Siberian Mathematical Journal, Vol. 52, No. 5, pp. 796–801, 2011
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VERTEX DECOMPOSITIONS OF SPARSE GRAPHS INTO AN INDEPENDENT VERTEX SET AND A SUBGRAPH OF MAXIMUM DEGREE AT MOST 1

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Abstract: A graph G is $(1,0)$-*colorable* if its vertex set can be partitioned into subsets V_1 and V_0 so that in $G[V_1]$ every vertex has degree at most 1, while $G[V_0]$ is edgeless. We prove that every graph with maximum average degree at most $\frac{12}{5}$ is $(1,0)$-colorable. In particular, every planar graph with girth at least 12 is $(1,0)$-colorable. On the other hand, we construct graphs with the maximum average degree arbitrarily close (from above) to $\frac{12}{5}$ which are not $(1,0)$-colorable.

In fact, we prove a stronger result by establishing the best possible sufficient condition for the $(1,0)$-colorability of a graph G in terms of the minimum, $Ms(G)$, of $6|V(A)| - 5|E(A)|$ over all subgraphs A of G. Namely, every graph G with $Ms(G) \geq -2$ is proved to be $(1,0)$-colorable, and we construct an infinite series of non-$(1,0)$-colorable graphs G with $Ms(G) = -3$.

Keywords: planar graphs, coloring, girth

1. Introduction

A graph G is called $(d_1,\dots,d_k)$-*colorable* if the vertex set of G can be partitioned into subsets $V_1,\dots,V_k$ so that the graph $G[V_i]$ induced by the vertices of V_i has the maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of the proper k-coloring (when $d_1 = \dots = d_k = 0$) and d-improper k-coloring (when $d_1 = \dots = d_k = d \geq 1$).

The proper and d-improper colorings have been widely studied. In particular, it was shown by Appel and Haken [1, 2] that every planar graph is 4-colorable, i.e. $(0,0,0,0)$-colorable. Cowen, Cowen, and Woodall [3] proved that every planar graph is 2-improperly 3-colorable, i.e. (2,2,2)-colorable. This latter result was extended by Havet and Sereni [4] to sparse graphs that are not necessarily planar: For every $k \geq 0$, every graph G with $\mathrm{mad}(G) < \frac{4k+4}{k+2}$ is k-improperly 2-colorable, i.e. (k,k)-colorable.

Recall that $\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$ is the maximum average degree over the subgraphs of G. The *girth* $g(G)$ of G is the length of a shortest cycle in G. The degree of a vertex v will be denoted by $d(v)$.

The problem of $(0,0)$-coloring is simple, since the odd cycle C_{2n-1} has $\mathrm{mad}(C_{2n-1}) = 2$ and is not $(0,0)$-colorable, whereas, on the other hand, if $\mathrm{mad}(G) < 2$, then G has no cycles, and so G is bipartite, i.e., $(0,0)$-colorable.

In this paper, we focus on the $(1,0)$-coloring of a graph, i.e. partitioning the vertices of a graph into subsets V_1 and V_0 so that every vertex in V_1 is adjacent to at most one vertex in V_1, while the vertices in V_0 are pairwise nonadjacent. (In what follows, we say that the vertices in $G[V_1]$ are *colored with color* 1, and the vertices of $G[V_0]$, by *color* 0.)

Glebov and Zambalaeva in [5] proved that every planar graph G with $g(G) \geq 16$ is $(1,0)$-colorable. This was strengthened by Borodin and Ivanova [6] by proving that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(1,0)$-colorable, which implies, in particular, that every planar graph G with $g(G) \geq 14$ is $(1,0)$-colorable.

The first author was supported by the Russian Foundation for Basic Research (Grants 08–01–00673 and 09–01–00244). The second author was supported by the NSF grant DMS–0965587 and the Ministry for Education and Science of the Russian Federation (Grant 14.740.11.0868).

Novosibirsk and Urbana. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 52, No. 5, pp. 1004–1011, September–October, 2011. Original article submitted July 15, 2010.

For each integer $k \geq 2$, Borodin et al. [7] proved that every graph G with $\text{mad}(G) < \frac{3k+4}{k+2} = 3 - \frac{2}{k+2}$ is $(k, 0)$-colorable and, on the other hand, for all $k \geq 2$ constructed non-$(k, 0)$-colorable graphs with mad arbitrarily close to $\frac{3k+2}{k+1} = 3 - \frac{1}{k+1}$, and also non-$(1, 0)$-colorable graphs with mad arbitrarily close to $\frac{17}{7}$.

The purpose of this paper is to prove

Theorem 1. *Every graph G with $\text{mad}(G) \leq \frac{12}{5}$ is $(1, 0)$-colorable, and the restriction on $\text{mad}(G)$ is sharp.*

The second part of Theorem 1 means that there exist non-$(1, 0)$-colorable graphs G with $\text{mad}(G)$ arbitrarily close to $\frac{12}{5}$.

Since each graph G embedded in a surface with a nonnegative Euler characteristic (i.e., the plane, projective plane, torus or Klein bottle) satisfies $\text{mad}(G) \leq \frac{2g(G)}{g(G)-2}$, from Theorem 1 we have

Corollary 1. *Each graph G embedded in a surface with a nonnegative Euler characteristic is $(1, 0)$-colorable if $g(G) \geq 12$.*

As proved in [7], there exist non-$(1, 0)$-colorable planar graphs with girth 7. Along with Corollary 1, this leads to the following

Problem 1. *Find the smallest natural number g such that every planar graph with girth at least g is $(1, 0)$-colorable.*

Now consider a refinement of the parameter $\text{mad}(G)$ for graphs G with $\text{mad}(G)$ close to $\frac{12}{5}$. For each graph A, let $\rho(A) = 6|V(A)| - 5|E(A)|$ and call this amount the *sparseness* of A. Define the *minimum sparseness* $Ms(G)$ of a graph G to be the minimum $\rho(A)$ over all subgraphs A of G. Thus, $\text{mad}(G) \leq \frac{12}{5}$ is equivalent to $Ms(G) \geq 0$.

We prove Theorem 1 in the following stronger form:

Theorem 2. *Each graph G with $Ms(G) \geq -2$ is $(1, 0)$-colorable, and there are infinitely many non-$(1, 0)$-colorable graphs G with $Ms(G) = -3$.*

2. Proving the Sharpness of Restrictions in Theorems 1 and 2

We now construct non-$(1, 0)$-colorable graphs G_p with $Ms(G_p) = -3$ for all $p \geq 1$ and with $\text{mad}(G_p)$ tending to $\frac{12}{5}$ as p grows.

Let $p \geq 1$ be an integer. Let G_p be the graph obtained from p independent 3-cycles $x_i y_i z_i$, where $1 \leq i \leq p$, by adding paths $y_i y'_i x'_{i+1} x_{i+1}$, where $d(y'_i) = d(x'_{i+1}) = 2$ whenever $1 \leq i \leq p - 1$, followed by adding 3-cycles $x_1 x'_1 x''_1$ and $y_p y'_p y''_p$, where $d(x'_1) = d(x''_1) = d(y'_p) = d(y''_p) = 2$ (see Fig. 1 for $p = 3$).

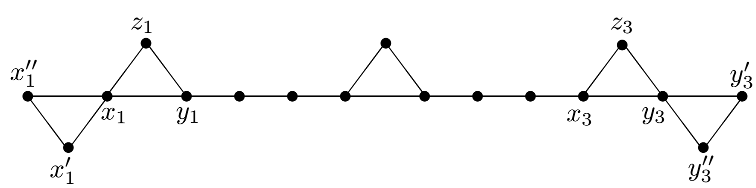

Fig. 1. The graph G_3.

The following simple observation is useful:

Claim 1. *In every $(1, 0)$-coloring of the 3-cycle C_3, precisely two vertices are colored with 1. In particular, each vertex in C_3 has a neighbor in it colored with 1.*

Suppose that G_p has a $(1, 0)$-coloring c. Since x_1 belongs to two 3-cycles, we have $c(x_1) = 0$ by Claim 1. But then $c(y_1) = c(z_1) = 1$, which implies that $c(y'_1) = 0$, and hence $c(x'_2) = 1$. As $x'_2 x_2 \in E(G_p)$, it follows by Claim 1 that x_2 has a neighbor colored with 1 in $\{y_2, z_2\}$, so that $c(x_2) = 0$. Repeating this argument, we see that $c(x_3) = \cdots = c(x_p) = 0$, and so $c(y_p) = c(z_p) = 1$; but y_p has another neighbor colored with 1 in $\{y'_p, y''_p\}$; a contradiction.

Finally, it is easy to check that

$$Ms(G_p) = \rho(G_p) = 6(5p+2) - 5(6p+3) = -3,$$

and

$$\text{mad}(G_p) = \frac{2|E(G_p)|}{|V(G_p)|} = \frac{12p+6}{5p+2} = \frac{12}{5} + \frac{6}{5(5p+2)}.$$

3. Proving the Main Statement in Theorem 2

A vertex of degree d (respectively, at least d or at most d) is called a *d-vertex* (respectively, a *d^+-vertex* or *d^--vertex*). By a *k-path* we mean a path in which all k internal vertices have degree 2, while both terminal vertices have degree at least 3. By a *$(k_1, k_2, \ldots, k_t)$-vertex* we mean a t-vertex that is incident with k_1-,k_2-, $\ldots$, k_t-paths.

A 3-cycle is *special* if it has at least two 2-vertices.

We say that a graph G is *smaller* than a graph G' if either $|V(G)| < |V(G')|$ or $|V(G)| = |V(G')|$ and G has more special 3-cycles than G'.

Let G be a smallest counterexample to Theorem 2. Clearly, G is connected and has no pendant vertices. Since $G \neq C_3$, each special 3-cycle in G actually has precisely two 2-vertices.

3.1. Structural properties of the minimum counterexample. Note that the *butterfly graph BF*, which consists of two special triangles xyz and $xy'z'$ with vertex x in common, has $\rho(BF) = 6 \times 5 - 5 \times 6 = 0$. It is not hard to check that every proper subgraph H' of BF has $\rho(H') > 0$, so that $Ms(BF) = 0$.

Lemma 1. *Every subgraph H of G such that $|V(H)| \geq 5$ and $BF \neq H \neq G$ has $\rho(H) \geq 1$.*

Proof. Suppose that $\rho(H) \leq 0$. Note that $G - H$ has no vertex adjacent to at least two vertices of H. Indeed, if v were such a vertex, then we would have $\rho(V(H) + v) \leq \rho(H) + 6 - 2 \times 5 \leq -4$, which is impossible. Since H is a subgraph of G, it has a $(1, 0)$-coloring c_0. We construct a graph $G^* = G^*(H)$ from G as follows:

(a) add to $G - H$ a copy H^* of the butterfly graph with 3-cycles $h_0 h'_0 h''_0$ and $h_0 h_1 h'_1$;

(b) join each vertex $w \in V(G - H)$ adjacent to a vertex colored with 1 in c_0 by an edge to h_1;

(c) join each vertex $w \in V(G - H)$ adjacent to a vertex colored with 1 in c_0 by an edge to h_0.

If $|V(H)| \geq 6$, then G^* has fewer vertices than G. Now check that if $|V(H)| = 5$, then $|V(G^*)| = |V(G)|$ but G^* has more special 3-cycles than G, since $H \neq H^*$.

Indeed, since $-2 \leq \rho(H) \leq 0$, we have $|E(H)| = 6$. As the complete graph K_4 has $\rho(K_4) = -6$, our H does not contain K_4. So, among the possible $H \neq BF$ we have only the complete bipartite graph $K_{2,3}$, the 5-cycle with a chord, and $K_4 - e$ with a pendant vertex (attached to $K_4 - e$ in one of two possible ways). Besides, a special 3-cycle in G cannot have only one vertex outside H, since such a vertex had to be adjacent to more than one vertex of H, while a special 3-cycle in G having at most one vertex in H is special in G^*, too. On the other hand, G^* has a special 3-cycle that does not belong to G. Thus G^* is smaller than G.

We prove now that

$$Ms(G^*) \geq -2. \tag{1}$$

Suppose that $A^* \subseteq G^*$ and $\rho(A^*) \leq -3$. Let $B = V(A^*) - H^*$, $H' = V(A^*) \cap H^*$, and let e^* edges join B to H'. Then

$$-3 \geq \rho(A^*) = \rho(G^*[B]) + \rho(H^*[H']) - 5e^*. \tag{2}$$

For $G' := G[B \cup H]$ we similarly obtain

$$\rho(G') \leq \rho(G^*[B]) + \rho(H) - 5e^*,$$

since each edge joining B to H' in G^* corresponds to an edge joining B to H in G. As $\rho(H^*[H']) \geq Ms(BF) \geq 0$ and $\rho(H) \leq 0$, this implies due to (2) that $Ms(G) \leq \rho(G') \leq \rho(A^*) \leq -3$; a contradiction. Thus (1) is proved.

Since G^* is smaller than G, it follows by (1) that there is a $(1,0)$-coloring c^* of G^*. Note that $c^*(h_0) = 0$, since h_0 belongs to two 3-cycles. Hence, all neighbors of h_0, including h_1 and h_1', are colored with 1 in c^*, while all neighbors of h_1, except h_1', are colored with 0. Thus, the restriction of c^* to $G^* \setminus \{h_0, h_1\}$ combined with the coloring c_0 yields a $(1,0)$-coloring of G. $\square$

Lemma 2. *If* 2*-vertices x and y in G are adjacent, then there is a* 3*-cycle xyz.*

Proof. Suppose that there is a 2-path $wxyz$, where $w \neq z$ and $d(x) = d(y) = 2$. Let a graph G^* be obtained from G by deleting the edge yz and adding the edge wy. As G^* has more special 3-cycles than G (due to the presence of the 3-cycle wxy), G^* is smaller than G.

If $Ms(G^*) \geq -2$, then due to the minimality of G, graph G^* has a $(1,0)$-coloring c^*. Define a coloring c of G as follows: (a) put $c(v) := c^*(v)$ for all $v \neq x, y$; (b) put $c(y) \neq c(z)$; (c) if $c(y) = c(w)$ or $c(w) = 0$, then put $c(x) \neq c(w)$; (d) if $c(z) = 1$ and $c(y) = 0$ (recall that due to Claim 1 at least one of x and y is colored with 1 in c^*), then put $c(x) := 1$. By construction, c is a $(1,0)$-coloring; a contradiction.

So, $Ms(G^*) \leq -3$, which means that G^* has a subgraph A^* with $\rho(A^*) \leq -3$. This can happen only if $\{w, x, y\} \subset V(A^*)$ (since $\rho(A^* \setminus \{x, y\}) \geq -2$, and each pendant vertex contributes $6 \times 1 - 5 \times 1 = 1$ to $\rho(A^*)$).

Thus we are done unless there is a subgraph A in $G' = G \setminus \{x, y\}$ such that

$$\rho(A) \leq \rho(A^*) - 6 \times 2 + 5 \times 3 = 0.$$

Note that $z \notin G'$, since otherwise the subgraph A^+ of G on the vertex set $V(A) \cup \{x, y\}$ has $\rho(A^+) = \rho(A) + 6 \times 2 - 5 \times 3 \leq -3$, which is impossible. By Lemma 1, either A is the butterfly graph or $|V(A)| \leq 4$.

By symmetry, z must also belong to a certain subgraph B of G such that $\rho(B) \leq 0$ and $V(B) \cap \{w, z\} = \{z\}$, with the same properties as A.

For the subgraphs $G'[A \cup B]$ and $G'[A \cap B]$ of G' on the vertex sets $A \cup B$ and $A \cap B$, respectively, it is not hard to check that

$$\rho(G'[A \cup B]) + \rho(G'[A \cap B]) \leq \rho(A) + \rho(B) \leq 0.$$

We have $\rho(G'[A \cup B]) \geq 1$, since otherwise $\rho(G[A \cup B \cup \{x, y\}]) \leq 0 + 6 \times 2 - 5 \times 3 = -3$. Hence, $\rho(G'[A \cap B]) \leq -1$, which implies by Lemma 1 that $|V(G'[A \cap B])| = 4$ and $|E(G'[A \cap B])| = 5$. But then A and B are not butterflies, so that $\rho(A) \geq 1$ and $\rho(B) \geq 1$ by Lemma 1; a contradiction. $\square$

Corollary 2. *G has no k-paths for $k \geq 3$.*

Lemma 3. *No* 3*-vertex in G belongs to a special* 3*-cycle.*

Proof. Suppose that a 3-vertex x lies in a special 3-cycle xyz and is adjacent to a vertex $w \notin \{y, z\}$. Take a $(1,0)$-coloring of $G \setminus \{x, y, z\}$, color x other than w, and then it is easy to color y and z. $\square$

3.2. Discharging. By the assumption on $Ms(G)$, we have

$$\sum_{v \in V(G)} (5d(v) - 12) \leq 4.$$

Let the *initial charge* of each vertex v of G be equal to $\mu(v) = 5d(v) - 12$, and let the *final charge* $\mu^*(v)$ be determined by the following rule:

R1. Every 2-vertex that belongs to a 1-path P gets charge 1 from each of the ends of P, while each 2-vertex that belongs to a special triangle gets charge 2 from the neighbor vertex of degree greater than 2.

If $d(v) = 2$, then $\mu^*(v) = 0$ by R1.

Claim 2. *Every* 3^+*-vertex* v *has* $\mu^*(v) \geq 1$, *unless* v *is a* $(1,1,1)$*-vertex, in which case* $\mu^*(v) = 0$.

Proof. If $d(v) = 3$, then by Lemma 3, either v makes at most two donations by R1, in which case $\mu^*(v) \geq 5 \times 3 - 12 - 2 \times \frac{4}{3} > 0$, or v is adjacent to three 2-vertices, in which case $\mu^*(v) = 0$.

If $d(v) \geq 4$, then $\mu^*(v) \geq 5d(v) - 12 - d(v) \times 2 = 3(d(v) - 4) \geq 0$. The equality here is attained only if $d(v) = 4$ and v is incident with two special triangles; but then G degenerates into the butterfly graph BF, which is trivially $(1,0)$-colorable. Thus, each 4^+-vertex v has $\mu^*(v) > 0$. $\square$

Since

$$\sum_{v \in V} \mu^*(v) = \sum_{v \in V} \mu(v) \leq 4, \tag{3}$$

it follows from Claim 2 that each vertex v of G has $\mu^*(v) \leq 4$.

An edge is *hard* if it joins the two 3^+-vertices. Let $h(v)$ be the number of hard edges at v, and let $s(v)$ be the number of special 3-cycles containing v.

Consider a partial $(1,0)$-coloring c of G in which all 3^+-vertices are colored with 1, all 2-vertices in 1-paths are colored with 0, and the 2-vertices in the special 3-cycles are uncolored. The only obstacle for immediate making c into a desired $(1,0)$-coloring of G is the presence in G of at least one vertex v with

$$h(v) + s(v) \geq 2. \tag{4}$$

Indeed, if (4) fails for each v, then c has no vertex colored with 1 and adjacent to more than one vertex colored with 1, and so we can easily extend c to the 2-vertices of the special 3-cycles.

We now make sure however that if (4) holds for at least one v, then (3) already fails.

Case 1. $d(v) \geq 6$. Here $\mu^*(v) \geq 5d(v) - 12 - d(v) \times 2 = 3(d(v) - 4) \geq 5$, which contradicts (3).

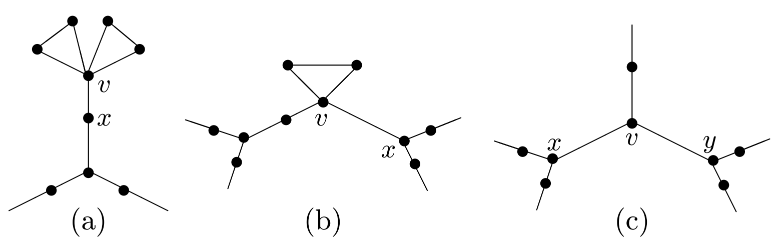

Fig. 2. The concluding reducible configurations in G.

Case 2. $d(v) = 5$. If $h(v) \geq 1$ or $s(v) \leq 1$, then $\mu^*(v) \geq 5$ contrary to (3). Suppose that $s(v) = 2$ and v is adjacent to a 2-vertex x not in a special 3-cycle (see Fig. 2a). Then $\mu^*(v) = 4$, which means due to Claim 2 that all other 3^+-vertices in G must have $\mu^* = 0$, i.e. they should be $(1,1,1)$-vertices. This makes it possible to color v with 0, and all its neighbors, including x, with 1 and to get a desired $(1,0)$-coloring of G.

Case 3. $d(v) = 4$. As we remember, $s(v) \leq 1$. If v is a 4-vertex incident with at least two hard edges, then $\mu^*(v) \geq 4$, and each adjacent 3^+-vertex has $\mu^* \geq 1$ due to Claim 2, which contradicts (3). Hence, (4) implies that $h(v) = 1$ and $s(v) = 1$. Let vx be a hard edge incident with v (see Fig. 2b). Then $\mu^*(v) = 3$ and $\mu^*(x) \geq 1$; therefore, as we did at the end of considering Case 2, we may color v with 0, and all its neighbors with 1.

Case 4. $d(v) = 3$. Lemma 3 implies that $s(v) = 0$. If $h(v) = 3$, then $\mu^*(v) = 3$ and $\mu^*(x) \geq 1$ for each vertex x adjacent to v, contrary to (3). Therefore, we have $h(v) = 2$ by (4), and let v be adjacent to the 3^+-vertices x and y (see Fig. 2c). Here $\mu^*(v) = 2$, $\mu^*(x) \geq 1$, and $\mu^*(y) \geq 1$, and so using Claim 2 we can again color v with 0, and color the only 2-vertex adjacent to v with 1.

Theorem 2 is proved.

In conclusion, the authors express their sincere gratitude to the referee for the remarks on the initial version of this paper.

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