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A REFINEMENT OF A RESULT OF CORRÁDI AND HAJNAL

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Corrádi and Hajnal proved that for every $k \geq 1$ and $n \geq 3k$, every n-vertex graph with minimum degree at least $2k$ contains k vertex-disjoint cycles. This implies that every $3k$-vertex graph with maximum degree at most $k-1$ has an equitable k-coloring. We prove that for $s \in \{3,4\}$ if an sk-vertex graph G with maximum degree at most k has no equitable k-coloring, then G either contains K_{k+1} or k is odd and G contains $K_{k,k}$. This refines the above corollary of the Corrádi-Hajnal Theorem and also is a step toward the conjecture by Chen, Lih, and Wu that for $r \geq 3$, the only connected graphs with maximum degree at most r that are not equitably r-colorable are $K_{r,r}$ (for odd r) and K_{r+1}.

1. Introduction

A vertex coloring of G is *equitable* if any two color classes differ in size by at most one; otherwise it is *inequitable*. Such colorings, and, more generally, colorings with bounded sizes of color classes have applications in the mutual exclusion scheduling problem, scheduling in communication systems, construction timetables, and round-a-clock scheduling (see [3,19,20]).

One of the first results on equitable coloring was a theorem by Corrádi and Hajnal [5].

Theorem 1 ([5]). *For every positive integer k and $n \geq 3k$, each n-vertex graph with minimum degree at least $2k$ contains k vertex-disjoint cycles.*

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Formally, the theorem is not about equitable colorings, but one of its corollaries is:

Corollary 2 ([5]). *For every positive integer k and $n=3k$, each n-vertex graph with maximum degree at most $k-1$ has an equitable k-coloring.*

This result was later generalized by Hajnal and Szemerédi [6] as follows.

Theorem 3 ([6]). *For every positive integer r, each graph with $\Delta(G)\le r$ has an equitable $(r+1)$-coloring.*

This theorem has interesting applications in extremal combinatorial and probabilistic problems, see e.g., [1,2,7,17,18]. For a shorter proof of Theorem 3, see [8] or [13]; for an algorithm see [12].

It is natural to ask which graphs G with $\Delta(G)=r\ge 3$ have equitable r-colorings. Certainly such graphs are r-colorable and so do not contain the complete graph K_{r+1}. Note that if r is odd, then the complete bipartite graph $K_{r,r}$ has no r-equitable coloring. Chen, Lih and Wu [4] proposed the following strengthening of Theorem 3 and Brooks' Theorem.

Conjecture 4 (3). *If G is an r-colorable graph with $\Delta(G)\le r$, then either G has an equitable r-coloring or $\omega(G)\ge r+1$ or both $r=2$ and G contains an odd cycle or both r is odd and G contains $K_{r,r}$.*

By Brooks' Theorem this is equivalent to:

Conjecture 5. *If G is an r-colorable graph with $\Delta(G)\le r$, then either G has an equitable r-coloring or r is odd and G contains $K_{r,r}$.*

Some partial cases of Conjecture 5 were proved in [4,16,21,22,15,11]. In particular, Chen, Lih and Wu [4] proved that the conjecture holds for $r=3$:

Theorem 6 ([4]). *Let G be a connected graph with $\Delta(G)\le 3$. Then G has no equitable 3-coloring if and only if $G=K_4$ or $G=K_{3,3}$.*

Furthermore, the authors proved the the conjecture also holds for $r=4$:

Theorem 7 ([11]). *Let G be a 4-colorable graph with $\Delta(G)\le 4$. Then G has an equitable 4-coloring.*

In this paper, we attack the Chen-Lih-Wu Conjecture from the other direction: we consider graphs with large maximum degree (in comparison with the number of vertices). Kostochka, Pelsmajer and West [14] proved

Theorem 8 ([14]). *Let G be a graph with $\chi(G),\Delta(G),\frac{1}{2}|G|\le r$. If r is even or G does not contain $K_{r,r}$ then G is equitably r-choosable. In particular, G has an equitable r-coloring.*

Our main result is

Theorem 9. *Let G be a graph with $\chi(G), \Delta(G), \frac{1}{4}|G| \leq r$. If r is even or G does not contain $K_{r,r}$ then G has an equitable r-coloring.*

This result can be considered as a refinement of Corollary 2 and an improvement of the equitable coloring version of Theorem 8. The rest of this paper is devoted to the proof of Theorem 9. The following theorem will be needed.

Theorem 10 (Theorem 4 of [10]). *If G is an r-colorable graph with $\Delta(G) \leq r$ that contains $Q = K_{r,r}$ then G has an equitable r-coloring if and only if $G-Q$ has an inequitable* r-coloring.

Most of our notation is standard. Here we review some possible exceptions. Let $G=(V,E)$ be a graph with $A, B \subseteq V$. Let $|G| := |V|$ and $\|G\| := |E|$. Set $E(A,B) := \{ab \in E\colon\ a \in A \text{ and } b \in B\}$. When we write $ab \in E(A,B)$ we tacitly assume that $a \in A$ and $b \in B$ even though the edge ab is symmetric. Put $\|A,B\| := |E(A,B)|$ and $\|a,B\| := \|\{a\},B\|$. Set $\|A\| := \|G[A]\|$.

2. Setup and Tools

Suppose that Theorem 9 fails, and let G be a counterexample. Then $\chi(G), \Delta(G), \frac{1}{4}|G| \leq r$, and if r is odd then $K_{r,r} \nsubseteq G$, but G has no equitable r-coloring for some integer r. We may assume that $|G|$ is divisible by r, since if not, adding a disjoint clique of $r-(|G| \bmod r)$ new vertices will result in another counter example. Choose r as small as possible; then choose $G=(V,E)$ with $|G| = rs$ for some integer $s \leq 4$ with s as small as possible, and subject to this $\|G\|$ as small as possible. Then the theorem holds for H if $\Delta(H) < r$, and also if both $\Delta(H) = r$ and $|H| \leq |G| - r$. Since $\chi(G) \leq r$, G does not contain K_{r+1}. Moreover, G does not contain $K_{r,r}$: If r is odd this is by hypothesis; if r is even, it is by the minimality of G, since $K_{r,r,}$ would be a component with an equitable r-coloring. By Theorem 7, $r \geq 5$, and by Theorem 8, $s \geq 3$.

A *nearly equitable* coloring is a coloring such that every color class has the same size s except for one *small* class V^- with size $s-1$ and one *large* class V^+ with size $s+1$. The following lemma (Theorem 2 in [15]) is used to show that G has a nearly equitable r-coloring.

Lemma 11 ([15]). *Let H be a graph with $\chi(H), \Delta(H) \leq r$. Let $u \in V(H)$ and f be any r-coloring of $G-u$ with color classes $V_1, \dots, V_r$. Then there is an r-coloring of G with color classes $W_1, \dots, W_r$ such that $|W_i| = |V_i|$ for all but one i.*

Proposition 12. *G has a nearly equitable r-coloring.*

Proof. Let $xy \in E$. By the minimality of G, $G-xy$ has an equitable r-coloring. Thus $G-x$ has an r-coloring with one class of size $s-1$ and all other classes of size s. Since G does not have an equitable r-coloring, Lemma 11 implies that G has a nearly equitable r-coloring. ∎

Let f be a nearly equitable coloring of G with color classes $V^- = V_1,\dots,V_r = V^+$. Construct an auxiliary digraph $\mathcal{H} := \mathcal{H}(G,f)$ as follows. The vertices of $\mathcal{H}$ are the color classes $V_1,\dots,V_r$. A directed edge $V'V''$ belongs to $E(\mathcal{H})$ if some vertex $x \in V'$ has no neighbors in V''. In this case we say that x *is movable to* V'' *and that* x *witnesses* the edge $V'V''$. Call a color class V_i of f *accessible* if V^- is reachable from V_i in the digraph $\mathcal{H}$. By definition, V^- is accessible. Let $\mathcal{A} := \mathcal{A}(f)$ denote the family of accessible classes, $\mathcal{B}$ denote the family of inaccessible classes, $A := \bigcup \mathcal{A}$, and $B := \bigcup \mathcal{B} = V - A$. If $V_r \in \mathcal{A}$ then switching witnesses along a path from V_r to V^- yields an equitable r-coloring; so $V_r \in \mathcal{B}$. Let $a := |\mathcal{A}|$ and $b := |\mathcal{B}| = r-a$. Then $|A| = as-1$ and $|B| = bs+1$. A vertex $v \in V_i$ is *movable* if it is movable to some accessible class; otherwise it is unmovable.

Call a nearly equitable r-coloring an *optimal coloring* if a is as big as possible. Fix an optimal coloring $f = (V^- = V_1,\dots,V_r = V^+)$, where the accessible classes $V_1,\dots,V_a$ are ordered (by breadth-first search) so that V^- is reachable from each accessible V_i by a path in $\mathcal{H}[V_1,\dots,V_i]$. An accessible class V_i is *terminal* if V^- can be reached from every accessible class $V_j \in \mathcal{A}-V_i$ by a path in $\mathcal{H}-V_i$. For example, V_a is terminal. Class V_1 is terminal if and only if $a=1$. Let $\mathcal{A}' = \mathcal{A}'(f)$ be the set of terminal classes, $A' := \bigcup \mathcal{A}'$ and $a := |\mathcal{A}'|$.

The next two propositions illustrate the utility of these definitions. The structural properties of $\mathcal{H}[A]$ allow us to find other useful equitable a-colorings of $G[A]$, and to bound the degree of $G[B]$. In previous papers this degree bound on $G[B]$ allowed us to apply the minimality of G to obtain an equitable b-coloring of $G[B-y]$ for any vertex $y \in B$. In the current setting this is more subtle when b is odd, since $G[B]$ may contain $K_{b,b}$.

Proposition 13. *Let u be a movable vertex in a terminal class Z. Then $G[A \setminus Z+u]$ has an equitable $(a-1)$-coloring.*

Proof. Suppose u is movable to $X \in \mathcal{A}-Z$. Since Z is terminal, there exists a path $\mathcal{P} = X\dots V^-$ in $\mathcal{H}-Z$. Moving u to X and shifting witnesses along $\mathcal{P}$ yields an equitable $(a-1)$-coloring of $G[A \setminus Z+u]$. ∎

Proposition 14. *$\Delta(G[B]) \le b$.*

Proof. Every vertex in B has a neighbor in each of the a classes in $\mathcal{A}$; so it has at most b neighbors in B. ∎

Lemma 15. $a \geq 2$.

Proof. Suppose not; so $a=1$. Every vertex in B has a neighbor in A. So

$$(r-1)s+1 \leq |B| \leq \|A,B\| \leq r|A| = r(s-1).$$

Thus $5 \leq r \leq s-1 \leq 3$, a contradiction. ∎

Proposition 16. *Every $Z \in \mathcal{A}$ has at least $|Z|-s+2$ movable vertices.*

Proof. If $Z=V^-$ set $Z^- := Z$. Otherwise $Z \in \mathcal{A}-V^-$, and so there exists a path $\mathcal{P}=ZX\dots V^-$ in $\mathcal{H}$. Put $Z^- := Z-u$, where u witnesses the edge ZX. In either case, $G[A \backslash Z^-]$ has an equitable $(a-1)$-coloring. To finish, we show that some vertex $z \in Z^-$ is movable to a class in $\mathcal{A}-Z$.

Suppose not. Then $\|z, A \setminus Z^-\| \geq a-1$, and so $\|z, B \cup Z^-\| \leq b+1$ for every $z \in Z^-$. Also $\|y,B\| \leq b$ and $\|y, B \cup Z^-\| \leq b+1$ for every vertex $y \in B$. Set $G' = G[B \cup Z^-]$. Then $|G'| = (b+1)s$ and $\Delta(G'), \chi(G') \leq b+1$. Since $a \geq 2$, $b+1 < r$. If G' has an equitable $(b+1)$-coloring then G has an equitable r-coloring, a contradiction. Otherwise the minimality of G implies $b+1$ is odd and there exists $Q = K_{b+1,b+1} \subseteq G'$. Moreover, Theorem 10 implies that every $(b+1)$-coloring of $G'-Q$ is equitable.

Since Z^- is independent, there exists $y \in Q \cap B$. Let $\{X, Y\}$ be a bipartition of Q with $y \in Y$. Since $\|y,B\| \leq b < |X|$, there exists $x \in X \cap Z^-$. Since Z^- is independent, $N_Q(x) = Y \subseteq B$. So every $x' \in X$ satisfies $\|x,B\| = b+1$. Thus $X \subseteq Z^-$. Since

$$|Z^- \setminus X| = s-1-(b+1) < s-(b+1) < s+1-(b+1) \leq |V^+ \setminus Y|,$$

$G'-Q$ has an inequitable $(b+1)$-coloring, a contradiction. ∎

For each vertex $x \in Z \in \mathcal{A}$, define the *weight* of an edge $xy \in E(Z,B)$ to be $w(xy) = \frac{1}{\|y,Z\|}$ and the *charge* of a vertex $x \in Z$ to be

$$ch(x) := \sum_{y \in B \cap N(x)} w(xy).$$

Then

$$\sum_{x \in Z} ch(x) = \sum_{x \in Z} \sum_{y \in B \cap N(x)} w(xy) = \sum_{y \in B} \sum_{x \in Z \cap N(y)} w(xy) = bs+1. \tag{2.1}$$

We say that $y \in B$ is a *c-neighbor* of x if $w(xy)=c$. In particular, if $w(xy) = c = 1$, we call x a *solo neighbor* of y, y a *solo neighbor* of x, and xy a solo edge. A vertex is a *solo vertex*, if it has a solo neighbor. Call a vertex $y \in B$ *good* if $B-y$ has an equitable b-coloring.

Proposition 17. *Let $z \in A'$ be a solo vertex, with a good solo neighbor y. Then z is not movable, $\|z,A\| \geq a-1$ and $\|z,B\| \leq b+1$.*

Proof. Suppose $z \in Z \in \mathcal{A}'$ is movable to some class $W \in \mathcal{A}-Z$. By Proposition 13, there exists an equitable (a–1)-coloring of $G[A \backslash Z+z]$. Since zy is solo, $Z-z+y$ is independent. Since y is good, $G[B-y]$ has an equitable b-coloring. So G has an equitable r-coloring, contradicting the minimality of G. ∎

Proposition 18. *Suppose $z \in Z \in \mathcal{A}'$ and z has a good solo neighbor y. Then y is adjacent to every $y' \in S_z$.*

Proof. Suppose not; say y is nonadjacent to $y' \in S_z - y$. Move z out of Z and y into $Z-z$. Since y is good, $G[B-y]$ has an equitable b-coloring g_B. If z is movable to one of the classes Y of g_B then doing so yields a new nearly equitable r-coloring g of G with large class $Y+z$. Moreover, $\mathcal{A}(f) \subseteq \mathcal{A}(g)$, since Z is terminal and z is not movable by Proposition 17; also the class $Y = g^{-1}(y') \in \mathcal{A}(g)$, since y' is movable to $Z-z+y$. This is a contradiction, since f is optimal. Otherwise, z has exactly one neighbor in every class of g_B. Then moving z into the class of y', y' into $Z-z+y$, and applying Proposition 13 to $G[A \setminus Z+u]$ for some movable $u \in Z$, yields an equitable r-coloring of G. ∎

Proposition 19. *If $z \in Z \in \mathcal{A}'$ has a good solo neighbor y and a $\frac{1}{2}$-neighbor y' that is adjacent to a movable vertex $u \in Z$ then y' is adjacent to y.*

Proof. Suppose $yy' \notin E$. First move z out of Z and y into $Z-z$. Since y is good, $G[B-y]$ has an equitable b-coloring g_B. If possible, move z into one of the classes of g_B. This yields a nearly equitable coloring g of G, and $\mathcal{A}(f) \subseteq \mathcal{A}(g)$. Moreover, y' now has a movable solo neighbor u, contradicting Proposition 17. Otherwise, z has exactly one neighbor in every class of g_B. Move z to the class of y', y' to $Z-z+y$, and u to some class of $W \in \mathcal{A}$; then shift witnesses along a W,V^--path in $\mathcal{H}[\mathcal{A}-Z]$, which is possible because Z is terminal. This yields an equitable r-coloring of G. ∎

For a class $X \in \mathcal{A}$, let $\mathcal{T}(X)$ be set of classes in $\mathcal{A}$–X from which there are no paths to V^- in digraph $\mathcal{H}$–X. Then $\mathcal{T}(X) = \emptyset$ if and only if X is terminal. Moreover, if $X' \in \mathcal{T}(X)$ then $\mathcal{T}(X') \subsetneq \mathcal{T}(X)$: Suppose $X'' \in \mathcal{T}(X')$. Then every X'',V^--path $\mathcal{P}$ in $\mathcal{H}$ contains X', and since $X' \in \mathcal{T}(X)$, $\mathcal{P}$ also contains X. So $X'' \subseteq \mathcal{T}(X)$, and the inclusion is strict since $X' \in \mathcal{T}(X) \setminus \mathcal{T}(X')$. It follows that

(2.2) if X is nonterminal then $\mathcal{T}(X)$ contains a terminal class.

Choose $X_0 \in \mathcal{A} \setminus \mathcal{A}'$ such that $|\mathcal{T}(X_0)|$ is minimum, and set $\mathcal{A}'' = \mathcal{T}(X_0)$. As usual, set $A'' := \bigcup \mathcal{A}''$, and $a'' := |\mathcal{A}''|$. Note that since X_0 is nonterminal, $a'' > 0$. Also, for all $w \in W \in \mathcal{A}''$

$$\|w, A\| \geq a - a'' - 1; \qquad \|w, B\| \leq b + a'' + 1. \tag{2.3}$$

Proposition 20. *$\mathcal{A}'' \subseteq \mathcal{A}'$. In particular, $1 \leq a'' \leq a'$.*

Proof. Since $X_0 \in \mathcal{A} - \mathcal{A}'$, $|\mathcal{A}''| \neq \emptyset$. By the minimality of $\mathcal{T}(X_0)$, every $X_1 \in \mathcal{T}(X_0)$ is terminal. Thus $\mathcal{A}'' \subseteq \mathcal{A}'$, and $1 \leq a'' \leq a'$. ∎

Proposition 21. *Every $x \in A''$ satisfies $\|x, A\| \geq a - a'' - 1$.*

Proof. Suppose $x \in A''$. Then there exists $X \in \mathcal{A}''$ with $x \in X$. Since $X \in \mathcal{T}(X_0)$, it has no out-neighbors in $\mathcal{A} \setminus (\mathcal{T}(X_0) + X_0)$. It follows that x has a neighbor in each of these $a - a'' - 1$ classes. ∎

3. Main Proof

In this section we use the tools established in the previous section to complete the proof of Theorem 9.

Proposition 22. *Every vertex in B is good. Moreover, if $b \geq 3$ then G contains no $K_{b,b}$.*

Proof. By construction, $\chi(G[B]) \leq b$, and Proposition 14 implies $\Delta(G[B]) \leq b$. So by the minimality of G it suffices to show that B does not contain $K_{b,b}$ when $b \geq 3$.

Suppose $Q := K_{b,b} \subseteq G[B]$ and $b \geq 3$. First note that every vertex $v \in Q$ is good: Since $|G[B] - Q|$ is not divisible by b, the minimality of G and Theorem 10 imply $G[B] - Q$ has an equitable b-coloring with one big class. This can be extended to an equitable b-coloring of $G[B] - v$: Since one of $b, b-1$ is odd, $Q - v = K_{b-1,b}$ has an equitable b-coloring with one small class. Since $Q - v$ is a component of $G - v$, its small class can be combined with the large class of $G[B] - Q$ and each of the $b-1$ large classes of $Q - v$ can be combined with one of the $b-1$ small classes of $G - Q$ to obtain an equitable b-coloring.

Since Q is b-regular, every $v \in Q$ satisfies $\|v, A\| = a$. So v has a solo neighbor in every accessible class. Let $Z \in \mathcal{A}'$, and let $z \in Z$ be a solo neighbor of $v \in Q$. Since $\|z, B\| \leq b + 1 < 2b = |Q|$, there exists a vertex $v' \in Q - N(z)$. So v' has a solo neighbor $z' \in Z - z$. Since v and v' are good, Proposition 17 implies z and z' are not movable. By Proposition 16, the remaining two vertices $Z - z - z'$ are both movable, and so not adjacent to vertices of Q.

Choose notation so that $\|z,Q\| \geq \|z',Q\|$; so $\|z,Q\| \geq b \geq 3$. Since v is good, Proposition 18, implies S_z is a clique. Since Q is bipartite, $|S_z \cap Q| \leq 2$, a contradiction. ∎

Corollary 23. *Let $x \in Z \in \mathcal{A}'$. If x is unmovable then*

$$\|x,B\| \leq b+1 \text{ and } ch(x) \leq \frac{1}{2}(\|x,B\| + |S_x|) \leq b + \frac{1}{2}; \tag{3.1}$$

if x is movable then

$$ch(x) \leq \frac{1}{2}\|x,B\|; \text{ and} \tag{3.2}$$

if x is movable and $Z \in \mathcal{A}''$ then

$$\|x,B\| \leq b + a'' + 1. \tag{3.3}$$

Proof. First suppose x is unmovable. Then $\|x,B\| \leq d(x) - \|x,A\| \leq b+1$. By Proposition 22, every vertex of B is good. Thus by Proposition 18, $|S_x|$ is a clique with $|S_x| \leq \chi(B) = b$.

Now suppose x is movable. Since it is good, Proposition 17 implies x is not solo. Thus $|S_x| = 0$. So $ch(x) \leq \frac{1}{2}\|x,B\|$. If $Z \in \mathcal{A}''$ then $\|x,A\| \geq a - a'' - 1$ by Proposition 21, and so $\|x,B\| \leq b + a'' + 1$. ∎

Proposition 24. *Let $Z \in \mathcal{A}'$. The average charge c of the movable vertices in Z is greater than b.*

Proof. Suppose $c \leq b$. Let M be the set of movable vertices in Z. By Proposition 16, $|M| \geq 2$. Since the total charge of vertices in Z is $bs+1$, and $ch(x) \leq b + \frac{1}{2}$ for $x \in Z \setminus M$,

$$(s - |M|)(b + \frac{1}{2}) \geq \sum_{x \in Z \setminus M} ch(x) \geq sb + 1 - c|M| \geq sb + 1 - b|M|.$$
$$s - 2 \geq |M| \geq 2.$$

Thus $s = 4$, $|M| = 2$, and there are two unmovable vertices z, z' with $ch(z) = ch(z') = b + \frac{1}{2}$. Moreover, each of z and z' has b solo neighbors and one $\frac{1}{2}$-neighbor. Let y be the $\frac{1}{2}$-neighbor of z. Since all vertices of B are good (Proposition 22), Proposition 18 implies that S_z and $S_{z'}$ are b-cliques; by definition they are disjoint. Since $\omega(G[B]) \leq \chi(G[B]) \leq b$, there exist $y' \in S_z$ with $yy' \notin E$. By Proposition 19, the other neighbor of y in Z is not movable, and so must be z'.

Since y is good, $G[B] - y$ has an equitable b-coloring g. Since

$$\|y,B\| \leq r - \|y, A \setminus Z\| - \|y,Z\| \leq r - (a-1) - 2 = b - 1,$$

we can move y into one of the classes Y of g. Let $w \in Y \cap S_z$ and $w' \in Y \cap S_{z'}$; they exist since S_z and $S_{z'}$ are b-cliques. Then

$$\left\|\{z,z'\}, Y \setminus \{y,w,w'\}\right\| = 0 = \left\|\{y,w,w'\}, Z \setminus \{z,z'\}\right\|.$$

Let $u \in M$. Obtain an equitable r-coloring of G by moving z, z' to $Y \setminus \{y,w,w'\}$, moving y,w,w' to $Z \setminus \{z,z',u\}$, and using Proposition 13 to equitably $(a-1)$-color $A-Z+u$. Thus G is not a counterexample, a contradiction. ∎

Corollary 25. *Every $Z \in \mathcal{A}'$ contains a vertex x satisfying $\|x,B\| \geq 2b+1$.*

Proof. Let x be a movable vertex with charge greater than b. By Corollary 23, $\|x,B\| \geq 2b+1$. ∎

Corollary 26. $b \leq a'' \leq a'$.

Proof. By Corollary 25 there exists $u \in Z \in \mathcal{A}''$ with $\|u,B\| \geq 2b+1$. Using (3.3),

$$2b+1 \leq \|u,B\| \leq b+a''+1.$$

So $b \leq a''$. ∎

Proposition 27. $b \geq a'$.

Proof. Suppose not; then $b+1 \leq a'$. For $y \in B$, let $\sigma(y)$ be the number of solo neighbors of y in A'. Then each $y \in B$ satisfies

$$\begin{aligned} a+b \geq d(y) &\geq a+a'-\sigma(y)+\|y,B\| \\ \sigma(y) &\geq a'-b+\|y,B\|. \end{aligned}$$

Let I be a maximum independent set in B. Since $|I| \geq s$ and $a'-b-1 \geq 0$,

$$\begin{aligned} \sum_{y \in I} \sigma(y) &\geq \sum_{y \in I} (a'-b+\|y,B\|) \geq |I|(a'-b)+|B|-|I| \\ &\geq s(a'-b-1)+bs+1 \geq a's-s+1. \end{aligned} \tag{3.4}$$

If $Z \in \mathcal{A}'$ then it has two movable vertices, and so at most $s-2$ solo vertices. Thus there are at most

$$a's-2a' \leq a's-2(b+1) \leq a's-s$$

solo vertices in A'. Combining this with (3.4), there exists a vertex in a terminal class that has two nonadjacent solo neighbors in B, contradicting Propositions 18 and 22. ∎

Recall that a, b, a', a'' etc. are defined with respect to a fixed, but arbitrary, optimal coloring f. So Corollary 26 and Proposition 27 apply to *all* optimal colorings. Now we apply this observation.

Proposition 28. *$a-1=a''=a'=b$. Moreover, $2a-1=r=2b+1$, $a\geq 3$ and $a'\geq 2$.*

Proof. The first sentence together with $r\geq 5$ implies the second. By Corollaries 26 and 27, $a''\leq a'\leq b\leq a''$. So $a''=a'=b$ and $\mathcal{A}''=\mathcal{A}'$. It remains to show $a'=a-1$.

Let $\mathcal{P}=X_0\dots V^-$ be a path $\mathcal{H}$. We first show that $V(\mathcal{P})\cup\mathcal{A}'=\mathcal{A}$. Inclusion follows immediately from definitions; so consider $W\in\mathcal{A}$. If W is terminal then $W\in\mathcal{A}'=\mathcal{A}''=\mathcal{T}(X_0)$. Otherwise, W is nonterminal, $W\notin\mathcal{T}(X_0)$, and by (2.3)

$$\emptyset\neq\mathcal{T}(W)\cap\mathcal{A}'\subseteq\mathcal{A}'=\mathcal{A}''=\mathcal{T}(X_0).$$

Suppose $X_1\in\mathcal{T}(W)\cap\mathcal{A}'$, and $\mathcal{Q}=W\dots V^-$ is a path in $\mathcal{H}$ avoiding X_0, which is possible because $W\notin\mathcal{T}(X_0)$. Every X_1,V^--path in $\mathcal{H}$ contains X_0 and W. Fix $\mathcal{R}=X_1\dots V^-$. Then $X_0\in\mathcal{R}$ and $W\in X_1\mathcal{R}X_0\mathcal{P}V^-$. If $W\in X_1\mathcal{R}X_0$ then $X_1\mathcal{R}W\mathcal{Q}V^-$ avoids X_0, a contradiction. So $W\in X_0\mathcal{P}V^-$. Thus $\mathcal{A}\subseteq V(\mathcal{P})\cup\mathcal{A}'$.

So if $\mathcal{A}'\neq\mathcal{A}$ then $X_0\neq V^-$, and there exists an edge $X_2V^-\in\mathcal{P}$ with some witness w. Obtain a new optimal coloring g with small class X_2-w by moving w from X_2 to V^-. Then after identifying V^- with V^-+w and X_2 with X_2-w,

$$E(\mathcal{H}(f)-V^-)+V^-X_2\subseteq E(\mathcal{H}(g)).$$

So $\mathcal{A}(f)\subseteq\mathcal{A}(g)$: for every $W\in V(\mathcal{P})$, we have $W\mathcal{P}X_2\subseteq\mathcal{H}(g)$, and every class in $\mathcal{T}(X_0)$ has the same path to X_0 in $\mathcal{H}(g)$ as in $\mathcal{H}(f)$. Similarly, $\mathcal{A}'(f)\subseteq\mathcal{A}'(g)$ and $V^-\subseteq\mathcal{A}'(g)\setminus\mathcal{A}(f)$, contradicting $a'(g)=b(g)$. We conclude that $\mathcal{A}=\mathcal{A}'+V^-$. ∎

Corollary 29. *Every vertex $x\in X\in\mathcal{A}'$ satisfies*

$$b-\frac{1}{2}\leq ch(x)\leq\frac{r}{2}=b+\frac{1}{2}. \tag{3.5}$$

If x is unmovable then

$$b\leq\|x,B\|\leq b+1; \tag{3.6}$$

if x is movable then

$$2b-1\leq\|x,B\|\leq r=2b+1. \tag{3.7}$$

Proof. The upper bounds follow from Corollary 23. The total charge of vertices in X is $bs+1$. Since no vertex in X has charge greater than $b+\frac{1}{2}$ and $s\le 4$, every vertex has charge at least $b-\frac{1}{2}$. This implies the lower bounds of (3.6) and (3.7). ∎

Now choose an optimal coloring f such that

$$\|B\| \text{ is maximum.} \tag{3.8}$$

Proposition 30. *Suppose $z\in Z\in\mathcal{A}'$ is solo and $y\in S_z$. Then (a) $\|z,B\|-2\le \|y,B\|$. Moreover, if $\|y,B\|\le\|z,B\|-2$ for some $y\in S_z$ then (b) $\|z,B\|=b+1$ and (c) z has a $\frac{1}{2}$-neighbor w. In any case, (d) $||y,B||\ge b-1$.*

Proof. Suppose $\|y,B\|\le\|z,B\|-2$ for some $y\in S_z$. By (3.6), $b\le\|z,B\|\le b+1$, and $\|y,B\|\le\|z,B\|-2\le b-1$. Let $\beta=b-\|y,B\|$; so $\beta\ge 1$. Move y out. Since y is good, there exists an equitable b-coloring g of $B-y$. Then there exist distinct classes $Y_1,\dots,Y_\beta$ of g such that Y_i+y is independent for all $i\in[\beta]$. If we could move z to a class of g then doing so, and moving y to $Z-z$, would yield an optimal coloring h of G with

$$\|B(h)\| \ge \|B(f)\| - \|y,B(f)\| + \|z,B(f)\| - \|zy\| \ge \|B(f)\| + 1,$$

contradicting (3.8), the maximality of $\|B\|$. So z must have a neighbor in every class of g. Since z is also adjacent to y, $\|z,B\| = b+1$, proving (b). Moreover, z has a unique neighbor in each class of g. Let $w_i \in N(z)$ with $w_i\in Y_i$ for $i\in[\beta]$. By Proposition 18, $w_i\notin S_z$, since $w_iy\notin E$. If w_i is movable to another class W of g, then we can move w_i to W, z to Y_i-w_i, and y to $Z-z$ to obtain an optimal coloring, again contradicting (3.8). So $\|w_i,B\|\ge b-1$. Since w_i also has at least two neighbors in Z, and at least $a-1$ neighbors in $A\setminus Z$, these bounds are tight, i.e., $\|w_i,A\setminus Z\| = a-1$, $\|w_i,Z\| = 2$, and $\|w_i,B\|=b-1$. In particular, w_i is a $\frac{1}{2}$-neighbor of z, proving (c).

Suppose (a) fails. Then $\beta\ge 2$. By Proposition 19 the second neighbor in Z of w_i is unmovable. Thus by Proposition 16 z' is the unique unmovable vertex in $Z-z$. Since z' is not movable, $b\le\|z',B\|\le b+1$ and it is not a witness for a Z,V^--path in $\mathcal{H}$.

Suppose z' is movable to a class W of g. Then moving z to Y_1, z' to W and both w_1 and y to $Z-z-z'$ would yield an optimal coloring g' with

$$\begin{aligned}\left\|B(g')\right\| &\ge \|B(f)\| - \|\{y,w_1\},B(f)\| + \left\|\{z,z'\},B(f)\right\| - \left\|\{y,w_1\},\{z,z'\}\right\|\\ &\ge \|B(f)\| - (2b-\beta-1) + (b+1+\left\|z',B(f)\right\|) - 3\\ &= \|B(f)\| + \beta - 1 \ge \|B(f)\| + 1,\end{aligned}$$

contradicting (3.8).

Finally, suppose z' has a neighbor in each class of g; so it has exactly one in every class, except at most one. Since $\beta \geq 2$, there exists $i \in [2]$ with $\|z', Y_i\| = 1$. Let y' be the unique neighbor of z' in Y_i and $u \in Z - z - z'$; then u is movable. Moving y, w_i, y' to $Z \setminus \{z, z', u\}$, z, z' to $Y_i \setminus \{w_i, y'\}$, and using Proposition 13 to equitably $(a-1)$-color $A - Z + u$ yields an equitable r-coloring of G, a contradiction. So, (a) holds.

If $\|z, B\| = b+1$, then (d) follows from (a). Otherwise, (d) follows from (b) and (3.6). ∎

Definition 31. Call a vertex $w \in A$ *heavy* if $\|w, B\| = r$. Let W be the set of heavy vertices. Set

$$U := \bigcup_{w \in W} B \cap N(w), I := \bigcap_{w \in W} B \cap N(w), \text{ and } \overline{I} := U \setminus I.$$

By definition, a heavy vertex has no neighbors in A. Since $r = 2b+1$, by Corollary 25, every terminal class contains a heavy vertex.

Proposition 32. *Suppose* $s \leq 4$. *Then* $\|B\| < \frac{s-2}{2} b^2$.

Proof. Suppose, for a contradiction, $\|B\| \geq \frac{s-2}{2} b^2$. For $y \in B$, let $\sigma(y)$ denote the number of solo neighbors of y in A. Since

$$a + b \geq d(y) \geq 2a - \sigma(y) + \|y, B\|,$$

Proposition 28 implies $\sigma(y) \geq 1 + \|y, B\|$. So there are

$$\sum_{y \in B} (1 + \|y, B\|) = |B| + 2\|B\| \geq bs + 1 + (s-2)b^2$$

solo edges. By Proposition 16 and 17, each terminal Z has at most $s-2$ solo vertices, each of which is incident to at most b solo edges by Proposition 18; so Z is incident to at most $(s-2)b$ solo edges. By definition, V^- is incident to at most $|B| = bs+1$ solo edges. So each of these $a = b+1$ bounds is tight. Consider a terminal class Z with a heavy vertex u, another movable vertex u', and remaining vertices $x_1, \ldots, x_{s-2}$. Each x_i must be incident to b solo edges by tightness. Moving u to V^- yields another optimal coloring f' that satisfies (3.8) and has a small class $V^-(f') = Z - u$. Again, $Z - u$ must be incident to $bs+1$ solo edges; so there are $2b+1$ new solo edges. Each x_i is still incident to the same b old solo edges. If $y \in B$ is incident to a new solo edge then it is a $\frac{1}{2}$-neighbor of u. By Corollary 19, its other $\frac{1}{2}$-neighbor is not solo. So it must be u'. Thus u' shares $r = 2b+1$ neighbors in B with u. So u' is heavy and $N(u') = N(u) \subseteq B$. Moreover, $N(u') = I$: Certainly

$I \subseteq N(u')$. If $y \in N(u') \setminus I$ then there exists a heavy vertex w with $wy \notin E$. Then switching w with u in f makes u' a movable vertex with solo neighbor y, contradicting Proposition 17, and proving the assertion. The assertion holds for each movable vertex in each terminal class. Moreover, it holds for $V^- + u$ in the new coloring, and so V^- also has a heavy vertex. So $W \cup I$ induces $K_{r,r}$, a contradiction. ∎

Proposition 33. *Every terminal class Z has at least two heavy vertices, and V^-contains one heavy vertex.*

Proof. Suppose Z does not have two heavy vertices. Let $u, u' \in Z$ with u heavy and u' movable. Then $ch(u) + ch(u') \leq 2b + \frac{1}{2}$. If $s = 3$ then the remaining vertex x satisfies $ch(x) = b + \frac{1}{2}$. Thus x has a $\frac{1}{2}$-neighbor y whose other neighbor in Z is movable, contradicting Proposition 19. Otherwise, let $Z = \{x, x', u, u'\}$. Then $ch(x) + ch(x') \geq 2b + \frac{1}{2}$. So we may assume x satisfies

$$|S_x| = b \text{ and } \|x, B\| = b + 1.$$

Case 1. $|S_{x'}| = b$. By Proposition 18, $\|S_x\| = \binom{b}{2} = \|S_{x'}\|$. By Proposition 30, either each $y \in S_x$ satisfies $\|y, B\| \geq \|x, B\| - 1 = b$ or x has a $\frac{1}{2}$-neighbor w. In the former case, each $y \in S_x$ is incident to an edge in $E(S_x, B \setminus S_x)$. Thus

$$\|B\| \geq 2\binom{b}{2} + b \geq b^2,$$

a contradiction.

In the latter case, the second neighbor of w in Z is x'. Since w is good, $B - w$ has an equitable b-coloring g. Since $\|w, Z\| = 2$ and $\|w, A \setminus Z\| \geq a - 1$, we have $\|w, B\| \leq b - 1$. Thus there exists a class Y in g with $Y + w$ independent. Since S_x and $S_{x'}$ are disjoint b-cliques and $\|z, B\| = b + 1 = \|z', B\|$, there exist distinct and unique vertices $y \in S_x \cap Y$ and $y' \in S_{x'} \cap Y$. By Proposition 13, $A \setminus Z + u$ has an equitable $(a-1)$-coloring. Combining this with g restricted to $B \setminus Y - w$ and the independent sets $Y - y - y' + x + x'$ and $Z - u - x - x' + w + y + y'$ yields an equitable r-coloring of G, a contradiction.

Case 2. $|S_{x'}| < b$. Since $ch(x') \geq b$,

$$|S_{x'}| = b - 1 \text{ and } \left\|x', B\right\| = b + 1 \text{ and } ch(x') = b.$$

So $\|S_x\| = \binom{b}{2}$ and $\|S_{x'}\| = \binom{b-1}{2}$. Moreover, the remaining neighbors of x and x' in B are $\frac{1}{2}$-neighbors. Since x has only one $\frac{1}{2}$-neighbor, x' has a $\frac{1}{2}$-neighbor w' whose other neighbor $u^* \in Z$ is movable. By Proposition 19, $S_{x'} + w'$ is a clique. Let w be the $\frac{1}{2}$-neighbor of x. Then $wy \notin E$ for some $y \in S_x$. So the other neighbor of w is x'. As in Case 1, $B-w$ has an equitable b-coloring g with a class Y such that $Y+w$ is independent. Since S_x and $S_{x'}+w'$ are disjoint b-cliques and $\|z,B\|=b+1=\|z',B\|$, there exist distinct and unique vertices $y \in S_x \cap Y$ and $y' \in (S_{x'}+w') \cap Y$. Moreover, $Y-y-y'+x+x'$ and $Z-x-x'-u^*+w+y+y'$ are independent, and $A \setminus Z+u^*$ has an equitable $(a-1)$-coloring. Combining all this yields an equitable r-coloring, a contradiction.

So every terminal class has two heavy vertices. Moreover, moving a heavy vertex w from a terminal class to V^- yields a new optimal coloring in which V^-+w is terminal. So V^- must have a heavy vertex. ∎

Proposition 34. *Every $y \in \overline{I}$ satisfies $\|y,W\|=1$.*

Proof. Suppose $y \in \overline{I}$ has neighbors $w_1, w_2 \in W$. By the definition of $\overline{I}$, it also has a nonneighbor $w_0 \in W$. By Proposition 33, W contains a subset W' that has 2 vertices in each $Z \in \mathcal{A}'$ and one vertex in V^-. By switching heavy vertices within A, we may assume $w_0, w_1, w_2 \in W'$.

Suppose that $Z-W'$ has no neighbors of y for some $Z \in \mathcal{A}$. Then we construct an equitable a-coloring g of $G[A+y]$ by adding y and w_0 to $Z-W'$ and distributing the vertices of $W'-w_0$ between the classes $X-W'$, $X \in \mathcal{A}-Z$, equitably. Since y is good, we complete g to an equitable r-coloring of G, a contradiction.

Thus, y has at least a neighbors in $A-W'$. Then it has at least $r-a=b$ nonneighbors in W', let w_0, w_0' be two of such nonneighbors. Since y has 2 neighbors in W', there is $Z \in \mathcal{A}$ such that $Z-W'$ has exactly one neighbor of y, say z. Consider a nearly equitable r-coloring g of G obtained from f by rearranging the vertices of W' between the classes in $\mathcal{A}$ so that Z contains w_0 and w_0'. Since no $w \in W'$ has neighbors in A, g is optimal. Since $Z \in \mathcal{A}'(g)$ and z is the solo neighbor of y in Z, by Proposition 30 (d), $||y,B|| \geq b-1$. But y also has at least a neighbors in $A-W'$ and two neighbors in W', a contradiction. ∎

We are now ready for our final contradiction. Let $W' \subseteq W$ have size r. Each $w \in W'$ satisfies $\left\|w,\overline{I}\right\|=r-|I|$. Using Proposition 34,

$$r(r-|I|) = \left\|W',\overline{I}\right\| = |\overline{I}| \leq |B|-|I| \leq r-1+r-|I|$$
$$(r-1)(r-|I|-1) \leq 0;$$

so $r-1 \leq |I|$. If $|I| \geq r$ then $G[I \cup W']$ contains $K_{r,r}$, a contradiction. Thus $|I|=r$–1, and so $|\overline{I}|=r$. So $E(\overline{I},W')$ is a matching of the form $\{i_h w_h \colon h \in [r]\}$. Also, I is independent since each $y \in I$ satisfies $\|x,W\|=r$.

By the minimality of G, there is an equitable r-coloring g of $G-I-W'$ with one big class X^+ of size $s-1$ and all other classes of size $s-2$. Since

$5 \le r = |W'|$ there exists $h \in [r]$ with $i_h \notin X^+$; choose notation so that $h = r = 2b+1$. Then $X^+ + w_r$ is an independent s-set. For $j \in [b]$, greedily add w_{2j} and w_{2j-1} to a class X_j of g so that $i_{2j}, i_{2j-1} \notin X_j$ and $X_j \neq X_l$ for all $l < j$. This is possible since $1 + (b-1) + 2 < 2b + 1 = r \ge 5$. Finally, let $X_{b+1}, \dots, X_{2b}$ be the remaining classes of g and let $\{\{y_{2j}, y_{2j-1}\} \colon j \in [b]\}$ be a partition of the independent set I. For each $j \in [b]$ add y_{2j}, y_{2j-1} to X_{b+j} to obtain an equitable r-coloring of G. This contradicts the minimality of G, and completes the proof of Theorem 9.

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