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ORE'S CONJECTURE FOR $k=4$ AND GRÖTZSCH'S THEOREM

ALEXANDR KOSTOCHKA*, MATTHEW YANCEY†

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A graph G is *k-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$–colorable. Let $f_k(n)$ denote the minimum number of edges in an n-vertex k-critical graph. In a very recent paper, we gave a lower bound, $f_k(n) \geq F(k,n)$, that is sharp for every $n \equiv 1 \pmod{k-1}$. It is also sharp for $k=4$ and every $n \geq 6$. In this note, we present a simple proof of the bound for $k=4$. It implies the case $k=4$ of two conjectures: Gallai in 1963 conjectured that if $n \equiv 1 \pmod{k-1}$ then $f_k(n) = \frac{(k+1)(k-2)n-k(k-3)}{2(k-1)}$, and Ore in 1967 conjectured that for every $k \geq 4$ and $n \geq k+2$, $f_k(n+k-1) = f(n) + \frac{k-1}{2}(k - \frac{2}{k-1})$. We also show that our result implies a simple short proof of Grötzsch's Theorem that every triangle-free planar graph is 3-colorable.

1. Introduction

A *proper k-coloring*, or simply *k-coloring*, of a graph $G=(V,E)$ is a function $f\colon V \to \{1,2,\dots,k\}$ such that for each $uv \in E$, $f(u) \neq f(v)$. A graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number*, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable. A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. Then every k-critical graph has chromatic number k and every k-chromatic graph contains a k-critical subgraph.

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The only 1-critical graph is K_1, and the only 2-critical graph is K_2. The only 3-critical graphs are the odd cycles. Let $f_k(n)$ be the minimum number of edges in a k-critical graph with n vertices. Since $\delta(G) \geq k-1$ for every k-critical n-vertex graph G, $f_k(n) \geq \frac{k-1}{2}n$ for all $n \geq k$, $n \neq k+1$. Equality is achieved for $n=k$ and for $k=3$ and n odd. In 1957, Dirac [2] asked to determine $f_k(n)$ and proved that for $k \geq 4$ and $n \geq k+2$, $f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}$. The bound is tight for $n=2k-1$. Gallai [5] found exact values of $f_k(n)$ for $k+2 \leq n \leq 2k-1$:

Theorem 1 (Gallai [5]). *If $k \geq 4$ and $k+2 \leq n \leq 2k-1$, then*

$$f_k(n) = \frac{1}{2}\left((k-1)n + (n-k)(2k-n)\right) - 1.$$

He also proved that $f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n$ for all $k \geq 4$ and $n \geq k+2$. Based on his description of k-critical graphs with exactly one vertex of degree at least k, Gallai [4] conjectured that *if* $n \equiv 1 \pmod{k-1}$ *then* $f_k(n) = \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}$.

Ore observed that Hajós' construction implies

$$f_k(n+k-1) \leq f_k(n) + \frac{(k-2)(k+1)}{2} = f_k(n) + (k-1)(k - \frac{2}{k-1})/2, \tag{1}$$

which yields that $\phi_k := \lim_{n\to\infty} \frac{f_k(n)}{n}$ exists and satisfies $\phi_k \leq \frac{k}{2} - \frac{1}{k-1}$. Ore [10] also conjectured that for every $n \geq k+2$, in (1) equality holds.

More detail on known results about $f_k(n)$ and Ore's Conjecture the reader can find in [7][Problem 5.3] and our recent paper [9]. In [9] we proved the following bound.

Theorem 2. *If $k \geq 4$ and G is k-critical, then*

$$|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil.$$

In other words, if $k \geq 4$ and $n \geq k, n \neq k+1$, then

$$f_k(n) \geq F(k,n) := \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil. \tag{2}$$

This bound is exact for $k=4$ and every $n \geq 6$. For every $k \geq 5$, the bound is exact for every $n \equiv 1 \pmod{k-1}$, $n \neq 1$ and settles the above Gallai's Conjecture. In particular, $\phi_k = \frac{k}{2} - \frac{1}{k-1}$ for every $k \geq 4$. The result also confirms the above conjecture by Ore from 1967 for $k=4$ and every $n \geq 6$ and also for $k \geq 5$ and all $n \equiv 1 \pmod{k-1}$, $n \neq 1$. One of the corollaries of Theorem 2 is a short proof of the following theorem due to Grötzsch [6]:

Theorem 3 ([6]). *Every triangle-free planar graph is* 3*-colorable.*

The original proof of Theorem 3 is somewhat sophisticated. There were subsequent simpler proofs (see, e.g. [11] and references therein), but Theorem 2 yields a half-page proof. A disadvantage of this proof is that the proof of Theorem 2 itself is not too simple. The goal of this note is to give a simpler proof of the case $k=4$ of Theorem 2 and to deduce Grötzsch's Theorem from this result. Note that even the case $k=4$ was a well-known open problem (see, e.g. [8][Problem 12] and recent paper [3]). Some further consequences for coloring planar graphs are discussed in [1].

In Section 2 we prove Case $k=4$ of Theorem 2 and in Section 3 deduce Grötzsch's Theorem from it. Our notation is standard. In particular, $\chi(G)$ denotes the chromatic number of graph G, $G[W]$ is the subgraph of a graph or digraph G induced by the vertex set W. For a vertex v in a graph G, $d_G(v)$ denotes the degree of vertex v in graph G, $N_G(v)$ is the set of neighbors of v. If the graph G is clear from the context, we drop the subscript.

2. Proof of Case $k=4$ of Theorem 2

Definition 4. For $R\subseteq V(G)$, define *the potential of* R to be $\rho_G(R)=5|R|-3|E(G[R])|$. When there is no chance for confusion, we will use $\rho(R)$. Let $P(G)=\min_{\emptyset\neq R\subseteq V(G)}\rho(R)$.

Fact 5. *We have* $\rho_{K_1}(V(K_1)) = 5$, $\rho_{K_2}(V(K_2)) = 7$, $\rho_{K_3}(V(K_3)) = 6$, $\rho_{K_4}(V(K_4))=2$.

By definition, we have the following.

Fact 6. *Let* G *be a graph and* $A,B,C\subseteq V(G)$ *be such that* $A\supset B$ *and* $A\cap C=\emptyset$. *Then* $\rho_G(A-B)=\rho_G(A)-\rho_G(B)+3|E_G(A-B,B)|$ *(equivalently,* $\rho_G(A\cup C)=\rho_G(A)+\rho_G(C)-3|E_G(A,C)|$*).*

Note that $|E(G)|<\frac{5|E(G)|-2}{3}$ is equivalent to $\rho(V(G))>2$. Let G be a vertex-minimal 4-critical graph with $\rho(V(G))>2$. This implies that

(3) if $|V(H)|<|V(G)|$ and $P(H)>2$, then H is 3-colorable.

Definition 7. For a graph G, a set $R\subset V(G)$ and a 3-coloring ϕ of $G[R]$, the graph $Y(G,R,\phi)$ is constructed as follows. First, for $i=1,2,3$, let R'_i denote the set of vertices in $V(G)-R$ adjacent to at least one vertex $v\in R$ with $\phi(v)=i$. Second, let $X=\{x_1,x_2,x_3\}$ be a set of new vertices disjoint from $V(G)$. Now, let $Y=Y(G,R,\phi)$ be the graph with vertex set $V(G)-R+X$, such that $Y[V(G)-R]=G-R$ and $N(x_i)=R'_i\cup(X-x_i)$ for $i=1,2,3$.

Claim 8. *Suppose $R \subset V(G)$, and ϕ is a 3-coloring of $G[R]$. Then $\chi(Y(G,R,\phi)) \geq 4$.*

Proof. Let $G' = Y(G,R,\phi)$. Suppose G' has a 3-coloring $\phi'\colon V(G') \to C = \{1,2,3\}$. By construction of G', the colors of all x_i in ϕ' are distinct. So we may assume that $\phi'(x_i) = i$ for $1 \leq i \leq 3$. By construction of G', for all vertices $u \in R'_i$, $\phi'(u) \neq i$. Therefore, $\phi|_R \cup \phi'|_{V(G)-R}$ is a proper coloring of G, a contradiction. ∎

Claim 9. *There is no $R \subsetneq V(G)$ with $|R| \geq 2$ and $\rho_G(R) \leq 5$.*

Proof. Let $2 \leq |R| < |V(G)|$ and $\rho(R) = m = \min\{\rho(W)\colon W \subsetneq V(G), |W| \geq 2\}$. Suppose $m \leq 5$. Then by Fact 5, $|R| \geq 4$. Since G is 4-critical, $G[R]$ has a proper coloring $\phi\colon R \to C = \{1,2,3\}$. Let $G' = Y(G,R,\phi)$. By Claim 8, G' is not 3-colorable. Then it contains a 4-critical subgraph G''. Let $W = V(G'')$. Since $|R| \geq 4 > |X|$, $|V(G'')| < |V(G)|$. So, by the minimality of G, $\rho_{G'}(W) \leq 2$. Let $X' = W \cap X$. Since G is 4-critical by itself, every proper subgraph of G is 3-colorable and so $X' \neq \emptyset$. Since $0 < |X'| \leq 3$, by Fact 5, $\rho_{G'}(X') \geq 5$. Since $|E_{G'}(W-X',X')| \leq |E_{G'}(W-X',X)| = |E_G(W-X',R)|$, by Fact 6,

$$
\begin{aligned}
(4) \qquad \rho_G((W-X')+R) &= \rho_G(W-X') + \rho_G(R) - 3|E_G(W-X',R)| \\
&= \rho_{G'}(W-X') + m - 3|E_{G'}(W-X',X)| \\
&\leq \rho_{G'}(W) - \rho_{G'}(X') + 3|E_{G'}(W-X',X')| + m - 3|E_{G'}(W-X',X)| \\
&\leq \rho_{G'}(W) - \rho_{G'}(X') + m \leq 2 - 5 + m.
\end{aligned}
$$

Since $W - X + R \supset R$, $|W - X + R| \geq 2$. Since $\rho_G(W - X + R) < \rho_G(R)$, by the choice of R, $W - X + R = V(G)$. But then $\rho_G(V(G)) \leq m - 3 \leq 2$, a contradiction. ∎

Claim 10. *If $R \subsetneq V(G)$, $|R| \geq 2$ and $\rho_k(R) \leq 6$, then R is a K_3.*

Proof. Let R have the smallest $\rho(R)$ among $R \subsetneq V(G)$, $|R| \geq 2$. Suppose $m = \rho(R) \leq 6$ and $G[R] \neq K_3$. Then $|R| \geq 4$. By Claim 9, $m = 6$.

Let $R_* = \{u_1, \ldots, u_s\}$ be the set of vertices in R that have neighbors outside of R. Because G is 2-connected, $s \geq 2$. Let $H = G[R] + u_1u_2$. Since $R \neq V(G)$, $|V(H)| < |V(G)|$. By the minimality of $\rho(R)$, for every $U \subseteq R$ with $|U| \geq 2$, $\rho_H(U) \geq \rho_G(U) - 3 \geq \rho_G(R) - 3 \geq 3$. Thus $P(H) \geq 3$, and by (3), H has a proper 3-coloring ϕ with colors in $C = \{1,2,3\}$. Let $G' = Y(G,R,\phi)$. Since $|R| \geq 4$, $|V(G')| < |V(G)|$. By Claim 8, G' is not 3-colorable. Thus G' contains a 4-critical subgraph G''. Let $W = V(G'')$. By the minimality of $|V(G)|$, $\rho_{G'}(W) \leq 2$. Since G is 4-critical by itself, $W \cap X \neq \emptyset$. Let $X' = W \cap X$. By Fact 5, if $|X'| \geq 2$ then similarly to (4), $\rho_{k,G}(W - X' + R) \leq \rho_{G'}(W) - 6 + 6 \leq 2$, a contradiction again. So, we may assume that $X' = \{x_1\}$. Then again as in (4),

$$(5) \qquad \rho_G(W - \{x_1\} + R) \leq (\rho_{G'}(W) - 5) + \rho_G(R) \leq \rho_G(R) - 3.$$

By the minimality of $\rho_G(R)$, $W - \{x_1\} + R = V(G)$. This implies that $W = V(G') - X + x_1$.

Let $R_1 = \{u \in R_* : \phi(u) = \phi(x_1)\}$. If $|R_1| = 1$, then $\rho_G(W - x_1 \cup R_1) = \rho_H(W) \leq 2$, a contradiction. Thus, $|R_1| \geq 2$. Since R_1 is an independent set in H and $u_1u_2 \in E(H)$, we may assume that $u_2 \notin R_1$. Then $E_{G'}(W - x_1, X - x_1) \neq \emptyset$. So, in this case repeating the argument of (4), instead of (5) we have

$$\begin{aligned}\rho_G(W - \{x_1\} + R) &\leq \rho_{G'}(W) - 5 + \rho_G(R) - 3|E_{G'}(W - x_1, X - x_1)| \\ &\leq \rho_G(R) - 6 \leq 0.\end{aligned}$$

∎

Claim 11. *G does not contain $K_4 - e$.*

Proof. If $G[R] = K_4 - e$, then $\rho_G(R) = 5(4) - 3(5) = 5$, a contradiction to Claim 10. ∎

Claim 12. *Each triangle in G contains at most one vertex of degree 3.*

Proof. By contradiction, assume that $G[\{x_1, x_2, x_3\}] = K_3$ and $d(x_1) = d(x_2) = 3$. Let $N(x_1) = X - x_1 + a$ and $N(x_2) = X - x_2 + b$. By Claim 11, $a \neq b$. Define $G' = G - \{x_1, x_2\} + ab$. Because $\rho_G(W) \geq 6$ for all $W \subseteq G - \{x_1, x_2\}$ with $|W| \geq 2$, and adding an edge decreases the potential of a set by 3, $P(G') \geq \min\{5, 6-3\} = 3$. So, by (3), G' has a proper 3-coloring ϕ' with $\phi'(a) \neq \phi'(b)$. This easily extends to a proper 3-coloring of G. ∎

Claim 13. *Let $xy \in E(G)$ and $d(x) = d(y) = 3$. Then both, x and y are in triangles.*

Proof. Assume that x is not in a K_3. Suppose $N(x) = \{y, u, v\}$. Then $uv \notin E(G)$. Let G' be obtained from $G - y - x$ by gluing u and v into a new vertex $u * v$. Since $|V(G')| < |V(G)|$, G' is smaller than G. If G' has a 3-coloring $\phi' : V(G') \to C = \{1, 2, 3\}$, then we extend it to a proper 3-coloring ϕ of G as follows: define $\phi|_{V(G)-x-y-u-v} = \phi'|_{V(G')-u*v}$, then let $\phi(u) = \phi(v) = \phi'(u*v)$, choose $\phi(y) \in C - (\phi'(N(y) - x))$, and $\phi(x) \in C - \{\phi(y), \phi(u)\}$.

So, $\chi(G') \geq 4$ and G' contains a 4-critical subgraph G''. Let $W = V(G'')$. Since G'' is smaller than G, $\rho_{G'}(W) \leq 2$. Since G'' is not a subgraph of G, $u*v \in W$. Let $W' = W - u*v + u + v + x$. Then $\rho_G(W') \leq 2 + 5(2) - 3(2) = 6$, since

$G[W']$ has two extra vertices and at least two extra edges in comparison with G''. Because $y \notin W'$, we have $W' \neq V(G)$, and therefore by Claim 10, W induces a K_3 in G. This contradicts our assumption that x is not in a K_3. ∎

By Claims 12 and 13, we have

(6) Each vertex with degree 3 has at most 1 neighbor with degree 3.

We will now use discharging to show that $|E(G)| \geq \frac{5}{3}|V(G)|$, which will finish the proof to Case $k=4$ of Theorem 2. Each vertex begins with charge equal to its degree. If $d(v) \geq 4$, then v gives charge $\frac{1}{6}$ to each neighbor with degree 3. Note that v will be left with charge at least $\frac{5}{6}d(v) \geq \frac{10}{3}$. By (6), each vertex of degree 3 will end with charge at least $3+\frac{2}{6}=\frac{10}{3}$. ∎

3. Proof of Theorem 3

Let G be a plane graph with fewest elements (vertices and edges) for which the theorem does not hold. Then G is 4-critical and in particular 2-connected. Suppose G has n vertices, e edges and f faces.

Case 1. G has no 4-faces. Then $5f \leq 2e$ and so $f \leq 2e/5$. By this and Euler's Formula $n-e+f=2$, we have $n-3e/5 \geq 2$, i.e., $e \leq \frac{5n-10}{3}$, a contradiction to Theorem 2.

Case 2. G has a 4-face (x,y,z,u). Since G has no triangles, $xz, yu \notin E(G)$. If the graph G_{xz} obtained from G by gluing x with z has no triangles, then by the minimality of G, it is 3-colorable, and so G also is 3-colorable. Thus G has an x,z-path (x,v,w,z) of length 3. Since G itself has no triangles, $\{y,u\} \cap \{v,w\} = \emptyset$ and there are no edges between $\{y,u\}$ and $\{v,w\}$. But then G has no y,u-path of length 3, since such a path must cross the path (x,v,w,z). Thus the graph G_{yu} obtained from G by gluing y with u has no triangles, and so, by the minimality of G, is 3-colorable. Then G also is 3-colorable, a contradiction. ∎

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Alexandr Kostochka

University of Illinois
at Urbana–Champaign,
Urbana, IL 61801, USA and
Sobolev Institute of Mathematics,
Novosibirsk 630090, Russia.
kostochk@math.uiuc.edu.

Matthew Yancey

Department of Mathematics,
University of Illinois, Urbana,
IL 61801, USA.
yancey1@illinois.edu.