots v_8)$ forces v_{22} to have its third neighbor in G'. But then Lemma 8 for P forces this neighbor to be in C, a contradiction. If $v_{20}v_{24} \in E(G')$, then the path $(v_8v_9\dots v_{19}v_1v_2\dots v_7v_{25}v_{24}v_{20}v_{21}v_{22}v_{23})$ forces v_{23} to have its third neighbor in G'. But then Lemma 8 for this path forces this neighbor to be in C, a contradiction.

Hence v_{25} (and by symmetry v_{20}) has at most one neighbor in C.

Case 1.2: Each of v_{20} and v_{25} has exactly one neighbor in C. Lemma 8 for P restricts the second neighbor of v_{25} in H to one of v_{21} and v_{22}. Similarly, the second neighbor of v_{20} in H is in $\{v_{23}, v_{24}\}$. If $v_{20}v_{23} \in E(G')$ and $v_{25}v_{21} \in E(G')$, then the path $(v_1v_2\dots v_{20}v_{23}v_{22}v_{21}v_{25}v_{24})$ forces v_{24} to have its third neighbor in G'. There is no room for this neighbor in H, so it is in C. Hence the set $\{v_{21}, v_{24}\}$ dominates all G' but a P_{18}. If $v_{20}v_{23} \in E(G')$, and $v_{25}v_{22} \in E(G')$, then the set $\{v_{20}, v_{25}\}$ dominates the set $\{v_{19}, v_{20},...,v_{25}\}$ leaving only a P_{18} undominated. If $v_{20}v_{24} \in E(G')$ and $v_{25}v_{21} \in E(G')$, then the path $(v_1v_2\dots v_{20}v_{24}v_{25}v_{21}v_{22}v_{23})$ forces v_{23} to have its third neighbor in G'. By Lemma 8 for P, this neighbor is in C. Hence the set $\{v_{21}, v_{23}\}$ dominates all but a P_{18}. Finally, suppose that $v_{20}v_{24} \in E(G')$ and $v_{25}v_{22} \in E(G')$. In our case, v_{25} has a neighbor v_i in C. Then the path $(v_{i+1}v_{i+2}\dots v_{19}v_1v_2\dots v_iv_{25}v_{22}v_{23}v_{24}v_{20}v_{21})$ forces v_{21} to have its third neighbor in G'. Lemma 8 for the path $(v_{i+1}v_{i+2}\dots v_{19}v_1v_2\dots v_iv_{25}v_{24}\dots v_{20})$ forces this neighbor to be in C. Then the set $\{v_{21}, v_{24}\}$ dominates all but a P_{18}.

Case 1.3: Vertex v_{25} has no neighbors in C. By Lemma 8, $N(v_{25}) = \{v_{24}, v_{22}, v_{21}\}$. Then both, v_{23} and v_{24} are (H, v_{20})-distant, and hence at least one of them has a neighbor in C. Thus we have Case 1.2.

Case 2: $r = 20$. Let $\mathrm{dist}_C(x, y)$ denote the distance on C between the vertices x and y. Suppose that $i, j \geq 21$ and that v_i is an (H, v_j)-distant vertex. If $v_{i'}$ is a neighbor of v_i in C, and $v_{j'}$ is a neighbor of v_j in C, then the maximality of r and Lemma 8 imply that

$$\mathrm{dist}_C(v_{i'}, v_{j'}) \in \{7, 10\}. \tag{10}$$

Case 2.1: Some (H, v_{21})-distant vertex, say v_{25}, has two neighbors in C. We claim that

(11) an (H, v_{25})-distant vertex has a neighbor in C distinct from v_{20}.

Indeed, otherwise v_{21} has a neighbor in H distinct from v_{22}. It could be only v_{24}. Then v_{23} is (H, v_{25})-distant and cannot have 3 neighbors in H. This proves (11).

By (11) and (10), v_{25} cannot be adjacent to both of v_7, and v_{13}. So we may assume that $v_{25}v_7, v_{25}v_{10} \in E(G')$. By (10), a neighbor of $H - v_{25}$ in C distinct from v_{20} can be only v_{17}. Then the path $(v_{21}v_{22} \dots v_{25}v_7v_6 \dots v_1v_{20}v_{19} \dots v_8)$ forces v_8 to have the third neighbor in G'. By Lemma 11 with $x = v_{20}$, $y = v_{10}$, and $z = v_8$, this third neighbor is in $\{v_1, v_2, \dots, v_6\}$. By Lemma 8 for the paths $(v_9v_8 \dots v_1v_{20}v_{19} \dots v_{10}v_{25}v_{24} \dots v_{21})$ and $(v_8v_9 \dots v_{17}$ H $v_7v_6 \dots v_1v_{20}v_{19}v_{18})$, this third neighbor is either v_1 or v_4. If $v_8v_1 \in E(G')$, then the hamiltonian cycle $(v_1v_2 \dots v_7v_{25}v_{24} \dots v_8)$ contradicts the maximality of r. If $v_8v_4 \in E(G')$, then the cycle $(v_4v_5v_6v_7v_{25}v_{24} \dots v_8)$ forces $r \geq 22$.

Case 2.2: No (H, v_{21})-distant vertex has two neighbors in C. We claim that

(12) an (H, v_{21})-distant vertex, say v_{25}, has a neighbor in C.

Indeed, otherwise v_{25} has 3 neighbors in H, which implies $v_{25}v_{22}, v_{25}v_{21} \in E(G)$. Then v_{24} is (H, v_{21})-distant and has no room in H for the third neighbor. This proves (12). Suppose that v_j is the neighbor of v_{25} in C. By Lemma 8 for P, the neighbor of v_{25} in $H - v_{24}$ is either v_{21} or v_{22}. Similarly, the neighbor of v_{21} in $H - v_{22}$ is either v_{24} or v_{25}.

Case 2.2.1: $v_{25}v_{22} \in E(G')$. Then $v_{25}v_{21} \notin E(G')$ and hence $v_{21}v_{24} \in E(G')$. The path $(v_{25}v_{22}v_{21}v_{24}v_{23})$ shows that v_{23} is (H, v_{25})-distant. Also, v_{23} is (H, v_{21})-distant. Since v_{23} cannot have the third neighbor in H, it has a neighbor, v_i, in C. Since $r = 20$, $\min\{\mathrm{dist}_C(v_i, v_{20}), \mathrm{dist}_C(v_j, v_{20}), \mathrm{dist}_C(v_i, v_j)\} \leq 6$, a contradiction to (10).

Case 2.2.2: $v_{25}v_{21} \in E(G')$. In this case, v_{22} is (H, v_{21})-distant and by Lemma 8 has no third neighbor in H. Therefore, v_{22} has a neighbor, v_h, in C. Similarly, v_{23} is (H, v_{22})-distant and hence has a neighbor, v_ℓ, in C and v_{24} is (H, v_{25})-distant and hence has a neighbor, v_q, in C. By (10) for v_h and v_{20}, and for v_{20} and v_j, $\mathrm{dist}_C(v_h, v_{20}) \equiv 1 \pmod 3$ and $\mathrm{dist}_C(v_{20}, v_j) \equiv 1 \pmod 3$. Vertices v_h, v_{20} and v_j partition C into three paths that we will call $P_{j,h}$, $P_{j,20}$, and $P_{20,h}$, where P_{i_1,i_2} connects v_{i_1} with v_{i_2} and does not contain v_{i_3} for distinct $i_1, i_2, i_3 \in \{j, h, 20\}$. Since $20 \equiv 2 \pmod 3$, the number of edges in $P_{j,h}$ is $0 \pmod 3$. If $v_\ell \notin V(P_{j,h})$, then $\mathrm{dist}_C(v_\ell, v_j) \equiv 1 \pmod 3$ and hence v_q cannot have $\mathrm{dist}_C(v_\ell, v_q) \equiv 1 \pmod 3$ and $\mathrm{dist}_C(v_q, v_j) \equiv 1 \pmod 3$ at the same time. So, $v_\ell \in V(P_{j,h})$. But then by the maximality of r, $P_{j,h}$ has at least 7 edges. Since each of $P_{j,20}$ and $P_{20,h}$ also has at least 7 edges, this is impossible for the 20-cycle C.

Case 3: $r = 22$. If v_{24} has its third neighbor in G', then v_{24} dominates all G' but a P_{21} which can be dominated by 7 vertices. Thus v_{24}'s third neighbor is outside of G'. Also if $v_{23}v_{25} \in E(G')$, then v_{23} dominates all but a P_{21}. Thus we may assume that each of v_{23} and v_{25} has exactly two neighbors in C. These four neighbors of v_{23} and v_{25} partition C into four paths. Suppose that the lengths of these paths are a, b, c, and d.

Case 3.1: The two neighbors of v_{23} in C and the two neighbors of v_{25} in C alternate on C for *each* representation of G' as a lasso with $r = 22$. We may assume that $v_{25}v_d, v_{23}v_{d+a}, v_{25}v_{22-b} \in E(G')$, $c = 22 - a - b - d$ and that $d = \max\{a, b, c, d\}$. By the maximality of r, $\min\{a, b, c, d\} \geq 4$ and hence $d = \max\{a, b, c, d\} \leq 22 - a - b - c \leq 10$. So, by Lemma 8,

(13) each of a, b, c, d is in $\{4, 6, 7, 9, 10\}$.

Case 3.1.1: $d \geq 8$. If $d = 10$, then by (13), $a = b = c = 4$ and the set $\{v_2, v_5, v_8, v_{12}, v_{16}, v_{20}, v_{23}, v_{25}\}$ dominates G'. If $d = 9$, then $a + b + c = 13$, which contradicts (13). By (13), $d \neq 8$.

Case 3.1.2: $d = 7$. By (13), $\{a, b, c\} = \{4, 4, 7\}$. By symmetry, there are two subcases: either $(d, a, c, b) = (7, 4, 4, 7)$ or $(d, a, c, b) = (7, 4, 7, 4)$. If $(d, a, c, b) = (7, 4, 4, 7)$, then the set $\{v_2, v_5, v_9, v_{13}, v_{17}, v_{20}, v_{23}, v_{25}\}$ dominates G'. If $(d, a, c, b) = (7, 4, 7, 4)$, then the set
$\{v_2, v_5, v_9, v_{13}, v_{16}, v_{20}, v_{23}, v_{25}\}$ dominates G'.

Case 3.1.3: $d \leq 6$. If $d \leq 5$ then by the maximality of d, we have $a + b + c + d \leq 20 < 22$. So, $d = 6$. By (13), $\{a, b, c\} = \{4, 6, 6\}$. So by symmetry we may assume that $(d, a, c, b) = (6, 6, 6, 4)$. Then the cycle $(v_1v_2 \ldots v_{18}v_{25}v_{24}v_{23}v_{22})$ with the handle v_{19}, v_{20}, v_{21} is a new lasso L with $r = 22$. By our assumption, the neighbors of v_{21} and the neighbors of v_{20} also alternate along the cycle in L. Since $d = 6$, each such adjacent pair of such neighbors along the cycle in L must be at distance 4 or 6. Since only one such distance can be 4, v_{19} is adjacent to v_6, but v_6 already has 3 neighbors.

Case 3.2: There exists a representation of G' as a lasso with $r = 22$ such that the neighbors of v_{23} along C, and the neighbors of v_{25} along C do not alternate. We may assume that $v_{23}v_d, v_{25}v_{d+a}, v_{25}v_{22-b} \in E(G')$, and $c = 22 - a - b - d$. We may assume further that $d \geq c$ and $a \leq b$. By the maximality of r, $a, b \geq 4$ and $c, d \geq 2$. Similarly to (8) in the proof of Lemma 18, we have

$$(14) \qquad b \geq a \geq 4, \quad a, b, a + c, b + c, a + d, b + d \notin \{5, 8, 11, 14, 17\}, \text{ and } 2 \leq c \leq d.$$

By (14), $d \leq 22 - 4 - 4 - 2 = 12$.

Case 3.2.1: $d = 12$. By (14), $a = b = 4$, and $c = 2$. By Lemma 9 for C and Lemma 8 for the paths $(v_{17}v_{18} \ldots v_{25}v_{16}v_{15} \ldots v_1)$ and $(v_{17}v_{16} \ldots v_{12}v_{23}v_{24}v_{25}v_{18}v_{19} \ldots v_{22}v_1v_2 \ldots v_{11})$, the third neighbor of v_{17} is either v_{13} or v_{21}. Assume by symmetry that $v_{17}v_{13} \in E(G')$. Since v_{15} has its third neighbor in G', by Lemma 11 with $x = v_{13}$, $y = v_{17}$, and $z = v_{15}$, $G'[C]$ has a dominating set of size 7 and hence G' has a dominating set of size 8.

Case 3.2.2: $d = 11$. By (14), $a = b = 4$, and $c = 3$. Then Lemma 9 for C and Lemma 8 for the path $(v_{16}v_{17} \ldots v_{25}v_{15}v_{14} \ldots v_1)$ forces the third neighbor of v_{17} to be amongst $v_2, v_5, v_8, v_{14}, v_{20}$, and v_{21}. If $v_{17}v_{21} \in E(G)$, then since v_{19} has its third neighbor in G', Lemma 11 with $x = v_{17}$, $y = v_{21}$, and $z = v_{19}$ yields a dominating set in $G'[C]$ of size 7. If $v_{17}v_i \in E(G')$ for $i \in \{2, 5, 8, 14\}$, then the set $\{v_{17}, v_{19}, v_{22}, v_{25}\}$ dominates 13 vertices and leaves only a collection of paths whose lengths are divisible by 3. So, in this case G' can be dominated by 8 vertices. If $v_{17}v_{20} \in E(G')$, then v_{20} dominates all but a P_{21} in G', and hence G' has a dominating set of size 8.

Case 3.2.3: $d = 10$. In this case, $a + b + c = 12$, and no combination of values for a, b, and c satisfies (14): if $a = 4$, then $b + c = 8$, a contradiction; otherwise $6 \leq a \leq b$, and $a + b + c \geq 14$.

Case 3.2.4: $d = 9$. By (14), $(a, b, c) \in \{(4, 4, 5), (4, 6, 3), (4, 7, 2)\}$.

Case 3.2.4.1: $a = b = 4$, and $c = 5$. Let v_i be the third neighbor of v_{17}. By Lemma 9 for C and Lemma 8 for the paths $(v_{14}v_{15} \ldots v_{25}v_{13}v_{12} \ldots v_1)$ and $(v_{17}v_{16} \ldots v_9v_{23}v_{24}v_{25}v_{18}v_{19} \ldots v_{22}v_1v_2 \ldots v_8)$, $i \in \{10, 14, 21\}$. Since v_{19} has its third neighbor in G', if $v_{17}v_{21} \in E(G')$, then Lemma 11 with $x = v_{17}$, $y = v_{21}$, and $z = v_{19}$ yields a dominating set of $G'[C]$ of size 7. Suppose that $v_{17}v_{10} \in E(G')$. Since v_{12} has a common neighbor with v_{25}, it has a third neighbor v_j. By Lemma 11 with $x = v_{17}$, $y = v_{10}$, and $z = v_{12}$, $j \in \{14, 15, 16\}$. By Lemma 9 for C, $j \neq 14$. Then the cycle $(v_1v_2 \ldots v_9v_{23}v_{24}v_{25}v_{13}v_{14} \ldots v_jv_{12}v_{11}v_{10}v_{17}v_{18} \ldots v_{22})$ contradicts the maximality of r. So $v_{17}v_{14} \in E(G')$. The path $(v_{23}v_{24}v_{25}v_{13}v_{12} \ldots v_1v_{22}v_{21} \ldots v_{17}v_{14}v_{15}v_{16})$ forces v_{16} to have its third neighbor in G'. By Lemma 8 for this path and Lemma 9 for C, this third neighbor is in $\{v_1, v_4, v_7, v_{10}, v_{20}\}$. If the neighbor is in $\{v_1, v_4, v_7, v_{20}\}$, then the set $\{v_1, v_4, v_7, v_9, v_{11}, v_{14}, v_{20}, v_{25}\}$ dominates G'. Hence $v_{16}v_{10} \in E(G')$. Symmetry then forces $v_{15}v_{21} \in E(G')$, and the set $\{v_1, v_4, v_7, v_{10}, v_{13}, v_{18}, v_{21}, v_{23}\}$ dominates G'.

Case 3.2.4.2: $a = 4, b = 6, c = 3$. Then v_{14} has its third neighbor in G', and by Lemma 9 for C and Lemma 8 for the paths $(v_{14}v_{15} \dots v_{25}v_{13}v_{12} \dots v_1)$ and $(v_{15}v_{14} \dots v_9v_{23}v_{24}v_{25}v_{16}v_{17} \dots v_{22}v_1v_2 \dots v_8)$, this neighbor is in $\{v_1, v_4, v_7, v_{10}, v_{17}, v_{20}\}$. Since v_{12} has its third neighbor in G', Lemma 11 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$ eliminates v_{10} as the third neighbor. Now for each of the remaining vertices v_i, Lemma 11 with $x = v_i$, $y = v_{14}$, and $z = v_9$ yields a dominating set of $G'[C]$ of size 7.

Case 3.2.4.3: $a = 4, b = 7, c = 2$. Then v_{14} has its third neighbor in G'. By Lemma 9 for C and Lemma 8 for the paths $(v_{14}v_{13} \dots v_9v_{23}v_{24}v_{25}v_{15}v_{16} \dots v_{22}v_1v_2 \dots v_8)$ and $(v_{14}v_{15} \dots v_{25}v_{13}v_{12} \dots v_1)$, this neighbor is in $\{v_{10}, v_{18}, v_{21}\}$. Since both of v_{12}, and v_{16} have third neighbors in G', Lemma 11 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$ and with $x = v_{14}$, $y = v_{18}$, and $z = v_{16}$ forces $v_{14}v_{21} \in E(G')$. This then forces the hamiltonian cycle $(v_{14}v_{13} \dots v_1v_{22}v_{23}v_{24}v_{25}v_{15}v_{16} \dots v_{21})$ contradicting the maximality of r.

Case 3.2.5: $d = 8$. By (14), $(a, b, c) \in \{(4, 4, 6), (4, 7, 3)\}$.

Case 3.2.5.1: $a = b = 4, c = 6$. Since $v_{25}v_{12} \in E(G')$, v_{13} has its third neighbor in G'. By Lemma 9 for C and Lemma 8 for the paths $(v_{17}v_{16} \dots v_8v_{23}v_{24}v_{25}v_{18}v_{19} \dots v_{22}v_1v_2 \dots v_7)$ and $(v_9v_{10} \dots v_{18}v_{25}v_{24}v_{23}v_8v_7 \dots v_1v_{22}v_{21}v_{20}v_{19})$, this neighbor is in $\{v_9, v_{10}, v_{16}, v_{17}\}$. Since v_{11} has its third neighbor in C, by Lemma 11 with $x = v_9$, $y = v_{13}$, and $z = v_{11}$, $v_9v_{13} \notin E(G')$. If $v_{13}v_{10} \in E(G')$, then the set $\{v_2, v_5, v_8, v_{10}, v_{15}, v_{18}, v_{21}, v_{25}\}$ dominates G'. If $v_{13}v_{16} \in E(G')$, then symmetry gives $v_{17}v_{14} \in E(G')$. Thus Lemma 9 for C and Lemma 8 for the paths $(v_{15}v_{14}v_{17}v_{16}v_{13}v_{12} \dots v_1v_{22}v_{21} \dots v_{18}v_{25}v_{24}v_{23})$ and $(v_{15}v_{16}v_{13}v_{14}v_{17}v_{18} \dots v_{22}v_1v_2 \dots v_{12}v_{25}v_{24}v_{23})$ eliminates all possible neighbors of v_{15}. The last possibility is that $v_{13}v_{17} \in E(G')$. The path $P' = (v_{23}v_{24}v_{25}v_{18}v_{19} \dots v_{22}v_1v_2 \dots v_{13}v_{17}v_{16}v_{15}v_{14})$ forces v_{14} to have its third neighbor, say v_i, in G'. By Lemma 8 for the path $(v_{13}v_{14} \dots v_{25}v_{12}v_{11} \dots v_1)$ and for P', $i \in \{2, 5, 11, 20, 21\}$. By Lemma 11 with $x = v_{14}$, $y = v_i$, and $z = v_{22}$, $i \notin \{2, 5, 11\}$. So, $i \in \{20, 21\}$. If $i = 20$, then the set $\{v_3, v_6, v_9, v_{12}, v_{16}, v_{20}, v_{22}, v_{25}\}$ dominates G'. If $i = 21$, then the cycle $(v_{18}v_{19}v_{20}v_{21}v_{14}v_{15}v_{16}v_{17}v_{13}\ v_{12} \dots v_1v_{22}v_{23}v_{24}v_{25})$ contradicts the maximality of r.

Case 3.2.5.2: $a = 4, b = 7, c = 3$. Since $v_{25}v_{12} \in E(G')$, v_{13} has its third neighbor in G'. Lemma 9 for C and Lemma 8 for the paths $(v_{14}v_{13} \dots v_8v_{23}v_{24}v_{25}v_{15}v_{16} \dots v_{22}v_1v_2 \dots v_7)$, and $(v_{13}v_{14} \dots v_{22}v_1v_2 \dots v_8v_{23}v_{24}v_{25}v_{12}v_{11}v_{10}v_9)$ forces this neighbor to be in $\{v_3, v_6, v_9\}$. Since v_{11} has its third neighbor in G', by Lemma 11 with $x = v_9$, $y = v_{13}$, and $z = v_{11}$, $v_9v_{13} \notin E(G')$. Finally, for $i = 3, 6$, Lemma 11 with $x = v_{13}$, $y = v_i$, and $z = v_8$ eliminates the remaining possible neighbors for v_{13}.

Case 3.2.6: $d = 7$. In this case, by (14), $a = b = 6$, and $c = 3$. So, v_{14} has its third neighbor in G'. Lemma 9 for C and Lemma 8 for the paths $(v_{15}v_{14} \dots v_1v_{22}v_{23}v_{24}v_{25}v_{16}v_{17} \dots v_{21})$ and $(v_{14}v_{15} \dots v_{25}v_{13}v_{12} \dots v_1)$, forces this neighbor to be in $\{v_1, v_4, v_{10}, v_{17}, v_{20}\}$. Since v_{12} has its third neighbor in C, by Lemma 11 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$, $v_{14}v_{10} \notin E(G')$. Furthermore, for $i = 17, 20$, Lemma 11 with $x = v_{14}$, $y = v_i$, and $z = v_{22}$ yields that the possible neighbor of v_{14} is either v_1 or v_4. If $v_{14}v_1 \in E(G')$, then the hamiltonian cycle $(v_{25}v_{24} \dots v_{14}v_1v_2 \dots v_{13})$ contradicts the maximality of r. If $v_{14}v_4 \in E(G')$, then by symmetry $v_{15}v_3 \in E(G)$ and the 23-cycle $(v_1v_2v_3v_{15}v_{14}v_4v_5 \dots v_{13}v_{25}v_{16}v_{15} \dots v_{22})$ contradicts the maximality of r.

Case 3.2.7: $d = 6$. By (14), $(a, b, c) \in \{(4, 6, 6), (4, 7, 5), (4, 9, 3), (4, 10, 2), (6, 6, 4), (6, 7, 3), (7, 7, 2)\}$.

Case 3.2.7.1: $a = 4, b = c = 6$. Since $v_{10}v_{25} \in E(G')$, v_{11} has its third neighbor in G'. Lemma 9 for C and Lemma 8 for the paths $(v_{11}v_{12}\dots v_{25}v_{10}v_9v_8\dots v_1)$ and $(v_{15}v_{14}\dots v_1v_{22}v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{21})$, this neighbor is in $\{v_1, v_4, v_7, v_{14}, v_{15}, v_{17}, v_{20}\}$. Since v_9 has its third neighbor in G', by Lemma 11 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_{11}v_7 \notin E(G')$. Also for $i \in \{4, 1, 20, 17\}$, Lemma 11 with $x = v_{11}$, $y = v_i$, and $z = v_6$ eliminates v_i as a neighbor of v_{11}. Thus v_{11} is adjacent to either v_{14} or v_{15}.

Case 3.2.7.1.1: $v_{11}v_{15} \in E(G')$. Then v_{14} has its third neighbor in G'. By Lemma 9 for C and Lemma 8 for and the paths $(v_{15}v_{14}\dots v_6v_{23}v_{24}v_{25}v_{16}\dots v_{22}v_1v_2\dots v_5)$ and $(v_{14}v_{13}v_{12}v_{11}v_{15}v_{16}v_{17}\dots v_{25}v_{10}v_9\dots v_1)$, this neighbor is in $\{v_1, v_4, v_7, v_{17}, v_{20}\}$. By Lemma 11 with $x = v_6$, $y = v_{16}$, and $z = v_{14}$, v_{14} is not adjacent to v_i for $i \in \{1, 4, 17, 20\}$. Thus $v_{14}v_7 \in E(G')$, and the hamiltonian cycle $(v_1v_2\dots v_6v_{23}v_{24}v_{25}v_{10}v_9v_8v_7v_{14}v_{13}v_{12}v_{11}v_{15}v_{16}\dots v_{22})$ contradicts the maximality of r.

Case 3.2.7.1.2: $v_{11}v_{14} \in E(G')$ and by symmetry $v_5v_2 \in E(G')$. Then v_{15} has a neighbor in G', and by Lemma 9 for C and Lemma 8 for the path $(v_{15}v_{14}\dots v_{10}v_{25}v_{24}v_{23}v_{22}v_1v_2\dots v_9)$, this neighbor is in $\{v_3, v_9, v_{12}, v_{18}, v_{19}, v_{21}\}$. Since v_{17} has its third neighbor in G', by Lemma 11 with $x = v_{15}$, $y = v_{19}$, and $z = v_{17}$, $v_{15}v_{19} \notin E(G')$. If $v_{15}v_3 \in E(G')$, then the 23-cycle $(v_1v_2v_3v_{15}v_{14}\dots v_6v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{22})$ contradicts the maximality of r. If $v_{15}v_9 \in E(G')$, then the 23-cycle $(v_1v_2\dots v_9v_{15}v_{14}\dots v_{10}v_{25}v_{16}v_{17}\dots v_{22})$ contradicts the maximality of r. If $v_{15}v_{12} \in E(G')$, then the path $(v_{23}v_{24}v_{25}v_{10}v_9\dots v_1v_{22}v_{21}\dots v_{15}v_{12}v_{11}v_{14}v_{13})$ forces v_{13} to have the third neighbor in G'. Then Lemma 8 for this path, C, and the path $(v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{22}v_1v_2\dots v_{11}v_{14}v_{15}v_{12}v_{13})$ eliminates all possible neighbors of v_{13}. If $v_{15}v_{21} \in E(G')$, then the cycle $(v_1v_2\dots v_{15}v_{21}v_{20}\dots v_{16}v_{25}v_{24}v_{23}v_{22})$ contradicts the maximality of r. Thus, $v_{15}v_{18} \in E(G')$ and, by symmetry, $v_1v_{20} \in E(G')$. Then the set $\{v_2, v_5, v_7, v_{10}, v_{13}, v_{16}, v_{20}, v_{23}\}$ dominates G'.

Case 3.2.7.2: $a = 4, b = 7, c = 5$. Since $v_6v_{23} \in E(G')$, v_5 has its third neighbor in G'. By Lemma 8 for C and Lemma 11 with $x = v_{22}$, $y = v_{10}$, and $z = v_5$, this neighbor is in $\{v_1, v_2, v_8, v_9\}$. Since v_7 has its third neighbor in G', by Lemma 11 with $x = v_5$, $y = v_9$, and $z = v_7$, $v_5v_9 \notin E(G')$. If $v_8v_5 \in E(G')$, then v_8 dominates all but a P_{21}, hence v_5 is adjacent to either v_1 or v_2.

Case 3.2.7.2.1: $v_5v_1 \in E(G')$. Then v_2 has its third neighbor in G', and by Lemma 9 for C and Lemma 11 with $x = v_{22}$, $y = v_{10}$, and $z = v_2$, this neighbor is either v_8 or v_9. If $v_2v_8 \in E(G')$, then v_8 dominates all but a P_{21} and hence $v_2v_9 \in E(G')$. Then the hamiltonian cycle $(v_1v_{22}v_{21}\dots v_{10}v_{25}v_{24}v_{23}v_6v_7v_8v_9v_2v_3v_4v_5)$ contradicts the maximality of r.

Case 3.2.7.2.2: $v_5v_2 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_6v_7\dots v_{22}v_1v_2v_5v_4v_3)$ forces v_3 to have its third neighbor in G'. By Lemma 9 for C and Lemma 8 for this path, this third neighbor is one of v_9, v_{12}, v_{18}, and v_{21}. If this neighbor is in $\{v_{12}, v_{18}, v_{21}\}$, then the set $\{v_2, v_3, v_7, v_{10}, v_{23}\}$ dominates all but a P_9 or but a P_3 and a P_6. In both cases, G' can be dominated by 8 vertices. Hence $v_3v_9 \in E(G')$. In this case, the hamiltonian cycle $(v_1v_2v_5v_4v_3v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11}\dots v_{22})$ contradicts the maximality of r.

Case 3.2.7.3: $a = 4, b = 9, c = 3$. Since $v_{10}v_{25} \in E(G')$, v_{11} has its third neighbor in G'. By Lemma 9 for C and Lemma 8 for the paths $(v_{12}v_{11}\dots v_6v_{23}v_{24}v_{25}v_{13}v_{14}\dots v_{22}v_1v_2\dots v_5)$ and $(v_7v_8\dots v_{13}v_{25}v_{24}v_{23}v_6v_5\dots v_1v_{22}v_{21}\dots v_{14})$, this neighbor is either v_7 or v_8. Since v_9 has its third neighbor in G', by Lemma 11 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_{11}v_7 \notin E(G')$. Hence $v_{11}v_8 \in E(G')$, and so v_8 dominates all but a P_{21} in G'.

Case 3.2.7.4: $a = 4, b = 10, c = 2$. Since v_5 is a neighbor of v_d, it has its third neighbor, say v_i, in G'. By Lemma 8 for P, $i \in \{1, 2, 8, 9, 11, 14, 15, 17, 18, 20, 21\}$. By Lemma 11 with $x = v_{10}, y = v_{22}, z = v_5$, $i \in \{1, 2, 8, 9\}$. If $v_5v_9 \in E(G')$, the hamiltonian cycle $(v_1v_2\dots v_5v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11}\dots v_{22})$ contradicts the maximality of r. If v_8 has its third neighbor in G' then v_8 dominates all but a P_{21} in G'. Hence $i \in \{1, 2\}$.

Case 3.2.7.4.1: $v_5v_1 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_6v_7 \dots v_{22}v_1v_5v_4v_3v_2)$ forces v_2 to have its third neighbor, say v_j, in G'. By Lemma 8 for this path $j \in \{8, 9, 11, 14, 15, 17, 18, 20, 21\}$. By Lemma 11 with $x = v_{10}$, $y = v_{22}$, $z = v_2$, $j \in \{8, 9\}$. Since v_8 does not have its third neighbor in G', $v_2v_9 \in E(G')$. Then the hamiltonian cycle $(v_1v_5v_4v_3v_2v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11} \dots v_{22})$ contradicts the maximality of r.
Case 3.2.7.4.2: $v_5v_2 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_6v_7 \dots v_{22}v_1v_2v_5v_4v_3)$ forces v_3 to have its third neighbor, say v_j, in G'. By Lemma 8 for this path and P, $j \in \{8, 9, 11, 14, 15, 17, 18, 20, 21\}$. By Lemma 11 with $x = v_{12}, y = v_{22}, z = v_j$, $j \in \{8, 9, 11, 15, 18, 21\}$. Since v_8 does not have its third neighbor in G', $j \in \{9, 11, 15, 18, 21\}$. If $v_3v_9 \in E(G')$, the hamiltonian cycle $(v_1v_2v_5v_4v_3v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11} \dots v_{22})$ contradicts the maximality of r. If $v_3v_{11} \in E(G')$, the cycle $(v_1v_2v_5v_4v_3v_{11}v_{10} \dots v_6v_{23}v_{24}v_{25}v_{12}v_{13} \dots v_{22})$ contradicts the maximality of r. If $v_3v_{21} \in E(G')$, the 23-cycle $(v_1v_2v_5v_4v_3v_{21}v_{20} \dots v_6v_{23}v_{22})$ contradicts the maximality of r. Hence $j \in \{15, 18\}$. Since v_7 is a neighbor of v_d, it has its third neighbor, say v_h, in G'. By Lemma 8 for P and the path $(v_7v_8 \dots v_{12}v_{25}v_{24}v_{23}v_6v_5 \dots v_{22}v_{21} \dots v_{13})$, $h \in \{11, 13, 16, 19\}$. Since v_9 has its third neighbor in G', by Lemma 11 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_{11}v_7 \notin E(G')$. If $v_{13}v_7 \in E(G')$ the hamiltonian path $(v_{12}v_{11} \dots v_7v_{13}v_{14} \dots v_{22}v_1 \dots v_6v_{23}v_{24}v_{25})$ contradicts the maximality of r. So $h \in \{16, 19\}$. If $h = j + 1$, then the 23-cycle $(v_1v_2v_5v_4v_3v_jv_{j-1} \dots v_{10}v_{25}v_{24}v_{23}v_6v_7v_hv_{h+1} \dots v_{22})$ contradicts the maximality of r. If $j = 15$ and $h = 19$, the set $\{v_2, v_3, v_7, v_{10}, v_{13}, v_{17}, v_{21}, v_{24}\}$ dominates G'. Hence $j = 18$, $h = 16$, and the set $\{v_5, v_7, v_{10}, v_{14}, v_{18}, v_{20}, v_{22}, v_{25}\}$ dominates G'.
Case 3.2.7.5: $a = b = 6, c = 4$. Since v_{11} has its third neighbor in G', by Lemma 9 for C and Lemma 8 for the paths $(v_{11}v_{10} \dots v_6v_{23}v_{24}v_{25}v_{12}v_{13} \dots v_{22}v_1v_2 \dots v_5)$, and $(v_{15}v_{14} \dots v_6v_{23}v_{24}v_{25}v_{16}v_{17} \dots v_{22}v_1v_2 \dots v_5)$ this neighbor is in $\{v_7, v_8, v_{15}\}$. Since v_{13} has its third neighbor in G', by Lemma 11 with $x = v_{11}$, $y = v_{15}$, and $z = v_{13}$, $v_{11}v_{15} \notin E(G')$.
Case 3.2.7.5.1: $v_{11}v_7 \in E(G')$. The path $(v_{23}v_{24}v_{25}v_{12}v_{13} \dots v_{22}v_1v_2 \dots v_7v_{11}v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. By Lemma 8 for this path and the paths $(v_7v_8 \dots v_{16}v_{25}v_{24}v_{23}v_6v_5 \dots v_1v_{22}v_{21} \dots v_{17})$, $(v_{15}v_{14} \dots v_6v_{23}v_{24}v_{25}v_{16}v_{17} \dots v_{22}v_1v_2 \dots v_5)$, and $(v_8v_9v_{10}v_{11}v_7v_6v_{23}v_{24}v_{25}v_{12}v_{13} \dots v_{22}v_1v_2 \dots v_5)$, this neighbor is v_{15}. Then the 23-cycle $(v_1v_2 \dots v_7v_{11}v_{10}v_9v_8v_{15}v_{14}v_{13}v_{12}v_{25}v_{16}v_{17} \dots v_{22})$ contradicts the maximality of r.
Case 3.2.7.5.2: $v_{11}v_8 \in E(G')$. The path $(v_{23}v_{24}v_{25}v_{12}v_{13} \dots v_{22}v_1v_2 \dots v_8v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By Lemma 8 for this path and Lemma 9 for C, this neighbor is in $S = \{v_2, v_5, v_{15}, v_{18}, v_{21}\}$. For each $v_i \in S$ except v_{15}, the set $\{v_2, v_5, v_6, v_{11}, v_{14}, v_{18}, v_{21}, v_{25}\}$ dominates G'. So, $v_9v_{15} \in E(G')$. Then the 23-cycle $(v_1v_2 \dots v_8v_{11}v_{10}v_9v_{15}v_{14}v_{13}v_{12}v_{25}v_{16}v_{17} \dots v_{22})$ contradicts the maximality of r.
Case 3.2.7.6: $a = 6, b = 7, c = 3$. Let v_i be the third neighbor of v_{14}. By Lemma 9 for C and Lemma 8 for the paths $(v_{21}v_{20} \dots v_{12}v_{25}v_{24}v_{23}v_{22}v_1v_2 \dots v_{11})$, $(v_{14}v_{13} \dots v_6v_{23}v_{24}v_{25}v_{15}v_{16} \dots v_{22}v_1v_2 \dots v_5)$, and $(v_{13}v_{14} \dots v_{25}v_{12}v_{11} \dots v_1)$, $i \in \{18, 21\}$. Since now v_{16} has its third neighbor in G', by Lemma 11 with $x = v_{14}$, $y = v_{18}$, and $z = v_{16}$, $v_{14}v_{18} \notin E(G')$. Hence $v_{14}v_{21} \in E(G')$ and the hamiltonian cycle $(v_1v_2 \dots v_{14}v_{21}v_{20} \dots v_{15}v_{25}v_{24}v_{23}v_{22})$ contradicts the maximality of r.
Case 3.2.7.7: $a = b = 7, c = 2$. Then v_{14} has its third neighbor in G'. By Lemma 9 for C and Lemma 8 for the paths $(v_{14}v_{13} \dots v_6v_{23}v_{24}v_{25}v_{15}v_{16} \dots v_{22}v_1v_2 \dots v_5)$, and $(v_{14}v_{15} \dots v_{25}v_{13}v_{12} \dots v_1)$, this neighbor is in $\{v_7, v_{10}, v_{18}, v_{21}\}$. By symmetry, we may assume that it is in $\{v_7, v_{10}\}$. Since v_{12} has its third neighbor in G', Lemma 11 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$ eliminates v_{10} as a possible neighbor of v_{14}. Thus $v_{14}v_7 \in E(G')$, and the hamiltonian cycle $(v_1v_2 \dots v_6v_{23}v_{24}v_{25}v_{13}v_{12} \dots v_7v_{14}v_{15} \dots v_{22})$ contradicts the maximality of r.
Case 3.2.8: $d = 5$. By (14), $(a, b, c) \in \{(4, 10, 3), (7, 7, 3)\}$.

Case 3.2.8.1: $a = 4, b = 10, c = 3$. Then v_{10} has its third neighbor in G'. If this neighbor lies on the 19-cycle $(v_5v_4 \dots v_1v_{22}v_{21} \dots v_{12}v_{25}v_{24}v_{23})$, the set $\{v_7, v_{10}\}$ dominates all but a P_{18}, hence this neighbor is in $\{v_6, v_7, v_8\}$. By Lemma 9 for C, v_8 cannot be this neighbor. Since v_8 has its third neighbor in C, if $v_{10}v_6 \in E(G')$, Lemma 11 with $x = v_{10}$, $y = v_6$, and $z = v_8$ gives a dominating set of $G'[C]$ of size 7. Hence $v_{10}v_7 \in E(G')$. Then the set $\{v_2, v_5, v_7, v_{12}, v_{15}, v_{18}, v_{21}, v_{25}\}$ dominates G'.

Case 3.2.8.2: $a = b = 7, c = 3$. Since $v_{25}v_{12} \in E(G')$, v_{13} has its third neighbor in G'. By Lemma 9 for C and Lemma 8 for the paths $(v_{14}v_{13} \dots v_5v_{23}v_{24}v_{25}v_{15}v_{16} \dots v_{22}v_1v_2 \dots v_4)$ and $(v_{13}v_{14} \dots v_{22}v_1v_2 \dots v_5v_{23}v_{24}v_{25}v_{12}v_{11} \dots v_6)$, this neighbor is in $\{v_3, v_6, v_9, v_{16}, v_{19}\}$. Lemma 11 with $x = v_5$, $y = v_{15}$, and $z = v_3$ shrinks the list to $\{v_6, v_9\}$. Since v_{11} has its third neighbor in G', by Lemma 11 with $x = v_9$, $y = v_{13}$, and $z = v_{11}$ yields $v_9v_{13} \notin E(G')$. Hence $v_{13}v_6 \in E(G')$. Now hamiltonian cycle $(v_1v_2 \dots v_5v_{23}v_{24}v_{25}v_{12}v_{11} \dots v_6v_{13}v_{14} \dots v_{22})$ contradicts the maximality of r.

Case 3.2.9: $d = 4$. By (14), $(a, b, c) = (6, 9, 3)$. Let v_i be the third neighbor of v_5. By Lemma 11 with $x = v_{22}$, $y = v_{10}$, and $z = v_5$, $i \le 9$. Then Lemma 8 for the path $(v_5v_6 \dots v_{10}v_{25}v_{24}v_{23}v_4v_3v_2v_1v_{22}v_{21} \dots v_{11})$, and Lemma 9 for C further yield that $i \in \{1, 8, 9\}$. Since v_3 has its third neighbor in G', by Lemma 11 with $x = v_1$, $y = v_5$, and $z = v_3$, $v_5v_1 \notin E(G')$. If $v_5v_9 \in E(G')$, then by the same argument, v_8 is adjacent to one of v_1 and v_5. So $v_8v_1 \in E(G')$. Then the 23-cycle $(v_1v_2v_3v_4v_{23}v_{22} \dots v_9v_5v_6v_7v_8)$ contradicts the maximality of r. Thus, $v_5v_8 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_4v_3v_2v_1v_{22}v_{21} \dots v_8v_5v_6v_7)$ forces v_7 to have its third neighbor, say v_i, in G'. By Lemma 8 for this path and Lemma 9 for C, $i \in \{1, 11, 14, 17, 20\}$. If $v_7v_1 \in E(G')$, then the 23-cycle $(v_1v_2v_3v_4v_{23}v_{22} \dots v_8v_5v_6v_7)$ contradicts the maximality of r. If $i \in \{11, 14, 17, 20\}$, then the set $\{v_2, v_5, v_{10}, v_{11}, v_{14}, v_{17}, v_{20}, v_{23}\}$ dominates G'.

Case 3.2.10: $d = 3$. Since $a \le b$ and $c \ge 2$, $a \le (22 - 3 - 2)/2 = 8.5$. So by (14), $a \in \{4, 6, 7\}$.

Case 3.2.10.1: $a \in \{4, 7\}$. Since v_2 shares a neighbor with v_{23}, it has its third neighbor, say v_i in G'. By Lemma 11 with $x = v_{22}$, $y = v_{a+d}$, and $z = v_2$, $4 \le i \le d + a - 1 \le 9$. By Lemma 8 for P, $i \ne 4, 7$. If $d + a - 2 \le i \le d + a - 1$, then the cycle $(v_{23}v_{24}v_{25}v_{d+a}v_{d+a+1} \dots v_{22}v_1v_2v_iv_{i-1} \dots v_3)$ contradicts the maximality of r. This means that $a = 7$ and $5 \le i \le 6$. The edge v_2v_5 contradicts Lemma 11 with $x = v_3$, $y = v_{10}$, and $z = v_5$. So $v_2v_6 \in E(G)$, a contradiction to Lemma 11 with $x = v_2$, $y = v_6$, and $z = v_4$.

Case 3.2.10.2: $a = 6$. Then by (14), $c = 3$. Since v_1 shares a neighbor with v_{23}, it has its third neighbor, say v_i in G'. By Lemma 11 with $x = v_{12}$, $y = v_3$, and $z = v_1$, $13 \le i \le 21$. By Lemma 8 for P, $i \ne 15, 18, 21$. If $13 \le i \le 14$, then the cycle $(v_{12}v_{11} \dots v_1v_iv_{i+1} \dots v_{25})$ contradicts the maximality of r. Lemma 11 with $x = v_{12}$, $y = v_{22}$, and $z = v_i$, shows that $i \ne 20, 17$. So $i \in \{16, 19\}$. The same lemma with $x = v_i$, $y = v_1$, and $z = v_{21}$, shows that the third neighbor of v_{21} is some v_j with $i + 1 \le j \le 19$. It follows that $i = 16$ and $17 \le j \le 19$. By Lemma 8 for P, $j \ne 19$. By the symmetry between v_1 and v_{11}, $v_{11}v_{18} \in E(G)$ and hence $j = 17$. But then symmetrically v_{13} also is adjacent to v_{17}, a contradiction.

Case 3.2.11: $d = 2$. Since $c \le d$, $c = 2$, and again no triple (a, b, c) satisfies (14).

Case 4: $r = 23$. Since v_{25} is the endpoint of a hamiltonian path in P, it has two neighbors in C. This forces v_{24} to be the endpoint of another hamiltonian path, and so v_{24} also has two neighbors in C. By the maximality of r, the distance on C between any neighbor of v_{24} and any neighbor of v_{25} is at least 3. Then Lemma 8 for P and the path $(v_{25}v_{24}v_{23}v_1v_2 \dots v_{22})$ forces the neighbors of v_{25} in C to be in $\{v_4, v_7, v_{10}, v_{13}, v_{16}, v_{19}\}$. By symmetry, we conclude that

(15) the distance on C between any neighbor of v_{24} and any neighbor of v_{25} is in $\{4, 7, 10\}$.

In particular, since each of these values is 1 modulo 3, the neighbors of v_{24}, and v_{25} cannot alternate around C. So, we may assume that $v_{25}v_d, v_{25}v_{d+a}, v_{24}v_{23-b} \in E(G')$, and $c = 23 - a - b - d$. We may assume further that $d \le c$ and $a \le b$. In particular, $d + a \le 11$ and hence $d \in \{4, 7\}$. Furthermore, since a is divisible by 3, $d+a \le 10$. As a neighbor of v_d, v_{d+1} has its third neighbor, say v_i, in C. By Lemma 11 with $x = v_{23}$, $y = v_{d+a}$, and $z = v_{d+1}$, $i \le d + a - 1$. If $i = 1$, then the hamiltonian cycle $(v_d v_{d-1} \dots v_1 v_{d+1} v_{d+2} \dots v_{25})$ contradicts the maximality of r. By Lemma 8 for the path $(v_{d+1}v_{d+2} \dots v_{23}v_1v_2 \dots v_d v_{25}v_{24})$, $i \notin \{2, 5, d + 3, d + 6\}$. By Lemma 8 for the path $(v_{d+a-1}v_{d+a-2} \dots v_1v_{23}v_{22} \dots v_{d+a}v_{25}v_{24})$, $i \ne 3, d-1$. Summarizing and remembering that $d \in \{4, 7\}$ and $d + a \le 10$, we have

$$\text{if } d = 4, \text{ then } 8 \le i \le d + a - 1 \le 9; \text{ if } d = 7, \text{ then } i = 4. \tag{16}$$

Case 4.1: $d = 4$. By above, $d + a \in \{7, 10\}$. So, by (16), $i \in \{8, 9\}$.

Case 4.1.1: $i = 9$. The path $(v_{24}v_{25}v_4v_3v_2v_1v_{23}v_{22} \dots v_9v_5v_6v_7v_8)$ forces v_8 to have its third neighbor, say v_j, in G'. By Lemma 8 for this path, $j \ne 2, 6$. By Lemma 11 with $x = v_{23}$, $y = v_{10}$, and $z = v_8$, $j \le 9$. Thus, $j \in \{1, 3\}$. If $j = 1$, then the hamiltonian cycle $(v_8v_7v_6v_5v_9v_{10} \dots v_{25}v_4v_3v_2v_1)$ contradicts the maximality of r. Thus, $v_3v_8 \in E(G')$. Then the set $\{v_3, v_6, v_{10}, v_{11}, v_{14}, v_{17}, v_{20}, v_{23}\}$ dominates G'.

Case 4.1.2: $i = 8$. By (15), $c \in \{4, 7\}$. The path $P' = (v_7v_6v_5v_8v_9 \dots v_{25}v_4v_3v_2v_1)$ forces v_7 to have its third neighbor, say v_j, in G'.

Case 4.1.2.1: $c = 4$. By the symmetry between v_5 and v_9, $v_6v_9 \in E(G')$. By Lemma 8 for P' and the symmetric path $(v_7v_8v_9v_6v_5 \dots v_1v_{23}v_{22} \dots v_{14}v_{24}v_{25}v_{10}v_{11}v_{12}v_{13})$, we have $j \in \{3, 11, 17, 20\}$. By symmetry, we may assume that either $j = 11$ or $j = 17$. Then the set $\{v_1, v_4, v_9, v_{11}, v_{14}, v_{17}, v_{18}, v_{21}\}$ dominates G'.

Case 4.1.2.2: $c = 7$. Recall that v_j is the third neighbor of v_7. By Lemma 8 for P, P', and the path $(v_7v_6v_5v_8v_9v_{10}v_{25}v_{24}v_{17}v_{18} \dots v_{23}v_1v_2v_3v_4)$, we have $j \in \{1, 11, 14\}$. If $j = 1$, then the hamiltonian cycle $(v_1v_2v_3v_4v_{25}v_{24} \dots v_8v_5v_6v_7)$ contradicts the maximality of r. If $j = 11$, then the 24-cycle $(v_1v_2v_3v_4v_{25}v_{10}v_9v_8v_5v_6v_7v_{11}v_{12} \dots v_{23})$ contradicts the maximality of r. Finally, if $j = 14$, then the set $\{v_2, v_7, v_8, v_{12}, v_{16}, v_{19}, v_{22}, v_{25}\}$ dominates G'.

Case 4.2: $d = 7$. By above, $d + a = 10$. By (16), $i = 4$. The path $P' = (v_1v_2v_3v_4v_8v_9 \dots v_{25}v_7v_6v_5)$ forces v_5 to have its third neighbor, say v_j, in G'. By Lemma 11 with $x = v_{23}$, $y = v_7$, and $z = v_5$, $j \le 6$. Thus, $j \in \{1, 2, 3\}$. Lemma 8 for P' and for the path $(v_9v_8 \dots v_1v_{23}v_{22} \dots v_{10}v_{25}v_{24})$ yields $j \ne 2$ and $j \ne 3$, respectively. So, $v_5v_1 \in E(G')$. Now the cycle $(v_1v_2v_3v_4v_8v_9 \dots v_{25}v_7v_6v_5)$ contradicts the maximality of r. □

Acknowledgment. The authors are grateful to V. A. Tashkinov for many helpful comments and suggestions.

References

[1] M. Blank, *An estimate of the external stability of a graph without pendant vertices*, Prikl. Math. i Programmirovanie, **10** (1973) 3–11.

[2] M. Cropper, D. Greenwell, A.J.W. Hilton, and A.V. Kostochka, *The domination number of cubic hamiltonian graphs*, AKCE J. Graphs Combin., **2** (2005), 137–144.

[3] K. Kawarabayashi, M. Plummer, and A. Saito, *Domination in a graph with a 2-factor*, J. of Graph Theory, **52** (2006), 1–6.

[4] A. K. Kelmans, *Counterexamples to the Cubic Graph Domination Conjecture*, submitted.

[5] A. V. Kostochka and B. Y. Stodolsky, *On domination in connected cubic graphs*, Discrete Math., **304** (2005), 45–50.

[6] A. V. Kostochka and B. Y. Stodolsky, *An upper bound on domination number of n-vertex connected cubic graphs*, to appear in Discrete Math.

[7] C. Löwenstein and D. Rautenbach, *Domination in graphs of minimum degree at least two and large girth*, Graphs Combin., **24** (2008), 37–46.

[8] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Coll. Publ. **3**, 1962.

[9] M. Plummer, *personal communication*.

[10] B. Reed, *Paths, stars, and the number three*, Combin. Probab. Comput. **5** (1996), 277–295.

[11] B. Y. Stodolsky, *On domination in* 2*-connected cubic graphs*, submitted.

Alexandr Vasil'evich Kostochka
Department of Mathematics, University of Illinois,
Urbana, IL 61801, USA and Institute of Mathematics,
630090, Novosibirsk, Russia
E-mail address: kostochk@math.uiuc.edu

Christopher Stocker
Department of Mathematics, University of Illinois,
Urbana, IL 61801, USA
E-mail address: stocker2@illinois.edu