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On 1-improper 2-coloring of sparse graphs

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ABSTRACT

A graph G is $(1, 1)$-colorable if its vertices can be partitioned into subsets V_1 and V_2 such that every vertex in $G[V_i]$ has degree at most 1 for each $i \in \{1, 2\}$. We prove that every graph with maximum average degree at most $\frac{14}{5}$ is $(1, 1)$-colorable. In particular, it follows that every planar graph with girth at least 7 is $(1, 1)$-colorable. On the other hand, we construct graphs with maximum average degree arbitrarily close to $\frac{14}{5}$ (from above) that are not $(1, 1)$-colorable.

In fact, we establish the best possible sufficient condition for the $(1, 1)$-colorability of a graph G in terms of the minimum, ρ_G, of $\rho_G(S) = 7|S| - 5|E(G[S])|$ over all subsets S of $V(G)$. Namely, every graph G with $\rho_G \geq 0$ is $(1, 1)$-colorable. On the other hand, we construct infinitely many non-$(1, 1)$-colorable graphs G with $\rho_G = -1$. This solves a related conjecture of Kurek and Ruciński from 1994.

1. Introduction

A graph G is called *improperly* $(d_1, \ldots, d_k)$*-colorable*, or just $(d_1, \ldots, d_k)$*-colorable*, if the vertex set of G can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of *proper* k*-coloring* (when $d_1 = \cdots = d_k = 0$) and d*-improper* k*-coloring* (when $d_1 = \cdots = d_k = d \geq 1$).

The first result on d-improper colorings with $d > 0$ belongs to Gerencsér [13], who proved that every graph G with *maximum degree* $\Delta(G)$ is 1-improperly $(\lfloor \frac{\Delta(G)}{2} \rfloor + 1)$-colorable; in particular, every subcubic graph is $(1, 1)$-colorable. This was extended by Lovász [17] as follows: every graph G is $(d_1, \ldots, d_k)$-colorable whenever $(d_1+1)+\cdots+(d_k+1) \geq \Delta(G)+1$. These bounds are attained by the complete graphs.

As shown by Appel and Haken [1,2], every planar graph is 4-colorable, i.e. $(0, 0, 0, 0)$-colorable. Cowen, Cowen, and Woodall [11] proved that every planar graph is 2-improperly 3-colorable, i.e. $(2, 2, 2)$-colorable.

Another important extension of proper coloring was introduced by Vizing [19] and, independently, by Erdős, Rubin, and Taylor [12]. Suppose that for each list $L(v)$ of colors admissible for v such that $|L(v)| \geq k$, there is a proper coloring in which a color of vertex v is taken from $L(v)$; then G is k*-choosable*. Clearly, if $L(v)$ is the same set of cardinality k for all vertices, then we have the case of proper k-coloring.

Borodin and Kostochka [8] extended the notion of $(d_1, \ldots, d_k)$-colorability as follows: Let f_i, $1 \leq i \leq s$, be functions from $V(G)$ to the non-negative integers. A graph G is called $(f_1, \ldots, f_s)$*-choosable* if $V(G)$ can be partitioned into subsets $V_1, \ldots, V_s$ such that each vertex $v \in V_i$ (i.e., colored with i), where $1 \leq i \leq s$, has strictly fewer than $f_i(v)$ neighbors in V_i. Clearly, if

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$f_i(v) \equiv d_i + 1$ for all $v \in V(G)$, $1 \le i \le s$, then we have the case of $(d_1, \dots, d_s)$-colorability. Note also that if $f_i(v) = 0$ then v cannot be colored with i by definition, so k-choosability is precisely the case of $(f_1, \dots, f_s)$-choosability if $f_i(v) \in \{0, 1\}$ for all $v \in V(G)$, $1 \le i \le s$ and $\sum_{v \in V(G)} f_i(v) \ge k$ for all $1 \le i \le s$. Indeed, it suffices to define the set of admissible colors at v as follows: $L(v) = \{i : f_i(v) = 1, 1 \le i \le s\}$. More generally, if $f_i(v) \in \{0, t+1\}$, for all $v \in V(G)$ and $1 \le i \le s$, where t is a non-negative integer, then we have the case of *t-improper k-choosability*, provided that $\sum_{v \in V(G)} f_i(v) \ge k(t+1)$ for all $1 \le i \le s$. The theorem of Lovász [17] was extended in [8] as follows: If $(f_1(v)+1) + \dots + (f_s(v)+1) \ge d(v) + 1$ for each $v \in V(G)$, where $d(v)$ is the degree of v, then G is $(f_1, \dots, f_s)$-choosable.

A natural measure of sparseness for a graph G is $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$, the maximum over the average degrees of the subgraphs of G. For planar graphs G the sparseness can be measured in terms of the *girth*, $g(G)$, which is the length of a shortest cycle in G. It is an easy consequence of Euler's formula that each planar graph G satisfies $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$.

We now survey the known results on probably the simplest version of improper coloring, namely improper colorings of sparse graph with two colors, and, more generally, k-improper 2-choosability. Mihók [18] constructed a planar graph that is not (k, k)-colorable for arbitrarily large k. Havet and Sereni [15] proved, for every $k \ge 0$, that every graph G with $\text{mad}(G) < \frac{4k+4}{k+2}$ is k-improperly 2-choosable, i.e. (k, k)-choosable.

For non-negative integers j and k, let $F(j, k)$ denote the supremum of x such that every graph G with $\text{mad}(G) \le x$ is (j, k)-colorable. It is easy to see that $F(0, 0) = 2$. Indeed, since the odd cycle C_{2n-1} has $\text{mad}(G) = 2$ and is not $(0, 0)$-colorable, $F(0, 0) \le 2$. On the other hand, each graph with $\text{mad}(G) < 2$ has no cycles and therefore is bipartite, i.e., $(0, 0)$-colorable.

Glebov and Zambalaeva [14] proved that every planar graph G with $g(G) \ge 16$ is $(0, 1)$-colorable. This was strengthened by Borodin and Ivanova [3]: they proved that every graph G with $\text{mad}(G) < \frac{7}{3}$ is $(0, 1)$-colorable, which implies that every planar graph G with $g(G) \ge 14$ is $(0, 1)$-colorable. Borodin and Kostochka [9] proved that $F(0, 1) = \frac{12}{5}$. In particular, this implies that every planar graph G with $g(G) \ge 12$ is $(0, 1)$-colorable.

For each integer $k \ge 2$, Borodin et al. [5] proved that every graph G with $\text{mad}(G) < \frac{3k+4}{k+2} = 3 - \frac{2}{k+2}$ is $(0, k)$-colorable. On the other hand, for all $k \ge 2$ [5] presents non-$(0, k)$-colorable graphs with mad arbitrarily close to $\frac{3k+2}{k+1} = 3 - \frac{1}{k+1}$.

Recently, it was proved by Borodin et al. [6] that every graph G with $\text{mad}(G) < \frac{10k+22}{3k+9}$, where $k \ge 2$, is $(1, k)$-colorable. On the other hand, [6] presents a construction of non-$(1, k)$-colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$.

Borodin and Kostochka [10] obtained an exact result for a wide range of j and k: if $j \ge 0$ and $k \ge 2j+2$ then $F(j, k) = 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$. In particular, together with [9], this yields exact values for $F(0, k)$ for every k.

From [10] we easily deduce:

Corollary 1. *Let G be a planar graph, then G is:*

(i) $(0, 2)$-*colorable if* $g(G) \ge 8$,
(ii) $(0, 4)$-*colorable if* $g(G) \ge 7$,
(iii) $(1, 4)$-*colorable if* $g(G) \ge 6$, *and*
(iv) $(2, 6)$-*colorable if* $g(G) \ge 5$.

Borodin et al. [5] constructed a planar graph with girth 6 which is not $(0, k)$-colorable for any k, and proved that every planar graph G with $g(G) \ge 7$ is $(0, 8)$-colorable, and if $g(G) \ge 8$ then G is $(0, 4)$-colorable. It follows from [6] that every planar graph G with $g(G) \ge 7$ is $(1, 2)$-colorable, and every one with $g(G) \ge 6$ is $(1, 5)$-colorable. Borodin et al. [7] also proved, among other results, that planar graphs with girth 5 are $(2, 13)$- and $(3, 7)$-colorable. Note that all these bounds are now strengthened by Corollary 1. Still, we suspect that Corollary 1 can be further improved. Also, the result of Havet and Sereni [15] yields that every planar graph G with $g(G) \ge 5$ (respectively, $g(G) \ge 6$, and $g(G) \ge 8$) is $(4, 4)$-choosable (respectively, $(2, 2)$-choosable, and $(1, 1)$-choosable).

The purpose of this paper is to prove Theorems 2 and 4.

Theorem 2. *Every graph G with* $\text{mad}(G) \le \frac{14}{5}$ *is $(1, 1)$-colorable, and the restriction on* $\text{mad}(G)$ *is sharp.*

Corollary 3. *Every planar graph G with $g(G) \ge 7$ is $(1, 1)$-colorable.*

Note that Theorem 2 and Corollary 3 improve the above mentioned sufficient conditions for the $(1, 1)$-colorability due to Havet and Sereni [15]: $\text{mad}(G) \le \frac{8}{3}$ for arbitrary graph G and $g(G) \ge 8$ if G is planar. Borodin and Ivanova [4] proved that every graph G with $g(G) \ge 7$ and $\text{mad}(G) < \frac{14}{5}$ can be partitioned into two strong linear forests (each connected component of such forests is allowed to have at most two edges). Clearly, this result also follows from Theorem 2.

A. Pokrovskiy pointed out that Theorem 2 has an application to sparse *vertex Ramsey* graphs. We say that $G \overset{v}{\rightarrow} (H_1, \dots, H_k)$ if for every partition of the vertex set of G into subsets $V_1, \dots, V_k$ there exists i such that H_i is a subgraph of $G[V_i]$. Let $m_{cr}(H_1, \dots, H_k) = \inf\{\text{mad}(F) : F(H_1, \dots, H_k)\}$.

It is clear that a graph is $(d_1, \dots, d_k)$-colorable if and only if $G \overset{v}{\not\rightarrow} (K_{1,d_1+1}, \dots, K_{1,d_k+1})$. Furthermore, $F(j, k) = m_{cr}(K_{1,j+1}, K_{1,k+1})$. Borodin and Kostochka's results [9,10] directly state exact values for $m_{cr}(K_{1,j+1}, K_{1,\ell+1})$ if $\ell \ge 2j+2$ or $j = 0$. Kurek

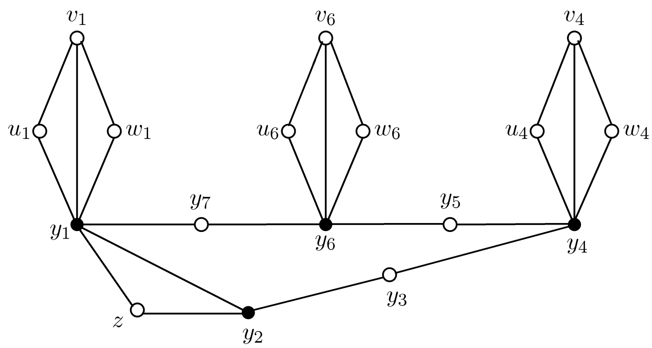

Fig. 1. A non-(1, 1)-colorable graph G_3 with $\rho_{G_3} = -1$.

and Ruciński [16] proved that

$$\sum_{i=1}^{k} \max_{H_i' \leq H_i} \delta(H_i') \leq m_{cr}(H_1, \ldots, H_k) \leq 2 \sum_{i=1}^{k} \max_{H_i' \leq H_i} \delta(H_i').$$

As a corollary, it follows that $m_{cr}(K_s, \ldots, K_s) = k(s-1)$. However, $m_{cr}(H_1, \ldots, H_k)$ is still unknown in general.

In the same paper, Kurek and Ruciński showed that $8/3 \leq m_{cr}(K_{1,2}, K_{1,2}) \leq 14/5$. Ruciński offered a 400,000 PLZ cash prize for the exact value of $m_{cr}(K_{1,2}, K_{1,2})$. Theorem 2 proves that $m_{cr}(K_{1,2}, K_{1,2}) = 14/5$.

In Theorem 4 we use a refined measure of sparseness employed in [9,10]. For $S \subseteq V(G)$, let $\rho_G(S) = 7|S| - 5|E(G[S])|$. This is the *potential of S*. When there is no chance for confusion, we may use the notation $\rho(S)$. In fact, we establish the best possible sufficient condition for the (1, 1)-colorability of a graph G in terms of the minimum, ρ_G, of $\rho_G(S) = 7|S| - 5|E(G[S])|$, over all non-empty subsets S of $V(G)$.

Theorem 4. *Every graph G with $\rho_G \geq 0$ is (1, 1)-colorable. On the other hand, there are infinitely many non-(1, 1)-colorable graphs G with $\rho_G = -1$.*

2. Sharpness of the restrictions in Theorems 2 and 4

We construct non-(1, 1)-colorable graphs G_p with $\rho_{G_p} = -1$ for all $p \geq 1$ and with $\text{mad}(G_p)$ tending to $\frac{14}{5}$ as p grows.

Let $p \geq 1$ be an integer. Let G_p be the graph obtained from a cycle $C_{2p+1} = y_1 \cdots y_{2p+1}$ as follows. For each $i \in \{1, 4, 6, \ldots, 2p\}$, we add a path $u_i v_i w_i$ and edges $y_i u_i$, $y_i v_i$, and $y_i w_i$. Also we add a vertex z and edges zy_1 and zy_2 (see Fig. 1 for $p = 3$).

Suppose that c is a (1, 1)-coloring of G_p. The following simple observation is useful: one of the vertices u_i, v_i, and w_i is colored the same as y_i whenever $i \in \{1, 4, 6, \ldots, 2p\}$. This implies, in particular, that $c(y_4) = c(y_6)$ because otherwise y_5 cannot be colored. Hence, $c(y_4) = c(y_6) = \cdots = c(y_{2p}) = c(y_1)$. It follows that each of the vertices z, y_2, and y_3 is colored differently from y_1 and y_4. Therefore, $c(z) = c(y_2) = c(y_3)$, a contradiction. Finally, it is easy to check that $\rho_{G_p} = 7 \times (2p + 1 + 3p + 1) - 5 \times (2p + 1 + 5p + 2) = -1$ and $\text{mad}(G_p) = \frac{2(7p+3)}{5p+2} \to \frac{14}{5}$ as $p \to \infty$.

3. Preliminaries

The structure of the proof of Theorem 4 is as follows:

(1) we will describe all non-trivial subgraphs of a minimum counterexample with potential at most 3 as belonging to a finite set of special graphs,
(2) assuming the absence of special graphs, we will give structural results concerning a subgraph G' of a minimum counterexample with $\rho_{G'} \geq 4$, and
(3) we conclude the proof with discharging.

A similar method was used in [9,10]. In particular, an introduction to this method is [9], where the argument has fewer technicalities.

If G' is a pendant block and w is the unique cut vertex in G' of G, then w is the *base* of G'. A *flag* is a pendant block isomorphic to $K_4 - e$, where the base is one of the vertices of degree 3 in $K_4 - e$. A flag *attached* at a vertex u is a flag whose base has been glued to u. (In Fig. 1, we see flags attached at bases y_1, y_4, and y_6.) The significance of a flag F in a graph G attached at u is that in each (1, 1)-coloring, there is a neighbor of u in F that is colored with the same color. Moreover, all other neighbors of u in G are colored with the other color. For the rest of the paper, we will assume that at most one flag is attached to each vertex. If two flags are attached to one vertex, then that subgraph is isomorphic to G_1 from Section 2.

Unimportant vertices in a graph are (a) vertices of degree at most 1, (b) vertices of degree 2 contained in a triangle, (c) vertices of degree 3 contained in a flag. *Semi-important vertices* are vertices of degree 2 not contained in a triangle.

All other vertices are *important*. (In Fig. 1, we see unimportant vertices u_i, v_i, w_i, z, semi-important vertices y_3, y_5, y_7, and important vertices y_1, y_2, y_4, and y_6.)

We say that a graph H is *smaller* than a graph G (and denote this by $H \prec G$) if (i) G has more important vertices than H, or (ii) G and H have the same number of important vertices and G has more semi-important vertices than H, or (iii) G and H have the same amounts of important and semi-important vertices, and $|V(G)| > |V(H)|$, or (iv) G has the same number of vertices in each class as H and

$$\sum_{u \in V(H)} d(u)^2 > \sum_{v \in V(G)} d(v)^2.$$

Note that by this definition, *if H is a proper subgraph of G, then H is smaller than G.* Let G be a counterexample to the main statement in Theorem 4 smallest with respect to the order above. In particular, $\rho_G \geq 0$. It is easy to see that G is connected, G has no separating edges, and hence $\delta(G) \geq 2$. A set $W \subseteq V(G)$ will be called an *i-set* if $\rho_G(W) = i$.

By a *B-subgraph* we mean the six-vertex subgraph of G obtained from a flag and a triangle by gluing the base of the flag to a vertex of the triangle. The base and one other vertex in the triangle are considered as *special*. The point of this is that in each $(1, 1)$-coloring of B, each of the special vertices has neighbors of both colors, and they have distinct colors.

A *super-flag* W is a B-subgraph of G such that only special vertices of W may have neighbors in G outside of W. In Fig. 1, the subgraph induced by $\{v_1, w_1, u_1, y_1, y_2, z\}$ is a super-flag in which y_1 and y_2 are special. Recall once more that in each $(1, 1)$-coloring of a super-flag W, each of the special vertices has neighbors of both colors, and they have distinct colors.

Let G' be a graph, $A \cup B = V(G')$, ϕ_A be a coloring of $G'[A]$, and ϕ_B be a coloring of $G'[B]$. If $\phi_A(u) = \phi_B(u)$ for all $u \in A \cap B$, then we define $\phi_A \cup \phi_B$ to be a coloring ϕ of G' such that $\phi(w) = \phi_A(w)$ when $w \in A$ and $\phi(v) = \phi_B(v)$ otherwise.

Lemma 5. *No super-flag is a separating set.*

Proof. Suppose that a super-flag W is a separating set. Let $V(G) - W = X \cup Y$ such that $E(X, Y) = \varnothing$.

By the minimality of G, let f_X be a coloring of $X \cup W$ and f_Y be a coloring of $Y \cup W$. Let y_1 and y_2 be the special vertices of W. By the symmetry of the coloring, we can assume that $f_X(y_1) = f_Y(y_1)$ and $f_X(y_2) = f_Y(y_2)$. Furthermore, if $v \in N(y_1) - W$ then $(f_X \cup f_Y)(v) \neq f(y_1)$ and if $u \in N(y_2) - W$ then $(f_X \cup f_Y)(u) \neq f(y_2)$.

Let f be a coloring such that if $u \in X \cup W$ then $f(u) = f_X(u)$ and $f(u) = f_Y(u)$ otherwise. Then f is a $(1, 1)$-coloring of G. □

Lemma 6. *If $xy \in E(G)$, then $\max\{d(x), d(y)\} \geq 3$. Furthermore, if xy is not part of a flag then x or y is important.*

Proof. If $d(x) = d(y) = 2$, then let g' be a $(1, 1)$-coloring on $G - x - y$. Let $ux, vy \in E(G)$. We create a $(1, 1)$-coloring g on G by setting $g|_{G-x-y} = g'$, $g(x) \neq g'(u)$, and $g(y) \neq g'(v)$. This is a contradiction.

Without loss of generality, assume that $d(x) \geq 3$. If x is not important, then $d(x) = 3$ and x is in a flag. But then xy is a part of that flag. □

Recall that the potential of a vertex set S of a graph G' is $\rho_{G'}(S) = 7|S| - 5|E(G'[S])|$. It follows that $\rho_{K_1}(V(K_1)) = 7$, $\rho_{K_2}(V(K_2)) = 9$, $\rho_{K_3}(V(K_3)) = 6$, and $\rho_{K_4}(V(K_4)) = -2$. The minimum potential over all non-empty subsets of $V(G')$ is $\rho_{G'}$. Let G be a smallest graph, with respect to the order of graphs defined above, such that $\rho_G \geq 0$ and G is not $(1, 1)$-colorable. By minimality, every proper subgraph of G is $(1, 1)$-colorable.

Fact 7. *Let G' be a graph and $A, B, C \subseteq V(G')$ be such that $A \supset B$ and $A \cap C = \emptyset$. Then $\rho_{G'}(A - B) = \rho_{G'}(A) - \rho_{G'}(B) + 5|E_{G'}(A - B, B)|$ (equivalently, $\rho_{G'}(A \cup C) = \rho_{G'}(A) + \rho_{G'}(C) - 5|E_G(A, C)|$).*

We will use Fact 7 throughout the rest of the paper.

4. Sets with potential 2

Proposition 8. *Let $\varnothing \neq T \subsetneq V(G)$ be a set such that either* (i) $\rho(T) \in \{0, 1\}$, *or* (ii) $\rho(T) = 2$ *and for every $T' \subsetneq T$, $\rho(T') \geq 3$. Let F be a flag in G. Under these assumptions,*

(a) $\delta(G[T]) \geq 2$,
(b) *if $w \notin T$ then $|N(w) \cap T| \leq 1$,*
(c) *either $F \subset T$ or $F \cap T = \varnothing$, and*
(d) *every vertex u in T incident to an edge uv leaving T is important.*

Proof. Because $\rho(T) \leq 6$, we have that $|T| \geq 2$.

If $v \in T$ is such that $|N(v) \cap T| \leq 1$, then $\rho(T - v) \leq \rho(T) - 7 + 5$. If $\rho(T) \leq 1$, then this contradicts the assumption that $\rho_G \geq 0$. If $rho(T) = 2$, then this contradicts that $\rho(T - v) \geq 3$ under assumption (ii). This proves (a).

Let $w \notin T$ with $|N(w) \cap T| \geq 2$. Then $\rho(T + w) \leq \rho(T) + 7 - 10 \leq -1$, which contradicts that $\rho_G \geq 0$. This proves (b).

If $|T \cap F| = 1$, then $T \cup F$ has three more vertices and at least five more edges than T. So in this case $\rho(T \cup F) \leq \rho(T) - 5 \cdot 5 + 7 \cdot 3 \leq 2 - 25 + 21 = -2$, which is a contradiction. Similarly, if $|T \cap F| = 2$, then $\rho(T \cup F) \leq \rho(T) - 5 \cdot 4 + 7 \cdot 2 \leq 2 - 20 + 14 = -4$, and if $|T \cap F| = 3$, then $\rho(T \cup F) \leq \rho(T) - 5 \cdot 2 + 7 \leq 2 - 10 + 7 = -1$. This proves (c).

Let $uv \in E(G)$ with $u \in T$ and $v \notin T$. Due to (a), we have $d(u) \geq 3$. Therefore, if u is not important, then it must be in a flag. But if u is in a flag F, then by (c), F is contained in T, which implies that $d(u) \geq 4$. □

Proposition 9. *Let B be a super-flag with special vertices y_1 and y_2. There is no $S \subsetneq V(G)$ such that $\rho(S) \le 2$ and $B \prec G[S]$.*

Proof. By way of contradiction, let $S \subsetneq V(G)$ be a set with potential at most 2. Let $f : S \to \{\alpha, \beta\}$ be a (1, 1)-coloring of $G[S]$. Let S_α be the set of vertices in S colored α, and S_β be the set of vertices in S colored β. Let $N_\alpha = N(S_\alpha) \cap (V(G) - S)$ and $N_\beta = N(S_\beta) \cap (V(G) - S)$. Let G' be defined as follows:

$$V(G') = V(G) - S + V(B)$$

and

$$E(G') = E(G - S) + E(B) + \{uy_1 : u \in N_\alpha\} + \{vy_2 : v \in N_\beta\}.$$

Because $B \prec G[S]$ and the unimportant vertices of G are unimportant in G', it follows that $G' \prec G$. Suppose that there exists a $T \subseteq V(G')$ with $\rho(T) \le -1$. Because such a T was not in G, we have $T \cap B \neq \varnothing$. We may assume that $B \subset T$, because a quick examination of all possibilities will show that $\rho_{G'}(T \cup B) \le \rho_{G'}(T)$. But then $\rho_G(T - B + S) \le \rho_{G'}(T)$, which is a contradiction. Therefore G' has a coloring g'.

By the symmetry of the coloring we may assume that $g'(y_1) = \alpha$. By the configuration of a super-flag, it follows that $g'(y_2) = \beta$. Moreover, it must be the case that $g'(u) = \beta$ for all $u \in N_\alpha$ and $g'(v) = \alpha$ for all $v \in N_\beta$. Therefore the coloring g on G defined by $g = g' \cup f$ is a (1, 1)-coloring. □

Proposition 10. *Suppose that $\varnothing \neq S \subsetneq V(G)$ with potential at most 2 and $x \in S$. Under these conditions $|S| \ge 5$ and $E(S - x, V(G) - S) \neq \varnothing$.*

Proof. The smallest solution to the Diophantine relation $7|S| - 5|E(G[S])| \in \{0, 1, 2\}$ that is realized by a simple graph is $|S| = 5$, $|E(G[S])| = 7$.

By way of contradiction, let $S \subsetneq V(G)$ have potential at most 2 and $E(S - x, V(G) - S) = \emptyset$. Let $G' = G - (S - x) + F$, where F is a flag attached to x. Because $\delta(G) \ge 2$, and by Lemma 6, there must be at least two important vertices in S. Hence $G' \prec G$.

Suppose that there is a $T \subset V(G')$ with $\rho_{G'}(T) \le -1$. Because $\rho_G \ge 0$, it follows that $T \cap (F - x) \neq \varnothing$. We may assume that $F \subset T$, because a quick examination of all possibilities will show that $\rho_{G'}(T \cup F) \le \rho_{G'}(T)$. But then $\rho_G(T - (F - x) + (S - x)) \le -1 + 4 + 2 - 7$, which contradicts $\rho_G \ge 0$. By the minimality of G, we have that G' has a (1, 1)-coloring g'.

Without loss of generality, assume that $g'(x) = \alpha$. By construction, it must be the case that for all $u \in N(x) \cap (V(G) - S)$, we have $g'(u) = \beta$. Let f be a (1, 1)-coloring of $G[S]$ with $f(x) = \alpha$. Then $g = f \cup g'$ is a (1, 1)-coloring of G. □

Corollary 11. *If $\varnothing \neq S \subsetneq V(G)$ has potential at most 2, then S contains exactly two important vertices and no semi-important vertices. Furthermore, $|S| \le 6$.*

Proof. Let $S' \subset S$ be a minimal non-empty set with potential at most 2. By Propositions 8.d and 10, S' contains at least two important vertices. By Proposition 9, S contains at most two important vertices and four unimportant vertices. □

Proposition 12. *There is no $S \subsetneq V(G)$ such that $\rho(S) = 0$ and $|S| = 5$.*

Proof. Suppose that such an S exists. By Corollary 11, S has important vertices $\{x_1, x_2\}$ and unimportant vertices $\{z_1, z_2, z_3\}$. If $G[S]$ contains a flag, then $|E(G[S])| \le 6$ and $\rho(S) \ge 5$. Therefore each of $\{z_1, z_2, z_3\}$ has degree 2 and is in a triangle. By Lemma 6, the z_i's are not adjacent to each other. Therefore $N(z_1) = N(z_2) = N(z_3) = \{x_1, x_2\}$ and $x_1x_2 \in E(G)$.

Define $G' = G - S + F$, where the base of F is v, and $N(v) = (N(x_1) \cup N(x_2)) - S$. If there exists a $T \subset V(G')$ with $\rho_{G'}(T) \le -1$ then because $\rho_G \ge 0$, it follows that $F \cap T \neq \varnothing$. We may assume that $F \subset T$, because a quick examination of all possibilities will show that $\rho_{G'}(T \cup F) \le \rho_{G'}(T)$. But then $\rho_G(T - F + S) \le -1 - 3 + 0 \le -1$, which is a contradiction. By construction, $G' \prec G$. By minimality of G, there is a coloring g' of G'.

Without loss of generality, let $g'(v) = \alpha$. For all $u \in N(v) - F$, $g'(u) = \beta$. Create a coloring g of G as follows: Set $g|_{V(G)-S} = g'|_{V(G')-F}$, $g(x_1) = g(x_2) = \alpha$, and $g(z_1) = g(z_2) = g(z_3) = \beta$. This is a (1, 1)-coloring of G. □

Lemma 13. *If $\varnothing \neq S \subsetneq V(G)$ has potential at most 2, then $G[S]$ is a super-flag.*

Proof. Let $\{y_1, y_2\}$ be the important vertices in S described by Corollary 11.

By Corollary 11 and Proposition 12, we conclude that if $G[S]$ is not a super-flag then $|S| = 6$ and S contains four vertices of degree 2 contained in a triangle. By Lemma 6, the neighborhood of each unimportant vertex is $\{y_1, y_2\}$ and $y_1y_2 \in E(G)$. But then $G[S]$ contains nine edges and $\rho(S) \le -3$, a contradiction. So $G[S]$ is a super-flag. □

As a corollary to Lemma 13, if $G' \subsetneq G$, then $\rho_{G'} \ge 2$. Because adding one vertex and two edges reduces the potential by 3, we immediately have the following corollary.

Corollary 14. *Let $\emptyset \neq T \subset V(G)$ and $x_1, x_2 \notin T$. Let $G' = G - T + x'$, where $N(x') = \{x_1, x_2\}$. If x_1 and x_2 are not in the same super-flag, then $\rho_{G'} \ge 0$.*

5. Sets with potential 3

A 3-set is *standard* if it is a flag or has at least $|V(G)|-1$ vertices. The goal of this section is to prove that all 3-sets of G are standard. We will do this through a sequence of smaller statements on a nonstandard 3-set X with fewest vertices in G. Let X_0 denote the set of vertices in X that have neighbors outside of X.

Proposition 15. *$G[X]$ is connected and $\delta(G[X]) \geq 2$.*

Proof. If $G_1, \dots, G_k$ are connected components of $G[X]$ and $k \geq 2$, then for some i, $\rho_G(V(G_i)) \leq \lfloor 3/k \rfloor \leq 1$, a contradiction to Lemma 13. If $x \in X$ and $d(x) \leq 1$, then $\rho_G(X-x) \leq 3-7+5 = 1$, a contradiction to Lemma 13. □

Proposition 16. *No vertex outside X has more than one neighbor in X.*

Proof. Suppose that $z \in V(G)-X$ has at least two neighbors in X. Let $X' = X+z$. Since X is nonstandard, $X' \neq V(G)$, but $\rho(X') \leq \rho(X)+7-2\cdot 5 = 0$, a contradiction to Lemma 13. □

Corollary 17. *Each $x \in X_0$ is important.*

Proof. By Proposition 15 and the definition of X_0, $d(x) \geq 3$. Suppose that x is a 3-vertex in a flag F. Then exactly one vertex of F is outside of X, and this vertex has at least two neighbors in F, a contradiction to Proposition 16. □

Proposition 18. *$G[X]$ has no $(1,1)$-coloring in which all vertices in X_0 have the same color.*

Proof. Suppose that $G[X]$ has a $(1,1)$-coloring f such that $f(x) = 1$ for each $x \in X_0$. Let G' be obtained from $G-X$ by adding a new flag F with base x_0 and adding an edge zx_0 for each $z \in V(G)-X$ that had a neighbor in X. Since the potential of F is the same as that of X in G, we have $\rho_{G'}(A) \geq 0$ for every $A \subseteq V(G')$. By Lemma 6 and Proposition 15, X had more than one important vertex. And any important vertex in $V(G')-x_0$ was important in G. So, G' is smaller than G. Thus, G' has a $(1,1)$-coloring g. We may assume that $g(x_0) = 1$. Since x_0 has a neighbor of color 1 in F, it has no such neighbors in $V(G')-F$. So $g|_{V(G)-X} \cup f$ is a $(1,1)$-coloring of G. □

Proposition 19. *Each $x \in X_0$ has a neighbor in $V(G)$ that is either important or semi-important.*

Proof. Suppose that a vertex $z \in V(G)-X$ adjacent to $x \in X_0$ is unimportant. Since $\delta(G) \geq 2$, by Proposition 16 there is a $w \in N(x) \cap N(z) \cap (V(G)-X)$. By Lemma 6, w is important unless w is in a flag F. By Proposition 16, $F \cap X = \{x\}$. This is a contradiction, because $\rho(X+F) \leq 3+21-25 = -1$. □

Proposition 20. *$|X_0| \geq 3$.*

Proof. By Proposition 18, $|X_0| \geq 2$ and if $X_0 = \{x_1, x_2\}$, then $f(x_1) \neq f(x_2)$ for every $(1,1)$-coloring f of $G[X]$. By definition, $\{x_1, x_2\}$ is a separating set. If x_1 and x_2 are the special vertices of some super-flag H of G, then H is also a separating set. But that contradicts Lemma 5. Therefore vertices x_1 and x_2 are not in the same super-flag of G.

Let G_0 be obtained from $G[X]$ by adding a vertex y adjacent to x_1 and x_2. Adding a vertex of degree 2 to a subgraph decreases its potential by 3; therefore by Lemma 13 the potential of every subgraph of $G[X]$ is still non-negative. We claim that G_0 is smaller than G. Indeed, by Proposition 19, each of x_1 and x_2 has a neighbor in $V(G)-X$ that is not unimportant, and by Proposition 16 they are different. But y is either semi-important or unimportant. Thus, G_0 is smaller than G and hence has a $(1,1)$-coloring f. By Proposition 18, $f(x_1) \neq f(x_2)$. By symmetry we may assume that $f(x_1) = f(y) = 1$ and $f(x_2) = 2$. Let G_1 be obtained from $G-(X-x_1-x_2)$ by adding an edge x_1x_2 (if it is not in $E(G)$) and placing a flag F on x_2. Note that the potential of $G_1[F]$ is 3 and that of $G_1[F+x_1]$ is $3+7-5 = 5$. So, G_1 has no sets of negative potential. Thus if G_1 is smaller than G, then it has a $(1,1)$-coloring g. Since x_2 has a neighbor of its color in F, $g(x_1) \neq g(x_2)$. Therefore, renaming colors such that $g(x_1) = 1$ and $g(x_2) = 2$, we would have that $f|_X \cup g|_{V(G)-X}$ is a $(1,1)$-coloring of G. Hence G_1 is not smaller than G. In this case, all vertices in $X-x_1-x_2$ are unimportant and $|X| \leq 5$. Since no unimportant vertex can be an intermediate vertex in a shortest path and $G[X]$ is connected, $x_1x_2 \in E(G)$. So $\rho_G(\{x_1, x_2\}) = 14-5 = 9$. Adding a flag to a set decreases the potential by 4, adding a 2-vertex decreases it by 3, and adding two adjacent 2-vertices generates a subgraph forbidden by Lemma 6. Because $\rho_G(X) = 3$, it follows that $X - \{x_1, x_2\}$ is two 2-vertices adjacent to x_1 and x_2. But then $G[X]$ does have a $(1,1)$-coloring f with $f(x_1) = f(x_2)$, a contradiction to Proposition 18. □

Proposition 21. *In every $(1,1)$-coloring of $G[X]$, each vertex in X_0 has neighbors of both colors.*

Proof. Suppose that in $(1,1)$-coloring f of $G[X]$, a vertex $x \in X_0$ has no neighbors of color 1. Then by Proposition 15, $f(x) = 1$. By Proposition 19, x has a neighbor $z \in V(G)-X$ that is not unimportant. Let G' be obtained from $G-X$ as follows:

(a) add a flag F attached to z;
(b) add a super-flag Y with special vertices y_1 and y_2;
(c) for each $v \in V(G)-X-z$ that is adjacent to a vertex of color i in f, join v by an edge to y_i.

By Proposition 16, there will be no confusion with (c).

Case 1: G' has a $(1, 1)$-coloring g. We claim that $f \cup g$ is a $(1, 1)$-coloring. Indeed, if $g(z) = 2$, then this follows from the construction of G'. Moreover, if $g(z) = 1$, then because of F, z has no neighbor of color 1 in $G' - (F - z)$. So, even though vertices x and z of color 1 are adjacent to each other, they do not have other neighbors of color 1 in G.

Case 2: G' has no $(1, 1)$-colorings. By Proposition 20, X contains at least three important vertices. So, since z was not unimportant, G' is smaller than G. Thus G' has a set Z with $\rho_{G'}(Z) \le -1$.

Case 2.1: $Z \cap Y \neq \varnothing$. The subgraph of a super-flag with smallest potential is the whole super-flag, so we may assume that $Y \subseteq Z$. If $z \notin Z$, then $F \cap Z = \varnothing$, and

$$\rho_G(X \cup (Z - Y)) \le \rho_{G'}(Z) - \rho_{G'}(Y) + \rho_G(X) \le -1 - 2 + 3 = 0,$$

a contradiction to Lemma 13. So $z \in Z$. Then $\rho_{G'-(F-z)}(Z - (F - z)) \le -1 - 21 + 25 = 3$. So, because of the edge xz,

$$\rho_G(X \cup (Z - (F - z) - Y)) \le \rho_{G'-(F-z)}(Z - (F - z)) - \rho_{G'}(Y) + \rho_G(X) - 5|E(X, \{z\})| \le 3 - 2 + 3 - 5 = -1,$$

which contradicts the assumption that $\rho_G \ge 0$.

Case 2.2: $Z \cap Y = \varnothing$. Then $z \in Z$. By the same calculation as in Case 2.1, $\rho_G(Z - (F - z)) \le 3$ and

$$\rho_G\,(X \cup (Z - (F - z))) \le \rho_G(X) + \rho_G(Z - (F - z)) - 5|E(X, Z - (F - z))| \le 3 + 3 - 5 = 1.$$

By Lemma 13, we get $X \cup (Z - (F - z)) = V(G)$. But since there are at least two edges between X and $V(G) - X$, we then have $\rho_G(V(G)) \le 3 + 3 - 10 = -4$, which contradicts the assumption that $\rho_G \ge 0$. □

Proposition 22. *For every $x, x' \in X_0$ such that x and x' are not in the same super-flag of G, there is a $(1, 1)$-coloring f of $G[X]$ such that $f(x) = f(x')$.*

Proof. Let G'' be obtained from $G[X]$ by adding a new vertex v adjacent to x and x'. By Corollary 14, $\rho_{G''} \ge 0$. By Proposition 19, G'' is smaller than G. So by the minimality of G, G'' has a $(1, 1)$-coloring f. By Proposition 21, both x and x' have neighbors of both colors in $G'' - v$. Thus $f(v) \neq f(x)$ and $f(v) \neq f(x')$. It follows that $f(x) = f(x')$. □

Lemma 23. *Graph G has no nonstandard 3-sets.*

Proof. Suppose that X is a minimum size nonstandard 3-set and X_0 is the set of vertices in X adjacent to $V(G) - X$. Since $G[X]$ is smaller than G, it has a $(1, 1)$-coloring f_1. Let X_1 be the set of vertices $x \in X_0$ with $f_1(x) = 1$ and $X_2 = X_0 - X_1$. By changing the names of colors if needed, we may assume that $|X_1| \ge |X_2|$. Since $|X_0| \ge 3$, we have $|X_1| \ge 2$. By Proposition 18, $X_2 \neq \varnothing$. Let $x, x' \in X_1$ and $y \in X_2$. Since each super-flag has only two important vertices, y and x are not in the same super-flag or y and x' are not in the same super-flag. So by Proposition 22, there is a $(1, 1)$-coloring f_2 of $G[X]$ such that $f_2(y) \in \{f_2(x), f_2(x')\}$. Thus, we have proved that

$$\text{there is a } (1, 1)\text{-coloring } f_2 \text{ of } G[X] \text{ distinct from } f_1. \quad (1)$$

Let $Y_1 = \{z \in X_0 : f_1(z) = f_2(z)\}$ and $Y_2 = X_0 - Y_1$. By switching the names of the colors in f_2, we can achieve that

$$|Y_1| \ge |Y_2|. \quad (2)$$

Case 1: All vertices in Y_2 have the same color in f_1. Let G_1 be obtained from $G - X$ by

(a) adding a flag F with base y_0,
(b) adding a copy H of a super-flag disjoint from F with special vertices y_1 and y_2,
(c) adding the edge (z, y_0) for every $z \in V(G) - X$ that is adjacent to some $w \in Y_2$, and
(d) adding the edge (z, y_{3-j}) for every $z \in V(G) - X$ that is adjacent to some $w \in Y_1$ with $f_1(w) = j$, for $j = 1, 2$.

Suppose first that G_1 has a $(1, 1)$-coloring g. We may assume that $g(y_1) = 1$. Then $g \cup f_1$ or $g \cup f_2$ is a $(1, 1)$-coloring of G.

So, G_1 has no $(1, 1)$-colorings. Suppose now that $\rho_{G_1}(Z) \le -1$ for some $Z \subseteq V(G_1)$ and Z has the smallest potential in G_1. Then Z either contains F or is disjoint from F, and similarly either contains H or is disjoint from H. By the construction of G_1, it contains F or H.

Case 1.1: $F \subset Z$ and $H \subset Z$. Then the potential of $(Z - F - H) \cup X$ in G is at most

$$\rho_{G_1}(Z) - \rho_{G_1}(F) - \rho_{G_1}(H) + \rho_G(X) \le -1 - 3 - 2 + 3 \le -3,$$

which contradicts the assumption that $\rho_G \ge 0$.

Case 1.2: $F \subset Z$ and $H \cap Z = \varnothing$. Following the calculation of Case 1.1, the potential of $(Z - F) \cup X$ in G is at most $-1 - 3 + 3 \le -1$, a contradiction.

Case 1.3: $F \cap Z = \varnothing$ and $H \subset Z$. Following the calculation of Case 1.1, the potential of $(Z - H) \cup X$ in G is at most $\rho_{G_1}(Z) - 2 + 3 \le 0$. If $(Z - H) \cup X \neq V(G)$, this contradicts Lemma 13. If $(Z - H) \cup X = V(G)$, then we did not take into account the contribution of edges connecting Y_2 with $V(G) - X$. So, in this case, the potential of $(Z - H) \cup X = V(G)$ is at most

$$\rho_{G_1}(Z) - \rho_{G_1}(H) + \rho_G(X) - 5|E(Y_2, V(G) - X)| \le -1 - 2 + 3 - 5 \le -5,$$

a contradiction.

Thus, G_1 satisfies the conditions of the theorem and has no $(1, 1)$-colorings. By the choice of G, it cannot be smaller than G. Let us check how this may happen. By Corollary 17, every vertex in X_0 is important. Since we have added to $V(G)-X$ at most three important vertices, we conclude that $X_0 = \{x, x', y\}$ and every other vertex of X is unimportant. By Proposition 15, $G[X]$ is connected and because no unimportant vertex can be an intermediate vertex in a shortest x, x'-path (or x, y-path or x', y-path), it follows that $G[X_0]$ is connected. So, $G[X_0]$ has either two or three edges.

Case 1.A: $G[X_0]$ has three edges. Then $\rho(X_0) = 6$. Since adding a 2-vertex decreases the potential by 3 and adding a flag decreases it by 4, the only way to get $\rho(X) = 3$ is that we obtain X by adding one 2-vertex. In this case, $G[X] = K_4 - e$. If we color the vertices of degree 3 in $G[X]$ with color 1, and the vertices of degree 2 in $G[X]$ with color 2, then the latter will have only neighbors of color 1, a contradiction to Proposition 21.

Case 1.B: $G[X_0]$ has two edges. This means that $G[X_0]$ is a path of length 2, say (x_1, x_2, x_3), where $\{x_1, x_2, x_3\} = \{x, x', y\}$. Then $\rho(X_0) = 11$ and $\rho(X) - \rho(X_0) = -8$. Since the only way to express 8 as a sum of 3's and 4's is $8 = 4 + 4$, $G[X]$ is obtained from $G[X_0]$ by adding two flags. Then there is a $(1, 1)$-coloring h of $G[X]$ such that $h(x_1) = h(x_3) \neq h(x_2)$. So, the vertex in X_0 not contained in a flag will have neighbors of only one color, a contradiction to Proposition 21.

Case 2: The set Y_2 has vertices of both colors in f_1. Then $2 \leq |Y_2| \leq |Y_1|$. The proof almost repeats the one for Case 1 with more case analysis at the end. Let G_2 be obtained from $G - X$ by

(a) adding two disjoint copies H_1 and H_2 of a super-flag with special vertices $y_{1,1}$ and $y_{1,2}$ in H_1 and $y_{2,1}$ and $y_{2,2}$ in H_2, and
(b) adding the edge $(z, y_{i,3-j})$ for every $z \in V(G) - X$ that is adjacent to some $w \in Y_i$ with $f_1(w) = j$, for all $i, j \in \{1, 2\}$.

Suppose first that G_2 has a $(1, 1)$-coloring g. We may assume that $g(y_{1,1}) = 1$. Then $g \cup f_1$ or $g \cup f_2$ is a $(1, 1)$-coloring of G.

So, G_2 has no $(1, 1)$-colorings. Suppose now that $\rho_{G_1}(Z) \leq -1$ for some $Z \subseteq V(G_2)$ and Z has the smallest potential in G_2. Then for $i = 1, 2$, Z either contains H_i or is disjoint from H_i. By construction, Z contains H_1 or H_2.

Case 2.1: $H_1 \subset Z$ and $H_2 \subset Z$. Then the potential of $(Z - H_1 - H_2) \cup X$ in G is at most

$$\rho_{G_2}(Z) - \rho_{G_2}(H_1) - \rho_{G_2}(H_2) + \rho_G(X) \leq -1 - 2 - 2 + 3 \leq -2,$$

which contradicts the assumption that $\rho_G \geq 0$.

Case 2.2: $H_1 \cap Z = \varnothing$ and $H_2 \subset Z$. Following the calculation of Case 2.1, the potential of $(Z - H_2) \cup X$ in G is at most $-1-2+3 \leq 0$. If $(Z-H_2)\cup X \neq V(G)$, this contradicts Lemma 13. If $(Z-H_2)\cup X = V(G)$, then we did not take into account the contribution of edges connecting Y_1 with $V(G) - X$. So, in this case, the potential of $(Z - H_2) \cup X = V(G)$ is at most

$$\rho_{G_2}(Z) - \rho_{G_2}(H_2) + \rho_G(X) - 5|E(Y_1, V(G) - X)| \leq -1 - 2 + 3 - 5 \leq -5,$$

a contradiction.

Case 2.3: $H_2 \cap Z = \varnothing$ and $H_1 \subset Z$. The case is symmetric to Case 2.2.

Thus, G_2 satisfies the conditions of the theorem and has no $(1, 1)$-colorings. By the minimality of G, G_2 is not smaller than G. Let us check how this may happen. We have added to $V(G) - X$ only four important vertices. Using the same logic as at the end of Case 1, we conclude that $|X_0| = 4$, every other vertex of X is unimportant, and $G[X_0]$ is connected. So, $G[X_0]$ has three, four or five edges.

Case 2.A: $G[X_0]$ has five edges. Then $\rho(X_0) = 3$ and hence $X = X_0$. In this case, $G[X] = G[X_0] = K_4 - e$. So we get a contradiction to Proposition 21 exactly as at the end of Case 1.A.

Case 2.B: $G[X_0]$ has four edges. Then $\rho(X_0) - \rho(X) = 5$ and there is no way to express 5 as a sum of 4's and 3's.

Case 2.C: $G[X_0]$ has three edges. Then $\rho(X_0) - \rho(X) = 10$. Since the only way to express 10 as a sum of 3's and 4's is $10 = 4 + 3 + 3$, $G[X]$ is obtained from $G[X_0]$ by adding one flag and two unimportant 2-vertices. Note that $G[X_0]$ is either $K_{1,3}$ or P_4. By Proposition 15, if w is a leaf of $G[X_0]$, then

w belongs to a flag or is adjacent to an unimportant 2-vertex. (3)

Case 2.C.1: $G[X_0] = K_{1,3}$. Let x_0 be the vertex of degree 3 in $G[X_0]$ and x_1, x_2, x_3 be the remaining vertices in X_0. By (3) we may, up to reordering of x_1, x_2, x_3, assume that x_1 belongs to a flag, one unimportant 2-vertex is adjacent to x_0 and x_2, and one unimportant 2-vertex is adjacent to x_0 and x_2. Then there is a $(1, 1)$-coloring f of $G[X]$ such that all neighbors of x_0 have the same color, a contradiction to Proposition 18.

Case 2.C.2: $G[X_0] = P_4$. We may assume that this path is (x_1, x_2, x_3, x_4). By symmetry, we may assume that the vertex in the flag is either x_1 or x_2. Suppose first that it is x_2. Then by (3), one unimportant 2-vertex, say v_1, is adjacent to x_1 and x_2, and the other unimportant 2-vertex, say v_2, is adjacent to x_3 and x_4. Then we let $f(x_1) = f(v_1) = f(x_3) = f(v_2) = 1$ and $f(x_2) = f(x_4) = 2$. In this coloring, both neighbors of x_4 have color 1, a contradiction to Proposition 21.

So, x_1 belongs to the flag. By (3), there is an unimportant 2-vertex v_1 adjacent to x_3 and x_4. Let v_2 be the other unimportant 2-vertex. Let $f(x_1) = f(x_3) = 1$ and $f(x_2) = f(x_4) = f(v_1) = f(v_2) = 2$. Then either this coloring extends to a $(1, 1)$-coloring of $G[X]$ (by coloring the vertices in the flag), or v_2 is adjacent to x_3 and x_4. In the former case, all neighbors of x_3 are colored with 2, a contradiction to Proposition 21. In the latter case, we recolor v_2 with 1, and get a $(1, 1)$-coloring of $G[X]$ in which both neighbors of x_2 are colored with 1, again a contradiction to Proposition 21. □

Attaching a flag to a vertex reduces the potential by 4. Therefore we immediately get the following corollary from Lemmas 13 and 23.

Corollary 24. *Let $T \subset V(G)$ and $w \notin T$. Let $G' = G - T + F$, where F is a flag attached at w. If $|T| \geq 2$ and w is not in a flag or super-flag, then $\rho_{G'} \geq 0$.*

6. Reducible configurations

For every super-flag, exactly one of the two special vertices is in a flag. We will call the special vertex not in a flag the *secondary base* of the super-flag. We will show in Lemma 26 that every secondary base has degree at least 4.

Lemma 25. *Let each of u and v be a base of a flag or a secondary base of a super-flag. If u and v are not special vertices of the same super-flag, then the distance between u and v is at least 3.*

Proof. Let A_1 be the flag or super-flag attached at u and A_2 be the flag or super-flag attached at v. Let P be a shortest path between u and v. If $|V(P)| < 3$ then $\rho_G(A_1 + P + A_2) \leq 3 + 3 - 3$. Because $A_1 + P + A_2$ is neither a super-flag nor a flag, and by Lemmas 13 and 23, $A_1 + P + A_2 = V(G)$. But then G has a $(1, 1)$-coloring. □

Lemma 26. *Let $v \in V(G)$ be a vertex with degree 3. If v is adjacent to a vertex of degree 2, then v is in a flag.*

Proof. By way of contradiction, let $N(v) = \{x, y, z\}$ and v not be in a flag. Note that v is important and every neighbor of v with degree at least 3 is important.

Case 1: v is in a super-flag. If v is in a super-flag, has degree 3, and is not in a flag, then v is the secondary base of that super-flag. Without loss of generality, let x be outside of that super-flag, z be the other special vertex of the super-flag, and $N(y) = \{v, z\}$. Let F be the flag attached at z.

Case 1.1: $d(x) \geq 3$. Let $G' = G - y + y'$, where $N(y') = \{x, z\}$. Note that v is important in G, but v and y' are at most semi-important in G'. Therefore $G' \prec G$. By Corollary 14, $\rho_{G'} \geq 0$. Therefore we may find a function $g' : G' \to \{\alpha, \beta\}$ such that g' is a $(1, 1)$-coloring of G'.

Without loss of generality, assume that $g'(z) = \alpha$. Because F is attached to z in G', it follows that $g'(v) = g'(y') = \beta$. From this, we deduce that $g'(x) = \alpha$. We may generate a $(1, 1)$-coloring g of G by setting $g|_{V(G)-y} = g'_{V(G')-y'}$ and $g(y) = \beta$.

Case 1.2: $N(x) = \{v, a\}$. By Lemma 6, a is important. Let $S = F + v + y$; in other words, $G[S]$ is the super-flag containing v. Let $G' = G - \{v, x, y\} + F'$, where F' is a flag attached at a. Because v and a were important in G, $G' \prec G$.

By Lemma 25, a is not in a flag or super-flag. So by Corollary 24, $\rho_{G'} \geq 0$. By the minimality of G, there exists a $(1, 1)$-coloring g' of G'. We create a $(1, 1)$-coloring g of G by setting $g|_{G-v-x-y} = g'|_{G'-(F'-a)}$, $g(x) = g'(z)$, and $g(v) = g(y) \neq g'(z)$, which is a contradiction.

For Cases 2–4, assume that v is not in a super-flag.

Case 2: v is adjacent to exactly two neighbors of degree 2. Let $N(x) = \{v, a\}$ and $N(y) = \{v, b\}$. By Lemma 6, a and b are important. Without loss of generality, assume that $d(a) \geq d(b)$.

Case 2.1: $a \neq b$ and $b \neq z$. Let $G' = G - y + y'$, where $N(y') = \{v, a\}$. We claim that G' is smaller than G. Because a and b are important in G, they cannot be more important in G'. Suppose that y' is more important than y. Then y was in a triangle and $b = z$, which contradicts the assumption. Therefore G' is smaller by the condition on the degrees. By Corollary 14, $\rho_{G'} \geq 0$. By minimality of G, there exists a $(1, 1)$-coloring g' of G'.

Without loss of generality, let $g'(a) = \alpha$. Let g be a coloring of G where $g|_{V(G)-x-y-v} = g'|_{V(G)-x-y'-v}$.

- If $g'(z) = \beta$, then color $g(x) = \beta$, $g(v) = \alpha$, and $g(y) \neq g(b)$.
- If $g'(z) = \alpha$ and $g'(v) = \beta$ then either $g'(x) = \alpha$ or $g'(y') = \alpha$. Furthermore, for all $u \in N_{G'}(a) - \{x, y'\}$, $g'(u) = \beta$. Color $g(x) = \alpha$, $g(v) = \beta$, and $g(y) \neq g(b)$.
- If $g'(z) = g'(v) = g'(b) = \alpha$ then color $g(x) = g(y) = \beta$ and $g'(v) = \alpha$.
- If $g'(z) = g'(v) = \alpha$ and $g'(b) = \beta$, then color $g(x) = g(v) = \beta$ and $g(y) = \alpha$.

The above assumptions exhaust all possibilities for g'. Moreover, each provides a $(1, 1)$-coloring of G.

Case 2.2: $a \neq b$ and $b = z$. Let $G' = G - \{v, x, y\} + F$, where F is a flag attached at z. If z was in a flag F' in G, then F', v, b form a super-flag containing v, which contradicts the assumption that v is not in a super-flag. So by Corollary 24, $\rho_{G'} \geq 0$. Because v and z are important in G, we have $G' \prec G$.

By minimality of G, there exists a $(1, 1)$-coloring g' of G'. Note that if $w \in N_{G'}(z) - F$ then $g'(w) \neq g'(z)$. Let g be a coloring of G where $g|_{V(G)-x-y-v} = g'|_{V(G)-(F-z)}$, $g(x) = g(y) \neq g'(a)$, and $g(v) = g'(a)$. Either $g(z) = g(v)$ or $g(z) = g(y)$, but not both. Therefore there is only one neighbor with the same color for each vertex in g.

Case 2.3: $a = b$. Since v is not in a flag, $z \neq a$. Let $G' = G - y + y'$, where $N(y') = \{v, z\}$. Because y was half important and y' is unimportant, G' is smaller than G. By Corollary 14, $\rho_{G'} \geq 0$. By minimality of G, there exists a $(1, 1)$-coloring $g' : V(G') \to \{\alpha, \beta\}$.

Without loss of generality, let $g'(z) = \alpha$. Thus $g'(x) = \alpha$ or $\alpha \in \{g'(v), g'(y')\}$. If neither of these statements is true then $g'(v) = g'(x) = g'(y') = \beta$, and v is adjacent to two vertices with the same color in g'.

Let g be a coloring on G where $g|_{V(G)-x-y-v} = g'|_{V(G)-x-y'-v}$ and

- if $g'(x) = \alpha$ then $g(x) = \alpha$, $g(y) \neq g(a)$, and $g(v) = \beta$, or
- otherwise if $\alpha \in \{g'(v), g'(y')\}$, then $g(x) = g(y) \neq g'(a)$, and $g(v) = g'(a)$.

Then g is a $(1, 1)$-coloring.

Case 3: v is adjacent to exactly one vertex of degree 2. Let $N(x) = \{v, a\}$.

Case 3.1: $a \notin \{y, z\}$. Then x is semi-important. Let $G' = G - x + x'$, where $N(x') = \{y, z\}$. Because x' and v are at most semi-important in G', it follows that G' is smaller than G. If y and z are in the same super-flag Y, then $\rho_G(Y + v) \leq -1$, which is a contradiction. So by Corollary 24, $\rho_{G'} \geq 0$. By minimality of G, there exists a $(1, 1)$-coloring g' of G'. Without loss of generality, let $g'(y) = \alpha$.

If $g'(z) = \alpha$, then create a coloring $g|_{V(G)-x-v} = g'|_{V(G)-x'-v}$, $g(v) = \beta$, and $g(x) \neq g'(a)$. This is a $(1, 1)$-coloring of G.

So we may assume that $g'(z) = \beta$. Because g' is a $(1, 1)$-coloring, it follows that $g'(v) \neq g'(x')$, or else y or z will have two neighbors with the same color. Without loss of generality we may assume that $g'(v) = \beta$ and $g'(x') = \alpha$. Therefore all other neighbors of y have color β and all other neighbors of z have color α. We color G with coloring $g|_{V(G)-x-v} = g'|_{V(G)-x'-v}$, $g(x) \neq g'(a)$, and $g(v) = g'(a)$. Note that $g(v)$ may be the same as $g(y)$ or $g(z)$, but it will not be the same as both. Hence, g is a $(1, 1)$-coloring of G.

Case 3.2: $a = y$. Let $G' = G - x - v + F$, where F is a flag attached at y. Then $G' \prec G$. If y was in a flag F' in G, then F', v, x form a super-flag containing v, which contradicts the assumption that v is not in a super-flag. So by Corollary 24, $\rho_{G'} \geq 0$. By minimality of G, there exists a $(1, 1)$-coloring g' of G'.

Note that if $w \in N_{G'}(y) - F$ then $g'(w) \neq g'(y)$. Let g be a coloring of G where $g|_{V(G)-x-v} = g'|_{V(G)-(F-y)}$, $g(x) = g'(z)$, and $g(v) \neq g'(z)$. Either $g(y) = g(v)$ or $g(y) = g(x)$, but not both. Therefore each vertex in G has at most one neighbor with the same color in g, which is a contradiction because G is not $(1, 1)$-colorable.

Case 4: v is adjacent to three vertices of degree 2. Let $G' = G - v$. Since G' is a proper subgraph, it has a coloring g'. In G', x, y, and z all have degree 1. Without loss of generality, we may assume that each of them has no neighbors with the same color as themselves. Extend the coloring on G' to a coloring on G by coloring v the color that appears the least often in the list $(g'(x), g'(y), g'(z))$. Because some color appears at most once and that vertex has no other neighbors with the same color, this is a $(1, 1)$-coloring. □

Corollary 27. *If y is a secondary base of a super-flag in G, then $d(y) \geq 4$.*

Proof. By construction, each special vertex of a super-flag is adjacent to a vertex of degree 2. By Lemma 6, $d(y) \geq 3$. By Lemma 26, $d(y) \neq 3$. □

Lemma 28. *Let F be a flag in G. Then $|E(F, V(G) - F)| \geq 3$.*

Clearly $|E(F, V(G) - F)| \geq 2$, or else G would have a separating edge. In order to prove the above lemma, we will need the following result.

Proposition 29. *If F is a flag in G with base v and $N(v) - F = \{x, y\}$, then both x and y are important.*

Proof. By way of contradiction, assume that x is semi-important or unimportant. By Lemma 25, $d(x) \leq 2$. If x is in a triangle, then v and y are the special vertices of a super-flag and Proposition 10 is contradicted. Therefore x is semi-important and $N(x) = \{v, a\}$. Let $G' = G - F - x + F'$, where F' is a flag attached at a.

By Lemma 25, a is not in a flag or a super-flag. So by Corollary 24, $\rho_{G'} \geq 0$. In G, v is important and x is semi-important. No vertex is more important in G' than in G, so $G' \prec G$. By minimality of G, there exists a $(1, 1)$-coloring g' of G'.

We have $g'(a) \neq g'(z)$ for all $z \in (N_{G'}(a) - F')$. Then G has a $(1, 1)$-coloring g, where $g|_{G-F-x} = g'|_{G'-(F'-a)}$, $g(v) \neq g'(y)$ and $g(x) = g'(y)$. □

Proof of Lemma 28. Let v be in flag F and $N(v) - F = \{x, y\}$.

Case 1: $xy \notin E(G)$ and $N(x) \cap N(y) = \{v\}$. Let $G' = G - F - x - y + z$, where $N(z) = (N(y) \cup N(x)) - v$. If $u \in V(G') - z$ is important in G', then u is important in G. Because both x and y are important in G, we have $G' \prec G$.

If $T \subset V(G')$ such that $\rho_{G'}(T) \leq -1$, then $z \in T$. It follows that

$$\rho_G(T - z + x + y + F) = \rho_{G'}(T) + 7(6 - 1) - 5(7) = \rho_{G'}(T) \leq -1,$$

which contradicts the assumption that $\rho_G \geq 0$.

Therefore G' has a $(1, 1)$-coloring, g'. We can create a $(1, 1)$-coloring of g by setting $g|_{G-F-x-y} = g'|_{G'-z}$, $g(x) = g(y) = g'(z)$, and $g(v) \neq g'(z)$.

Case 2: $xy \in E(G)$ or there exists a w such that $w \in (N(x) \cap N(y)) - v$. Let $G' = G - F - x - y + z + F'$, where $N(z) = (N(y) \cup N(x)) - v$ and F' is a flag attached at z. If $u \in V(G') - z$ is important in G', then u is important in G. Because both x and y are important in G, we have $G' \prec G$.

If $T \subset V(G')$ such that $\rho_{G'}(T) \leq -1$, then $z \in T$. Because of edge xy or because $|\{wx, wy\}| = |\{wz\}| + 1$, we get one extra edge over Case 1, and so

$$\rho_G(T - F' + x + y + F) \leq \rho_{G'}(T) + 7(6 - 4) - 5(7 + 1 - 5) \leq -2,$$

which contradicts the assumption that $\rho_G \geq 0$.

Therefore G' has a $(1, 1)$-coloring, g'. Furthermore, for all $u \in N_{G'}(z) - F'$, $g'(u) \neq g'(z)$. We can create a $(1, 1)$-coloring of g by setting $g|_{G-F-x-y} = g'|_{G'-F'}$, $g(x) = g(y) = g'(z)$, and $g(v) \neq g'(z)$. □

Lemma 30. *If $v \in V(G)$ such that $N(v) = \{u_1, u_2, u_3, u_4\}$ and $N(u_i) = \{x_i, v\}$ for all i, then x_i is in a flag or a super-flag for all i.*

Proof. Without loss of generality, let v, u_i and x_j be as above, and assume that x_1 is not in flag or a super-flag. By Lemma 6, each x_j is important. Let $G' = G - \{v, u_1, u_2, u_3, u_4\} + F$, where F is a flag attached at x_1. By construction, $G' \prec G$. By Corollary 24, $\rho_{G'} \geq 0$. By minimality of G, there is a $(1, 1)$-coloring g' of G'.

Because of the flag F, for all $w \in N_G(x_1) - u_1$, we have $g'(w) \neq g'(x_1)$. We construct a $(1, 1)$-coloring g of G as follows:

- Set $g|_{G-\{v,u_1,u_2,u_3,u_4\}} = g'|_{G'-(F-x_1)}$.
- Set $g(u_i) \neq g'(x_i)$ for $i \in \{2, 3, 4\}$.
- Set $g(v)$ equal to the color that appears the least in the list $(g(u_2), g(u_3), g(u_4))$.
- Set $g(u_1) \neq g(v)$.

Then g is a $(1, 1)$-coloring of G, which is a contradiction. □

7. Proof of the theorem

By assumption, on G, we have

$$\sum_{v \in V(G)} (5d(v) - 14) \leq 0. \tag{4}$$

The *initial charge* of each vertex v of G is $\mu(v) = 5d(v) - 14$, and the *final charge* $\mu^*(v)$ is determined by applying the following rules:

R1. Every 2-vertex in a flag gets charge 4 from the base of the flag.
R2. Let x be the base of a flag F or the secondary base of a super-flag H. Every vertex adjacent to x *outside* of F or H gets charge 2.5 from x.
R3. Every 2-vertex u adjacent to a base x of a flag F, where u is not in F, gets from its other neighbor charge 1.5.
R4. Every 2-vertex u adjacent to a secondary base x of a super-flag H, where u is not in H, gets from its other neighbor charge 1.5.
R5. Every 2-vertex not adjacent to a base of a flag or a secondary base of a super-flag gets charge 2 from each of its neighbors.

By Lemma 25, the application of Rules 2–4 is well-defined. By Lemma 6, the application of Rule 5 is well-defined.

Lemma 31. *For every $v \in V(G)$, $\mu^*(v) \geq 0$. Moreover, if $d(v) \notin \{2, 4\}$, then $\mu^*(v) > 0$.*

Proof. Recall that $\delta(G) \geq 2$. Note that by Lemma 28, each base of a flag has degree at least 6. If x is the base of a flag F, then by R1, it gives charge 8 to the two 2-vertices in F and charge $2.5(d(x) - 3)$ to the neighbors outside of F. So

$$\mu^*(x) = 5d(x) - 14 - 8 - 2.5(d(x) - 3) = 2.5d(x) - 14.5 \geq 2.5 \cdot 6 - 14.5 = 0.5.$$

A 3-vertex v in a flag does nothing to any other vertex, so $\mu^*(v) = \mu(v) = 5d(v) - 14 = 1$. A 2-vertex v in a flag receives charge 4 from the base of the flag, so $\mu^*(v) = 5 \cdot 2 - 14 + 4 = 0$.

If w is the 2-vertex in a super-flag H that is not in a flag, then w gets charge 2.5 from the base of the flag in H, because w is not in the flag. Furthermore, w gets charge 1.5 from the secondary base of H, because w is in the same super-flag and hence R3 applies and not R2. So

$$\mu^*(w) = 5d(w) - 14 + 2.5 + 1.5 = 0.$$

Let y be a secondary base for a super-flag H. By R2, y receives 2.5 charge from the other special vertex y' of H (because y' is the base of a flag). By R3, y sends 1.5 charge to its other neighbor in H (see the discussion immediately above). By Corollary 27, every secondary base has degree at least 4. So

$$\mu^*(y) \geq 5d(y) - 14 - 1.5 + 2.5 - 2.5(d(y) - 2) = 2.5d(y) - 8 \geq 2.$$

For vertices not in flags or super-flags, we consider cases according to their degrees.

Case 1: $d(v) = 2$. Since by Lemma 6, v has no neighbors of degree 2, by Rules R2–R5, v gets from its neighbors the total charge at least 4, so $\mu^*(v) \geq 5 \cdot 2 - 14 + 4 = 0$.

Case 2: $d(v) = 3$. Since v is not in a flag, by Lemma 26, v has no adjacent 2-vertices, so $\mu^*(v) \geq \mu(v) = 5d(v) - 14 = 1$.

Case 3: $d(v) \geq 5$. Since v is not in a flag, it gives at most 2 to each neighbor. So,

$$\mu^*(v) = 5d(v) - 14 - 2d(v) = 3d(v) - 14 \geq 3 \cdot 5 - 14 = 1.$$

Case 4: $d(v) = 4$. Since v is not in a flag, it gives at most 2 charge to each neighbor of degree 2. So, if v has at most three neighbors of degree 2, then $\mu^*(v) \geq 5 \cdot 4 - 14 - 2 \cdot 3 = 0$. Suppose that all neighbors of v are 2-vertices. Then by Lemma 30, each of these neighbors has a neighbor in a flag or a super-flag. So, by Rules R3 and R4, v sends to each neighbor only 1.5 charge. Thus, in this case, $\mu^*(v) = 5 \cdot 4 - 14 - 1.5 \cdot 4 = 0$. □

Remark 1. By the proof of Case 4, if $d(v) = 4$ and $\mu^*(v) = 0$, then v has at least three neighbors of degree 2.

By the above lemma, in order for (4) to hold, we need $\mu^*(v) = 0$ for every $v \in V(G)$. Then by the same lemma, *G* has only vertices of degree 2 and 4. By Remark 1, each 4-vertex has at most one neighbor of degree 4, and by Lemma 6, each 2-vertex has no neighbors of degree 2. In such a graph *G*, if we color all 4-vertices with color 1 and all 2-vertices with color 2, then we get a $(1, 0)$-coloring of *G*, a contradiction.

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