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Describing faces in plane triangulations

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Dedicated to Douglas R. Woodall on the occasion of his 70th birthday

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ABSTRACT

Lebesgue (1940) proved that every plane triangulation contains a face with the vertex-degrees majorized by one of the following triples:

$$(3, 6, \infty), (3, 7, 41), (3, 8, 23), (3, 9, 17), (3, 10, 14), (3, 11, 13),$$
$$(4, 4, \infty), (4, 5, 19), (4, 6, 11), (4, 7, 9), (5, 5, 9), (5, 6, 7).$$

Jendrol' (1999) improved this description, except for $(4, 4, \infty)$ and $(4, 6, 11)$, to

$$(3, 4, 35), (3, 5, 21), (3, 6, 20), (3, 7, 16), (3, 8, 14), (3, 9, 14), (3, 10, 13),$$
$$(4, 4, \infty), (4, 5, 13), (4, 6, 17), (4, 7, 8), (5, 5, 7), (5, 6, 6)$$

and conjectured that the tight description is

$$(3, 4, 30), (3, 5, 18), (3, 6, 20), (3, 7, 14), (3, 8, 14), (3, 9, 12), (3, 10, 12),$$
$$(4, 4, \infty), (4, 5, 10), (4, 6, 15), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

We prove that in fact every plane triangulation contains a face with the vertex-degrees majorized by one of the following triples, where every parameter is tight:

$$(3, 4, 31), (3, 5, 21), (3, 6, 20), (3, 7, 13), (3, 8, 14), (3, 9, 12), (3, 10, 12),$$
$$(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

1. Introduction

The *degree* $d(v)$ of a vertex v ($r(f)$ of a face f) in a plane map M is the number of edges incident with it (loops are counted twice in $d(v)$, and cut-edges are counted twice in $r(f)$). By Δ and δ denote the maximum and minimum vertex degrees of M, respectively. A *k-vertex* (*k-face*) is a vertex (face) with degree k; a k^+*-vertex* has degree at least k, etc.

It is well known that each *normal* plane map, in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three, has a 5^--vertex and a 5^--face. From now on, M denotes a normal plane map.

As proved by Steinitz [31], 3-polytopes are in 1–1 correspondence with 3-connected planar graphs. Plane triangulations are triangulated 3-polytopes; in particular, plane triangulations have neither loops nor multiple edges.

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The *weight* of a face in M is the degree-sum of its boundary vertices, and $w(M)$, or simply w, denotes the minimum weight of 5^--faces in M.

Let a face f be incident with vertices $x_1, \dots, x_{r(f)}$, where $d(x_1) \le d(x_2) \le \cdots \le d(x_{r(x)})$. We say that f is a *face of type* $(k_1, \dots, k_{r(f)})$, or simply a $(k_1, \dots, k_{r(f)})$*-face*, where $k_1 \le \cdots \le k_{r(f)}$, if $d(x_1) = k_1$, $d(x_2) = k_2$, and $d(x_i) \le k_i$ whenever $3 \le i \le r(f)$. In other words, the boundary of a $(k_1, \dots, k_{r(f)})$-face has a k_1-vertex, another vertex of degree k_2, yet another vertex of degree at most k_3, and so on. By a $(k_1, k_2^-, k_3, \dots, k_{r(f)})$*-face* we mean a $(k_1, l_2, k_3, \dots, k_{r(f)})$-face with $k_1 \le l_2 \le k_2$, etc.

Back in 1940, Lebesgue [23] gave an approximate description of 5^--faces in normal plane maps.

Theorem 1 (*Lebesgue [23]*)**.** *Every normal plane map has a 5^--face of one of the following types:*

$$(3, 6^-, \infty), (3, 7, 41), (3, 8, 23), (3, 9, 17), (3, 10, 14), (3, 11, 13),$$
$$(4, 4, \infty), (4, 5, 19), (4, 6, 11), (4, 7, 9), (5, 5, 9), (5, 6, 7),$$
$$(3, 3, 3, \infty), (3, 3, 4, 11), (3, 3, 5, 7), (3, 4, 4, 5), (3, 3, 3, 3, 5).$$

Theorem 1, along with other ideas in Lebesgue [23], has a lot of applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [8,28,30]).

Some parameters of Lebesgue's Theorem 1 were improved for certain subclasses of plane graphs. In 1963, Kotzig [21] proved that every plane triangulation with $\delta = 5$ satisfies $w \le 18$ and conjectured that $w \le 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 2 (*Borodin [2]*)**.** *Every normal plane map with $\delta = 5$ has a $(5, 5, 7)$-face or a $(5, 6, 6)$-face, where all parameters are tight.*

Theorem 2 also confirmed a conjecture of Grünbaum [16] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [29]).

We note that a 3-polytope with $(4, 4, \infty)$-faces can have unbounded w, as follows from the n-pyramid. The same is true concerning $(3, 3, 3, \infty)$-faces: take the double $2n$-pyramid and delete all even upper spokes and all odd lower ones to obtain a quadrangulation having only $(3, 3, 3, 2n)$-faces.

For plane triangulations without 4-vertices, Kotzig [22] proved $w \le 39$, and Borodin [4], confirming Kotzig's conjecture in [22], proved $w \le 29$, which is best possible due to the dual of the twice-truncated dodecahedron. This was strengthened by Borodin [5] as follows: either there is a triangle of weight at most 17, or a triangle of weight at most 29 incident with a 3-vertex. Borodin [6] further shows that each triangulated 3-polytope without $(4^-, 4, \infty)$-faces satisfies $w \le 29$, and that for triangulations without $(4, 4, \infty)$-faces there is a sharp bound $w \le 37$.

Note that $29 = 3+5+21 = 3+6+20$, so already [4] implies that the terms $(3, 5, 21)$ and $(3, 6, 20)$ could be expected to appear in a tight description of faces in plane triangulations, where the sharpness of 20 in $(3, 6, 20)$ follows from the dual of the twice-truncated dodecahedron while the sharpness of 21 in $(3, 5, 21)$ is first established in the present paper (see Fig. 2). A similar remark concerns the tight term $(3, 4, 30)$ that comes from Borodin [6].

For arbitrary normal plane maps, Theorem 1 yields $w \le \max\{51, \Delta + 9\}$. Horňák and Jendrol' [17] strengthened this as follows: if there are neither $(4^-, 4, \infty)$-faces nor $(3, 3, 3, \infty)$-faces, then $w \le 47$. Borodin and Woodall [12] proved that forbidding $(3, 3, 3, \infty)$-faces implies $w \le \max\{29, \Delta + 8\}$.

Also, Horňák and Jendrol' [17] consider the minimum, w^*, of face weights over all faces instead of over only 5^--faces, as was being done before beginning with Lebesgue [23]. Clearly, $w^* \le w$. They proved [17] that any normal map avoiding $(4^-, 4, \infty)$-faces and $(3, 3, 3, \infty)$-faces satisfies $w^* \le 32$.

For quadrangulated 3-polytopes, Avgustinovich and Borodin [1] improved the description of 4-faces implied by Lebesgue's Theorem as follows: $(3, 3, 3, \infty)$, $(3, 3, 4, 10)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$.

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in a recent survey by Jendrol' and Voss [19], and also in [3,5,11–15,18,20,24–27,32].

In 2002, Borodin [7] strengthened nine parameters in Lebesgue's Theorem 1 without changing the others (the entries marked by an asterisk are best possible, see [7]).

Theorem 3 (*Borodin [7]*)**.** *Every normal plane map has a 5^--face of one of the following types:*

$$(3, 6^-, \infty^*), (3, 7^*, 22), (3, 8^*, 22), (3, 9^*, 15), (3, 10^*, 13), (3, 11^*, 12),$$
$$(4, 4, \infty^*), (4, 5^*, 17), (4, 6^*, 11), (4, 7^*, 8), (5, 5^*, 8), (5, 6, 6^*),$$
$$(3, 3, 3, \infty^*), (3, 3, 4^*, 11), (3, 3, 5^*, 7), (3, 4, 4, 5^*), (3, 3, 3, 3, 5^*).$$

In particular, to check the tightness of $(3, 6^-, \infty^*)$ in Theorems 1 and 3 we may use the following construction (Borodin [5]), derived from the double n-pyramid: join each vertex of a cycle $C_n = x_1 \dots x_n$ to n-vertices v_j with $1 \le j \le 2$, delete all edges $x_i x_{i+1}$ (addition modulo n), and for each i and j add a vertex $y_{i,j}$ joined to x_i, x_{i+1}, and v_j. In the 3-polytope obtained, every 3-face is incident with a 3-vertex, 6-vertex, and $2n$-vertex, while every 4^+-face is a 4-face incident with two 3-vertices and two 6-vertices.

Note that for plane triangulations the term $(3, 6^-, \infty)$ is not tight, as follows from Theorems 5 and 8 below.

We can see already from Lebesgue's Theorem 1 that if $\delta \ge 4$, then there is either a $(4, 4, \infty)$-face, or a 3-face of bounded weight. From Theorem 3 we have a bit more, and the ultimate result in this direction is as follows.

Theorem 4 (*Borodin–Ivanova [9]*)**.** *Every normal plane map without* 3*-vertices has a* 3*-face of one of the following types, where all parameters are sharp:*

$$(4, 4, \infty), (4, 5, 14), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

In 1999, Jendrol' [18] improved the description of faces that comes from Lebesgue's Theorem 1 for the case of plane triangulations, except $(4, 4, \infty)$ and $(4, 6, 11)$.

Theorem 5 (*Jendrol' [18]*)**.** *Every plane triangulation of order at least* 5 *has a face of one of the following types:*

$$(3, 4, 35), (3, 5, 21), (3, 6, 20), (3, 7, 16), (3, 8, 14), (3, 9, 14), (3, 10, 13),$$
$$(4, 4, \infty), (4, 5, 13), (4, 6, 17), (4, 7, 8), (5, 5, 7), (5, 6, 6).$$

The next conjecture was suggested by Jendrol' [18], and it also appears in a recent survey by Jendrol'–Voss [19, Conjecture 4.9].

Conjecture 6 (*Jendrol' [18]*)**.** *Every plane triangulation of order at least* 5 *has a face of one of the following types, where every parameter is tight:*

$$(3, 4, 30), (3, 5, 18), (3, 6, 20), (3, 7, 14), (3, 8, 14), (3, 9, 12), (3, 10, 12),$$
$$(4, 4, \infty), (4, 5, 10), (4, 6, 15), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

Recently, the first counterexample to Conjecture 6 was constructed in Borodin–Ivanova [10], as a corollary of the following theorem, which shows that $(4, 5, 11)$ can be attained.

Theorem 7 (*Borodin–Ivanova [10]*)**.** *Every plane triangulation with* $\delta \ge 4$ *has a face of one of the following types, where all parameters are sharp:*

$$(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

Comparing Theorems 4 and 7, we see that 3-faces are more restricted in the class of plane triangulations than in arbitrary normal plane maps.

The purpose of this paper is to characterize the faces of arbitrary plane triangulations.

Theorem 8. *Every plane triangulation of order at least* 5 *has a face of one of the following types:*

(Ta) $(3, 4, 31)$,	(Th) $(4, 4, \infty)$,	(Tl) $(5, 5, 7)$,
(Tb) $(3, 5, 21)$,	(Ti) $(4, 5, 11)$,	(Tm) $(5, 6, 6)$.
(Tc) $(3, 6, 20)$,	(Tj) $(4, 6, 10)$,	
(Td) $(3, 7, 13)$,	(Tk) $(4, 7, 7)$,	
(Te) $(3, 8, 14)$,		
(Tf) $(3, 9, 12)$,		
(Tg) $(3, 10, 12)$,		

Moreover, all parameters in (Ta)–(Tm) *are tight.*

In particular, we see that Theorem 8 extends or strengthens the above mentioned results in [2,4,6,10,18,21,22] and corrects the terms $(3, 4, 30)$, $(3, 5, 18)$, $(3, 7, 14)$, $(4, 5, 10)$, and $(4, 6, 15)$ in Conjecture 6.

2. The tightness of Theorem 8

The bounds in Theorem 8 are all sharp, as follows from the constructions in Figs. 1–6.

Namely, in Fig. 1 we see how to transform the snub dodecahedron (that is a 5-regular polyhedron in which every vertex is incident with one pentagon and four triangles) into a triangulation with all vertices having degree from 3, 4, 5, 6, 8, and at

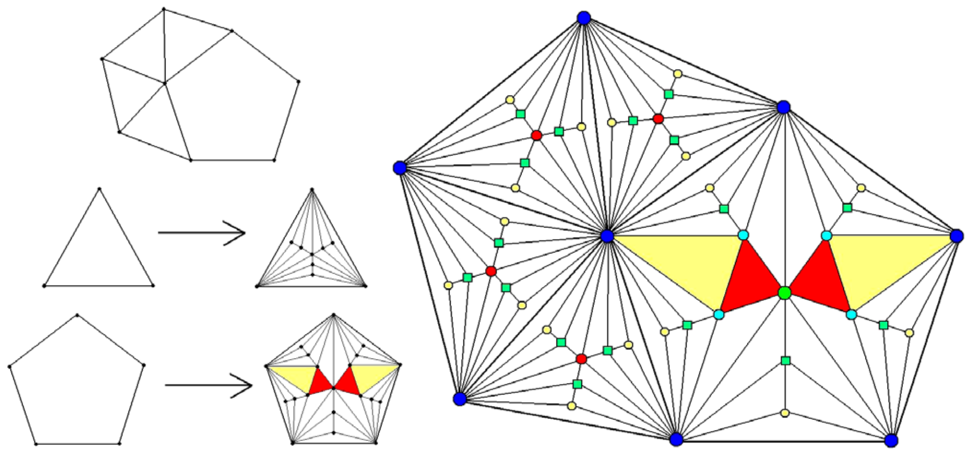

Fig. 1. A construction with no faces of types other than (3, 4, 31) in Theorem 8, showing the tightness of (Ta).

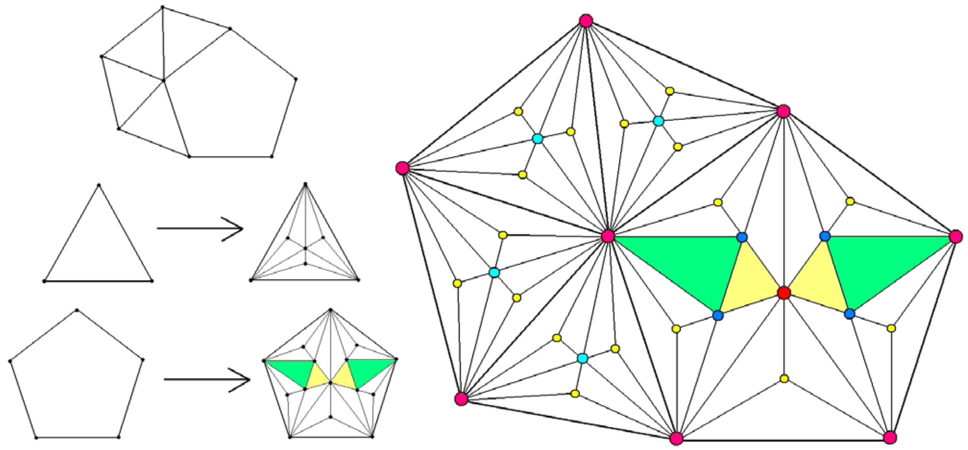

Fig. 2. A construction with only (3, 5, 21)-faces to justify (Tb).

least 31 and such that there are no 3-faces of the types mentioned in Theorem 8 other than (3, 4, 31). A similar construction with only (3, 5, 21)-faces, justifying the tightness of (Tb), is given in Fig. 2.

In Fig. 3 we see simple constructions showing the tightness of (Tc), (Te), (Tg), and (Tj)–(Tm). For (Tm), we start from the dodecahedron, for (Tc) and (Tj) from the icosahedron, and for (Tk) from the octahedron. To obtain a construction for (Te), we put a 3-vertex into each face of the previously obtained construction confirming the tightness of (Tk). A triangulation justifying (Tl) is obtained by gluing two copies shown in Fig. 3(Tl) along the outside cycle. Recall that the tightness of (Th) follows from the above mentioned double pyramid.

Fig. 4 represents a replacement for each face of the icosahedron (a 5-regular triangulation on twelve vertices) such that the resulting triangulation has vertices of degree 3, 7, and at least 13 only. More specifically, the corner vertices have degree 15, and there are three 13-vertices, each incident with three faces avoiding 3-vertices (shadowed). Furthermore, if a face is incident with a 3-vertex, then it is incident with a 7-vertex and a 13^+-vertex. This construction confirms the tightness of (Td).

In Fig. 5 we see one eighth of a construction derived from the octahedron that has only (3, 9, 12)-faces and confirms the tightness of (Tf).

Finally, Fig. 6 represents a plane triangulation which arises from the snub dodecahedron and confirms the tightness of (Ti) in Theorem 8.

3. Proving the main statement of Theorem 8

A face is *hard* if it is not incident with a 3-vertex. Suppose T^* is a counterexample to Theorem 8 with the fewest hard faces.

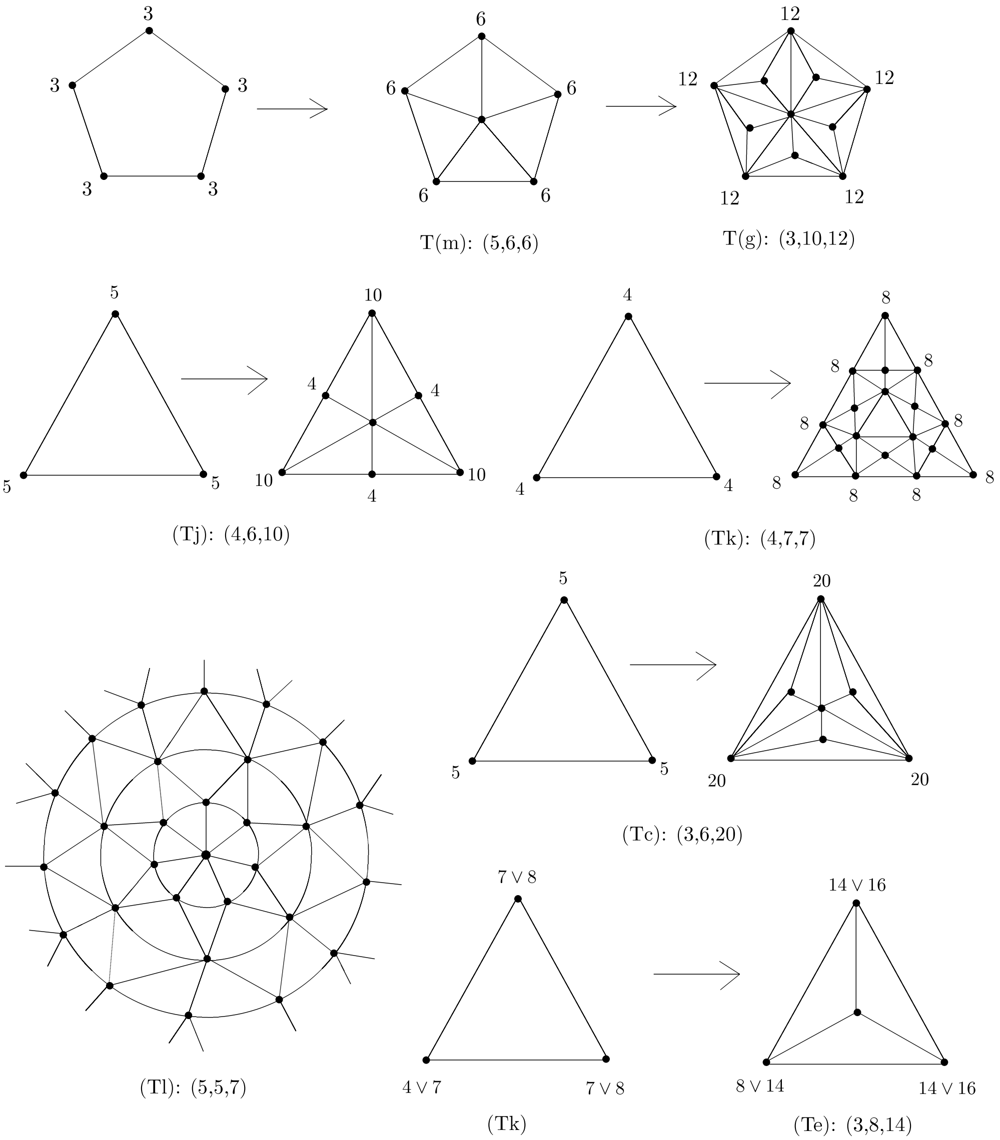

Fig. 3. Constructions showing the tightness of (Tc), (Te), (Tg), and (Tj)–(Tm).

3.1. Simple structural properties of the counterexample T^*

By $v_1, \ldots, v_{d(v)}$ we denote the neighbors of a vertex v in a cyclic order.

We will use the following simple structural properties of T^*.

(SP1) *No* 3*-vertex is adjacent to a* 3*-vertex.*

Indeed, $T^* \neq K_4$ and T^* has no multiple edges.

(SP2) *A* 4*-vertex has at most one neighbor of degree* 3.

This follows from the absence of loops and multiple edges in T^*.

(SP3) *A* $(2k+1)$*-vertex* v *with* $2 \le k \le 5$ *cannot have neighbors* v_1 and v_{2k-1} *of degree* 3.

Indeed, suppose $d(v_1) = d(v_{2k-1}) = 3$. Note that since $d(v_{2k})$ and $d(v_{2k+1})$ are sufficiently large due to (Tb), (Td), (Tf), and (Tg), adding a vertex z in the face $vv_{2k}v_{2k+1}$ followed by joining z to v, v_{2k}, and v_{2k+1} results in a new counterexample with fewer hard faces than T, a contradiction.

(SP4) *A* $(2k+1)$*-vertex* v *with* $2 \le k \le 5$ *cannot have* k *neighbors of degree* 3.

This follows immediately from (SP3).

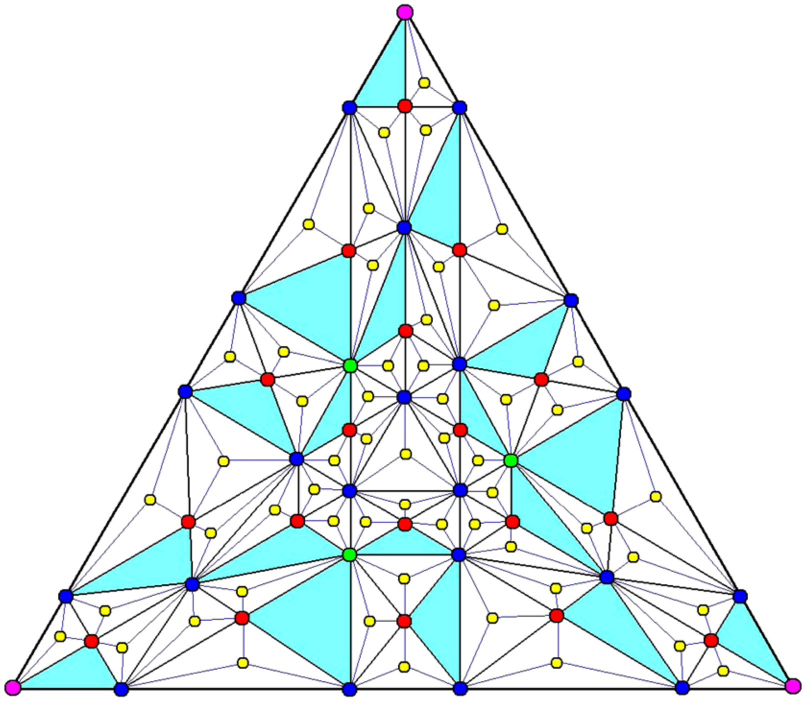

Fig. 4. One twentieth of the icosahedron-like triangulation with only (3, 7, 13)-faces confirming the tightness of (Td).

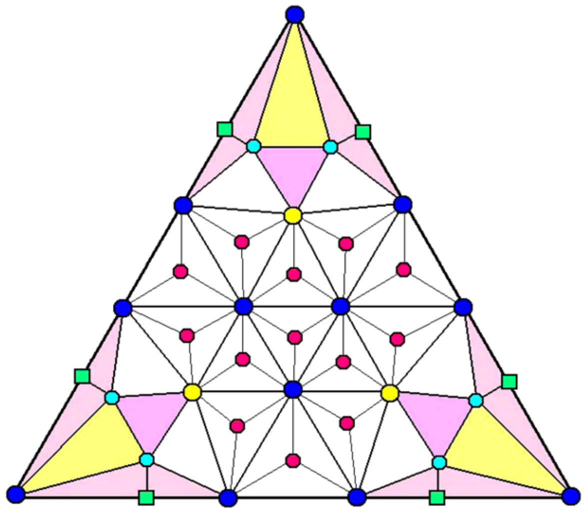

Fig. 5. This replacement for every face of the octahedron produces only (3, 9, 12)-faces, as required in (Tf).

3.2. Discharging

The sets of vertices, edges, and faces of T^* are denoted by V, E, and F, respectively. Euler's formula $|V| - |E| + |F| = 2$ for T^* implies

$$\sum_{v\in V}(d(v) - 6) + \sum_{f\in F}(2r(f) - 6) = -12. \tag{1}$$

We assign a *charge* $\mu(v) = d(v) - 6$ to every vertex v and $\mu(f) = 0$ to every face f, so only 5^--vertices have a negative charge. Using the properties of T^* as a counterexample, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

First we give a few definitions concerning 7-vertices.

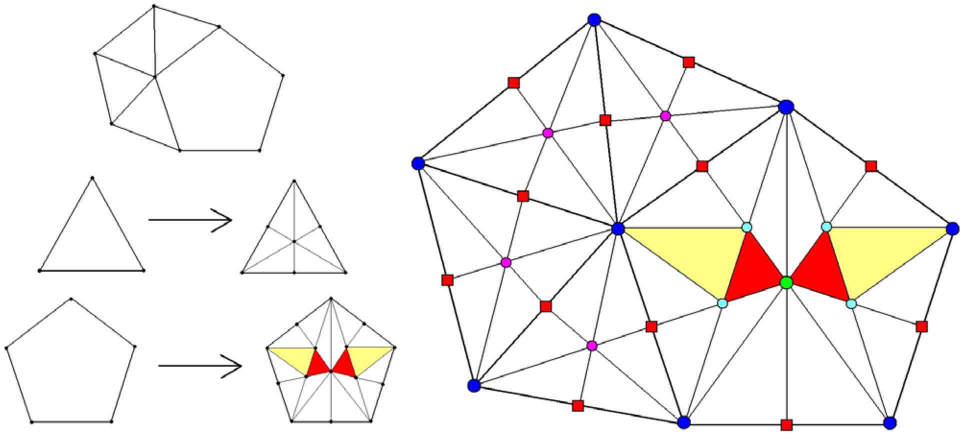

Fig. 6. A construction (Borodin–Ivanova [9]) with only (4, 5, 11)-faces, as in (Ti).

A 7-vertex v is *poor* if $d(v_1) = 4$, $4 \le d(v_3) \le 5$, and $d(v_5) = 5$. By a 7_p-*vertex* we mean a poor vertex. A 7_p^4-*vertex* or 7_p^5-*vertex* stands for a poor vertex with $d(v_3) = 4$ or $d(v_3) = 5$, respectively. A 7_p-vertex is *coupled* if it is adjacent to a 7_p-vertex (a coupled 7_p^4-vertex is shown in Fig. 7(R4b1); for a coupled 7_p^5-vertex see Fig. 7(R4b2)).

A 7-vertex v is *bad* if $d(v_4) = d(v_6) = 3$, $d(v_1) = 7$, $d(v_2) = 5$, and there is a face v_2v_3z with $z \ne v$ and $d(z) \ge 5$ (see Fig. 7(R4c)). Note that $d(v_3) \ge 14$ and $d(v_7) \ge 14$ here due to (Td).

We use the following rules of discharging (see Fig. 7):

R1. Every 3-vertex v receives the following charge from its neighbors:
- (a) $\frac{3}{2}$ from each of v_2 and v_3 if $d(v_1) \le 6$;
- (b) $\frac{2}{3}$ from v_1 and $\frac{7}{6}$ from each of v_2 and v_3 if $d(v_1) = 7$;
- (c) $\frac{1}{2}$ from v_1 and $\frac{5}{4}$ from each of v_2 and v_3 if $d(v_1) = 8$;
- (d) $\frac{3}{4}$ from v_1 and $\frac{9}{8}$ from each of v_2 and v_3 if $d(v_1) = 9$;
- (e) $\frac{4}{5}$ from v_1 and $\frac{11}{10}$ from each of v_2 and v_3 if $d(v_1) = 10$;
- (f) 1 from each neighbor if v has no 10^--neighbors.

R2. If T is a face uvw with $d(v) = 4$ and $d(u) \ge 8$, then v receives from u through T:
- (a) $\frac{1}{2}$ if $d(w) \le 6$;
- (b) $\frac{1}{4}$ if $d(w) = 7$, and v also receives $\frac{1}{2}$ from w along the edge wv;
- (c) $\frac{1}{4}$ if $d(w) \ge 8$ (and $\frac{1}{4}$ from w by symmetry), with the following exception (c*).
- (c*) If $d(v_1) = 3$, $d(v_2) \ge 32$, $d(v_3) \ge 13$, and $d(v_4) \ge 32$, then v receives $\frac{1}{2}$ from v_3 through each of the faces v_2vv_3 and v_3vv_4, and nothing from v_2 and v_4 through these faces.

R3. If T is a face uvw with $d(v) = 5$ and $d(u) \ge 8$, then v receives from u through T:
- (a) $\frac{1}{8}$ if $d(w) = 5$ and $d(u) \le 11$;
- (b) $\frac{1}{4}$ if $d(w) \ge 6$ and $d(u) \le 11$;
- (c) $\frac{1}{4}$ if $d(w) = 5$ and $d(u) \ge 12$;
- (d) $\frac{1}{2}$ if $d(w) \ge 6$ and $d(u) \ge 12$, except for (d*);
- (d*) $\frac{1}{4}$ if $d(w) = 7$ and there is a face $uv'w$ with $d(v') = 3$.

R4. A 7-vertex v gives to its 5-neighbor v_2 the following charge.
- (a) If v is neither bad nor coupled poor while $d(v_1) \ge 6$, then
- (a1) $\frac{1}{4}$ when $d(v_3) \ge 8$, or
- (a2) $\frac{1}{3}$ when $d(v_3) \le 7$ and $d(v_1) \le 7$.
- (b) If v and v_1 are coupled poor 7-vertices (and hence $d(v_6) = 4$), then
- (b1) $\frac{1}{8}$ if $d(v_4) = 4$, or
- (b2) $\frac{3}{8}$ if $d(v_4) = 5$, in which case $\frac{1}{4}$ is also given by v to v_4.
- (c) $\frac{1}{6}$ if v is a bad vertex.

R5. A 7-vertex v receives the following charge from a 8^+-vertex v_2 through the face v_1vv_2:
- (a) $\frac{1}{4}$ if $d(v_1) = 4$;
- (b) suppose $d(v_3) = 3$, $d(v_1) = 5$, and there is a face $v_1v'v_2$ with $v' \ne v$; then
- (b1) $\frac{1}{3}$ if $d(v') \le 4$, or
- (b2) $\frac{1}{4}$ if $d(v_4) \ge 5$;

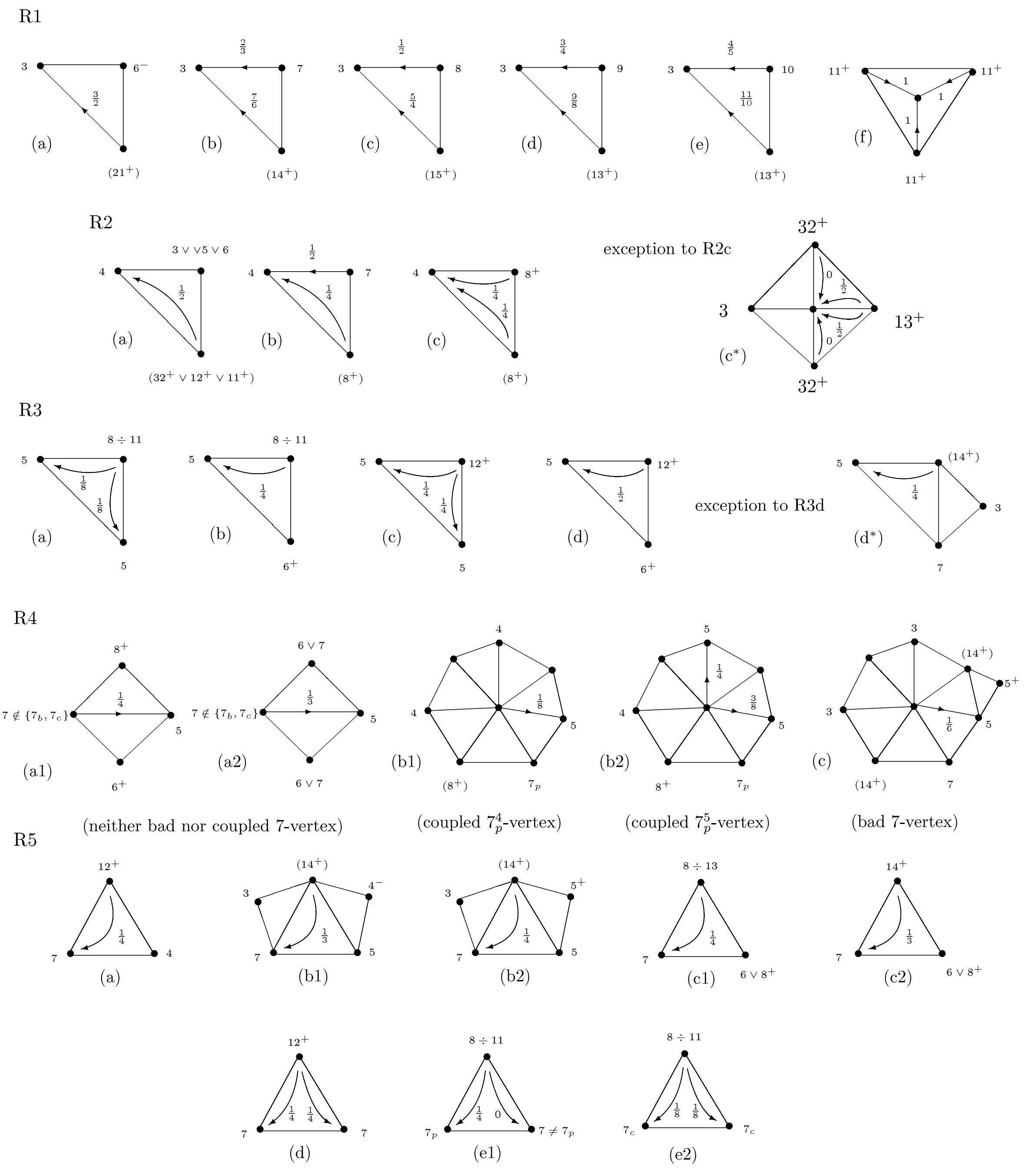

Fig. 7. Rules of discharging.

(c) suppose $d(v_1) = 6$ or $d(v_1) \geq 8$; then

(c1) $\frac{1}{4}$ if $d(v_2) \leq 13$, or

(c2) $\frac{1}{3}$ if $d(v_2) \geq 14$;

(d) $\frac{1}{4}$ if $d(v_1) = 7$ and $d(v_2) \geq 12$;

(e) suppose $d(v_1) = 7$ and $d(v_2) \leq 11$, then

(e1) $\frac{1}{4}$ if v_1 is poor but not coupled, or

(e2) $\frac{1}{8}$ otherwise.

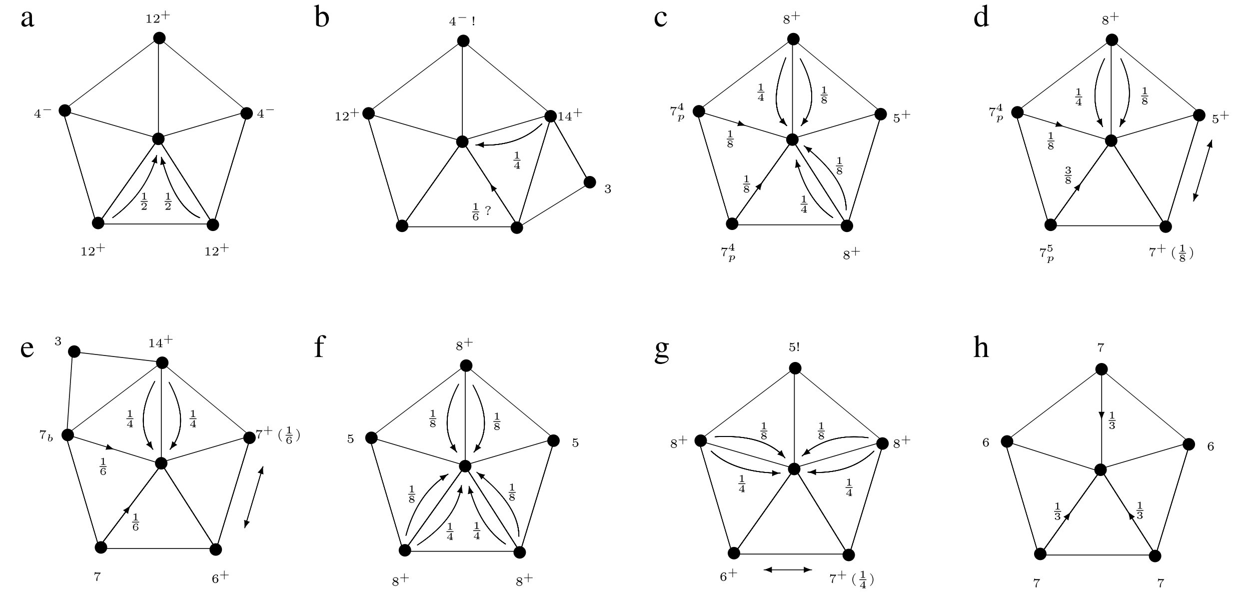

Fig. 8. To Case 3.

3.3. *Proving $\mu'(x) \geq 0$ whenever $x \in V \cup F$*

If f is a face in T^*, then f does not participate in discharging, and so $\mu'(v) = \mu(f) = 2 \times 3 - 6 = 0$.

Now let v be a vertex in T^*.

Case 1. $d(v) = 3$. Since v receives the total of precisely 3 from its neighbors by R1, we have $\mu'(v) = 3 - 6 + 3 = 0$.

Case 2. $d(v) = 4$. By R2, v receives $\frac{1}{2}$ through each incident face not incident with a 7-vertex. If $d(v_1) = 7$, then v receives $\frac{1}{2}$ from v_1 and $\frac{1}{4}$ through each of the faces v_1vv_2 and v_1vv_4, so we have $\mu'(v) = 4 - 6 + 2 = 0$.

Case 3. $d(v) = 5$. Since $\mu(v) = -1$, we have to check that v receives a total of at least 1 from its neighbors $v_1, \ldots, v_5$. Some situations arising in Case 3 are shown in Fig. 8.

Subcase 3.1. $d(v_1) \leq 4$ and $d(v_3) \leq 4$ (see Fig. 8(a)). Here, $d(v_4) \geq 12$ and $d(v_5) \geq 12$ by(Ti), so v receives $\frac{1}{2}$ from each of v_4 and v_5 through face vv_4v_5 by R3d.

Subcase 3.2. $d(v_2) \leq 4$ and $d(v_4) \geq 5$. Now $d(v_1) \geq 12$ and $d(v_3) \geq 12$. If $d(v_4) = 5$, then $d(v_5) \geq 8$ by (Tl), so v through face vv_1v_5 receives $\frac{1}{2}$ from v_1 by R3d and at least $\frac{1}{4}$ from v_5 by R3b. Furthermore, v receives $\frac{1}{4}$ from v_3 through face vv_3v_4 by R3c.

Suppose $d(v_4) \geq 6$ and $d(v_5) \geq 6$ (in fact either $d(v_4) \geq 7$ or $d(v_5) \geq 7$ due to (Tm)). By symmetry, it suffices to check that v receives at least $\frac{1}{2}$ from v_3 and v_4 together. If R3d* is not applied to v_3, then v receives $\frac{1}{2}$ from v_3 through face vv_3v_4 by R3d, so suppose it is (see Fig. 8(b)). If v receives at least $\frac{1}{4}$ from v_4, then we are done since v receives at least $\frac{1}{4}$ from v_3 by R3(c–d*). According to R4, this does not happen only if v_4 is either a coupled 7^4_p-vertex or a bad 7-vertex.

Note that v_4 is not poor since it has a 3-neighbor. Finally, suppose that v_4 is a bad 7-vertex (see Fig. 8(b) again). However, v_4 cannot give $\frac{1}{6}$ to v by R4c since this requires $d(v_2) \geq 5$, contrary to the above assumption.

From now on we assume that 5-vertex v has no 4^--neighbors.

Subcase 3.3. There is a donation of $\frac{1}{8}$ to v from a 7^4_p-vertex v_3 by R4b1 (see Fig. 8(c), (d)). Suppose v_2 is a 7_p-vertex, so that $d(v_4) \geq 8$. If v_2 is a 7^4_p-vertex (Fig. 8(c)), then $d(v_1) \geq 8$. Since v receives at least $\frac{1}{4} + \frac{1}{8}$ from each of v_1 and v_4 by R3(b,d,d*) and $\frac{1}{8}$ from each of v_2 and v_3 by R4b1, we have $\mu'(v) \geq 0$.

Now suppose that v_2 is a 7^5_p-vertex (Fig. 8(d)). Still v receives at least $\frac{1}{4} + 2 \times \frac{1}{8}$ from v_3 and v_4 together. Also v receives $\frac{3}{8}$ from v_2 by R4b2. We have to find $\frac{1}{8}$ more to be sure that $\mu'(v) \geq 0$, but at least one of v_1 and v_5 is a 7^+-vertex due to (Tm), and so it cannot give less than $\frac{1}{8}$ to v by R3 and R4.

Subcase 3.4. There is a donation of $\frac{1}{6}$ to v from a bad 7-vertex v_3 by R4c (see Fig. 8(e) for the final situation here). Suppose that $d(v_2) \geq 14$ and $d(v_4) = 7$ (as it was assumed, $d(v_1) \geq 5$). Note that v_2 gives at least $\frac{1}{4}$ to v through each of the faces v_2vv_1 and v_2vv_3 by R3(c–d*). Due to Subcase 3.3, v_4 gives at least $\frac{1}{6}$ to v. At least one of v_1 and v_5 is a 7^+-vertex due to the absence of (5, 6, 6)-faces, and so also gives at least $\frac{1}{6}$ to v. This yields $\mu'(v) \geq -1 + 2 \times \frac{1}{4} + 3 \times \frac{1}{6} = 0$, as desired.

Hereafter, we assume that each 7-neighbor gives at least $\frac{1}{4}$ to v according to R4. Of course, the same is true for each 8^+-neighbor of v due to R3, where in the worst case v receives $\frac{1}{8} + \frac{1}{8}$ through two faces by R3a.

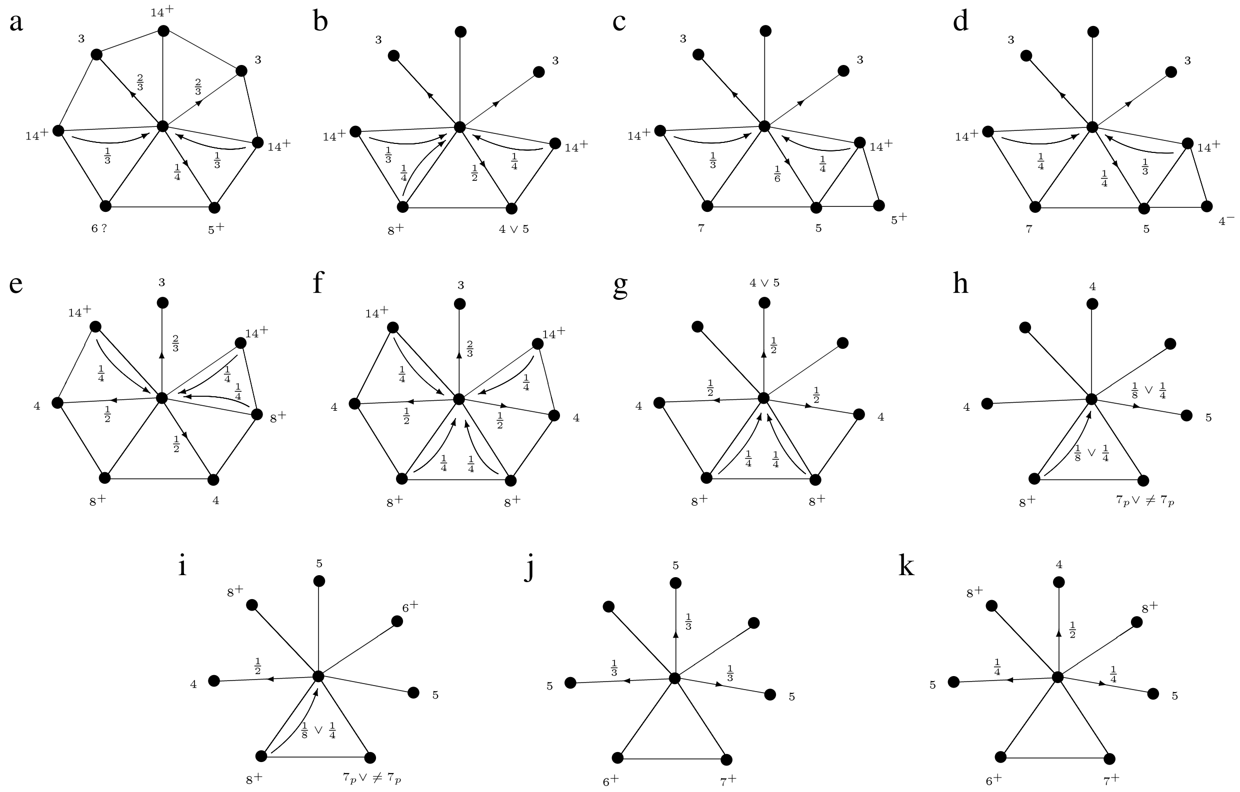

Fig. 9. Handling 7-vertices in Case 5.

Subcase 3.5. $d(v_1) = d(v_3) = 5$ (see Fig. 8(f)). Since $d(v_2) \geq 8$, $d(v_4) \geq 8$, and $d(v_5) \geq 8$ by (Tl), it follows that v receives at least $\frac{1}{4}$ from v_2 and at least $\frac{1}{4} + \frac{1}{8}$ from each of v_4 and v_5 through incident faces by R3, which implies $\mu'(v) \geq 0$.

Subcase 3.6. $d(v_2) = 5$, $d(v_4) \geq 6$, and $d(v_5) \geq 7$ (Fig. 8(g)). Recall that $d(v_1) \geq 8$ and $d(v_3) \geq 8$ due to (Tl). Now v receives at least $\frac{3}{8}$ from each of v_1 and v_3 and at least $\frac{1}{4}$ from v_4 or v_5, and we are done.

Subcase 3.7. v has no 5^--neighbors (see Fig. 8(h) for the final situation). If v has at least four 7^+-neighbors, then $\mu'(v) \geq -1 + 4 \times \frac{1}{4} = 0$. On the other hand, v has at least three 7^+-neighbors due to (Tm), so we can assume that $d(v_1) = d(v_3) = 6$, $d(v_2) \geq 7$, $d(v_4) \geq 7$, and $d(v_5) \geq 7$. If at least one of v_2, v_4, v_5 is a 8^+-vertex, then R3a is not applied to it, and we have $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$. Thus suppose $d(v_2) = d(v_4) = d(v_5) = 7$. Inspecting R4, we see that each of v_2, v_4, v_5 gives v either $\frac{1}{3}$ by R4a2 or, due to (Tk), $\frac{3}{8}$ by R4b2. This implies $\mu'(v) \geq -1 + 2 \times \frac{1}{3} = 0$.

Case 4. $d(v) = 6$. Since v does not participate in discharging, we have $\mu'(v) = \mu(v) = 0$.

Case 5. $d(v) = 7$ (see Fig. 9). We note that handling 7-vertices is the most difficult part of the proof of Theorem 8. By (SP4), v has at most two 3-neighbors. By our rules, v gives $\frac{2}{3}$ to each 3-neighbor (R1b), $\frac{1}{2}$ to each 4-neighbor (R2b), and at most $\frac{3}{8}$ to each 5-neighbor (R4).

Subcase 5.1. v has two 3-neighbors (Fig. 9(a)–(d)). By (SP3), we can assume that $d(v_1) = d(v_3) = 3$. It follows from (Td) that $d(v_4) \geq 14$ and $d(v_7) \geq 14$. Note that v receives by R5(a,b,c2,d,f) at least $\frac{1}{4}$ from each of v_4 and v_7 through faces v_4vv_5 and v_6vv_7, respectively.

Therefore, v has at least $\frac{3}{2}$ to discharge to its 5^--neighbors. Since v gives $\frac{4}{3}$ to its 3-neighbors and nothing to its 6^+-neighbors, we can assume that $4 \leq d(v_5) \leq 5$. Due to (Tk) and (Tl), we have $d(v_6) \geq 6$.

Due to (Tk) and (Tl), we have $d(v_6) \geq 6$ (see Fig. 9(a)). First suppose that $d(v_6) = 6$, and hence $d(v_5) = 5$ due to (Tj). Now v receives $\frac{1}{3}$ from v_7 by R5c2. Note that v is neither coupled (since it has a 3-neighbor), nor bad (since it has no 7-neighbor). Therefore, v gives $\frac{1}{4}$ to v_5 by R4a, which yields $\mu'(v) \geq 1 + \frac{1}{3} + \frac{1}{4} - 2 \times \frac{2}{3} - \frac{1}{4} = 0$.

On the other hand, if $d(v_6) \geq 8$ (Fig. 9(b)), then v receives at least $\frac{1}{4} + \frac{1}{3}$ from v_6 and v_7 through face v_6vv_7 by R5c1 and R5c2 and, as mentioned above, at least $\frac{1}{4}$ from v_4. Since v gives at most $\frac{1}{2}$ to v_5, this yields $\mu'(v) \geq 1 + \frac{1}{3} + 2 \times \frac{1}{4} - \frac{4}{3} - \frac{1}{2} = 0$.

Thus it remains to assume that $d(v_6) = 7$. Due to (Tk), we have $d(v_5) = 5$ (see Fig. 9(c), (d)).

Note that v can give at most $\frac{1}{4}$ to v_5 by R4 since $d(v_4) \geq 14$ and v is not coupled. Thus, the total expenditure of v is at most $2 \times \frac{2}{3} + \frac{1}{4}$. If v_5 is bad and hence receives only $\frac{1}{6}$ from v by R4c (see Fig. 9(c)), then $\mu'(v) \geq \frac{3}{2} - \frac{4}{3} - \frac{1}{6} = 0$.

There is only one reason why R4c is not applicable to v_5. Let x, y, v_6, v, v_4 be the neighbors of v_5 in a cyclic order. In these terms, this reason is $d(x) \le 4$ (see Fig. 9(d)). However, then v receives $\frac{1}{3}$ rather than $\frac{1}{4}$ from v_4 by R5b1, while v_5 still receives precisely $\frac{1}{4}$ from v by R4a, which implies $\mu'(v) \ge 1 + \frac{1}{4} + \frac{1}{3} - \frac{4}{3} - \frac{1}{4} = 0$.

Subcase 5.2. v has just one 3-neighbor, v_1 (see Fig. 9(e), (f)). Now $d(v_2) \ge 14$ and $d(v_7) \ge 14$. Still, v has at least $\frac{3}{2}$ to discharge to its neighbors. In particular, v has at least $\frac{5}{6}$ for its 4^+-neighbors. Since v is not poor, it can give at most $\frac{1}{3}$ to its 5-neighbors. Recall that v gives $\frac{1}{2}$ to every 4-neighbor. Thus the only problem to consider is that v has two 4-neighbors.

If $d(v_3) = d(v_5) = 4$ (see Fig. 9(e)), then $d(v_6) \ge 8$ due to (Tk), so v receives at least $3 \times \frac{1}{4}$ from the 8^+-vertices v_2, v_6, and v_7, and we have $\mu'(v) \ge 1 + 3 \times \frac{1}{4} - \frac{2}{3} - 2 \times \frac{1}{2} > 0$.

If $d(v_3) = d(v_6) = 4$ (see Fig. 9(f)), then $d(v_4) \ge 8$ and $d(v_5) \ge 8$, so v receives at least $4 \times \frac{1}{4}$ from its 8^+-neighbors, which yields $\mu'(v) > 0$.

Subcase 5.3. v has no 3-neighbors. In particular, v is not bad. We have nothing to prove unless v is adjacent to three 5^--vertices.

If $d(v_1) = d(v_5) = 4$ (see Fig. 9(g)), then $d(v_6) \ge 8$ and $d(v_7) \ge 8$, so v receives at least $\frac{1}{4} + \frac{1}{4}$ through 3-face vv_6v_7 by R5c2, which implies $\mu'(v) \ge 1 + 2 \times \frac{1}{4} - 3 \times \frac{1}{2} = 0$.

Now suppose $d(v_1) = 4$ and $d(v_5) = 5$, that is v is poor. Here, $d(v_6) \ge 6$ and $d(v_7) \ge 8$. First assume that $d(v_3) = 4$, so v is a 7_p^4-vertex (see Fig. 9(h)).

If v_6 is not a 7_p-vertex, which means that v is not coupled, then v receives at least $\frac{1}{4}$ from v_7 through face v_6vv_7 by R5(c1,c2,d,e1) and gives $\frac{1}{4}$ to v_5 by R4. This yields $\mu'(v) \ge 7 - 6 + \frac{1}{4} - 2 \times \frac{1}{2} - \frac{1}{4} = 0$.

If v_6 is a coupled a 7_p-vertex, then v receives at least $\frac{1}{8}$ from v_7 through face v_6vv_7 by R5(d,e2) and gives $\frac{1}{8}$ to v_5 by R4b1, so $\mu'(v) \ge 7 - 6 + \frac{1}{8} - 2 \times \frac{1}{2} - \frac{1}{8} = 0$.

If $d(v_3) = 5$, which means that v is a 7_p^5-vertex (no matter coupled or not, see Fig. 9(i)), then v receives at least $\frac{1}{8}$ by R5(c1–e2) and gives $\frac{1}{4}$ to v_3 by R4b2 if v is coupled or R4a otherwise and at most $\frac{3}{8}$ to v_4 by Rb2 or R4a, respectively. This implies that $\mu'(v) \ge 7 - 6 + \frac{1}{8} - \frac{1}{2} - \frac{1}{4} - \frac{3}{8} = 0$.

Finally, suppose that $d(v_1) = d(v_5) = 5$. If $d(v_3) = 5$ (Fig. 9(j)), then $\mu'(v) \ge 7 - 6 - 3 \times \frac{1}{3} = 0$ due to R4(a1,a2). Otherwise (Fig. 9(k)), $\mu'(v) \ge 7 - 6 - \frac{1}{2} - 2 \times \frac{1}{4} = 0$ due to R2b and R4a1.

Remark 1. Every vertex v with $8 \le d(v) \le 11$ sends at most $\frac{1}{4}$ through each incident hard face v_1vv_2 by R2(b,c), R3(a,b), and R5(c1,e1,e2), unless $d(v) = 11$, $d(v_1) = 4$, $d(v_2) = 5$, in which case v sends $\frac{1}{2}$ to v_1 by R2a (and nothing to v_2 by R1–R5).

Case 6. $d(v) = 8$. We may view the donation of $\frac{1}{2}$ by v by R1c to a 3-vertex v_2 as giving $\frac{1}{4}$ to v_2 through each of the non-hard faces v_1vv_2 and v_2vv_3. Due to Remark 1, under this convention v sends at most $\frac{1}{4}$ through each incident face, whence $\mu'(v) \ge 8 - 6 - 8 \times \frac{1}{4} = 0$.

In what follows, let n_3 be the number of 3-neighbors of v.

Case 7. $9 \le d(v) \le 10$. Note that v gives at most $\frac{1}{4}$ through any of $d(v) - 2n_3$ faces not incident with a 3-vertex due to Remark 1.

For $d(v) = 9$ we have $n_3 \le 3$ by (SP4), which implies $\mu'(v) \ge 9 - 6 - n_3 \times \frac{3}{4} - (9 - 2n_3) \times \frac{1}{4} = \frac{3-n_3}{4} \ge 0$ due to R1d.

Suppose $d(v) = 10$; then $\mu'(v) \ge 10 - 6 - n_3 \times \frac{4}{5} - (10 - 2n_3) \times \frac{2}{5} = 0$ in view of R1e.

Case 8. $d(v) = 11$. We may look at the donation of 1 by v to a 3-neighbor w by R1f as giving $\frac{1}{2}$ through each of the two faces incident with edge vw. If so, then v gives $\frac{1}{2}$ through face v_1vv_2 only if $d(v_1) = 3$ and $d(v_2) \ge 11$ by the so modified R1, or $d(v_1) = 4$ and $d(v_2) = 6$ by R2a. Furthermore, any other face receives from v at most $\frac{1}{4}$ due to Remark 1. Since $\mu(v) = 11 - 6 = 5$, we are done unless either every incident face receives $\frac{1}{2}$ from v or each of ten incident faces receives $\frac{1}{2}$ and the eleventh face receives a positive charge. So suppose this is the case.

If $n_3 = 0$, then v has two consecutive 6-neighbors, v_1 and v_2 say, but such a face v_1vv_2 receives nothing from v, a contradiction.

Thus we can assume that $n_3 \ge 1$ (see Fig. 10(a)).

Considering a maximal sequence $v_1, \dots, v_{2k+1}$ with $d(v_2) = \cdots = d(v_{2k}) = 3$, $k \le 4$, we find two distinct faces v_1vv_{11} and $v_{2k+1}vv_{2k+2}$ (as $n_3 \le 4$ due to (SP4)), each receiving less than $\frac{1}{2}$ from v since $d(v_1) \ge 11$ and $d(v_{2k+1}) \ge 11$ (in fact, $\frac{1}{4}$ by R2c, R3b, R5e2, or nothing otherwise), a contradiction.

Remark 2. Every vertex v with $d(v) \ge 12$ sends at most $\frac{1}{2}$ through each incident hard face v_2vv_3 by (combinations of) R2(a,c*), R3(c,d,d*), and R5(a,b2,c2,d), and sends $\frac{1}{4} + \frac{1}{3}$ by R3d* combined with R5b1 when $d(v) \ge 14$, $d(v_1) = 3$, $d(v_2) = 7$, $d(v_3) = 5$, and $d(v_4) \le 4$.

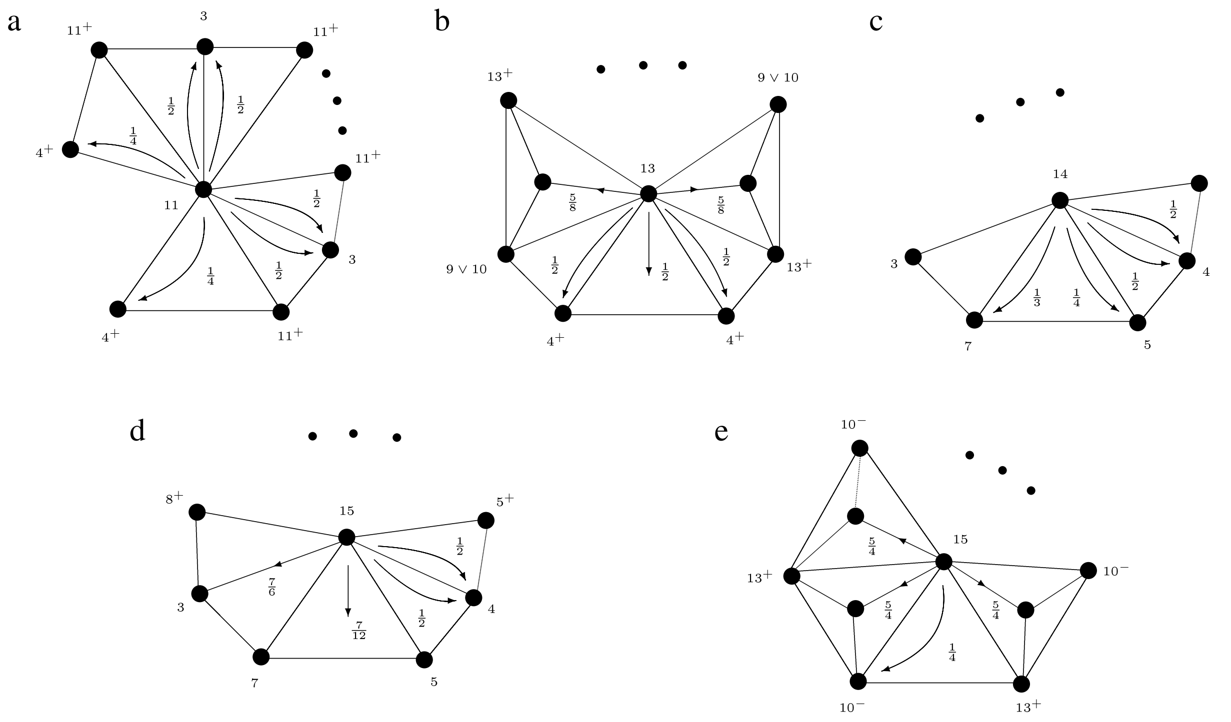

Fig. 10. Some situations in Cases 8 and 10–12.

Case 9. $d(v) = 12$. As in Case 8, every 3-neighbor receives 1 from v by R1f in view of (Tg), so we see from Remark 2 that v actually sends at most $\frac{1}{2}$ through every incident face. Thus $\mu'(v) \geq 12 - 6 - \frac{12}{2} = 0$.

Case 10. $d(v) = 13$. Now v sends at most $\frac{9}{8}$ to a 3-vertex by R1(d–f) due to (Ta)–(Te) and at most $\frac{1}{2}$ through a face not incident with a 3-vertex in view of Remark 2, including $\frac{1}{3}$ if R5c2 is in action. This implies that $\mu'(v) \geq 13 - 6 - n_3 \times \frac{9}{8} - (13 - 2n_3) \times \frac{1}{2} = \frac{4-n_3}{8}$, so we are already done if $n_3 \leq 4$. On the other hand, $n_3 \leq 6$ due to (SP1).

For $n_3 = 6$ it suffices to note that v has two consecutive 9^+-neighbors v_1 and v_2 by parity combined with (Ta)–(Te), which means that v does not give any charge through face v_1vv_2, and hence $\mu'(v) \geq 7 - 6 \times \frac{9}{8} > 0$.

Thus we can assume that $n_3 = 5$. Furthermore, we are done if, say, $d(v_1) \geq 11$, $d(v_2) = 3$, and $d(v_3) \geq 11$ because v_2 then receives only 1 from v by R1f, which implies that $\mu'(v) \geq 7 - 4 \times \frac{9}{8} - 1 - 3 \times \frac{1}{2} = 0$.

Due to (Ta)–(Te), we can now assume that every 3-neighbor of v is adjacent to a vertex of degree 9 or 10. On the other hand, if $d(v_i) = 3$ and $d(v_{i-1}) \leq 10$ (addition modulo 13), then $d(v_{i+1}) \geq 13$ due to (Tf) and (Tg) applied to the face $v_{i-1}v_iv_{i+1}$.

So let $d(v_4) = d(v_6) = \cdots = d(v_{12}) = 3$, $9 \leq d(v_3) \leq 10$, and $d(v_{13}) \geq 13$ (see Fig. 10(b)). We note that v sends either $\frac{1}{4}$ or nothing through the face v_2vv_3. Indeed, if $d(v_2) = 4$ then R2c is applied; if $d(v_2) = 5$ then R3c works; if $d(v_2) = 6$ or $d(v_2) \geq 8$, then no charge is transferred from v through the face v_2vv_3 by R1–R5; and if $d(v_2) = 7$ then $\frac{1}{4}$ is given by R5c2. Therefore, $\mu'(v) \geq 7 - 5 \times \frac{9}{8} - \frac{1}{4} - 2 \times \frac{1}{2} > 0$.

Case 11. $d(v) = 14$. As compared to $d(v) \leq 13$, now four new rules, R1b, R3d*, and R5(b1,b2), join the play. Namely, now v sends by R5b1 and R3d* as much as $\frac{1}{3} + \frac{1}{4}$ through a face v_1vv_2 when $d(v_1) = 5$, $d(v_2) = 7$, $d(v_3) = 3$, and $d(v_{14}) \leq 4$, where in fact $d(v_{14}) = 4$ due to (Td). If R5b1 is not applied, then each incident face avoiding 3-neighbors receives from v at most $\frac{1}{2}$, according to Remark 2. Also, v sends $\frac{7}{6}$ to v_2 when $d(v_1) = 7$ and $d(v_2) = 3$ (due to (Td) applied to the face $v_1v_2v_3$, we have $d(v_3) \geq 14$).

Note that $\mu'(v) \geq 14 - 6 - n_3 \times \frac{7}{6} - (14 - 2n_3) \times \frac{7}{12} = -\frac{1}{6}$. If R5b1 is applied (see Fig. 10(c)), then the 4-vertex v_{14} above is incident with two faces each taking $\frac{1}{2}$ from v rather than $\frac{7}{12}$, which implies that $\mu'(v) \geq 8 - 6 \times \frac{7}{6} - 2 \times \frac{1}{2} = 0$. So suppose R5b1 never applies to our v.

If $n_3 \leq 6$, then again there are at least two hard faces each taking at most $\frac{1}{2}$ from v, so $\mu'(v) \geq 0$. Thus suppose $n_3 = 7$. By parity combined with (Tg), there is a 3-neighbor of v surrounded by 11^+-vertices. This 3-vertex receives 1 from v by R1f rather than $\frac{7}{6}$ by R1b, and we are done.

Case 12. $15 \leq d(v) \leq 20$. By Remark 2, v gives strictly less than $\frac{5}{8}$ through every incident face. Also, v gives at most $\frac{5}{4}$ to every adjacent 3-vertex by R1(b–f) since applying R1a is forbidden by (Ta)–(Tc).

For $d(v) \geq 16$ we are already done since $\mu'(v) \geq d(v) - 6 - n_3 \times \frac{5}{4} - (d(v) - 2n_3) \times \frac{5}{8} = \frac{3(d(v)-16)}{8} \geq 0$, so suppose $d(v) = 15$.

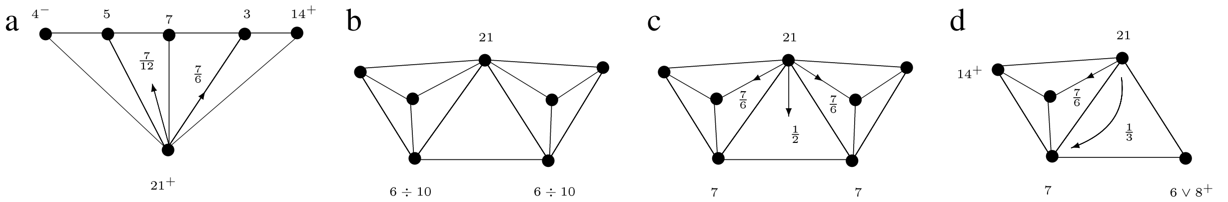

Fig. 11. To Case 13.

Now we have a rough estimation $\mu'(v) \geq -\frac{3}{8}$ and wish to improve it to $\mu'(v) \geq 0$ by saving $\frac{3}{8}$ with respect to the above mentioned level of donations of $\frac{5}{8}$ through hard faces and $\frac{5}{4}$ to 3-vertices.

First suppose a face v_2vv_3 conducts more than $\frac{1}{2}$ from v. As in Case 11, this happens only by R5b1, so we have $d(v_1) = 4, d(v_2) = 5, d(v_3) = 7$, and $d(v_4) = 3$ (see Fig. 10(d)). In fact, v_2vv_3 conducts $\frac{1}{3}$ to v_3 by R5b1 and $\frac{1}{4}$ to v_2 by R3d*. We can say that the saving caused by the face v_2vv_3 alone is $\frac{5}{8} - \frac{1}{3} - \frac{1}{4} = \frac{1}{8} - \frac{1}{12}$. Furthermore, v gives $\frac{7}{6}$ to v_4 rather than $\frac{1}{4}$, which results in saving of $\frac{1}{12}$ on v_4. Finally, face v_1vv_2 conducts $\frac{1}{2}$ and hence saves $\frac{1}{8}$.

Therefore, any application of R5b1 results in saving of $\frac{2}{8}$. Note that $d(v_5) \geq 8$ in view of (Td), so the saving of $\frac{1}{12}$ caused by v_4 should be attributed to face v_2vv_3 solely. The same is true for the face v_1vv_2; its saving of $\frac{1}{8}$ also cannot be counted twice and belongs to v_2vv_3 only.

Thus more than one application of R5b1 results in saving of at least $\frac{4}{8}$, and we are done. On the other hand, if the above application of R5b1 is unique for v, then we have another saving of $\frac{1}{8}$ caused by face v_1vv_{14}, where $d(v_{14}) \geq 5$ due to (Tb) again, as desired. (Informally speaking, any application of R5b1 saves $\frac{2}{8}$ on four consecutive faces and saves $\frac{3}{8}$ on five consecutive faces if R5b1 is applied just once.)

So from now on we can assume that R5b1 is not applied to our v. This means that every hard face conducts at most $\frac{1}{2}$ from v and hence saves at least $\frac{1}{8}$ for v. Due to parity, we can assume that a face v_1vv_2 with $d(v_1) \geq 4$ and $d(v_2) \geq 5$ is unique, for otherwise we already have nothing to prove. This means that $n_3 = 7$, so let $d(v_3) = d(v_5) = \cdots = d(v_{15}) = 3$ (see Fig. 10(e)).

If there is a v_{2k+1}, $1 \leq k \leq 7$, surrounded by three 11^+-vertices, then v_{2k+1} receives only 1 from v by R1f and thus saves $\frac{1}{4}$ for v. This yields a desired total saving of $\frac{1}{8} + \frac{1}{4}$.

Otherwise, we deduce by parity combined with (Tg) that $d(v_2) \leq 10$ and $d(v_1) \geq 13$ due to symmetry. Note that $d(v_2) \geq 7$ due to (Tc). If $d(v_2) \geq 8$ then v gives nothing through face v_1vv_2 by R1–R5, which saves $\frac{5}{8}$. So suppose $d(v_2) = 7$. Thus v_1vv_2 is as described in R5c2, and it takes away from v only $\frac{1}{3}$. Since v now gives $\frac{7}{6}$ to v by R1b rather than $\frac{5}{4}$ by R1c, this implies $\mu'(v) \geq 15 - 6 - 7 \times \frac{5}{4} - \frac{1}{3} = 0$.

Case 13. $21 \leq d(v) \leq 31$. A face v_2vv_3 incident with v is *single* if $d(v_2) \geq 4$ and $d(v_3) \geq 5$. Clearly, there are precisely $d(v) - 2n_3$ single (or hard, which is the same here) faces at v.

Note that every single face v_2vv_3 at v either receives at most $\frac{1}{2}$ from v or participates in R5b1 (we will call such faces, described in Remark 2, *bad singles* for brevity), in which case $d(v_1) = 3$, $d(v_2) = 7$, $d(v_3) = 5$, and $d(v_4) \leq 4$ (see Fig. 11(a)). Recall that v_2vv_3 conducts $\frac{7}{12}$ from v while v_1 receives $\frac{7}{6}$. We say that the 3-vertex v_1 is *associated* with a bad single v_2vv_3. Due to (Td), we have $d(v_{d(v)}) \geq 14$, so v_1 cannot be associated with two bad singles at v.

Due to R1, every 3-neighbor of v receives at most $\frac{3}{2}$ from v. Let n_3' be the number of 3-vertices associated with bad singles at v. Since $\frac{3}{2} + \frac{1}{2} - (\frac{7}{6} + \frac{7}{12}) = \frac{1}{4}$, we see that a bad single along with its associated 3-vertex even causes saving of $\frac{1}{4}$ for v with respect to the "normal" donation of $\frac{1}{2} + \frac{3}{2}$ to a hard face plus that to a 3-vertex. From this informal observation combined with $n_3 \leq \lfloor \frac{d(v)}{2} \rfloor$, we deduce that

$$\begin{aligned}\mu'(v) &\geq d(v) - 6 - (n_3 - n_3') \times \frac{3}{2} - n_3' \times \frac{7}{6} - (d(v) - 2n_3 - n_3') \times \frac{1}{2} - n_3' \times \frac{7}{12} \\ &\geq d(v) - 6 - n_3 \times \frac{3}{2} - (d(v) - 2n_3) \times \frac{1}{2} = \frac{d(v) - 12 - n_3}{2} \geq \frac{d(v) - 24}{4}.\end{aligned}$$

Thus we are already done if $d(v) \geq 24$. If $d(v) = 23$, then we have $\mu'(v) \geq \frac{23-12-n_3}{2} \geq 0$.

Suppose $d(v) = 22$. Since $\mu'(v) \geq \frac{22-12-n_3}{2}$, the only case to consider is $n_3 = 11$. Due to (Tg) and the oddness of $\frac{22}{2}$, there is a 3-neighbor v_2 of v such that $d(v_1) \geq 11$ and $d(v_3) \geq 11$. By R1f, v_2 receives as little as 1 from v. This improves our general rough estimation $\mu'(v) \geq \frac{22-12-11}{2} = -\frac{1}{2}$ above by $\frac{3}{2} - 1$ and hence proves that $\mu'(v) \geq 0$.

Finally, suppose $d(v) = 21$ (see Fig. 11(b)–(d)). Due to the rough estimation $\mu'(v) \geq \frac{9-n_3}{2}$, it suffices to assume that $n_3 = 10$, so we have $\mu'(v) \geq -\frac{1}{2}$.

Let $d(v_2) = d(v_4) = \cdots = d(v_{20}) = 3$. By (Tb), we have $d(v_1) \geq 6$ and $d(v_{21}) \geq 6$ (see Fig. 11(b)). In particular, we see that R5b1 is not applied to v_1vv_2. If in fact $d(v_1) \neq 7 \neq d(v_{21})$, then face v_1vv_{21} does not receive anything by R1–R5, and we have $\mu'(v) \geq 21 - 6 - 10 \times \frac{3}{2} = 0$.

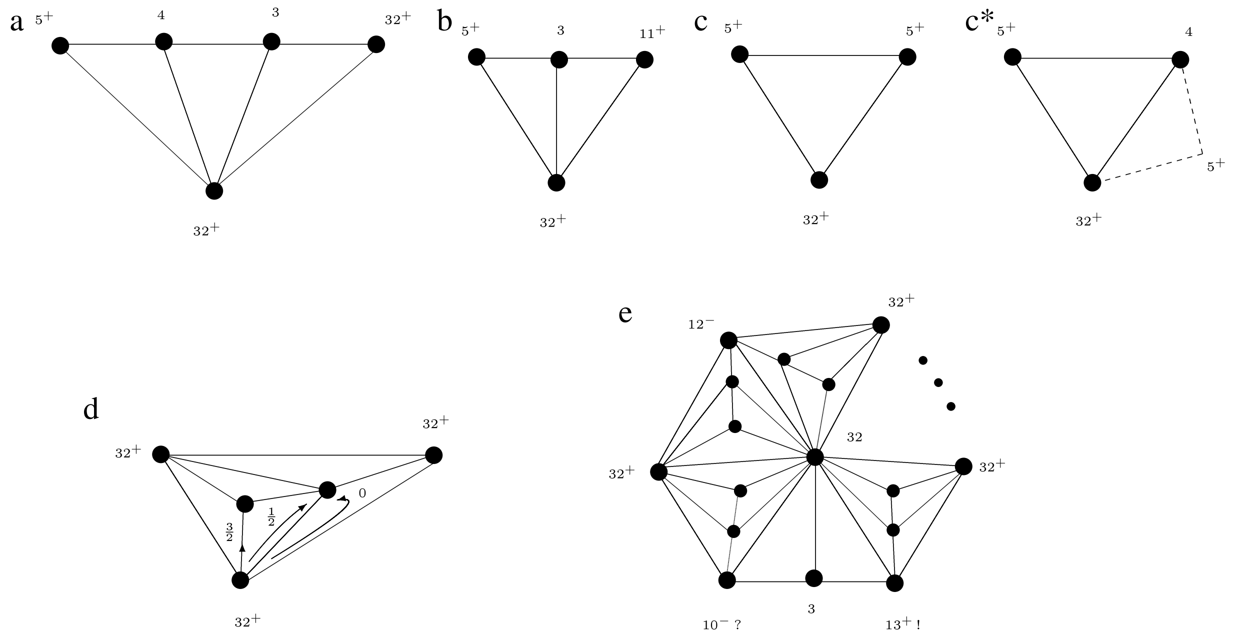

Fig. 12. To Case 14.

So suppose $d(v_1) = 7$ (see Fig. 11(c), (d)). Now v gives $\frac{7}{6}$ to v_2 rather than $\frac{3}{2}$ in our rough estimation. Thus v saves $\frac{1}{3}$ at v_2.

If $d(v_{21}) = 7$ (see Fig. 11(c)), then $\mu'(v) \geq -\frac{1}{2} + 2 \times \frac{1}{3} > 0$, as desired. If $d(v_{21}) \neq 7$ (see Fig. 11(d)), then v gives $\frac{1}{3}$ to v_1 through $v_1 v v_{21}$ by R5c2, which implies $\mu'(v) \geq 21 - 6 - 9 \times \frac{3}{2} - \frac{7}{6} - \frac{1}{3} = 0$.

Case 14. $d(v) \geq 32$. Finally, R1a becomes applicable to v in full strength, since a 3-neighbor of v can have a 4-neighbor. If $d(v_2) = 4$ and $d(v_3) = 3$ (which implies that $d(v_4) \geq 32$ due to (Ta)), then faces $v_1 v v_2$, $v_2 v v_3$ and $v_3 v v_4$ form a *triple receiver* from v, or a *triple* for brevity (see Fig. 12(a)). A *double* is a pair of faces $v_1 v v_2$, $v_2 v v_3$ with $d(v_1) \geq 5$, $d(v_2) = 3$, and, due to (Tg) and symmetry, $d(v_3) \geq 11$ (see Fig. 12(b)). A face $v_1 v v_2$ forms a *single* receiver if either $d(v_1) \geq 5$ and $d(v_2) \geq 5$ (see Fig. 12(c)), or $d(v_1) \geq 5$, $d(v_2) = 4$, and $d(v_3) \geq 5$ (see Fig. 12(c*)).

It follows from (SP1) combined with (Th) that each face incident with v belongs to precisely one receiver, so $3n_t + 2n_d + n_s = d(v)$, where n_t, n_d, and n_s are the numbers of corresponding receivers.

By our rules, every triple receives from v at most $\frac{1}{2} + \frac{1}{2} + \frac{3}{2}$, so each of the three faces gets at most $\frac{5}{6}$ on the average. A double receives at most $\frac{3}{2}$, so we can say that it saves for v at least $2 \times \frac{5}{6} - \frac{3}{2}$, which is $\frac{1}{6}$, with respect to the level $\frac{5}{6}$ of donation per face. Any single, except for that described in R5b1, receives at most $\frac{1}{2}$, and so saves at least $\frac{1}{3}$.

First suppose that R5b1 is applied to v, so we have a bad single described in Case 13 with $d(v_2) = 3$, $d(v_3) = 7$, $d(v_4) = 5$, and $d(v_5) \leq 4$. Here $v_3 v v_4$ is a single as defined in Case 14, while the two faces incident with edge vv_2 form a double receiver. Recall that v_2 receives $\frac{7}{6}$ from v, while v_3 and v_4 receive $\frac{1}{3}$ and $\frac{1}{4}$, respectively, through face $v_3 v v_4$. Since $\frac{7}{6} + \frac{1}{3} + \frac{1}{4} = 3 \times \frac{5}{6} - \frac{3}{4}$, our v saves at least $\frac{3}{4}$ on these two receivers. This implies that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{5}{6} + \frac{3}{4} \geq \frac{32-36}{6} + \frac{3}{4} > 0$. So from now on we assume that R5b1 is not applied to v.

For $d(v) \geq 36$ we have already nothing to prove as $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{5}{6} = \frac{d(v)-36}{6} \geq 0$. For the remaining cases $32 \leq d(v) \leq 35$ we should argue more carefully to prove that the total saving of all receivers always covers the deficiency $\frac{d(v)-36}{6} \geq 0$ of our v. (For example, if $d(v) = 32$, then it suffices to check that the total saving is at least $\frac{2}{3}$, while for $d(v) = 35$ a saving of $\frac{1}{6}$ is already enough.)

Subcase 14.1. $d(v) = 35$. It suffices to observe that $n_d + n_s \geq 1$ since $\frac{35}{3}$ is not an integer, which implies a saving of at least $\frac{1}{6}$.

Subcase 14.2. $d(v) = 34$. If $n_s \geq 1$, then v already saves at least $\frac{1}{3}$, and we are done. However, $n_s = 0$ implies that $n_d \geq 2$ because neither $\frac{34}{3}$ nor $\frac{34-1\times 2}{3}$ is an integer. This yields $\mu'(v) \geq \frac{34-36}{6} + 2 \times \frac{1}{6} = 0$.

Subcase 14.3. $d(v) = 33$. Now we have to save $\frac{3}{6}$, so suppose $n_s \leq 1$. If $n_s = 1$, then $n_d \geq 1$ since $\frac{33-1\times 1}{3}$ is not an integer, so we save at least $\frac{1}{3} + \frac{1}{6}$, as desired.

Suppose that $n_s = 0$. Since $\frac{33-k\times 2}{3}$ is not an integer when $k \in \{1, 2\}$, we are done unless $n_d = 0$. Thus the neighborhood of v is partitioned into 11 triples. Recall that a 3-vertex in a triple has a 32^+-neighbor by (Ta). Since $\frac{33}{3}$ is not even, there is a path $v_1 \cdots v_4$ such that $d(v_1) \geq 32$, $d(v_2) = 4$, $d(v_3) = 3$, and $d(v_4) \geq 32$ (see Fig. 12(d)). According to R2c*, our v

gives $0 + \frac{1}{2}$ to v_2 through faces v_1vv_2 and v_2vv_3, respectively, Also, v gives $\frac{3}{2}$ to v_3 along edge vv_3. Hence this triple saves $3 \times \frac{5}{6} - \frac{1}{2} - \frac{3}{2}$, that is $\frac{1}{2}$, as desired.

Subcase 14.4. $d(v) = 32$. Now we have to save $\frac{4}{6}$. If $n_s = 1$, then $n_d \geq 2$, which implies a total saving of at least $\frac{1}{3} + 2 \times \frac{1}{6}$. So suppose that $n_s = 0$. Since $\frac{32-k\times 2}{3}$ is not an integer whenever $k \in \{0, 2, 3\}$, we have either $n_d = 1$ or $n_d \geq 4$.

Thus we are done unless $n_d = 1$. Let us have a double receiver D defined on path $v_1v_2v_3$. Due to (Tg), we can assume that $d(v_1) \geq 5$, $d(v_2) = 3$, and $d(v_3) \geq 11$. If $d(v_1) \leq 10$, then D saves $2 \times \frac{5}{6} - \frac{3}{2}$, which is $\frac{1}{6}$. Otherwise, D alone saves all we need ($2 \times \frac{5}{6} - 1 = \frac{2}{3}$).

It remains to assume that $d(v_1) \leq 10$, which implies $d(v_3) \geq 13$ due to (Tg) (see Fig. 12(e)). By the same alternation argument as in Subcase 14.4 based on R2c*, we deduce that there is a triple receiver that receives nothing from v through a hard face by R2c*. Therefore, it saves $\frac{1}{2}$ for v in addition to $\frac{1}{6}$ already saved by D, and we are done.

Thus we have proved $\mu'(x) \geq 0$ for every $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 8.

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