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On almost $(k-1)$-degenerate $(k+1)$-chromatic graphs and hypergraphs

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ABSTRACT

Recall that a (hyper)graph is *d-degenerate* if each of its nonempty subgraphs has a vertex of degree at most d. Every d-degenerate (hyper)graph is (easily) $(d+1)$-colorable. A (hyper)graph is *almost d-degenerate* if it is not d-degenerate, but each of its proper subgraphs is d-degenerate. In particular, if G is almost $(k-1)$-degenerate, then after deleting any edge it is k-colorable. For $k \geq 2$, we study properties of almost $(k-1)$-degenerate (hyper)graphs that are not k-colorable. By definition, each such (hyper)graph is $(k+1)$-critical.

1. Introduction

A (proper) *k-coloring* of a hypergraph G is a mapping $f : V(G) \to \{1, \ldots, k\}$ such that no edge of G becomes monochromatic. If a hypergraph has a proper k-coloring, then it is called *k-colorable*. The *chromatic number*, $\chi(G)$, of a hypergraph G is the minimum k such that G is k-colorable.

A hypergraph is *k-critical* if it is not $(k-1)$-colorable but after deleting any edge or vertex it becomes $(k-1)$-colorable. In a way, understanding why (hyper)graphs have chromatic number at least k is equivalent to understanding the structure of all k-critical (hyper)graphs. Each 2-critical hypergraph has exactly one edge and no isolated vertices. The only 3-critical graphs are the odd cycles, but 3-critical hypergraphs have not been described. There should be no good characterization of 3-critical hypergraphs, since it is an *NP*-complete problem to check whether a given hypergraph is 2-colorable (see e.g. [4]).

Since there is no hope of describing all k-critical graphs for $k \geq 4$, it makes sense to describe such graphs with additional properties. The famous Brooks Theorem can be interpreted as the description of $(k+1)$-critical graphs with maximum degree k. There are a number of other results describing k-critical graphs with additional restrictions. This paper also gives results of this type.

Recall that a hypergraph is *d-degenerate* if each of its subgraphs has a vertex of degree at most d. Every d-degenerate hypergraph G is easily $(d+1)$-colorable: by definition, the vertices of any d-degenerate G can be ordered as $v_1, v_2, \ldots$ in such a way that for every i, the degree of v_i in the subgraph induced by the first i vertices is at most d. So we can simply color the vertices of G greedily in this order, and $d+1$ colors will suffice. Say that a (hyper)graph is *almost d-degenerate* if it is not d-degenerate but after deleting any edge it becomes d-degenerate. Being $(k-1)$-degenerate, every proper subgraph of an almost $(k-1)$-degenerate hypergraph is k-colorable. Hence, if an almost $(k-1)$-degenerate hypergraph G is not k-colorable, then it is $(k+1)$-critical. For brevity, we will call almost $(k-1)$-degenerate non-k-colorable hypergraphs *k-special*.

A natural problem is that of describing k-special (hyper)graphs. In particular, this problem (for graphs) is stated as Part 4 of Problem 4.1 in [5]. According to [5], the problem was posed by Borodin in 1974. By Brooks' Theorem, the k-special graphs with maximum degree k are only K_{k+1} for $k \geq 3$, odd cycles for $k = 2$ and single edges for $k = 1$. A couple of other examples of k-special graphs (for $k = 4$ and $k = 3$) are given in Fig. 1.

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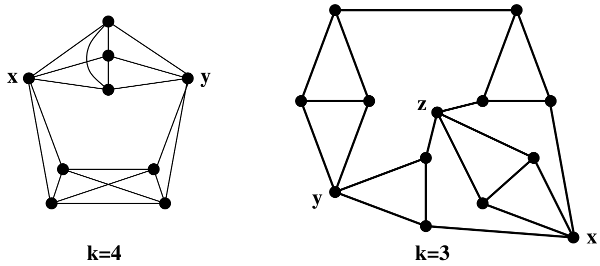

Fig. 1. Letters x, y, and z mark the vertices of degree greater than k.

The author worked on this problem in his Doctor of Science Thesis [6]. Recently, Pedersen [8] described (a) k-special graphs with at most two vertices of degree $k+1$ or greater, (b) k-special graphs with connectivity exactly 2 and (c) 3-special graphs with at most three vertices of degree 4 or greater.

It turns out that instead of describing k-special graphs it is easier and more natural to describe k-special hypergraphs. In this paper, based mainly on unpublished dissertation [6], the following is done:

(A) We give a description of the k-special hypergraphs with at most one vertex of degree $k+1$ or greater and show a way to construct all other k-special hypergraphs starting from these. This implies a polynomial algorithm for recognition of k-special hypergraphs.
(B) We obtain some restrictive properties of k-special hypergraphs for $k \geq 4$. In particular, every vertex of such hypergraph belongs to a K_k.
(C) We give an explicit description of 2-special hypergraphs.

The structure of the paper is as follows. In the next section we introduce relevant notation and cite some known results on the structure of $(k+1)$-critical graphs, and especially that of their subgraphs containing vertices of degree k. In Section 3 we use list coloring to describe so called Z_k-*graphs* that will be important bricks in the constructions of k-special hypergraphs. Then in Section 4 we deduce some properties of k-special hypergraphs that will allow us to prove the above statement in (A), in Section 5 we deduce that in (B), and in Section 6 we prove that in (C).

2. Preliminaries

A t-*edge* is an edge of cardinality t. Sometimes, we will call 2-edges *graph edges*, since a graph is a hypergraph whose every edge is a 2-edge. Similarly, we will say that e is a *hypergraph edge* if $|e| \geq 3$.

Let G be a hypergraph and $A \subseteq V(G)$. Then $G(A)$ is the hypergraph with vertex set A whose edges are the edges of G fully contained in A. The hypergraph $G[A]$ also has vertex set A, but the edge set is defined differently: $E(G[A]) := \{e \cap A : e \in E(G) \text{ and } |e \cap A| \geq 2\}$. Let

$$N_G(A) := \{v \in V(G) - A : \exists\, e \in E(G) \text{ such that } v \in e \text{ and } e \cap A \neq \emptyset\}.$$

An edge e in a connected hypergraph G is a *cut edge* if $G - e$ has $|e|$ components. A vertex v in a connected hypergraph G is a *cut vertex* if $G[V(G) - v]$ is not connected. By a *block* of a hypergraph G we mean an inclusion maximal connected subhypergraph B of G such that no vertex of B is a cut vertex of B. Any two blocks of G have at most one vertex in common and, by definition, a vertex of G is a cut vertex of G iff it is contained in more than one block of G. An end-block of G is a block that contains at most one cut vertex of G. By a *brick* we mean a block consisting of either one (hyper)edge, or an odd cycle with graph edges, or a complete graph. A connected hypergraph all of whose blocks are bricks is called a *Gallai tree*.

A $(k+1)$-critical (hyper)graph has no vertices of degree less than k. In view of this, vertices of degree k in a $(k+1)$-critical (hyper)graph G are called *low vertices* and other vertices of such G are *high*. The set of low vertices in a $(k+1)$-critical (hyper)graph G is denoted by $L(G)$ or simply by L, and the set of high vertices is denoted by $H(G)$ or simply by H. Recall that each k-special hypergraph is $(k+1)$-critical. Furthermore, by the definition of k-special hypergraphs, for each such hypergraph G, the hypergraph $G(H(G))$ does not contain any edge.

Gallai [3] proved that for every $(k+1)$-critical graph G, every component of $G(L(G))$ is a Gallai tree. We will heavily use an extension of this result to hypergraphs and list colorings from [7] stated in the next section (Theorem 2).

Sachs and Stiebitz [9,10] studied $(k+1)$-critical graphs G with special structures of $G(H(G))$ and $G(L(G))$. Stiebitz [11] proved the following fact conjectured by Gallai.

Theorem 1 (*[11]*). *For every $(k+1)$-critical graph G, the number of components of $G(H(G))$ does not exceed the number of components of $G(L(G))$.*

In particular, this means that for each k-special graph G, $|H(G)|$ is at most the number of components of $G(L(G))$. Note that in the examples in Fig. 1, the equality is attained: $G(L(G))$ has two components in the left example and three components in the right example.

Following Sachs and Stiebitz [9,10], we call a set M of blocks of a hypergraph G a *matching* if distinct blocks in M are vertex disjoint.

3. List coloring and Z_k-graphs

The notion of *list coloring* was introduced by Vizing [12] and independently later by Erdős, Rubin and Taylor [2]. Vizing introduced it to study total colorings, and Erdős, Rubin and Taylor were inspired by the Dinitz Conjecture on edge colorings of bipartite graphs. A *list* $\mathcal{L}$ for a hypergraph G is an assignment to each vertex $v \in V(G)$ of a set $\mathcal{L}(v)$ of *admissible colors for* v. For such a list $\mathcal{L}$, an $\mathcal{L}$*-coloring* is a proper coloring f of vertices of G such that $f(v) \in \mathcal{L}(v)$ for every $v \in V(G)$.

A list $\mathcal{L}$ for a hypergraph G is called a *degree list* if $|\mathcal{L}(v)| \geq \deg(v)$ for every $v \in V(G)$. We will use the following characterization of connected hypergraphs not colorable from degree lists.

Theorem 2 ([7]). *Let G be a connected hypergraph. Let $\mathcal{L}$ be a degree list for G such that G is not $\mathcal{L}$-colorable. Then:*

(1) $|\mathcal{L}(v)| = \deg(v)$ *for every* $v \in V(G)$*;*
(2) *every hyperedge (i.e. edge of size at least* 3*) in G is a cut edge, and therefore is a block of G;*
(3) *G is a Gallai tree.*

The next fact is well known (see [1,2] for graph versions).

Lemma 3. *Let G be a connected hypergraph. Let $\mathcal{L}$ be a degree list for G such that G is not $\mathcal{L}$-colorable. If $e \in E(G)$, $x, y \in e$, and x is not a cut vertex in G, then $\mathcal{L}(x) \subseteq \mathcal{L}(y)$.*

Proof. Let $n = |V(G)|$. Suppose $\beta \in \mathcal{L}(x) - \mathcal{L}(y)$. Since x is not a cut vertex in G, $G[V(G) - x]$ is connected. So, we can order the vertices of G, $v_1, \ldots, v_n$, in such a way that $v_1 = x$, $v_n = y$ and

$$\text{for each } 2 \leq i \leq n-1, \quad N_G(v_i) \cap \{v_{i+1}, \ldots, v_n\} \neq \emptyset. \tag{1}$$

For $i = 2, \ldots, n$, let E_i denote the set of edges $e \in E(G)$ such that v_i is the last vertex in e in the above ordering. We define a coloring f as follows: Let $f(v_1) = \beta$, and for $i = 2, \ldots, n$, choose $f(v_i) \in \mathcal{L}(v_i)$ so that no edge in E_i becomes monochromatic. We can do this for $i = 2, \ldots, n-1$ because of (1), and for $i = n$ because $\beta = f(v_1) \notin \mathcal{L}(v_n)$. Since each edge belongs to some E_i, f is an $\mathcal{L}$-coloring, a contradiction. □

A list $\mathcal{L}$ for a hypergraph G with maximum degree at most k is k*-natural* if $\mathcal{L}(v) \subseteq \{1, 2, \ldots, k\}$ for each $v \in V(G)$. For $\alpha \in \{1, \ldots, k\}$, a k-natural list $\mathcal{L}$ for a hypergraph G is (k, α)*-natural* if for each $v \in V(G)$,

$$\mathcal{L}_{k,\alpha}(v) := \begin{cases} \{1, \ldots, k\}, & \text{if } \deg_G(v) = k; \\ \{1, \ldots, k\} - \alpha, & \text{if } \deg_G(v) < k. \end{cases} \tag{2}$$

A connected hypergraph G with $\Delta(G) \leq k$ is a Z_k*-graph* if its vertices cannot be colored from the $(k, 1)$-natural list $\mathcal{L}_{k,1}$.

The following fact is a small extension of Brooks' Theorem [3].

Lemma 4. *Let $k \geq 2$ and G be a connected Z_k-graph distinct from K_{k+1}, a single edge, and an odd cycle. Then the blocks of G can be partitioned into two matchings $\mathcal{B}_1$ and $\mathcal{B}_2$ so that:*

(a) *the blocks in $\mathcal{B}_1$ cover $V(G)$;*
(b) *the blocks in $\mathcal{B}_2$ cover all vertices of degree k and only them;*
(c) *each block in $\mathcal{B}_2$ is a cut edge of G;*
(d) *each block in $\mathcal{B}_1$ is an edge if $k = 2$, an odd cycle if $k = 3$, and a K_k if $k \geq 4$.*

Proof. Let G be a vertex-minimal Z_k-graph for which the lemma does not hold. Let B_1 be a leaf-block in G. If $B_1 = G$, then by Theorem 2, G is a complete graph or an odd cycle, or a single edge, a contradiction. So, let $B_1 \neq G$ and v_1 be the cut vertex in B_1.

Case 1: $k \geq 4$. By Theorem 2, B_1 is $(k-1)$-regular and hence $B_1 = K_k$. Since $\Delta(G) \leq k$, there is only one block $B_2 \neq B_1$ containing v_1, and moreover, B_2 is a cut edge in G. Suppose $V(B_2) = \{v_1, \ldots, v_t\}$. Let $G_1 := G[V - V(B_1)]$ and

$$\mathcal{L}'_{k,1}(x) := \begin{cases} \{2, \ldots, k\}, & \text{if } t = 2 \text{ and } x = v_2; \\ \mathcal{L}_{k,1}(x), & \text{otherwise.} \end{cases}$$

If G_1 can be colored from $\mathcal{L}'_{k,1}$, then we can extend this coloring to an $\mathcal{L}_{k,1}$-coloring of G by coloring v_1 with 1 and the remaining vertices of B_1 with $2, \ldots, k$. So, G_1 is a Z_k-graph with fewer vertices than G. Thus the lemma holds for G_1, and if $t \geq 3$, then the edge $\{v_2, \ldots, v_t\}$ is a cut edge in G_1. Hence the lemma holds for G, as well.

Case 2: $k = 3$. The argument is the same; the only difference is that B_1 is an odd cycle.

Case 3: $k = 2$. By Theorem 2, B_1 is an edge. We essentially repeat the argument of Case 1. The only new obstacles are that it may happen that either $t \geq 3$ and the block $B'_2 = \{v_2, \ldots, v_t\}$ is in $\mathcal{B}_1(G_1)$, or $t = 2$ and v_2 belongs to a block in $\mathcal{B}_2(G_1)$. Note first that if $t = 2$, then $\deg_{G_1}(v_2) = 1$, and so v_2 cannot belong to a block in $\mathcal{B}_2(G_1)$. So suppose that $t \geq 3$ and the block $B'_2 = \{v_2, \ldots, v_t\}$ is in $\mathcal{B}_1(G_1)$. Because of the structure of G_1, we can construct an $\mathcal{L}'_{k,1}$-coloring f' of G_1 with the only monochromatic edge B'_2 as follows: we let $f'(v_i) = 2$ for all $v_i \in B'_2$, and for $j = 1, 2$ and each edge $D \in \mathcal{B}_j(G_1) - \{B'_2\}$, we color the cut vertex w_D closest to B'_2 with j and every other vertex of D with $3 - j$. We extend f' to G by coloring v_1 with 1 and all other vertices of B_1 with 2. □

Lemma 5. *Let $k \geq 2$ and G be a Z_k-graph. Let $\mathcal{L}$ be a k-natural degree list for G. If G is not $\mathcal{L}$-colorable, then $\mathcal{L}$ is (k, α)-natural for some $\alpha \in \{1, \dots, k\}$.*

Proof. Suppose that G is a vertex-minimal Z_k-graph for which the lemma does not hold and $\mathcal{L}$ is a k-natural degree list for G such that G is not $\mathcal{L}$-colorable. Let B_1 be a leaf-block in G. If $B_1 = G$, then by Lemma 3, the lemma holds. So, let $B_1 \neq G$ and v_1 be the cut vertex in B_1. By Lemma 4, the minimum degree of G is $k-1$, so for each $v \in V(G)$ of degree $k-1$ there is $\alpha_v \in \{1, \dots, k\}$ such that $\mathcal{L}(v) = \{1, \dots, k\} - \alpha_v$. Also by Lemma 4, v_1 belongs to exactly two blocks, and the block $B_2 \neq B_1$ containing v_1 is a cut edge. Suppose that this edge is $e = \{v_1, \dots, v_t\}$. Again by Lemma 4, each $v_i \in e$ has degree k in G, and so its list is $\{1, \dots, k\}$.

By Lemma 3, for all $v \in V(B_1) - v_1$, the lists are the same. Let α_0 denote the color missing in the lists of all $v \in V(B_1) - v_1$. Let $G' = G - e$ and $G_1, \dots, G_t$ be the components of G'. Define the list $\mathcal{L}'$ as follows:

$$\mathcal{L}'(w) := \begin{cases} \{1, \dots, k\} - \alpha_0, & \text{if } w = v_i \text{ for some } 1 \leq i \leq t; \\ \mathcal{L}(w) & \text{otherwise.} \end{cases} \tag{3}$$

If at least one G_i is $\mathcal{L}'$-colorable, then color this G_i from $\mathcal{L}'$ and every other G_j from $\mathcal{L}$. We can color them by Theorem 2(1). Note that v_1 must be colored with α_0, and so e will not be monochromatic. Thus we would get an $\mathcal{L}$-coloring of G, a contradiction. Hence no G_i is $\mathcal{L}'$-colorable. By the minimality of G, for each $1 \leq i \leq t$, the restriction of $\mathcal{L}'$ to G_i is (k, α_i)-natural for some α_i. By (3), $\alpha_1 = \cdots = \alpha_t = \alpha_0$. So $\mathcal{L}$ is (k, α_1)-natural for G. □

4. Simple properties of k-special hypergraphs

Let G be a k-special hypergraph. In this section, we derive some properties of such a G. The first three of them are so simple that they do not need proofs.

Property 1. $\delta(G) = k$.

Property 2. *G is $(k+1)$-critical.*

Property 3. *$G(H(G))$ is an independent set.*

Property 4. *Let $e \in E(G)$, $|e| \geq 3$ and $|e \cap L(G)| \geq 2$. Then $\widetilde{e} := e \cap L(G)$ is a cut edge in $G[L]$.*

Proof. Suppose that $\{v, w\} \subseteq \widetilde{e}$ and $G[L] - \widetilde{e}$ contains a path $(v_1, \dots, v_s)$ connecting $v_1 = v$ with $v_s = w$. Let G_0 be the hypergraph obtained from G by replacing e with $e - v$. Then $\deg_{G_0}(v) = k-1$. For $i = 1, \dots, s$, let $G_i := G_0 - \{v_1, v_2, \dots, v_i\}$. By construction,

$$\deg_{G_{i-1}}(v_i) \leq k-1 \quad \text{for } i = 1, \dots, s. \tag{4}$$

Since $G_0 - \{v_1, v_2, \dots, v_s\} = G - \{v_1, v_2, \dots, v_s\}$, we have $\chi(G_s) \leq k$. It follows from (4) that each k-coloring of G_s can be extended to a k-coloring of G_0. But $\chi(G_0) \geq \chi(G) = k+1$, a contradiction. □

Property 5. *Some component of $G[L(G)]$ is a Z_k-graph.*

Proof. Color all vertices in $H(G)$ with 1. By Property 3, this is a proper partial coloring of G. For every $v \in L(G)$, we let

$$\mathcal{L}^*(v) := \begin{cases} \{2, \dots, k\}, & \text{if there exists } e \in E \text{ such that } e - H(G) = \{v\}; \\ \{1, 2, \dots, k\}, & \text{otherwise.} \end{cases} \tag{5}$$

Since $\chi(G) = k+1$, there exists a component R of $G[L(G)]$ that is not colorable from $\mathcal{L}^*$. By definition, this component is a Z_k-graph. □

The components of $G[L(G)]$ that are Z_k-graphs will be called *Z_k-components*. By Lemma 5 we have:

Property 6. *If Z is a Z_k-component in $G[L(G)]$, then each coloring of $N_G(Z)$ that uses at least two and at most k colors can be extended to a proper k-coloring of $G[Z \cup N_G(Z)]$.*

If Z is a Z_k-component in $G[L(G)]$ with $|N_G(Z)| \geq 2$, then let $F(G, Z)$ denote the hypergraph obtained from $G - V(Z)$ by adding edge $e_Z := N_G(Z)$. Note that if $|H(G)| \geq 2$, then $|N_G(Z)| \geq 2$ for each Z_k-component Z in $G[L(G)]$, since $(k+1)$-critical hypergraphs have no cut vertices.

Property 7. *If $|H(G)| \geq 2$, then for each Z_k-component Z in $G[L(G)]$, the hypergraph $F(G, Z)$ is k-special.*

Proof. Let $G' := F(G, Z)$. By Property 6 and the fact that G is k-special, $\chi(G') \geq k+1$. Since G is almost $(k-1)$-degenerate, the proper subgraph $G' - e_Z$ of G is $(k-1)$-degenerate. Furthermore, let $e \in E(G') - e_z$. Since G is almost $(k-1)$-degenerate, there is an ordering $v_1, \ldots, v_n$ of the vertices of $G - e$ witnessing that $G - e$ is $(k-1)$-degenerate. Deleting from this sequence the vertices in Z we obtain a witness sequence for $G' - e$. It follows that G' is almost $(k-1)$-degenerate and hence is k-special. □

Property 8. *Let $k \geq 2$. Let R be a connected hypergraph that has a vertex w such that every other vertex of R has degree at most k and w is not a cut vertex. Let $L(R)$ denote the set of vertices in R of degree at most k. If $\chi(R) = k+1$, then the following conditions hold:*

(a) *$R[L(R)]$ is a Z_k-graph;*
(b) *for each $e \in E(R)$ with $w \in e$, either $e - w$ is a cut edge in $R[L(R)]$, or $|e| = 2$;*
(c) *$\delta(R) = k$.*

Moreover, if $k \geq 3$ and conditions (a)–(c) *hold for some connected hypergraph R and $w \in V(R)$ is such that every other vertex of R has degree at most k, then $\chi(R) = k+1$.*

Proof. Suppose $\chi(R) \geq k+1$. If $\deg_R(w) \leq k$, then by Lemma 4, either $R = K_{k+1}$ or $k = 2$ and R is an odd cycle. In both cases, the statements (a)–(c) hold. So suppose $\deg_R(w) \geq k+1$. Since w is not a cut vertex, $R[V(R) - w]$ is connected. So, for each edge $e \in E(R)$, the hypergraph $R - e$ is $(k-1)$-degenerate (and thus k-colorable). In this case, R is k-special and, in particular, $(k+1)$-critical. Thus by Property 5, in this case $R[L]$ is a Z_k-graph, i.e., (a) holds. Then by Property 4, (b) also holds. And $\delta(R) \geq k$ holds for each $(k+1)$-critical hypergraph R.

Suppose now that $k \geq 3$ and R satisfies (a)–(c). If $\deg_R(w) \leq k$, then by (a) and (c), $R = K_{k+1}$. So in this case the lemma holds. Otherwise, $\deg_R(w) \geq k+1$. By (b), for each vertex of degree $k-1$ in $R[L(R)]$, the edge wv is in $E(R)$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the block matchings in $R[L(R)]$ described in Lemma 4. If $k \geq 3$, then each cut edge in $R[L(R)]$ is a block in $\mathcal{B}_2$. And if we extend any number of these edges to w, we obtain a $(k+1)$-chromatic hypergraph. □

Remark 1. Properties 7 and 8 imply the following polynomial-time procedure for checking a hypergraph G to see whether it is k-special.

Let $L(G)$ denote the set of vertices of degree at most k. First, check whether G is almost $(k-1)$-degenerate. If not, then G is not k-special. Suppose, G is almost $(k-1)$-degenerate. If $|L(G)| \geq |V(G)| - 1$, then we use Property 8. Suppose G has at least two vertices of degree at least $k+1$. Then look at whether $G[L(G)]$ has Z_k-components. If it does not, then G is not k-special. Otherwise, choose any Z_k-component R and consider $F(G, R)$. It is k-special if and only if G is, and it has fewer vertices, and so we can iterate.

Remark 2. Properties 7, 6 and 8 also give a procedure for generating all k-special hypergraphs: Start from either a brick or a k-special hypergraph with only one high vertex (described in Property 8) and at each step first replace a (hyper)edge with a Z_k-component R (an operation opposite to taking $F(G, R)$) and then check whether the resulting hypergraph is almost $(k-1)$-degenerate. If it is, then it is k-special. And by Properties 7, 6 and 8, every k-special hypergraph can be produced this way.

5. On k-special hypergraphs for $k \geq 4$

The main result of this section is Theorem 6. It shows that not only do Z_k-components have a very special structure, but also the structure of the other components is restricted. And complete k-vertex subgraphs cover all vertices but one. In particular, the number of vertices of a k-special hypergraph is always 1 modulo k. The proof goes by induction. Roughly speaking, we replace one Z_k-component by a hyperedge and look at the structure of the smaller hypergraph obtained. We use the properties of this hypergraph to prove that the original hypergraph also possesses these properties.

Theorem 6. *Let $k \geq 4$. Let G be an n-vertex k-special hypergraph with $n > k+1$. Then:*

(i) *there is a vertex $w \in H(G)$ such that $V(G) - w = W_1 \cup \cdots \cup W_{(n-1)/k}$ where $G(W_j) = K_k$ for every $1 \leq j \leq (n-1)/k$ and $G(W_1 + w) = K_{k+1} - e$;*
(ii) *each block of $G[L(G)]$ is either a cut edge, or K_k, or K_{k-1}, and each vertex in $L(G)$ belongs to at most two blocks of $G[L(G)]$;*
(iii) *$|H(G)|$ equals the number of components of $G[L(G)]$.*

Proof. Suppose that the theorem fails, and let G be a vertex-minimal counter-example to the theorem. Since G is $(k+1)$-critical, it has no cut vertices. By Property 8, $|H(G)| \geq 2$; so by Property 5, some component R_1 of $G[L(G)]$ is a Z_k-component. Let $N_G(R_1) = \{v_1, \ldots, v_s\}$. By definition, $\{v_1, \ldots, v_s\} \subseteq H(G)$. Consider $G' := F(G, R_1)$. By Property 7, G' is k-special. By the minimality of G, G' satisfies (i) or $G' = K_{k+1}$. Recall that by Lemma 4(d), $V(R_1)$ is the union of vertex disjoint copies of K_k: $U_1, \ldots, U_m$. So, if G' satisfies (i), then we take w and the partition for it, and add $U_1, \ldots, U_m$. If $G' = K_{k+1}$, then $v_1v_2 = e_{R_1}$, and we can take $w := v_1$ (recall that $v_1 \in H(G)$) and the partition $V(G) = V(G') \cup U_1 \cup \cdots \cup U_m$. So, in both cases, (i) holds for G. We now need to prove (ii) and (iii).

Since G' is almost $(k-1)$-degenerate,

$$e_{R_1} \cap L(G') \neq \emptyset,$$

where $e_{R_1} = \{v_1, \dots, v_s\}$. We may assume that $e_{R_1} \cap L(G') = \{v_1, \dots, v_t\}$ and that R' is the component of $G'[L(G')]$ containing $v_1, \dots, v_t$.

Case 1: $s = 2$ and $G' = K_{k+1}$. Then $G - R_1 = K_{k+1} - v_1v_2$. As was mentioned above, $|H(G)| \geq 2$. Hence $H(G) = \{v_1, v_2\}$. This and Lemma 4 yield all the statements for G.

Case 2: $s \geq 3$. Then $G' \neq K_{k+1}$ and by the minimality of G, our theorem holds for G'. Let $1 \leq j \leq t$. By (i), v_j belongs to a K_k-subgraph of G'. Since $s \geq 3$, none of the edges of this subgraph incident to v_j are e_{R_1}. Since $v_j \in H[G]$, each $v \in V(G')$ joined by a 2-edge to v_j must be in $L[G']$. It follows that for each $j \in \{1, \dots, t\}$, the whole K_k-subgraph of G' containing v_j is in $L[G']$. So, by (ii) this K_k-subgraph of G' is a block of $G'[L(G')]$. Then (ii) will hold for G. By Property 4, the number of connected components in $G[L(G)]$ is greater by t than that of $G'[L(G')]$. But $H(G) = H(G') \cup \{v_1, \dots, v_t\}$. Hence (iii) also holds for G.

Case 3: $s = 2$ and $G' \neq K_{k+1}$. Again, by the minimality of G, our theorem holds for G'. Suppose first that $t = 2$. Since $H(G)$ is independent, the K_k-subgraphs of G' containing v_1 and v_2 are fully in $L(G')$. If these K_k-subgraphs are distinct, then v_1v_2 is a cut edge of G' and the argument of Case 2 works. So, we may assume that if $t = 2$, then

$$v_1v_2 \text{ belongs to a block } B_1 = K_k \text{ of } R', \text{ and } v_1 \text{ and } v_2 \text{ are cut vertices in } R'. \tag{6}$$

In other words, G contains a subgraph depicted in Fig. 2 (the numbers at the edges show how many edges connect the corresponding components of $G[L(G)]$ with v_1 or v_2).

Suppose now that $t = 1$, i.e., $v_2 \in H(G')$. If v_1 belongs to a block of R' isomorphic to K_k, then we argue as at the end of Case 2. Suppose that v_1 is in a K_{k-1}-block B' of R'. Since $v_1 \in H(G)$, every edge of G containing v_1 also contains a vertex in $L(G)$. Thus v_1v_2 is the only edge incident with v_1 that does not have other vertices in $L(G')$. So by (i) and (ii),

$$v_1 \text{ is in a block } B' = K_{k-1} \text{ of } R' \text{ and is a cut vertex of } R';\ v_2b \in E(G') \text{ for all } b \in B'. \tag{7}$$

In other words, G contains a subgraph depicted in Fig. 3.

Thus, by the above, for each Z_k-component R_1 in G, $|N_G(R_1)| = 2$, and for $G' := F(G, R_1)$, either (6) or (7) holds.

Case 3.1: $G[L(G)]$ has only one Z_k-component R_1. Then G' has only one Z_k-component, and it is R'. But in the case of (7) (see Fig. 3), R' is not a Z_k-component, since it has a K_{k-1}-block, and in the case of (6) (see Fig. 2), if R' is a Z_k-component, then R_3 and R_4 are Z_k-components in G.

Case 3.2: $G[L(G)]$ has distinct Z_k-components R_1 and P_1. Observe that in both cases, (6) and (7), $G[L(G)]$ has a block $B = K_{k-2}$ all vertices of which are joined by 2-edges with v_1 and v_2. By the minimality of G, neither $L(F(G, R_1))$ nor $L(F(G, P_1))$ contains a K_{k-2}-block. Hence this K_{k-2}-block is the same for R_1 and P_1. Let $N_G(R_1) = \{v_1, v_2\}$ and $N_G(P_1) = \{w_1, w_2\}$. Since $\deg_G(u) = k$ for all $u \in B$, $|\{v_1, v_2, w_1, w_2\}| \leq 3$. So, we may assume that $w_2 = v_2$ (but we do not assume that $v_2 \in L(G')$ and we may have w_1 instead of w_2; the proof does not change). If also $w_1 = v_1$, then for every k-coloring f of $G - R_1$, we have $f(v_1) \neq f(v_2)$, since otherwise we would not be able to extend the coloring to P_1. So in this case, by Property 6 we can extend f to a k-coloring of G, a contradiction. Thus $w_1 \neq v_1$ and block $B = K_{k-2}$ is a component of $G[L(G)]$ (see Fig. 4). Note that in Fig. 4 the roles of w_1 and w_2 could be swapped.

Let $a_1 := |E_G(v_1, V(G) - R_1 - R_2)|$, $a_2 := |E_G(v_1, R_1)|$, $a_3 := |E_G(v_2, V(G) - R_1 - R_2 - P_1)|$, $a_4 := |E_G(v_2, R_1)|$, $a_5 := |E_G(v_2, P_1)|$, $a_6 := |E_G(w_1, P_1)|$, $a_7 := |E_G(w_1, V(G) - P_1 - R_2)|$ (see Fig. 4).

If $a_2 \geq 2$ and $a_6 \geq 2$, then the minimum degree of $G'' := G(R_1 \cup R_2 \cup P_1 \cup \{v_1, v_2, w_1\})$ is k, and since G is almost $(k-1)$-degenerate, $G'' = G$. In this case, we can let $f(v_1) = f(w_1) = 1$, $f(v_2) = 2$, and by Property 6 extend this coloring to a k-coloring of G, a contradiction. So, $\min\{a_2, a_6\} = 1$. By symmetry, we assume

$$a_2 = 1.$$

Then to have $v_1 \in H(G)$, we need $a_1 \geq 2$. This means that R_1 is of type (7) (see Fig. 3), and so $a_3 + a_5 = 1$, which yields $a_3 = 0$ and $a_5 = 1$. Since $a_2 = 1$ and R_1 is a Z_k-component, we have $a_4 \geq k - 1$. Hence P_1 also is of type (7) and $a_7 = 1$ (see Fig. 5).

Similarly, since $a_5 = 1$ and P_1 is a Z_k-component, $a_6 \geq k - 1$. Now, if $a_1 \geq k - 1$, then $\delta(G - R_2) \geq k$, a contradiction. So, $2 \leq a_1 \leq k - 2$. Let $G_1 := G - R_1 - R_2 - P_1$. Since

$$\deg_{G_1}(v_1) + \deg_{G_1}(w_1) = a_1 + a_7 \leq k - 1,$$

there exists a k-coloring f of G_1 such that $f(v_1) = f(w_1) = 1$. Then we let $f(v_2) = 2$ and by Property 6 we can extend this coloring to a k-coloring of G. □

Remark 3. A similar result would hold also for $k = 3$, where the role of K_k above would be played by odd cycles. But the statement is somewhat messier than that of Theorem 6 (in particular, we do not have such a nice fact as that of the number of vertices always being 1 modulo k), and the proofs are significantly messier.

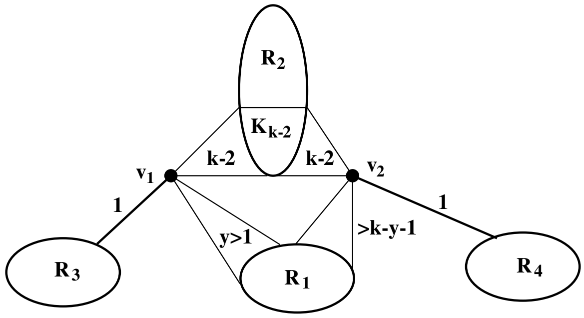

Fig. 2. $t = 2$.

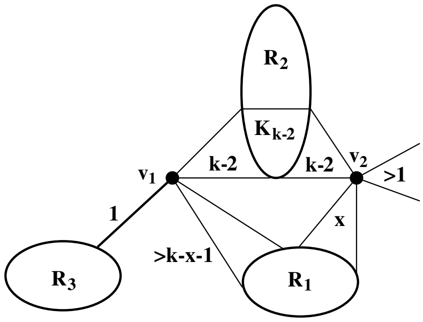

Fig. 3. $t = 1$.

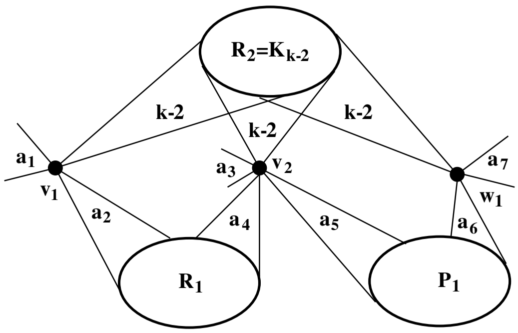

Fig. 4. $w_2 = v_2$.

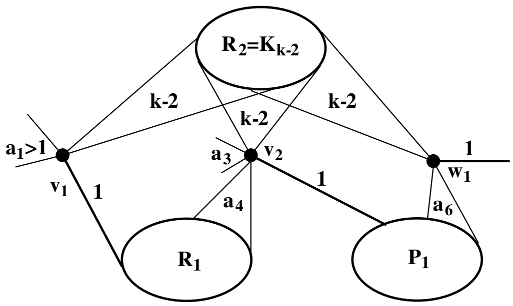

Fig. 5. Adjusted Fig. 4.

6. The structure of 2-special hypergraphs

The only 3-critical graphs are odd cycles, but there are many 3-critical hypergraphs and moreover, many 2-special hypergraphs. In particular by Property 8, from every Z_2-graph one can construct a 2-special hypergraph by adding one vertex w and some 2-edges and extending some cut edges to w. However, the structure of 2-special hypergraphs is more restricted than that for k-special hypergraphs when $k \geq 4$. First we derive some important properties in Lemma 7 and then use them to give the characterization of 2-special hypergraphs in Theorem 9.

Lemma 7. *Let G be a 2-special hypergraph with $|H(G)| \geq 2$. Then for each component R of $G[L(G)]$:*

(i) $|N_G(R)| = 2$;
(ii) *if* $N_G(R) = \{w_1, w_2\}$, *then* $\min\{\deg_{G(R\cup N(R))}(w_1), \deg_{G(R\cup N(R))}(w_2)\} = 1$ *and* $\min\{\deg_{G-R}(w_1), \deg_{G-R}(w_2)\} = 1$;
(iii) $|H(G)|$ *is equal to the number of components of* $G[L(G)]$.

Proof. Suppose that G is a minimum counter-example to the lemma. Let R_0 be a Z_2-component of $G[L(G)]$. Let $N(R_0) - R_0 = \{v_1, \dots, v_s\}$, $G_1 = F(G, R_0)$ and $e_{R_0} := \{v_1, \dots, v_s\}$. By Property 7, G_1 is 2-special. Suppose that vertices $v_1, \dots, v_t$ have degree 2 in G_1, and vertices $v_{t+1}, \dots, v_s$ have degree at least 3 in G_1. Since $G - R_0$ is 1-degenerate, $t \geq 1$. First we prove that

$$s \geq t + 1. \tag{8}$$

Indeed, each of the vertices $v_1, \dots, v_t$ belongs in G_1 to only one edge apart from e_{R_0}. It follows that

$$\deg_{G(R_0\cup\{v_1,\dots,v_s\})}(v_i) \geq 2 \quad \forall i = 1, \dots, t. \tag{9}$$

So, if $s = t$, then the minimum degree of $F = G(R_0 \cup \{v_1, \dots, v_s\})$ is 2. But every proper subgraph of G is 1-degenerate, a contradiction.

Let R' be the component of $G_1[L(G_1)]$ containing $v_1, \dots, v_t$. Since $\{v_{t+1}, \dots, v_s\} \subseteq N_{G_1}(R')$, by the minimality of G, $s-t \leq 2$ and if $s - t = 2$, then $\min\{\deg_{G_1(R'\cup\{v_{t+1},v_s\})}(v_{t+1}), \deg_{G_1(R'\cup\{v_{t+1},v_s\})}(v_s)\} = 1$ and $\min\{\deg_{G_1-R'}(v_{t+1}), \deg_{G_1-R'}(v_s)\} = 1$.

By Property 4 and (8), if $t \geq 2$, then $\{v_1, \dots, v_t\}$ is a cut edge in R'. This means that $v_1, \dots, v_t$ belong to different components $R'_1, \dots, R'_t$ of the hypergraph R'' obtained from R' by deleting edge $\{v_1, \dots, v_t\}$. Since $\deg_{R'_i}(v_i) = 1$, the hypergraph $R_i := G[R'_i - v_i]$ is a component of $G[L(G)]$ for every $i = 1, \dots, t$.

Case 1: $s - t = 2$. By symmetry, we may assume that $\deg_{G_1(R'\cup\{v_{t+1},v_s\})}(v_{t+1}) = 1$ and $\deg_{G_1-R'}(v_s) = 1$. Then for $i = 1, \dots, t$, $N_{G_1}(R_i) = \{v_i, v_s\}$. So, the minimum degree of $G - \bigcup_{i=1}^t R_i$ is 2, a contradiction.

Case 2: $s - t = 1$ and $H(G_1) = \{v_s\}$. Since G_1 is 3-critical, it has no cut vertices, and hence R' is the only component of $G_1[L(G_1)]$. Since G also has no cut vertices, for $i = 1, \dots, t$, $N_{G_1}(R_i) = \{v_i, v_s\}$. So, by (9) and the fact that v_s belongs to an edge intersecting R_0 and to an edge intersecting R_1, if $t \geq 2$, then the minimum degree in $G - R_2$ is 2, a contradiction. So, $t = 1$. Property 8 describes the structure of G_1, and G is obtained from G_1 by replacing edge v_1v_2 (where $\deg_{G_1}(v_1) = 2$ and $\deg_{G_1}(v_2) \geq 3$) with a Z_2-component attached to these vertices. Then $|H(G)| = 2$, $G[L(G)]$ has two components, R_0 and R_1, and the remaining statements of the lemma also hold.

Case 3: $s - t = 1$ and $|H(G_1)| \geq 2$. By the minimality of G, there is some $w \in V(G_1)$ such that $N_{G_1}(R') = \{v_s, w\}$. And some $x \in \{v_s, w\}$ belongs to only one edge in $G_1[R' \cup \{v_s, w\}]$.

Case 3.1: $x = w$. Then $\deg_{G-R_1}(w) \geq 2$. Recall that R_1 is a component of $G[L]$ such that $V(R_1) + v_1$ induces a component of $R'' = R' - e_{R_0}$. Since $N_{G_1}(R') = \{v_s, w\}$, $N_G(R_1) \subseteq \{v_1, v_s, w\}$. Since only one edge containing v_1 intersects $V(R_1)$ and $v_1 \in H(G)$, $\deg_{G-R_1}(v_1) \geq 2$. Furthermore, by (ii), $\deg_{G_1-R'}(v_s) = 1$ and v_s also belongs to an edge in $G(R_0 \cup \{v_1, \dots, v_s\})$, and we have $\deg_{G-R_1}(v_s) \geq 2$. So, since each vertex in $N_G(R_1)$ has degree at least 2 in $G - R_1$, the minimum degree of $G - R_1$ is 2, a contradiction to the fact that $G - R_1$ is 1-degenerate.

Case 3.2: $x = v_s$ and $t \geq 2$. Then $\deg_{G_1-R'}(w) = 1$. Since $\{v_1, \dots, v_t\}$ is a cut edge in $G_1[R']$, each edge in $G_1[R' + w]$ containing w intersects only one of $R_1, \dots, R_t$. So, we may assume that there is an edge e_1 containing w that is contained in $R_1 \cup \{v_1, w\}$. Since v_2 belongs to an edge in $G(R_0 \cup \{v_1, \dots, v_s\})$, and $N_G(R_2) \subseteq \{w, v_2\}$, the minimum degree of $G - R_2$ is 2, a contradiction.

Case 3.3: $x = v_s$ and $t = 1$. In particular, $s = 2$. By the minimality of G, the lemma holds for G_1. Recall that we obtain G from G_1 by replacing the graph edge v_1v_2 (where $\deg_{G_1}(v_1) = 2$ and $\deg_{G_1}(v_2) \geq 3$) with the component R_0 "hanging" on v_1 and v_2. Since G is 2-special, $|R'| \geq 2$, and hence G has one more vertex in H than G_1 and one more component in $G[L(G)]$, so (iii) holds for G. Since all components of $G_1[L(G_1)]$ apart from R' are also components of $G[L(G)]$, and each of R_0 and R_1 has neighborhood of size 2, (i) also holds for G. Since Cases 3.1 and 3.2 do not hold, Statement (ii) holds for R_0 and R_1. Since the degrees of v_2 and w in G are equal to their degrees in G_1, Statement (ii) holds also for all other components of $G[L(G)]$. □

Let G be a 2-special hypergraph with $|H(G)| \geq 2$. Construct the auxiliary digraph $D = D(G)$ with $V(D) = H(G)$ as follows. Every edge $e(R)$ of D corresponds to a component R of $G[L(G)]$. In view of Lemma 7, we may define $e(R)$ as w_1w_2, where $N_G(R) = \{w_1, w_2\}$, $\deg_{G-R}(w_1) = 1$ and $\deg_{G(R\cup N(R))}(w_2) = 1$. Since each $w \in H(G)$ has degree at least 3, and G is almost 1-degenerate, D has the following properties:

(a) for each $w \in V(D)$, $d^-(w) + 2d^+(w) \geq 3$;
(b) for every $w_1w_2 \in E(D)$, $d^-(w_1) = 1$.

The only digraph with properties (a) and (b) is a (directed) cycle. Thus D is a cycle. From the coloring point of view, if $|H(G)| \geq 2$, then a component R of $G[L(G)]$ can be of two types: either there is no proper 2-coloring of $G(R \cup N(R))$ such that the two vertices w_1 and w_2 in $N(R)$ have the same color, or there is no proper 2-coloring of $G(R \cup N(R))$ such that w_1 and w_2 have distinct colors. In the first case, we will say that R *is of edge type*, in the second case—that R *is of vertex type*.

Note that since G is critical, each component R of edge type has a proper 2-coloring of $G(R \cup N(R))$ such that w_1 and w_2 have distinct colors, and each component R of vertex type has a proper 2-coloring of $G(R \cup N(R))$ such that w_1 and w_2 have the same color. Thus, in order for G to be non-bipartite, the digraph $D(G)$ must have an odd number of edges corresponding to edge-type components.

Our last step is to understand what edge-type components and vertex-type components are. Edge-type components by definition are Z_2-components, and we know their structure from Lemma 4. Suppose that R is a vertex-type component and $N(R) = \{w_1, w_2\}$, where $\deg_{G(R\cup N(R))}(w_1) > 1$ and $\deg_{G(R\cup N(R))}(w_2) = 1$. Then $R' = R + w_2$ where w_2 has only one neighbor outside of R' that is of edge type. Thus we have the following lemma.

Lemma 8. *Let G be a 2-special hypergraph with $|H(G)| \geq 2$. If R is an edge-type component of $G[L(G)]$ with $N(R) = \{w_1, w_2\}$, then $R \cup \{w_1, w_2\}$ is obtained from a 2-special hypergraph P with $|H(P)| = 1$ by splitting the vertex w of degree at least 3 in P into two vertices, w_1 and w_2, so that the degree of w_2 is 1. The hyperedge that contains w_2 may also contain w_1. Similarly, if R is a vertex-type component of $G[L(G)]$ with $N(R) = \{w_1, w_2\}$, then R is obtained from a 2-special hypergraph P with $H(P) = \{w_1\}$ having a graph edge w_1w_2 by deleting this edge.*

Summarizing we have the following description of 2-special hypergraphs.

Theorem 9. *Let G be a 2-special hypergraph. If $H(G) = \emptyset$, then G is an odd cycle. If $H(G) = \{w\}$, then $G[L(G)]$ is a Z_2-graph, say G', and G is obtained from G' by adding the edge wx for each vertex x of degree 1 and including w in some cut edges of G'. Suppose that $|H(G)| = t \geq 2$. Then G is obtained from the graph cycle $C_t = (w_1, \dots, w_t)$ by replacing each edge w_iw_{i+1} (where the count is modulo t) with a subgraph R_i such that:*

(i) *the degree of w_i in R_i is at least 2, and the degree of w_{i+1} in R_i is 1;*
(ii) *each hypergraph $G[R_i - w_i - w_{i+1}]$ is a component of $G[L(G)]$ either of edge type or of vertex type;*
(iii) *the number of edge-type components is odd.*

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