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Every 3-polytope with minimum degree 5 has a 6-cycle with maximum degree at most 11

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ABSTRACT

Let $\varphi_P(C_6)$ (respectively, $\varphi_T(C_6)$) be the minimum integer k with the property that every 3-polytope (respectively, every plane triangulation) with minimum degree 5 has a 6-cycle with all vertices of degree at most k. In 1999, S. Jendrol' and T. Madaras proved that $10 \le \varphi_T(C_6) \le 11$. It is also known, due to B. Mohar, R. Škrekovski and H.-J. Voss (2003), that $\varphi_P(C_6) \le 107$.

We prove that $\varphi_P(C_6) = \varphi_T(C_6) = 11$.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of incident edges. A *k-vertex* (*k-face*) is a vertex (face) with degree k, a k^+*-vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but $d(x) \ge 3$ for every vertex or face x. As proved by Steinitz [20], the 3-connected plane graphs are planar representations of the convex three-dimensional polytopes, called hereafter 3-polytopes.

In this note, we consider the class $\mathbf{M_5}$ of NPMs with $\delta = 5$ and its subclasses $\mathbf{P_5}$ of 3-polytopes and $\mathbf{T_5}$ of plane triangulations. A cycle on k vertices is denoted by C_k, and S_k stands for a k-star centered at a 5-vertex. (So, S_k is a subgraph of M_5 on a 5-vertex and k vertices adjacent to it, where $0 \le k \le 5$.)

In 1904, Wernicke [21] proved that $M_5 \in \mathbf{M_5}$ implies, in M_5, the presence of a vertex of degree 5 adjacent to a vertex of degree at most 6. This result was strengthened by Franklin [8] in 1922 to the existence of a vertex of degree 5 with two neighbors of degree at most 6. In 1940, Lebesgue [15, p. 36] gave an approximate description of the neighborhoods of vertices of degree 5 in $\mathbf{T_5}$.

The *weight* $w_M(H)$ is the maximum over $M_5 \in \mathbf{M_5}$ of the minimum degree-sum of the vertices of H over subgraphs H of M_5. The *weights* $w_P(H)$ and $w_T(H)$ are defined similarly for $\mathbf{P_5}$ and $\mathbf{T_5}$, respectively.

The bounds $w_M(S_1) \le 11$ (Wernicke [21]) and $w_M(S_2) \le 17$ (Franklin [8]) are tight. It was proved by Lebesgue [15] that $w_M(S_3) \le 24$ and $w_M(S_4) \le 31$, which was improved much later to the following tight bounds: $w_M(S_3) \le 23$ (Jendrol'

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and Madaras [10]) and $w_M(S_4) \leq 30$ (Borodin and Woodall [6]). Note that $w_M(S_3) \leq 23$ readily implies $w_M(S_2) \leq 17$ and immediately follows from $w_M(S_4) \leq 30$ (it suffices to delete a vertex of maximum degree from a star of the minimum weight).

It follows from Lebesgue [15, p. 36] that $w_T(C_3) \leq 18$. In 1963, Kotzig [14] gave another proof of this fact and conjectured that $w_T(C_3) \leq 17$. (The bound 17 is easily shown to be tight.)

In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form, by proving $w_M(C_3) = 17$. Another consequence of this result is the confirming of a conjecture of Grünbaum [9] from 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [19]).

It also follows from Lebesgue [15, p. 36] that $w_T(C_4) \leq 26$ and $w_T(C_5) \leq 31$. In 1998, Borodin and Woodall [6] proved $w_T(C_4) = 25$ and $w_T(C_5) = 30$.

Now let $\varphi_M(H)$ ($\varphi_P(H)$, $\varphi_T(H)$) be the minimum integer k with the property that every normal plane map (3-polytope, plane triangulation) with minimum degree 5 has a copy of H with all vertices of degree at most k.

It follows from Franklin [8] that $\varphi_M(S_2) = 6$. From $w_M(C_3) = 17$ (Borodin [2]), we have $\varphi_M(C_3) = 7$. In 1996, Jendrol' and Madaras [10] proved $\varphi_M(S_4) = 10$ and $\varphi_T(C_4) = \varphi(C_5) = 10$. R. Soták (personal communication; see surveys of Jendrol' and Voss [12,13]) proved $\varphi_P(C_4) = 11$ and $\varphi_P(C_5) = 10$.

In 1999, Jendrol' et al. [11] obtained the following bounds: $10 \leq \varphi_T(C_6) \leq 11$, $15 \leq \varphi_T(C_7) \leq 17$, $15 \leq \varphi_T(C_8) \leq 29$, $19 \leq \varphi_T(C_9) \leq 41$, and $\varphi_T(C_p) = \infty$ whenever $p \geq 11$. Madaras and Soták [17] proved $20 \leq \varphi_T(C_{10}) \leq 415$.

For the broader class $\mathbf{P_5}$ (an easy induction proof shows that every planar triangulation on at least four vertices is 3-connected), it is known that $10 \leq \varphi_P(C_6) \leq 107$ due to Mohar et al. [18] (in fact, this bound is proved in [18] for all 3-polytopes with $\delta \geq 4$ in which no 4-vertex is adjacent to a 4-vertex), and $\varphi_P(C_7) \leq 359$ is due to Madaras et al. [16].

The purpose of our note is to prove that $\varphi_P(C_6) = \varphi_T(C_6) = 11$. This answers a question raised by Jendrol' et al. [11].

Theorem 1. *Every 3-polytope with minimum degree 5 has a 6-cycle such that each of its vertices has degree at most 11, and this bound is tight.*

Other structural results on $\mathbf{M_5}$, some of which have application to coloring, can be found in the papers already mentioned and in [3,4,7,16–18].

One of the ideas used in our proof is to look for a suitable 6-cycle not in the whole graph but in a carefully chosen portion of it. A similar approach to coloring problems on plane graphs is described in a survey [5, pp. 520–521], and it has been used by us several times, beginning with [1].

2. Proving the tightness of Theorem 1

We transform the octahedron (the 4-regular plane triangulation on six vertices) to a plane triangulation in which every 6-cycle goes through a vertex of degree at least 11, replacing each of the eight 3-faces of the octahedron by the configuration shown in Fig. 1.

More specifically, half of the image of every edge (partly invisible) of the octahedron starts at an "angular" 12-vertex, goes through an 11-vertex, cuts an edge between two 5-vertices, then goes through two 5-vertices, cuts another edge between two 5-vertices, and ends in a 12-vertex, the mid-point of the image of the edge. The graph obtained has only 5-, 11-, and 12-vertices. Furthermore, every 5-vertex belongs to a blue (shadowed) triangle. It is easily seen that the subgraph on 5-vertices does not contain 6-cycles.

Note that we could use instead of the octahedron any plane triangulation with $\delta \geq 4$ to obtain a plane triangulation with the desired property.

3. Proving the upper bound in Theorem 1

Suppose G' is a counterexample to the main statement of Theorem 1. Thus G' is a 3-polytope with $\delta = 5$ in which no 6-cycle avoids a 12^+-vertex.

By Euler's formula $|V'| - |E'| + |F'| = 2$ for G', we have

$$\sum_{v \in V'} (d(v) - 4) + \sum_{f \in F'} (d(f) - 4) = -8. \quad (1)$$

This implies that G' has a 3-face. So we may assume that the external face of G' is bounded by a 3-cycle with the vertex set T'.

A *special triangle* $T^* = t_1t_2t_3$ of G' is a 3-cycle of G' with the fewest vertices inside. We define G to be the subgraph of G' induced by the vertices inside T^*. The vertices of G are *internal*, and the vertices t_1, t_2, and t_3 are *special*.

By G^{**} we denote the subgraph of G' induced by the vertices of $G \cup T^*$. In particular, $T^* = T'$ when $G^{**} = G'$. In both cases, T^* is the boundary $\partial(f_\infty)$ of the external face f_∞ of G^{**}.

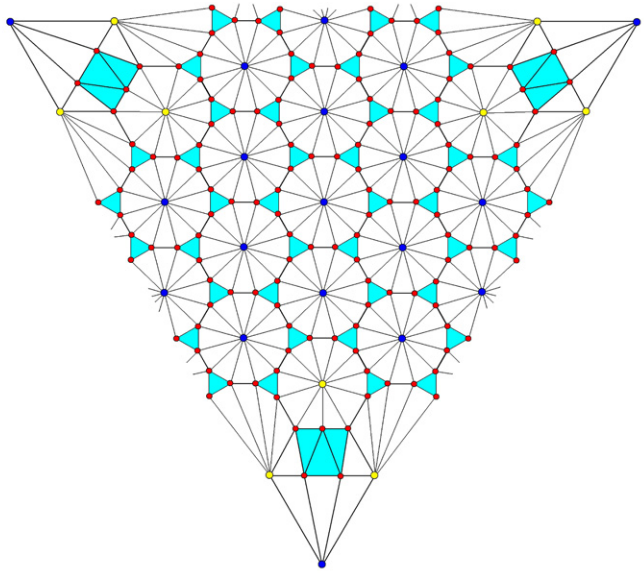

Fig. 1. One eighth of an octahedral triangulation with $\delta = 5$ and $\varphi(C_6) = 11$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To complete the proof of Theorem 1, it suffices to prove the following stronger fact.

Proposition 2. *G^{**} has a 6-cycle that consists of internal 11^--vertices.*

Suppose G^{**} is a counterexample to Proposition 2. By G^* we denote a counterexample to Proposition 2 with the most edges on the same set of vertices as G^{**}.

More specifically, consider a maximal sequence

$$G_0 = G^{**}, G_1, \ldots, G_I = G^*,$$

where the graph G_i whenever $1 \le i \le I$ is obtained from G_{i-1} by adding a diagonal such that at least one of its end-vertices either belongs to T^* or has degree at least 12 in G_{i-1}.

Note that we cannot create a 6-cycle consisting of internal 11^--vertices since we cannot create a new internal 11^--vertex at any step of constructing G^*. The other structural properties of G^* confirming that it is indeed a counterexample to Proposition 2 are discussed in Section 3.1.

In what follows, we prove that G^* cannot exist. This will imply that G^{**} cannot exist either, and thus complete the proofs of Proposition 2 and Theorem 1.

From now on, the degrees of vertices and faces of T^* are those in G^* rather than those in G'. Denote the sets of vertices, edges, and faces of G^* by V^*, E^*, and F^*, respectively. We put $V = V(G)$, so $V^* = V \cup \{t_1, t_2, t_3\}$. Euler's formula $|V^*| - |E^*| + |F^*| = 2$ for G^* yields

$$\sum_{v \in V}(d(v) - 6) + \sum_{v \in \{t_1,t_2,t_3\}}(d(v) - 2) + \sum_{f \in F^*}(2d(f) - 6) = 0. \quad (2)$$

We assign an *initial charge* $\mu(x)$ to x whenever $x \in V^* \cup F^*$ as follows: $\mu(v) = d(v) - 6$ if $v \in V$, $\mu(v) = d(v) - 2$ if $v \in \{t_1, t_2, t_3\}$, and $\mu(f) = 2d(f) - 6$ if $f \in F^*$. Note that only 5-vertices have a negative initial charge.

Using the properties of G^* as a counterexample to Proposition 2, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x) \ge 0$ is non-negative whenever $x \in V^* \cup F^*$. Also, we will show that $\mu'(t_1) > 0$. This will contradict the fact that the sum of the new charges is, by (2), equal to 0.

3.1. Structural properties of G^*

To state the simplest structural properties (SP1)–(SP5) of G^*, we need a few definitions.

By $v_1, \ldots, v_{d(v)}$ we denote the neighbors of a vertex v in a cyclic order. A vertex is *simplicial* if it is completely surrounded by 3-faces.

For brevity, an internal 11^--vertex of G_i whenever $0 \le i \le I$ is a *white* vertex, and a vertex v is *black* if $d(v) \ge 12$ or $v \in T^*$.

An edge is *strict* if it is incident with at least one black vertex. To *add a diagonal* means to add an edge joining two non-consecutive vertices on the boundary $\partial(f)$ of a 4^+-face f.

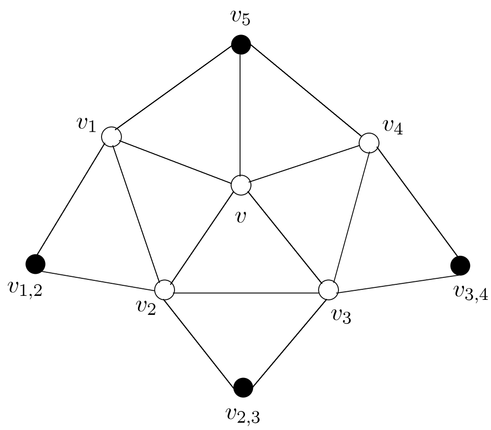

Fig. 2. A bad 5-vertex.

We first check that G_i, whenever $0 \le i \le I$, is a counterexample to Proposition 2. For $i = 0$ this was our assumption.

Suppose $i < I$ and u is a black vertex in $\partial(f) = \cdots uvw$ in G_i. Note that adding a diagonal uw inside f does not create a loop since G_i has no cut-vertices. If adding a diagonal uw creates multiple edges, then uvw is a separating cycle strictly enclosed in T^*, contrary to the definition of T^*.

Finally, we cannot create a 6-cycle consisting of white vertices in G_{i+1} since we cannot create a new white vertex by adding a strict diagonal uw.

A simplicial internal 5-vertex v is *poor* if it is adjacent to four white vertices $v_1, \dots, v_4$ and a black vertex v_5, and none of the faces $f_{1,2} = \cdots v_1v_2v_{1,2}$, $f_{2,3} = \cdots v_2v_3v_{2,3}$, and $f_{3,4} = \cdots v_3v_4v_{3,4}$, where $v \notin \{v_{1,2}, v_{2,3}, v_{3,4}\}$, is a 7^+-face.

A poor 5-vertex v is *bad* if $d(f_{1,2}) = d(f_{2,3}) = d(f_{3,4}) = 3$ and $v_{1,2}$, $v_{2,3}$, and $v_{3,4}$ are black vertices (see Fig. 2). If v is a bad vertex, then the vertex $v_{2,3}$ is *associated* with v.

In what follows, we will need the simple structural properties of G^* expressed by (SP1)–(SP5) and the slightly more involved properties of G^* presented in Lemma 3.

(SP1) The boundary $\partial(f)$ of any 4^+-face f is a cycle that consists of white vertices.
The first claim follows for G_0 from the absence of cut-vertices both in G' and in G_0. For $i > 0$, this follows from the fact that adding a diagonal cannot create a cut-vertex. The second claim follows from the maximality of G^* on the number of strict edges.

(SP2) If v is a black vertex, then v is simplicial.
Indeed, this is just a different way to express (SP1).

(SP3) There are no 6-faces in G^*.
This follows immediately from (SP1).

(SP4) A simplicial white 5-vertex v cannot be surrounded by five white vertices.
Indeed, otherwise we have a desired 6-cycle $vv_1 \cdots v_5$, a contradiction.

(SP5) No separating 3-cycle consists of white vertices.

This is immediate from the minimality of T^* on the number of internal vertices and the fact that adding a strict diagonal cannot create a white vertex, and hence it cannot create a cycle consisting of three white vertices.

Lemma 3. *If a* 5*-vertex is poor, then it is bad.*

Proof. Let v be a poor 5-vertex, and suppose v is not bad.

Case 1. There is a 5-face $f = v_iv_{i+1}xyz$ with $1 \le i \le 3$. By (SP1), $\partial(f)$ is a 5-cycle of white vertices. Since v is simplicial, $v \notin \partial(f)$, so the 6-walk $v_ivv_{i+1}xyz$ is a 6-cycle of white vertices, which contradicts (SP3).

Case 2. By symmetry between the edges v_1v_2 and v_4v_3, we can assume that there is a 4-face $f = v_1v_2xy$ (see Fig. 3). Arguing as above, we see that v_1vv_2xy is a 5-cycle of 11^--vertices. Now look at the 6-walk $C = v_1vv_3v_2xy$. Since the five vertices of C other than v_3 are all distinct, the only possibility for C not to be a cycle is to have degeneration into two 3-cycles with a common vertex, which happens only when $v_3 = y$. (Note that v_3 cannot coincide with x, for otherwise we have $d(v_2) < 5$ or the multi-edge v_2v_3.) However, then the set $\{v_2, v_3\}$ separates x from the rest of G^*, contrary to the 3-connectedness of G'.

Case 3. There is a 4-face $f = v_2v_3xy$. Now $v_2vv_4v_3xy$ is a forbidden 6-cycle, for otherwise $v_4 = y$, with a similar contradiction.

Case 4. By symmetry, suppose there is a 3-face $f = v_1v_2x$ where x is white (see Fig. 4). Since $vv_1 \cdots v_4$ is a 5-cycle consisting of white vertices, it suffices to check that $x \ne v_4$. However, if $x = v_4$, then we have a separating 3-cycle vv_1v_4 consisting of white vertices; this is a contradiction with (SP5).

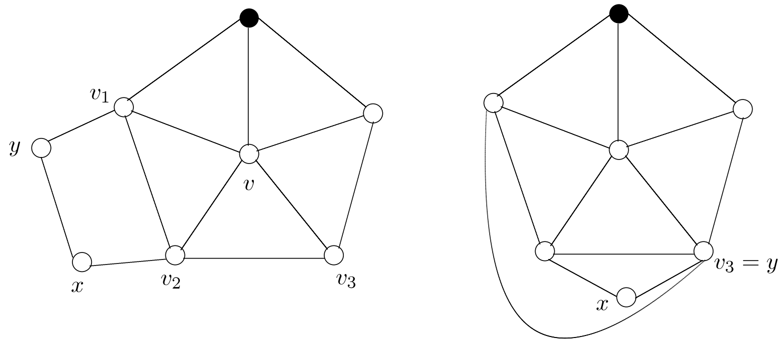

Fig. 3. Case 2 in Lemma 3.

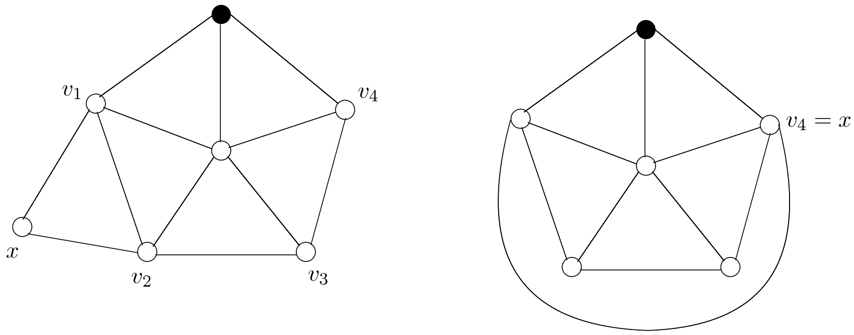

Fig. 4. Case 4 in Lemma 3.

Case 5. There is a 3-face $f = v_2v_3x$ with a white vertex x. Since $vv_1 \cdots v_4$ is a 5-cycle consisting of 11^--vertices, it suffices to check that $x \neq v$. Similarly, $v_1 \neq x$ because otherwise we have $d(v_2) < 5$ and also a white separating 3-cycle v_1vv_3. However, $x \neq v$ follows immediately from the absence of loops and multiple edges in G^*.

This completes the proof of Lemma 3.

3.2. *Discharging on G^**

We use the following rules of discharging (see Fig. 5):

R1. Every 4^+-face gives $\frac{1}{2}$ to every incident white 5-vertex.

R2. Every 7^+-face $f = v_1v_2 \cdots$ gives $\frac{1}{2}$ to every white 5-vertex v across every edge v_1v_2 such that vv_1v_2 is a 3-face.

R3. Every black vertex gives $\frac{1}{2}$ to every adjacent white 5-vertex.

R4. Every black vertex gives $\frac{1}{2}$ to every bad 5-vertex associated with it.

3.3. *Checking $\mu'(x) \geq 0$ for $x \in V^* \cup F^*$ and $\mu'(t_1) > 0$*

Case 1. $d(v) = 5$ and v is white.

Subcase 1.1. Vertex v has at least two donations of $\frac{1}{2}$ from 4^+-faces by R1–R2, adjacent black vertices by R3, and the associated black vertex if it exists by R4. Here, $\mu'(v) \geq 5 - 6 + 2 \times \frac{1}{2} = 0$, as desired. So from now on we assume that v has at most one donation by R1–R4.

Subcase 1.2. All vertices $v_1, \ldots, v_5$ are white. By Subcase 1.1 combined with R1, v is either simplicial or incident with a unique 4^+-face, say $f = \cdots v_1vv_2$. In both cases, we have a desired 6-cycle $vv_2 \cdots v_5v_1$, a contradiction.

Subcase 1.3. Vertex v_5 is a unique black neighbor of v. So v receives $\frac{1}{2}$ from v_5 by R3. Due to Subcase 1.1 combined with R1, our v is simplicial. If a path $P = v_1v_2v_3v_4$ is incident with a 7^+-face, then v receives $\frac{1}{2}$ across at least one of the edges of P by R2, contrary to Subcase 1.1. Hence, v is poor. By Lemma 3, v is bad, which means that v receives another $\frac{1}{2}$ from the associated black vertex by R4, contrary to Subcase 1.1 again.

Case 2. $6 \leq d(v) \leq 11$ and v is internal. Since v does not participate in R1–R4, we have $\mu'(v) = d(v) - 6 \geq 0$.

Case 3. $d(v) \geq 12$ and v is internal. Recall that v is black, which implies by (SP2) that v is simplicial. To show that the total donation of v to 5-vertices by R3 and R4 (see Fig. 5) is relatively small (in fact, at most $d(v) \times \frac{1}{2}$), we use the following argument.

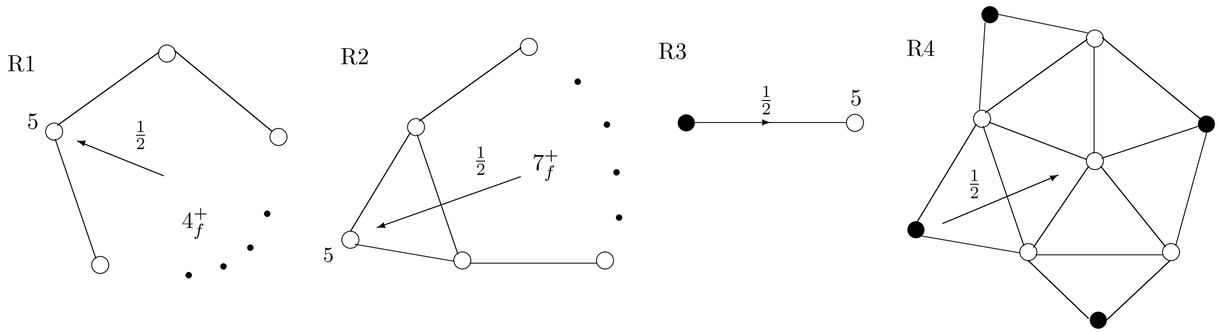

Fig. 5. Rules of discharging.

Let W_5 be the set of white 5-neighbors of v. By A we denote the set of bad 5-vertices associated with v. Recall that each vertex in $W_5 \cup A$ receives $\frac{1}{2}$ from v by R3 or R4.

We now show that

$(*)$ $|W_5 \cup A| \leq d(v)$.

If $a_j \in A$, then let a_j be associated with v with respect to edge v_jv_{j+1}. We relate a_j to two neighbors, $\phi(a_j)$ and $\varphi(a_j)$, of v that do not belong to W_5, as follows:

(i) If $v_j, v_{j+1} \in W_5$, then we put $\phi(a_j) = v_{j-1}$ and $\varphi(a_j) = v_{j+2}$ (hereafter, addition is modulo $d(v)$).
(ii) If $v_j \in W_5, v_{j+1} \notin W_5$, then $\phi(a_j) = v_{j-1}$ and $\varphi(a_j) = v_{j+1}$. By symmetry, this also handles the case $v_j \notin W_5, v_{j+1} \in W_5$.
(iii) If $v_j, v_{j+1} \notin W_5$, then $\phi(a_j) = v_j$ and $\varphi(a_j) = v_{j+1}$.

(The fact that $\{\phi(a_j), \varphi(a_j)\} \cap W_5 = \emptyset$ in (i)–(iii) follows directly from the definition of a bad 5-vertex.)

Since, on the other hand, no neighbor of v is related to more than two vertices in A, we have proved $(*)$. This implies that $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2} \geq 0$.

Case 4. $t_i \in T^*$. Recall that $d(t_i)$ is the number of vertices adjacent to t_i in G^*, including two vertices of T^*. Since t_i is black, it is simplicial due to (SP2). Note that $d(t_i) \geq 3$ since otherwise $T^* - t_i$ is a separating set in G' on two vertices, contrary to the 3-connectedness of G'. Recall that $\mu(t_i) = d(t_i) - 2$. Vertex t_i gives $\frac{1}{2}$ to at most $d(t_i) - 2$ adjacent white 5-vertices by R3 and to at most $d(t_i) - 3$ associated ones by R4. This implies $\mu'(t_i) \geq d(t_i) - 2 - \frac{2d(t_i)-5}{2} = \frac{1}{2}$, as desired.

Case 5. $f \in F^*$. If $d(f) = 3$, then $\mu'(f) = \mu(f) = 2 \times 3 - 6 = 0$ since f does not participate in discharging, no matter whether $f = f_\infty$ or not.

Suppose f is an internal face with $d(f) \geq 4$. Recall that $d(f) \neq 6$ by (SP3). If $4 \leq d(f) \leq 5$, then f participates only in R1, and we have $\mu'(f) \geq 2d(f) - 6 - \frac{d(f)}{2} = \frac{3(d(f)-4)}{2} \geq 0$. Finally, if $d(f) \geq 7$, then $\mu'(f) \geq 2d(f) - 6 - \frac{2d(f)}{2} > 0$ by R1 and R2.

Thus $\mu'(x) \geq 0$ whenever $x \in V^* \cup F^*$ and $\mu'(t_1) > 0$. A contradiction, $0 < 0$, with (2) shows that G^* cannot exist. This completes the proof of Proposition 2, and hence that of Theorem 1.

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