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Describing 3-paths in normal plane maps

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ABSTRACT

We prove that every normal plane map, as well as every 3-polytope, has a path on three vertices whose degrees are bounded from above by one of the following triplets: $(3, 3, \infty)$, $(3, 4, 11)$, $(3, 7, 5)$, $(3, 10, 4)$, $(3, 15, 3)$, $(4, 4, 9)$, $(6, 4, 8)$, $(7, 4, 7)$, and $(6, 5, 6)$. No parameter of this description can be improved, as shown by appropriate 3-polytopes.

1. Introduction

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. Let δ be the minimum vertex degree, and w_k be the minimum degree sum of a path on k vertices in an NPM or a graph. The degree of a vertex v or a face f, that is the number of edges incident with v or f (loops and cut-edges are counted twice), is denoted by $d(v)$ or $r(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc. An edge uv is an *(i, j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type* (i, j, k) if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$. A path uvw is a *off-(i, j, k)-path* if $d(u) \geq i$, $d(v) \geq j$, and $d(w) \geq k$.

Already in 1904, Wernicke [14] proved that every NPM M_5 with $\delta(M_5) = 5$ contains a 5-vertex adjacent to a 6^--vertex, and Franklin [7] strengthened this to the existence of at least two 6^--neighbors, which implies that M_5 satisfies $w_3 \leq 17$. Franklin's bound 17 is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

It follows from Lebesgue's results in [12] that each normal plane map has an edge $e = uw$ of *weight* $w(e) = d(u) + d(w)$ at most 14 (more specifically, a $(3, 11)$-, or $(4, 7)$-, or $(5, 6)$-edge, where bounds 7 and 6 are sharp). For 3-connected plane graphs, Kotzig [11] proved a precise result: $w_2 \leq 13$.

Note that $\delta(K_{2,t}) = 2$ and $w_2(K_{2,t}) = t+2$, so w_2 is unbounded if $\delta \leq 2$. In 1972, Erdős (see [8]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all planar graphs with $\delta \geq 3$. Barnette (see [8]) announced to have proved this conjecture, but

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the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [2]. More generally, Borodin [3–5] proved that every NPM contains a (3, 10)-, or (4, 7)-, or (5, 6)-edge (as easy corollaries of some stronger structural facts with applications to coloring).

Theorem 1 (*Ando, Iwasaki, Kaneko [1]*)**.** *Every* 3*-polytope satisfies* $w_3 \le 21$*, which is sharp.*

The sharpness of the bound $w_3 \le 21$ in Theorem 1 is witnessed by the Jendrol' construction [9] (see Fig. 5).

Jendrol' [10] proves that each 3-polytope has a 3-path uvw such that $\max\{d(u), d(v), d(w)\} \le 15$ (the bound is precise). Jendrol' [9] further shows that such a path must belong to one of ten types, in which $d(u)+d(v)+d(w)$ varies from 23 to 16:

Theorem 2 (*Jendrol' [9]*)**.** *Every* 3*-polytope has a* 3*-path of one of the following types:* (10, 3, 10), (7, 4, 7), (6, 5, 6), (3, 4, 15), (3, 6, 11), (3, 8, 5), (3, 10, 3), (4, 4, 11), (4, 5, 7), *or* (4, 7, 5).

Note that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to Steinitz's famous theorem [13]. The requirement of 3-connectedness is essential for the finiteness of w_3, as shown by the construction $K^*_{2,2t}$ in Fig. 1.

Borodin [6] showed that only the presence in a NPM of a $(K_4 - e)$-like configuration $K^*_{2,4}$, described as "two adjacent 3-vertices lying in two common 3-faces", is responsible for the unboundedness of w_3 in NPMs. The following refinement of Theorem 1 holds:

Theorem 3 (*Borodin [6]*)**.** *Every normal plane map without* $K^*_{2,4}$ *has*

(i) *either* $w_3 \le 18$ *or a vertex of degree* ≤ 15 *adjacent to two* 3*-vertices, and*
(ii) *either* $w_3 \le 17$ *or* $w_2 \le 7$.

As mentioned above, the bounds $w_3 \le 21$ and $w_3 \le 17$ are tight. It has been open whether the bound $w_3 \le 18$ in Theorem 3 is sharp or not; its sharpness now follows from Fig. 2.

In particular, Ando, Iwasaki, and Kaneko's [1] precise bound $w_3 \le 21$ is valid for all NPMs in which no two 3-vertices are adjacent:

Corollary 4 (*[6]*)**.** *Every normal plane map with* $w_2 > 6$ *has* $w_3 \le 21$.

Theorem 3 immediately implies that Franklin's precise bound $w_3 \le 17$ is valid for all normal plane maps with $\delta \ge 4$:

Corollary 5 (*[6]*)**.** *Every normal plane map without* 3*-vertices has* $w_3 \le 17$.

The upper bound in the following statement is also immediate:

Corollary 6 (*[6]*)**.** *Every* 3*-polytope with* $\delta \ge 4$ *has a path* uvw *such that* $\max\{d(u), d(v), d(w)\} \le 9$.

The bound 9 in Corollary 6 is sharp, as follows from Fig. 6(Te).

The purpose of this paper is to precisely describe 3-paths in normal plane maps (in particular, in planar graphs G with $\delta(G) \ge 3$ and in 3-polytopes) as follows:

Theorem 7. *Every normal plane map without two adjacent* 3*-vertices lying in two common* 3*-faces has a* 3*-path of one of the following types:*

(Ta) (3, 4, 11),
(Tb) (3, 7, 5),
(Tc) (3, 10, 4),
(Td) (3, 15, 3),
(Te) (4, 4, 9),
(Tf) (6, 4, 8),
(Tg) (7, 4, 7),
(Th) (6, 5, 6).

Moreover, none of the options (Ta)–(Th) *can be strengthened or dropped, as shown by certain triangular* 3*-polytopes.*

Corollary 8. *Every planar graph has a* 2^-*-vertex, or two adjacent* 3*-vertices, or a* 3*-path of one of the types* (Ta)–(Th) *described in Theorem* 7.

Corollary 9. *Every* 3*-polytope has a* 3*-path of one of the types* (Ta)–(Th) *described in Theorem* 7.

Clearly, Theorem 7 extends or refines Franklin's Theorem, Theorem 1, and Corollaries 4 and 6.

2. The tightness of Theorem 7

In Fig. 1, we see the graph $K^*_{2,2t}$, in which every path of three vertices contains a vertex of degree $2t$, where t can be arbitrarily large. This confirms the necessity of the assumption of Theorem 7.

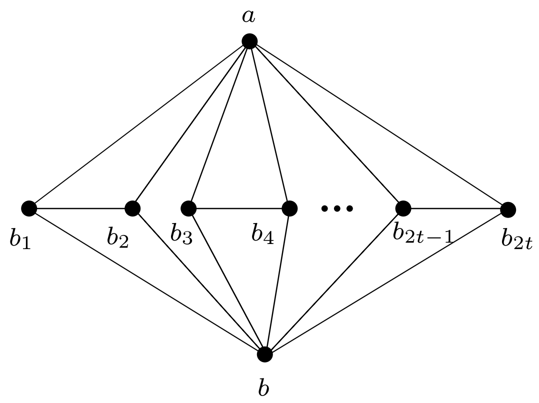

Fig. 1. The graph $K^*_{2,2t}$ with only off-$(3, 3, \infty)$-paths.

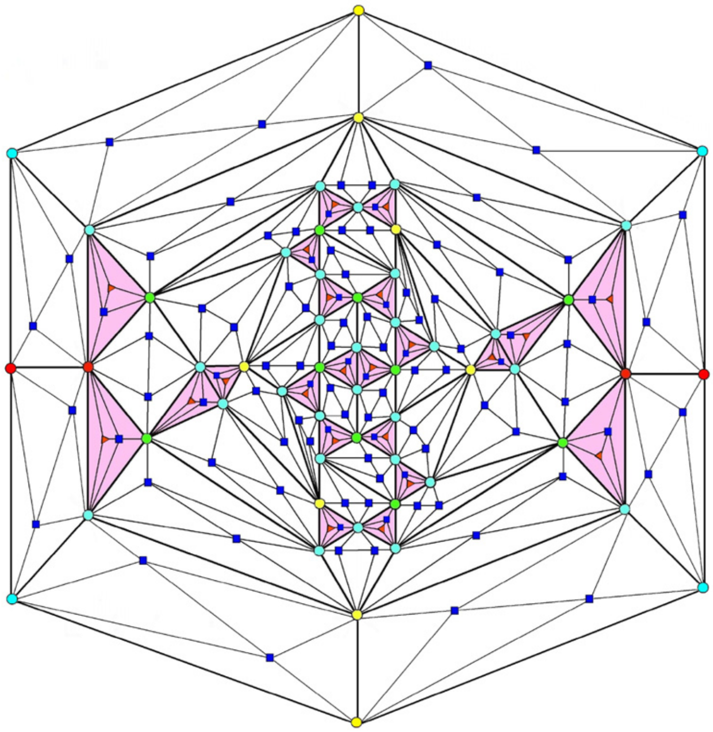

Fig. 2. A bit more than a half of a triangulation with only off-$(3, 4, 11)$-paths among those mentioned in (Ta)–(Th), which implies the tightness of (Ta).

The bounds in Theorem 7 are all tight, as the following examples show. For each of the alternatives (Ta)–(Th) in Theorem 7, we present a triangulation (see Figs. 2–6) that contains only one type of 3-paths allowed in (Ta)–(Th) and such that all upper bounds on the degrees of the corresponding vertices are attained.

(Ta). The picture in Fig. 2 can be treated as a semi-spherical map extended by the tropical belt. Namely, the outside cycle and the concentric cycle next to it are analogues of tropics, with the invisible equator going in the middle between them. In particular, the equator separates the paired 4-vertices.

It is easy to see that the triangulation obtained by gluing two such semi-spherical maps along the equator contains only vertices of degree 3, 4 and at least 11. Since no vertex is adjacent to two 3-vertices, there are only paths of type $(3, 4, 11)$ among those appearing in (Ta)–(Th) of Theorem 7. Moreover, all these paths of type $(3, 4, 11)$ are in fact off-$(3, 4, 11)$-paths. This proves that the bounds 4 and 11 in (Ta) are sharp.

(Tb). In Fig. 3, there are vertices of degree 3, 5, 7, and 8 only. We glue two such pictures along the equator, which goes through the outside 5-vertices and cuts the outgoing edges. The resulting triangulation has only off-$(3, 7, 5)$-paths among the 3-paths mentioned in (Ta)–(Th), which confirms the tightness of (Tb).

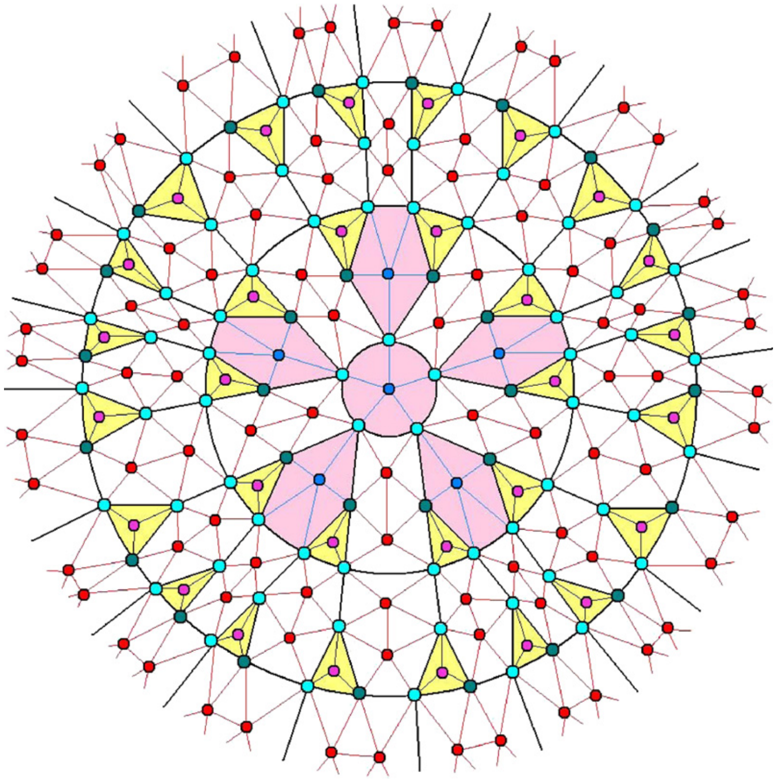

Fig. 3. A half of a triangulation with only off-(3, 7, 5)-paths confirming that (Tb) is tight.

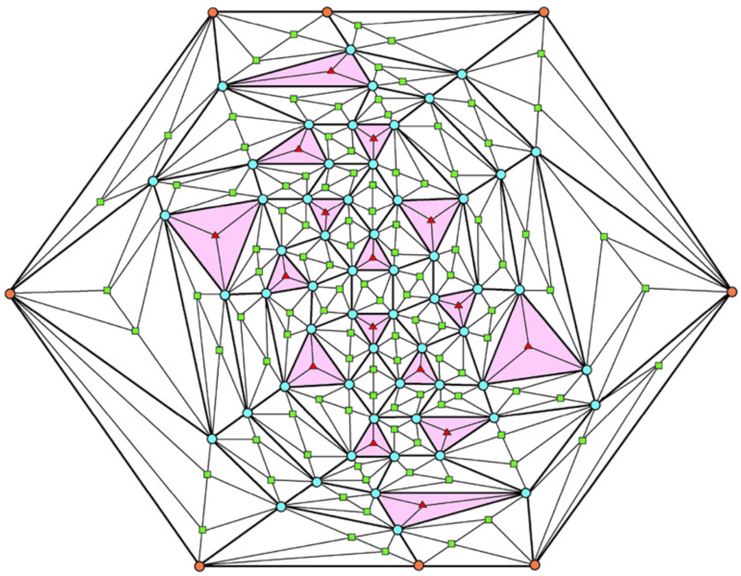

Fig. 4. A half of the off-(3, 10, 4)-triangulation proving the tightness of (Tc).

(Tc). In Fig. 4, we see a half of a plane triangulation. Its internal vertices have degrees 3, 4, or 10 only. No 3-vertex is adjacent to a 4^--vertex. The boundary vertices have the clockwise degree-sequence 97569756. We rotate the view by $\frac{\pi}{2}$ and glue the two halves along the boundary (equator).

Note that after gluing every equatorial vertex has degree at least 11 and is not adjacent to 3-vertices. As a result, we have only off-(3, 10, 4)-paths among the 3-paths mentioned in (Ta)–(Th). Thus (Tc) is tight.

(Td). Fig. 5 (Jendrol' [10]) shows how to transform the dodecahedron to a triangulation with vertices of degree 3, 4, and 15 only and such that every 3-path goes through a 15-vertex. This construction confirms the tightness of (Td).

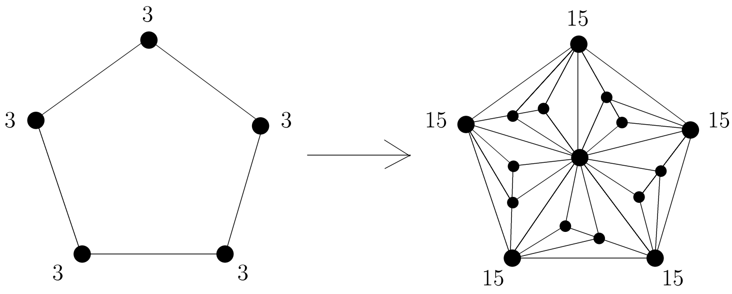

Fig. 5. The Jendrol' construction [10] with only off-(3, 15, 3)-paths confirming the tightness of (Td).

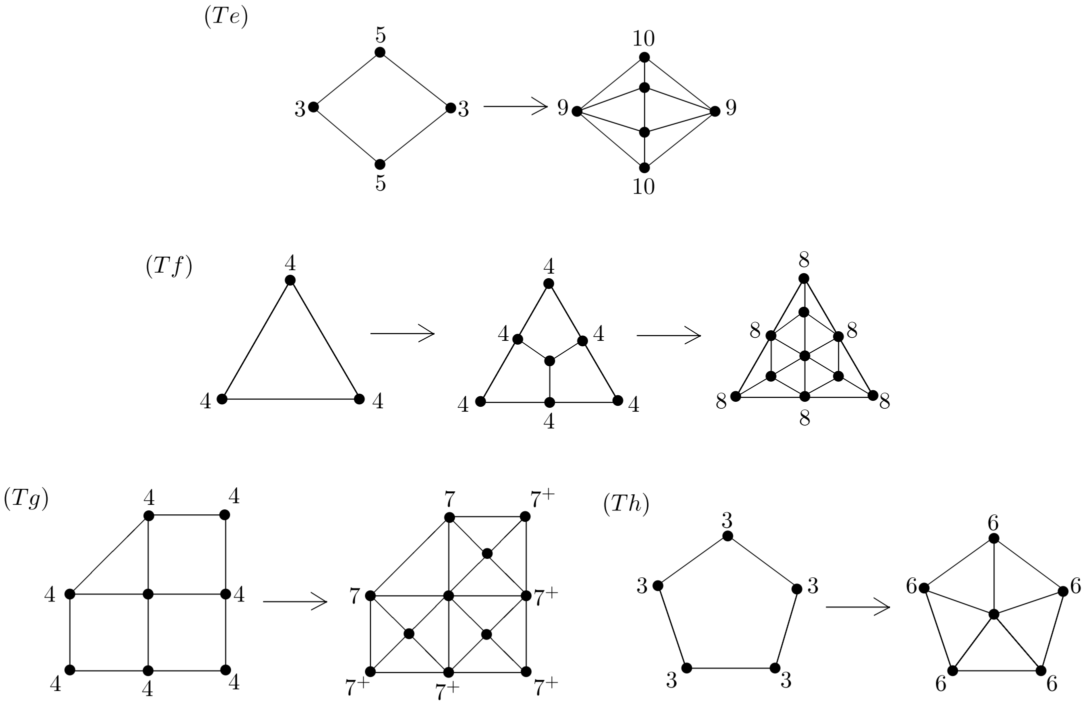

Fig. 6. Constructions pertaining to (Te)–(Th).

(Te)–(Th). Fig. 6 represents four simple constructions arising from Platonic and Archimedean solids. For (Tf) and (Th), we start from the octahedron and dodecahedron, respectively. The quadrangulation for (Te) is drawn forcedly. The same is true concerning the initial graph for (Tg), in which every vertex is incident with a 3-face and three 4-faces. These constructions confirm the tightness of (Te)–(Th), respectively.

3. Proving the main statement of Theorem 7

Suppose that M' is a counterexample to Theorem 7.

3.1. Constructing a triangular counterexample to Theorem 7

Let M be a counterexample on $V(M')$ with the greatest number of edges. From now on, by $d(v)$ of $v \in V(M)$ we mean the degree of v in M. We abbreviate the clause "since M does not contain a path xyz such that $d(x) \le i$, $d(y) \le j$ and $d(z) \le k$" to "by non-(i, j, k)!".

(A) *M is a triangulation.*

Suppose there is a 4^+-face $f = abc \dots$. Let b be a vertex of minimum degree among all vertices incident with f and, moreover, $d(a) \le d(c)$. It suffices to prove that $M + ac$ is also a counterexample to Theorem 7.

First observe that $M + ac$ cannot contain the $(K_4 - e)$-like configuration $K^*_{2,4}$ since $\delta(M) \ge 3$ and M does not contain $K^*_{2,4}$ by assumption. Secondly, suppose $M + ac$ has a forbidden path zac or acz. Clearly, this cannot happen if $z \ne b$ since a similar forbidden path should exist already in M due to the fact that $d(b) \le d(a) \le d(c)$ (namely, we could replace c or a in such a path, respectively, by b), a contradiction. Thus, we can assume that $z = b$.

Case 1. $d(b) = 3$. If $d(a) = 3$ then $d(c) \ge 12$ by non-(3, 4, 11)!, so the new paths bac and acb in $M + ac$ are either off-(3, 13, 4)-paths or off-(3, 4, 13)-paths, that is they are not forbidden. If $d(a) = 4$ then $d(c) \ge 10$ by non-(4, 4, 9)!, so the new paths in $M + ac$ are either off-(3, 11, 5)-paths or off-(3, 5, 11)-paths, so they are again allowed. Similarly, for $d(a) \in \{5, 6\}$ we have $d(c) \ge 7$ by non-(6, 5, 6)!, which means that the new paths are admissible because they are either off-(3, 6, 8)-paths or off-(3, 8, 6)-paths. The same is true if $7 \le d(a) \le d(c)$.

Case 2. $d(b) = 4$. If $d(a) = 4$ then $d(c) \ge 10$ by non-(4, 4, 9)!, and new off-(4, 11, 5)-paths or off-(4, 5, 11)-paths are admissible again. If $d(a) \ge 5$ then $d(c) \ge 8$ by non-(7, 4, 7)!, so we can create only off-(4, 6, 9)-paths or off-(4, 9, 6)-paths.

Case 3. $d(b) \ge 5$. Here, adding the edge ac results in new off-(5, 6, 8)-paths or off-(5, 8, 6)-paths, which are again not forbidden, as desired.

The next property follows immediately from (A) and the assumptions of our theorem:

(B) *No* 3*-vertex of* M *is adjacent to a* 3*-vertex.*

3.2. Discharging

Euler's formula $|V| - |E| + |F| = 2$ for M may be written as

$$\sum_{v \in V} (d(v) - 6) = -12. \tag{1}$$

Every vertex v contributes the *charge* $\mu(v) = d(v) - 6$ to (1), so only the charges of 5^--vertices are negative. Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

Throughout the paper, we denote the vertices adjacent to a vertex v in a cyclic order by $v_1, \dots, v_{d(v)}$.

The rules of discharging are as follows (see Fig. 7):

R1. *Suppose* $d(v) = 3$.

(a) *If* $4 \le d(v_1) \le 6$, *then each of* v_2, v_3 *gives* $\frac{3}{2}$ *to* v.

(b) *Otherwise, each of* v_1, v_2, *and* v_3 *gives* 1 *to* v.

Note that $d(v_2) \ge 12$ and $d(v_3) \ge 12$ if $d(v_1) = 4$ due to non-(3, 4, 11)!, and $d(v_2) \ge 9$ and $d(v_3) \ge 9$ if $5 \le d(v_1) \le 6$ due to non-(6, 4, 8)!. Now $\mu'(v)$ is completely determined for $d(v) = 3$, and clearly $\mu'(v) = \mu(v) + 3 = 0$.

R2. *Suppose* $d(v) = 4$.

(a) *If* $d(v_1) = 3$, *then* v_3 *gives* 1 *to* v, *and each of* v_2, v_4 *gives* $\frac{1}{2}$.
(b) *If* $d(v_1) = 4$, *then*:
 (b1) *if* $d(v_3) \le 11$, *then* v_3 *gives* $\frac{4}{5}$ *to* v, *and each of* v_2, v_4 *gives* $\frac{3}{5}$;
 (b2) *if* $d(v_3) \ge 12, d(v_2) \le 11$, *and* $d(v_4) \le 11$, *then* v_3 *gives* $\frac{4}{5}$ *to* v, *and each of* v_2, v_4 *gives* $\frac{3}{5}$;
 (b3) *if* $d(v_3) \ge 12, d(v_2) \ge 12$, *and* $d(v_4) \le 11$, *then* v_3 *gives* $\frac{9}{10}$ *to* v, v_2 *gives* $\frac{1}{2}$, *and* v_4 *gives* $\frac{3}{5}$;
 (b4) *if* $d(v_3) \ge 12, d(v_2) \ge 12$, *and* $d(v_4) \ge 12$, *then* v_3 *gives* 1 *to* v, *and each of* v_2, v_4 *gives* $\frac{1}{2}$.
(c) *If* $5 \le d(v_1) \le 6$, *then each of* v_2, v_3, v_4 *gives* $\frac{2}{3}$ *to* v.
(d) *Otherwise,* v *receives* $\frac{1}{2}$ *from each of* $v_1, \dots, v_4$.

Note that $d(v_i) \ge 12$ whenever $2 \le i \le 4$ if $d(v_1) = 3$ due to non-(3, 4, 11)!, and $d(v_i) \ge 10$ whenever $2 \le i \le 4$ if $d(v_1) = 4$ due to non-(4, 4, 9)!. Furthermore, if $5 \le d(v_1) \le 6$ then $d(v_i) \ge 9$ whenever $2 \le i \le 4$ due to non-(6, 4, 8)!. Otherwise, we can assume that $d(v_1) \le 7$ and $d(v_i) \ge 8$ whenever $2 \le i \le 4$ due to non-(7, 4, 7)!. Clearly, each 4-vertex v has $\mu'(v) \ge \mu(v) + 2 = 0$.

Let $n_k(v)$ be the number of k-neighbors of v. A 7-vertex v is *poor* if $n_3(v) = 0$ and $\frac{1}{2}n_4(v) + \frac{1}{4}n_5(v) > 1$. It is easy to check that if v is poor, then $n_4(v) = 3, n_5(v) = 0$, or $n_4(v) = 2, 1 \le n_5(v) \le 2$, or else $n_4(v) = 1, n_5(v) = 3$. All poor vertices are presented in Fig. 7 (R3 and R4).

R3. (a) *Normally, every* 5*-vertex* v *receives* $\frac{1}{4}$ *from every adjacent* 7^+*-vertex, with the following exceptions* (b)–(c).

(b) *If a face* vv_iv_{i+1} *is such that* $d(v_{i-1}) = 5$, v_i *is a poor* 7*-vertex, and the third vertex* w *in the face* $v_iv_{i+1}w$ *containing the edge* v_iv_{i+1} *is a* 4*-vertex, then* v *receives from* v_{i+1} *charge* $\frac{1}{2}$ *and from* v_i *nothing.*

(c) *If* v *is adjacent to two consecutive poor* 7*-vertices* v_2 *and* v_3 *such that* $n_4(v_2) = n_4(v_3) = 2$, *then it receives* $\frac{1}{8}$ *from each of* v_2, v_3, *and* $\frac{3}{8}$ *from each of* v_1, v_4.

Concerning R3c, we note that the vertex w different from v and lying in the face v_2v_3w satisfies $d(w) \ge 8$ due to non-(7, 4, 7)!.

R4. (a) *Suppose a poor* 7*-vertex* v *is adjacent to* 4*-vertices* v_2, v_4, *a* 5*-vertex* v_7, *and a* 6^+*-vertex* v_6; *then* v *receives through face* vv_5v_6 *from* v_5 *the charge* $\frac{1}{4}$ *if* v_6 *is not* 7*-poor, and* $\frac{1}{8}$ *otherwise (in the latter case,* v_6 *also gets* $\frac{1}{8}$ *from* v_5).

(b) *If a poor* 7*-vertex* v *is adjacent to* 4*-vertices* v_4, v_7, *and a* 5*-vertex* v_2, *then* v *receives* $\frac{1}{8}$ *from each of* v_5 *and* v_6 *through face* vv_5v_6.

(c) *If a poor* 7*-vertex* v *is adjacent to* 4*-vertices* v_2, v_4, v_7, *then* v *receives* $\frac{1}{4}$ *from each of* v_5 *and* v_6 *through face* vv_5v_6.

We have $d(v_5) \ge 8$ in R4a–R4c and $d(v_6) \ge 8$ in R4b–R4c by non-(7, 4, 7)!. Since a 6-vertex v does not participate in discharging, we have $\mu'(v) = \mu(v) = 6 - 6 = 0$.

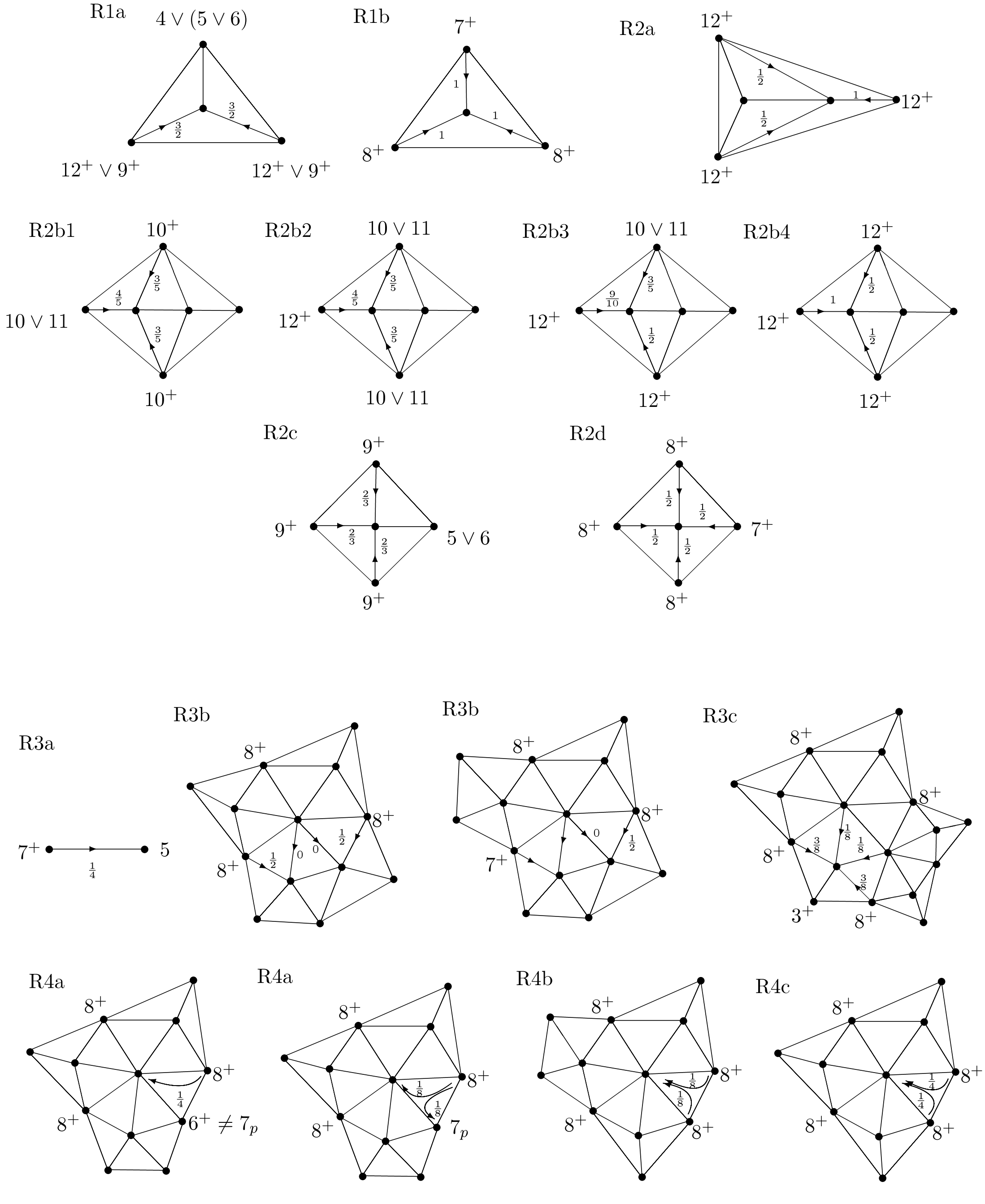

Fig. 7. Rules of discharging.

3.3. Checking $\mu'(v) \geq 0$ for $d(v) = 5$ and $d(v) \geq 7$

CASE 1. $d(v) = 5$. First suppose that v has only 6^+-neighbors. Then v has at least four 7^+-neighbors by non-(6, 5, 6)!, which implies that $\mu'(v) \geq \mu(v) + 4 \times \frac{1}{4} = 0$ by R3a or $\mu'(v) \geq \mu(v) + 2 \times \frac{1}{8} + 2 \times \frac{3}{8} = 0$ by R3c.

Suppose $d(v_1) \leq 5$. By non-(6, 5, 6)!, all of $v_2, \ldots, v_5$ are 7^+-vertices. By symmetry, it suffices to see that the total donation to v from v_2 and v_3 by R3 is $\frac{1}{2} = \frac{1}{2} + 0 = \frac{1}{4} + \frac{1}{4} = \frac{1}{8} + \frac{3}{8}$.

CASE 2. $d(v) = 7$. If v has a 3-neighbor, then v makes only one donation to its neighbors due to non-(3, 7, 5)!; namely, v gives 1 to its 3-neighbor by R1b.

Suppose v has no 3-neighbors. A simple case analysis based on R3b, R3c, and R4 shows that each poor vertex has $\mu'(v) = 0$. Finally, if v is not poor then $\mu'(v) \geq 7 - 6 - (\frac{1}{2}n_4(v) + \frac{1}{4}n_5(v)) \geq 0$ by R2d and R3a.

CASE 3. $d(v) = 8$. Now v can have at most one 3-neighbor by non-(3, 15, 3)!. First suppose v has no 3-neighbors. Our v can give:

(D1) at most $\frac{1}{4}$ to (or through) face vv_1v_8 by R4, where $d(v_1) \geq 6$, $d(v_8) \geq 7$,

(D2) at most $\frac{1}{2}$ to v_7, where $4 \leq d(v_7) \leq 5$, $d(v_8) \geq 7$, $d(v_6) \geq 6$ by R2d, R3, and

(D3) at most $2 \times \frac{3}{8}$ to 5-vertices v_7, v_6 by R3a, R3c with $d(v_5) \geq 7$, $d(v_8) \geq 7$.

The $\frac{1}{4}$-*normal donation* from v is more generous than the real donation by R1–R4: in the situation (D1), precisely $\frac{1}{4}$ is given; in (D2), precisely $\frac{1}{2}$ is given; and in (D3), still $2 \times \frac{3}{8}$ is given.

To estimate the total expenditure of v by R2d, R3, and R4, we share each donation evenly among one, two or three nearest incident faces, respectively. Since every incident face then receives from v at most $\frac{1}{4} : 1 = \frac{1}{2} : 2 = 2 \times \frac{3}{8} : 3 = \frac{1}{4}$, it follows that $\mu'(v) \geq 8 - 6 - 8 \times \frac{1}{4} = 0$.

Now suppose v has a 3-neighbor v_2. Then we can assume that $d(v_1) \geq 7$ by non-(6, 4, 8)! and $d(v_3) \geq 8$ by non-(7, 4, 7)! and symmetry. By non-(3, 15, 3)!, such a 3-neighbor is unique. Here, v is not adjacent to 4-vertices due to non-(3, 10, 4)!. This imposes two significant restrictions on donations from v. First, v cannot give $\frac{1}{2}$ or $\frac{3}{8}$ to a 5-neighbor by R3b and R3c. Secondly, v cannot send $\frac{1}{8}$ or $\frac{1}{4}$ to a poor 7-vertex through a "heavy triangle" by R4.

Thus our v gives 1 to its 3-vertex by R1b and at most $\frac{1}{4}$ to each 5-neighbor. Since v is adjacent to at most four 5-vertices by non-(6, 5, 6)!, we have $\mu'(v) \geq 2 - 1 - 4 \times \frac{1}{4} = 0$.

CASE 4. $d(v) = 9$. Again, v has at most one 3-neighbor and cannot have a 3-neighbor and a 4-neighbor at once. If v has a 3-neighbor, then there are at most five 5-neighbors by non-(6, 5, 6)!, so the argument in Case 3 combined with R1 yields $\mu'(v) \geq 9 - 6 - \frac{3}{2} - 5 \times \frac{1}{4} > 0$.

Now suppose v has no 3-neighbor. To resolve this case, it suffices to show that every face receives at most $\frac{1}{3}$ from v on the average since $9 - 6 - 9 \times \frac{1}{3} = 0$. If $d(v_1) \geq 6$, $4 \leq d(v_2) \leq 5$, and $d(v_3) \geq 6$, then we share the charge of at most $\frac{2}{3}$ received by v_2 from v by R2c evenly between faces v_1vv_2 and v_2vv_3, so each of them receives at most $\frac{1}{3}$. Now suppose $d(v_1) \geq 7$, $4 \leq d(v_2) \leq 5$, $d(v_3) = 5$, and $d(v_4) \geq 7$. If v_3 receives $\frac{1}{4}$ from v by R3a, then we share $\frac{2}{3} + \frac{1}{4}$ evenly among faces v_1vv_2, v_2vv_3, and v_3vv_4, as desired. So suppose v_3 receives $\frac{3}{8}$ from v by R3c, then v_4 is poor and v_5 is a 4-vertex surrounded by 7^+-vertices by non-(7, 4, 7)! and hence receives $\frac{1}{2}$ from v by R2d. We split the charge received by v_5 evenly between faces v_4vv_5 and v_5vv_6. Now we can share $\frac{2}{3} + \frac{3}{8} + \frac{1}{4}$, which is less than $4 \times \frac{1}{3}$, evenly among the four faces $v_1vv_2, \dots, v_4vv_5$. Finally, if v_1 is poor while $d(v_2) \geq 6$, then v_1vv_2 receives less than $\frac{1}{3}$ by R4, and we are done.

CASE 5. $d(v) = 10$. If v has a 3-neighbor, then 4-neighbors are still impossible due to non-(3, 10, 4)!, which means that we can again apply the argument in Case 3 combined with R1a. Since now v still has at most five 5-neighbors, $\mu'(v) \geq 10 - 6 - \frac{3}{2} - 5 \times \frac{1}{4} > 0$.

Assume that v has no 3-neighbors. By R2, R3, our v gives at most $\frac{4}{5}$ to each "single" 5^--neighbor (which is a 5^--vertex v_2 such that $d(v_1) \geq 6$ and $d(v_3) \geq 6$), and at most $2 \times \frac{3}{5}$ to each pair of "5^--twins" (that is vertices v_2, v_3 of degree at most 5).

To estimate the total expenditure of v, we undertake the averaging of transfers from v to its neighbors w.r.t. the level of $\frac{2}{5}$, which implies that every incident face takes at most $\frac{2}{5}$ from v on the average. Thus $\mu'(v) \geq 10 - 6 - 10 \times \frac{2}{5} = 0$.

CASE 6. $d(v) = 11$. Still, at most one 3-neighbor at v is possible. In contrast to Case 5, now v can have a 3-neighbor and a 4-neighbor simultaneously, but still no 4-neighbor of v is adjacent to a 3-vertex by non-(3, 4, 11)!.

If v has a 3-neighbor, then we can apply the $\frac{2}{5}$-argument to the donation from v. Suppose $d(v_1) \geq 9$, $d(v_2) = 3$, $d(v_3) = 5$, and $d(v_4) \geq 7$. If v_3 receives $\frac{1}{4}$ from v by R3a, then we have $\mu'(v) \geq 11 - 6 - \frac{3}{2} - \frac{1}{4} - 8 \times \frac{2}{5} > 0$. Now suppose v_3 receives $\frac{3}{8}$ from v by R3c, then v_4 is poor and v_5 is a 4-vertex surrounded by 7^+-vertices and hence receives $\frac{1}{2}$ from v by R2d. Now $\mu'(v) \geq 11 - 6 - \frac{3}{2} - \frac{1}{2} - \frac{3}{8} - 6 \times \frac{2}{5} > 0$. A similar situation arises when $d(v_1) \geq 9$, $d(v_2) = 3$, $d(v_3) = 6$. Now v_3 receives nothing from v, and $d(v_4) \geq 6$ due to non-(3, 7, 5)!. It can happen that v_4 is 7-poor, in which case v must give $\frac{1}{4}$ through face v_3vv_4 by R4a, so again $\mu'(v) \geq 11 - 6 - \frac{3}{2} - \frac{1}{4} - 8 \times \frac{2}{5} > 0$. If $d(v_1) \geq 7$, $d(v_2) = 3$, $d(v_3) \geq 8$, then we similarly have $\mu'(v) \geq 5 - 1 - 9 \times \frac{2}{5} > 0$ by R1b.

Finally, assume v has no 3-neighbors; then our $\frac{2}{5}$-averaging yields $\mu'(v) \geq 5 - 11 \times \frac{2}{5} > 0$.

To estimate the total donation of v whenever $d(v) \geq 12$, we use the following notions.

A face vv_1v_2 is a *single receiver* if $d(v_1) \geq 6$ and $d(v_2) \geq 6$. Note that a single receiver gets nothing from v, unless at least one of v_1, v_2 is a poor 7-vertex, in which case vv_1v_2 receives at most $\frac{1}{4}$ from v by R4.

Faces vv_1v_2 and vv_2v_3 form a *double receiver* if $d(v_1) \geq 6$, $d(v_2) \leq 5$, and $d(v_3) \geq 7$, except for the case $d(v_1) = 6$ and $d(v_2) = 3$, where $d(v_3) \geq 9$ by non-(6, 4, 8)!. Note that v_2 in a double receiver gets at most 1 from v by R1b, R2, and R3. In fact, v_2 receives 1 only by R1b, R2a, R2b4.

A *triple receiver* consists of faces vv_1v_2, vv_2v_3, and vv_3v_4, where $d(v_1) \geq 7$, $d(v_4) \geq 7$ and either $d(v_2) = 3$, $d(v_3) = 6$, or $d(v_2) \leq 5$, $d(v_3) \leq 5$. A triple receiver gets at most $\frac{3}{2} + \frac{1}{2}$ from v by R1a, R2a–R2c, R3. Note that if $d(v_2) = 3$, $d(v_3) = 6$, then face vv_3v_4 gets nothing from v since v_4 cannot be poor by non-(3, 6, 5)!, which means that R4a is not applicable. It is easy to see that a triple receiver gets more than $\frac{3}{2}$ from v only if $d(v_2) = 3$ and $4 \leq d(v_3) \leq 5$.

CASE 7. $12 \leq d(v) \leq 15$. As before, at most one 3-neighbor at v is possible. Hence, there is at most one triple receiver getting more than $3 \times \frac{1}{2}$ (and at most 2) from v, while every double receiver gets at most $2 \times \frac{1}{2}$, and every single receiver gets at most $1 \times \frac{1}{2}$ (in fact, at most $\frac{1}{4}$). For $13 \le d(v) \le 15$, this simple observation already implies that $\mu'(v) \ge d(v) - 6 - d(v) \times \frac{1}{2} = \frac{d(v)-12}{2} > d(v) - 6 - 2 - (d(v) - 3) \times \frac{1}{2} = \frac{d(v)-13}{2} \ge 0$.

So suppose $d(v) = 12$ and $\mu'(v) < 0$. Here, we must argue more carefully. Thus we have a "bad" triple receiver getting more than $\frac{3}{2}$ from v. We want to check that our v saves at least $\frac{1}{2}$ on the other receivers with respect to the average level of donation of $\frac{1}{2}$ to each face at v.

We first prove that v has no single receiver vv_1v_2. Indeed, we are done if none of v_1, v_2 is a poor 7-vertex, since vv_1v_2 then gets nothing from v. So, suppose v_2 is poor. Then vv_1v_2 gets at most $\frac{1}{4}$ from v by R4, so it saves $\frac{1}{4}$. If so, then $d(v_3) = 4$ by R4 (see Fig. 7). Furthermore, v_3 has three 8^+-neighbors due to non-(7, 4, 7)!. This implies that v_3 gets $\frac{1}{2}$ from v by R2d. In other words, the single receiver at v_3 saves $\frac{1}{2}$, and we are done.

From now on, v has only double and triple receivers. Due to parity reasons, the number of triple receivers is even, and so we have a "good" triple receiver TR with the central vertices v_{i+1}, v_{i+2} (addition modulo 12), where $4 \le d(v_{i+1}) \le 5$, $4 \le d(v_{i+2}) \le 5$. As we see, TR gets at most $\frac{3}{5} + \frac{3}{5}$ from v, and so saves at least $3 \times \frac{1}{2} - 2 \times \frac{3}{5} = \frac{3}{10}$, with $\frac{3}{10}$ happening only in R2b1. Thus we can assume that there is precisely one good triple receiver.

Let us look at our TR more attentively. It cannot happen that $d(v_{i+1}) = d(v_{i+2}) = 5$, for otherwise it saves at least $3 \times \frac{1}{2} - 2 \times \frac{3}{8} > \frac{1}{2}$ by R3, and we are done. Now suppose $d(v_{i+1}) = 4$, $d(v_{i+2}) = 5$. Here, v_{i+1} receives $\frac{2}{3}$ by R2c and v_{i+2} receives either $\frac{1}{4}$ by R3a, or $\frac{3}{8}$ by R3c. In the former case, TR saves more than $\frac{1}{2}$. In the latter case, v_{i+3} is a poor 7-vertex, in which case $d(v_{i+4}) = 4$ (see R3c in Fig. 7). Due to non-(7, 4, 7)!, our v gives only $\frac{1}{2}$ to v_{i+4} by R2d, and so saves the desired $\frac{1}{2}$ already at v_{i+4}.

Thus we can assume that $d(v_{i+1}) = d(v_{i+2}) = 4$. If $d(v_i) \ge 12$ and $d(v_{i+3}) \ge 12$, then TR gets $2 \times \frac{1}{2}$ by R2b3, R2b4, and we are done. By symmetry, it suffices to assume that $10 \le d(v_i) \le 11$ (in which case v_i gets $\frac{3}{5}$ from v by R2b1), and to find an extra saving of $\frac{1}{10}$ at v_{i-1}.

We have two possibilities. First, suppose v_{i-1} belongs to the "bad" triple receiver BR, where $\{d(v_{i-1}), d(v_{i-2})\} = \{3, 5\}$ due to non-(3, 4, 11)!. In this case, BR receives at most $\frac{3}{2} + \frac{3}{8}$ by our rules, so it saves at least $\frac{1}{8}$, and this saving can be attributed to our v_{i+1}. Secondly suppose v_{i-1} belongs to a double receiver. Then $4 \le d(v_{i-1}) \le 5$, and v_{i-1} cannot receive as much as 1 from v by R2a or R2b4 since $10 \le d(v_i) \le 11$ by assumption. Hence, v_{i-1} receives at most $\frac{9}{10}$ from v by our rules. So, v_{i+1} and v_{i-1} together receive at most $\frac{3}{5} + \frac{9}{10} = \frac{3}{2}$, and we are done.

CASE 8. $d(v) \ge 16$. Note that every receiver gets at most $\frac{2}{3}$ from v per face. For $d(v) \ge 18$, this implies that $\mu'(v) \ge d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3} \ge 0$.

Suppose $16 \le d(v) \le 17$. Note that every double receiver saves at least $2 \times \frac{2}{3} - 1 = \frac{1}{3}$ with respect to the level of $\frac{2}{3}$. If single receiver vv_1v_2 gets nothing from v, then it already saves $\frac{2}{3}$, which implies that $\mu'(v) \ge d(v) - 6 - d(v) \times \frac{2}{3} + \frac{2}{3} = \frac{d(v)-16}{3} \ge 0$. If v_2 is a poor 7-vertex, then vv_1v_2 gets at most $\frac{1}{4}$ from v. Furthermore, v_3 is then a 4-vertex in a double receiver, and it gets $\frac{1}{2}$ from v, as mentioned above. This means that these two receivers together save at least $3 \times \frac{2}{3} - \frac{1}{4} - \frac{1}{2} > 1$, and we are done.

Finally, suppose there are no single receivers at v. If $d(v) = 17$, then there is at least one double receiver, and so $\mu'(v) \ge 17 - 6 - 17 \times \frac{2}{3} + \frac{1}{3} = 0$. If $d(v) = 16$, then there are at least two double receivers, and we have $\mu'(v) \ge 16 - 6 - 16 \times \frac{2}{3} + 2 \times \frac{1}{3} = 0$.

Thus we have proved $\mu'(v) \ge 0$ for every $v \in V$, which contradicts (1):

$$0 \le \sum_{v \in V} \mu'(v) = \sum_{v \in V} \mu(v) = -12.$$

This completes the proof of Theorem 7.

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