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Note

On the number of edges in a graph with no $(k+1)$-connected subgraphs

Anton Bernshteyn[a,*], Alexandr Kostochka[a,b]

[a] Department of Mathematics, University of Illinois at Urbana–Champaign, IL, USA
[b] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

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ABSTRACT

Mader proved that for $k \geq 1$ and $n \geq 2k$, every n-vertex graph with no $(k+1)$-connected subgraphs has at most $(1+\frac{1}{\sqrt{2}})(n-k)$ edges. He also conjectured that for n large with respect to k, every such graph has at most $\frac{3}{2}\left(k-\frac{1}{3}\right)(n-k)$ edges. Yuster improved Mader's upper bound to $\frac{193}{120}k(n-k)$ for $n \geq \frac{9k}{4}$. In this note, we make the next step towards Mader's Conjecture: we improve Yuster's bound to $\frac{19}{12}k(n-k)$ for $n \geq \frac{5k}{2}$.

1. Introduction

All graphs considered here are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. If $U \subseteq V(G)$, then $G[U]$ denotes the induced subgraph of G whose vertex set is U, and $G - U := G[V(G) \setminus U]$. For $v \in V(G)$, $N(v) := \{u \in V(G) : uv \in E(G)\}$ denotes the neighborhood of v in G.

Let $k \in \mathbb{N}$. Recall that a graph G is $(k+1)$-*connected* if, for every set $S \subset V(G)$ of size k, the graph $G[V(G) \setminus S]$ is connected and contains at least two vertices (so $|V(G)| \geq k+2$). Mader [1] posed the following question:

> What is the maximum possible number of edges in an n-vertex graph that does not contain a $(k+1)$-connected subgraph?

It is easy to see that for $k = 1$ the answer is $n-1$: every tree on n vertices contains $n-1$ edges and no 2-connected subgraphs, whereas every graph on n vertices with at least n edges contains a cycle, and cycles are 2-connected. Thus for the rest of the note we will assume $k \geq 2$.

The following construction due to Mader [2] gives an example of a graph with no $(k+1)$-connected subgraphs and a large number of edges. Fix k and n, and suppose that $n = kq + r$, where $1 \leq r \leq k$. The graph $G_{n,k}$ has vertex set $\bigcup_{i=0}^{q} V_i$, where the sets $V_0, \ldots, V_q$ are pairwise disjoint and satisfy the following conditions.

1. $|V_0| = \cdots = |V_{q-1}| = k$, while $|V_q| = r$.
2. V_0 is an independent set in $G_{n,k}$.
3. For $1 \leq i \leq q$, V_i is a clique in $G_{n,k}$.
4. Every vertex in V_0 is adjacent to every vertex in $\bigcup_{i=1}^{q} V_i$.
5. $G_{n,k}$ has no other edges.

* Corresponding author.
E-mail addresses: bernsht2@illinois.edu (A. Bernshteyn), kostochk@math.uiuc.edu (A. Kostochka).

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Note that V_0 is a separating set of size k and every component of $G_{n,k} - V_0$ has at most k vertices. It follows that $G_{n,k}$ has no $(k+1)$-connected subgraphs. A direct calculation shows that $G_{n,k}$ has at most $\frac{3}{2}\left(k - \frac{1}{3}\right)(n-k)$ edges, where the equality holds if n is a multiple of k. Mader [2] conjectured that this example is, in fact, best possible.

Conjecture 1 (*Mader [2]*)**.** *Let $k \geq 2$. Then for n sufficiently large, the number of edges in an n-vertex graph without a $(k+1)$-connected subgraph cannot exceed $\frac{3}{2}\left(k - \frac{1}{3}\right)(n-k)$.*

Mader himself proved Conjecture 1 for $k \leq 6$. Moreover, he showed that for all k, the weaker version of the conjecture, where the coefficient $\frac{3}{2}$ is replaced by $1 + \frac{1}{\sqrt{2}}$, holds. Yuster [4] improved this result by showing that the coefficient can be taken to be $\frac{193}{120}$.

Theorem 2 (*Yuster [4]*)**.** *Let $k \geq 2$ and $n \geq \frac{9k}{4}$. Then every n-vertex graph G with $|E(G)| > \frac{193}{120}k(n-k)$ contains a $(k+1)$-connected subgraph.*

Here we improve Yuster's bound, obtaining the value $\frac{19}{12}$ for the coefficient.

It turns out that for this problem, computations work out nicer if we "normalize" vertex and edge counts by assigning a weight $\frac{1}{k}$ to each vertex and a weight $\frac{1}{k^2}$ to each edge in a graph. Using this terminology, we can restate Conjecture 1 in the following way.

Conjecture 1′. *Let $k \geq 2$. Then for γ sufficiently large, every graph G with $\frac{1}{k}|V(G)| = \gamma$ and $\frac{1}{k^2}|E(G)| > \frac{3}{2}(\gamma - 1)$ contains a $(k+1)$-connected subgraph.*

Our main result in these terms is as follows.

Theorem 3. *Let $k \geq 2$. Then every graph G with $\frac{1}{k}|V(G)| = \gamma \geq \frac{5}{2}$ and $\frac{1}{k^2}|E(G)| > \frac{19}{12}(\gamma - 1)$ contains a $(k+1)$-connected subgraph.*

We follow the ideas of Mader and Yuster: Use induction on the number of vertices for graphs with at least $\frac{5}{2}k$ vertices. The hardest part is to prove the case when after deleting a separating set of size k, exactly one of the components of the remaining graph has fewer than $\frac{3}{2}k$ vertices, since the induction assumption does not hold for $n < \frac{5}{2}k$. New ideas in the proof are in Lemmas 8 and 9.

2. Proof of Theorem 3

We want to derive a linear in $(n-k)$ bound on the number of edges in a graph that does not contain $(k+1)$-connected subgraphs. But the bound becomes linear only for graphs with large number of vertices; while for small graphs the dependency is quadratic in $n-k$. The main difficulties we encounter are around the transition between the quadratic and linear regimes. To deal with small n, we use the following lemma due to Matula [3], whose bound is asymptotically exact for $n < 2k$.

Lemma 4 (*Matula [3]*)**.** *Let $k \geq 2$. Then every graph G with $|V(G)| = n \geq k+1$ and $|E(G)| > \binom{n}{2} - \frac{1}{3}((n-k)^2 - 1)$ contains a $(k+1)$-connected subgraph.*

We will use the following "normalized" version of this lemma.

Lemma 4′. *Let $k \geq 2$. Then every graph G with $\frac{1}{k}|V(G)| = \gamma > 1$, and*

$$\frac{1}{k^2}|E(G)| > \frac{1}{6}\left(\gamma^2 + 4\gamma - 2\right) \tag{1}$$

contains a $(k+1)$-connected subgraph.

Proof. Indeed, (1) yields

$$\begin{aligned}
|E(G)| &> \frac{k^2}{6}\left(\gamma^2 + 4\gamma - 2\right) \\
&= \binom{\gamma k}{2} - \frac{1}{3}((\gamma k - k)^2 - 1) + \frac{\gamma k}{2} - \frac{1}{3} \\
&> \binom{\gamma k}{2} - \frac{1}{3}((\gamma k - k)^2 - 1),
\end{aligned}$$

and we are done by original Matula's lemma. ∎

From now on, fix a graph G with $\frac{1}{k}|V(G)| = \gamma \geq \frac{5}{2}$ and $\frac{1}{k^2}|E(G)| > \frac{19}{12}(\gamma - 1)$, and suppose for contradiction that G does not contain a $(k+1)$-connected subgraph. Choose G to have the least possible number of vertices (so we can apply induction hypothesis for subgraphs of G). Since G itself is not $(k+1)$-connected, it contains a separating set $S \subset V(G)$ of size k. Let $A \subset V(G) \setminus S$ be such that $G[A]$ is a smallest connected component of $G - S$, and let $B := V(G) \setminus (S \cup A)$. Let $\alpha := \frac{1}{k}|A|$ and $\beta := \frac{1}{k}|B|$.

We start by showing that the graph G cannot be too small, using Matula's Lemma.

Lemma 5. $\gamma > 3$.

Proof. Suppose that $\gamma \leq 3$. Then, by Lemma 4′,

$$0 \leq \frac{1}{k^2}|E(G)| - \frac{19}{12}(\gamma - 1) \leq \frac{1}{6}\left(\gamma^2 + 4\gamma - 2\right) - \frac{19}{12}(\gamma - 1) = \frac{1}{12}(2\gamma^2 - 11\gamma + 15). \tag{2}$$

The function $g(\gamma) = 2\gamma^2 - 11\gamma + 15$ on the right-hand side of (2) is convex in γ. Hence it is maximized on the boundary of the interval $[\frac{5}{2}; 3]$. But it is easy to check that $g(\frac{5}{2}) = g(3) = 0$, hence it is nonpositive on the whole interval. Therefore, $\gamma > 3$. ■

All the edges in G either belong to the graph $G[S \cup B]$, or are incident to the vertices in A. The number of edges in $G[S \cup B]$ can be bounded either using Matula's lemma (which is efficient for $\beta \leq \frac{3}{2}$) or using the induction hypothesis (which can be applied if $\beta > \frac{3}{2}$). Hence the difficulty is in bounding the number of edges incident to the vertices in A.

The first step is to show that A cannot be too large, because otherwise we can use induction.

Lemma 6. $\alpha < \frac{3}{2}$.

Proof. If $\alpha \geq \frac{3}{2}$, then we can apply the induction hypothesis both for $G[S \cup A]$ and for $G[S \cup B]$, and thus obtain

$$\frac{1}{k^2}|E(G)| \leq \frac{19}{12}\alpha + \frac{19}{12}\beta = \frac{19}{12}(\alpha + \beta) = \frac{19}{12}(\gamma - 1). \quad ■$$

The next lemma shows that A cannot be too small either, since otherwise the total number of edges between the vertices in A and the vertices in $S \cup A$ is small.

Lemma 7. $\alpha > 1$.

Proof. Suppose that $\alpha \leq 1$. Then $\beta > 1$, since $\alpha + \beta + 1 = \gamma > 3$. If $\beta \geq \frac{3}{2}$, then using the induction hypothesis for $G[S \cup B]$, we get

$$\frac{1}{k^2}|E(G)| \leq \frac{1}{2}\alpha^2 + \alpha + \frac{19}{12}\beta \leq \frac{3}{2}\alpha + \frac{19}{12}\beta < \frac{19}{12}(\alpha + \beta) = \frac{19}{12}(\gamma - 1).$$

Thus $\beta < \frac{3}{2}$. Therefore, $\alpha > \frac{1}{2}$. In this case, applying Lemma 4′ to $G[S \cup B]$ reduces the problem to prove the inequality

$$\frac{1}{2}\alpha^2 + \alpha + \frac{1}{6}\left((\beta + 1)^2 + 4(\beta + 1) - 2\right) \leq \frac{19}{12}(\alpha + \beta),$$

which is equivalent to

$$6\alpha^2 + 2\beta^2 - 7\alpha - 7\beta + 6 \leq 0. \tag{3}$$

For α fixed, the left-hand side of (3) is monotone decreasing in β when $\beta < \frac{7}{4}$, so its maximum is attained at $\beta = 1$. Thus (3) will hold if the function $g_1(\alpha) = 6\alpha^2 - 7\alpha + 1$ is nonpositive. Since $g_1(\alpha)$ is a convex function, its maximum on the interval $[\frac{1}{2}; 1]$ is attained at one of the boundary points. We have

$$g_1\left(\frac{1}{2}\right) = 6 \cdot \left(\frac{1}{2}\right)^2 - 7 \cdot \frac{1}{2} + 1 = -1 < 0, \qquad \text{and} \qquad g_1(1) = 6 \cdot 1^2 - 7 \cdot 1 + 1 = 0. \quad ■$$

So we know that $1 < \alpha < \frac{3}{2}$. How can we bound the number of edges incident to the vertices in A? The ideas of Lemma 7 and of Lemma 4′ are not sufficient here. The solution is to combine them by applying Lemma 4′ only to the graph $G[A \cup (S \setminus S')]$, where S' is a subset of S with relatively few edges between A and S'. To obtain such set S', we will use Lemma 8, which asserts that there are many vertices in S that have not too many neighbors in A.

Lemma 8. *Let* $S_1 := \{v \in S : \frac{1}{k}|N(v) \cap A| \leq \frac{1}{2}(\alpha + 1)\}$. *Then* $\frac{1}{k}|S_1| > \frac{1}{3}$.

Proof. Suppose that $\frac{1}{k}|S_1| = \sigma \le \frac{1}{3}$. Let $G_1 := G[A \cup (S \setminus S_1)]$. Since G_1 is not $(k+1)$-connected, it has a separating set $T \subset V(G_1)$ of size k. Let X and Y form a partition of $V(G_1) \setminus T$ and be separated by T in G_1. Without loss of generality assume that $|X \cap A| \ge |Y \cap A|$. Then

$$\frac{1}{k}|X \cap A| \ge \frac{1}{2} \cdot \frac{1}{k}|A \setminus T| \ge \frac{1}{2}(\alpha - 1).$$

Hence if $v \in Y \cap S$, then

$$\frac{1}{k}|N(v) \cap A| \le \frac{1}{k}(|A| - |X \cap A|) \le \frac{1}{2}(\alpha + 1),$$

which means that $v \in S_1$. But that is impossible, since $S_1 \cap V(G_1) = \emptyset$. Thus $Y \cap S = \emptyset$, i.e. $Y \subset A$. In particular, since $|X \cap A| \ge |Y \cap A| = |Y|$, we have $\frac{1}{k}|Y| \le \frac{1}{2}\alpha$. Then

$$\frac{1}{k}|V(G) \setminus Y| = \alpha + \beta + 1 - \frac{1}{k}|Y| \ge \frac{1}{2}\alpha + \beta + 1 \ge \frac{5}{2},$$

so the induction hypothesis holds for $G - Y$, and

$$\frac{1}{k^2}|E(G - Y)| \le \frac{19}{12}\left(\frac{1}{k}|V(G - Y)| - 1\right).$$

Hence we are done if

$$\frac{1}{k^2}(|E(G)| - |E(G - Y)|) \le \frac{19}{12} \cdot \frac{1}{k}|Y|,$$

so assume that is not the case. Let $\mu := \frac{1}{k}|Y|$. Then

$$\frac{1}{2}\mu^2 + \mu(1 + \sigma) > \frac{19}{12}\mu,$$

so

$$\mu > \frac{7}{6} - 2\sigma.$$

We consider two cases.

Case 1: $X \cap S \ne \emptyset$. Let $v \in X \cap S$. Then v has more than $k \cdot \frac{1}{2}(\alpha + 1)$ neighbors in A, none of which belong to Y. Hence $\mu < \frac{1}{2}(\alpha - 1)$, and so

$$\frac{1}{2}(\alpha - 1) > \frac{7}{6} - 2\sigma.$$

Therefore,

$$\alpha > \frac{10}{3} - 4\sigma \ge \frac{10}{3} - 4 \cdot \frac{1}{3} = 2;$$

a contradiction.

Case 2: $X \cap S = \emptyset$. Then $S \setminus S_1 \subset T$, and the set $T \cap A$ separates X and Y in $G[A]$ and satisfies $\frac{1}{k}|T \cap A| = \frac{1}{k}(|T| - |T \cap S|) = 1 - (1 - \sigma) = \sigma$. Note that since $|Y| \le |X|$, we have

$$\frac{7}{6} - 2\sigma < \mu \le \frac{1}{2}(\alpha - \sigma),$$

so

$$\sigma > \frac{7}{9} - \frac{1}{3}\alpha > \frac{7}{9} - \frac{1}{3} \cdot \frac{3}{2} = \frac{5}{18}.$$

Now observe that

$$\frac{1}{k^2}|E(G[A])| \le \frac{1}{2}\alpha^2 - \mu(\alpha - \sigma - \mu).$$

Since $\frac{7}{6} - 2\sigma < \mu \le \frac{1}{2}(\alpha - \sigma)$, the latter expression is less than

$$\frac{1}{2}\alpha^2 - \left(\frac{7}{6} - 2\sigma\right) \cdot \left(\alpha + \sigma - \frac{7}{6}\right).$$

Hence $\frac{1}{k^2}(|E(G)| - |E(G-A)|)$ is less than

$$\frac{1}{2}(\alpha+1)\sigma + \alpha(1-\sigma) + \frac{1}{2}\alpha^2 - \left(\frac{7}{6} - 2\sigma\right) \cdot \left(\alpha + \sigma - \frac{7}{6}\right).$$

Case 2.1. $\beta \le \frac{3}{2}$. Then, after adding Matula's estimate for the number of edges in $G[S \cup B]$ and subtracting $\frac{19}{12}(\alpha+\beta)$, it is enough to prove that the following quantity is nonpositive:

$$\frac{1}{2}(\alpha+1)\sigma + \alpha(1-\sigma) + \frac{1}{2}\alpha^2 - \left(\frac{7}{6} - 2\sigma\right) \cdot \left(\alpha + \sigma - \frac{7}{6}\right) + \frac{1}{6}\left((\beta+1)^2 + 4(\beta+1) - 2\right) - \frac{19}{12}(\alpha+\beta),$$

which is equal to

$$\frac{1}{36}(18\alpha^2 + 54\alpha\sigma - 63\alpha + 6\beta^2 - 21\beta + 72\sigma^2 - 108\sigma + 67).$$

Note that for α and σ fixed, the last expression is monotone decreasing in β (recall that $\beta \le \frac{3}{2}$, while the minimum is attained at $\beta = \frac{7}{4}$), so its maximum is attained when $\beta = \alpha$, where it turns into

$$\varphi_1(\alpha, \sigma) = \frac{1}{36}(24\alpha^2 + 54\alpha\sigma - 84\alpha + 72\sigma^2 - 108\sigma + 67).$$

Since $\varphi_1(\alpha, \sigma)$ is convex in both α and σ, it attains its maximum at some point (α_0, σ_0), where $\alpha_0 \in \{1, \frac{3}{2}\}$ and $\sigma_0 \in \{\frac{5}{18}, \frac{1}{3}\}$. It remains to check the four possibilities:

$$\varphi_1\left(1, \frac{5}{18}\right) = -\frac{11}{162} < 0, \qquad \varphi_1\left(1, \frac{1}{3}\right) = -\frac{1}{12} < 0,$$
$$\varphi_1\left(\frac{3}{2}, \frac{5}{18}\right) = -\frac{125}{648} < 0, \quad \text{and} \quad \varphi_1\left(\frac{3}{2}, \frac{1}{3}\right) = -\frac{1}{6} < 0.$$

Case 2.2. $\beta > \frac{3}{2}$. Then, instead of using Matula's bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that the function

$$\varphi_2(\alpha, \sigma) = \frac{1}{2}(\alpha+1)\sigma + \alpha(1-\sigma) + \frac{1}{2}\alpha^2 - \left(\frac{7}{6} - 2\sigma\right) \cdot \left(\alpha + \sigma - \frac{7}{6}\right) - \frac{19}{12}\alpha$$

is nonpositive. Again, we only have to check the boundary values:

$$\varphi_2\left(1, \frac{5}{18}\right) = -\frac{49}{324} < 0, \qquad \varphi_2\left(1, \frac{1}{3}\right) = -\frac{1}{6} < 0,$$
$$\varphi_2\left(\frac{3}{2}, \frac{5}{18}\right) = -\frac{125}{648} < 0, \quad \text{and} \quad \varphi_2\left(\frac{3}{2}, \frac{1}{3}\right) = -\frac{1}{6} < 0.$$

This finishes the proof. ■

Now we can simply try to use as the set S' the set S_1 itself. This choice indeed gives a good bound if A is large, as the next lemma shows.

Lemma 9. $\alpha < \frac{4}{3}$.

Proof. Suppose that $\alpha \ge \frac{4}{3}$. Recall that $\sigma = |S_1| > \frac{1}{3}$. Using Lemma 4′ for $G[A \cup (S \setminus S_1)]$, we get that

$$\begin{aligned}\frac{1}{k^2}(|E(G)| - |E(G-A)|) &\le \frac{1}{6}\left((\alpha+1-\sigma)^2 + 4(\alpha+1-\sigma) - 2\right) + \frac{1}{2}(\alpha+1)\sigma \\ &= \frac{1}{6}(\alpha^2 + \alpha\sigma + 6\alpha + \sigma^2 - 3\sigma + 3).\end{aligned}$$

Case 1: $\beta \le \frac{3}{2}$. Adding Matula's estimate for $G[B \cup S]$ and subtracting $\frac{19}{12}(\alpha+\beta)$, we get

$$\begin{aligned}\frac{1}{k^2}|E(G)| - \frac{19}{12}(\alpha+\beta) &\le \frac{1}{6}(\alpha^2 + \alpha\sigma + 6\alpha + \sigma^2 - 3\sigma + 3) + \frac{1}{6}\left((\beta+1)^2 + 4(\beta+1) - 2\right) - \frac{19}{12}(\alpha+\beta) \\ &= \frac{1}{12}(2\alpha^2 + 2\alpha\sigma - 7\alpha + 2\beta^2 - 7\beta + 2\sigma^2 - 6\sigma + 12).\end{aligned}$$

Again, the maximum is attained when $\beta = \alpha$, so we should consider the expression

$$\varphi_3(\alpha, \sigma) = \frac{1}{6}(2\alpha^2 + \alpha\sigma - 7\alpha + \sigma^2 - 3\sigma + 6).$$

It is convex in both α and σ, so again it is enough to check the boundary points:

$$\varphi_3\left(\frac{4}{3}, \frac{1}{3}\right) = -\frac{1}{27} < 0, \qquad \varphi_3\left(\frac{4}{3}, 1\right) = -\frac{2}{27} < 0,$$
$$\varphi_3\left(\frac{3}{2}, \frac{1}{3}\right) = -\frac{7}{108} < 0, \quad \text{and} \quad \varphi_3\left(\frac{3}{2}, 1\right) = -\frac{1}{12} < 0.$$

Case 2: $\beta > \frac{3}{2}$. Then, instead of using Matula's bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that the function

$$\varphi_4(\alpha, \sigma) = \frac{1}{6}(\alpha^2 + \alpha\sigma + 6\alpha + \sigma^2 - 3\sigma + 3) - \frac{19}{12}\alpha = \frac{1}{12}(2\alpha^2 + 2\alpha\sigma - 7\alpha + 2\sigma^2 - 6\sigma + 6)$$

is nonpositive. The function is convex in both α and σ, so we check the boundary points:

$$\varphi_4\left(\frac{4}{3}, \frac{1}{3}\right) = -\frac{1}{18} < 0, \qquad \varphi_4\left(\frac{4}{3}, 1\right) = -\frac{5}{54} < 0,$$
$$\varphi_4\left(\frac{3}{2}, \frac{1}{3}\right) = -\frac{7}{108} < 0, \quad \text{and} \quad \varphi_4\left(\frac{3}{2}, 1\right) = -\frac{1}{12} < 0.$$

This finishes the proof. ∎

The next lemma is the final piece of the jigsaw. It shows that if A is small, we can still obtain the desired bound if we take the set S' to be slightly bigger than S_1.

Lemma 10. $\alpha > \frac{4}{3}$.

Proof. Suppose that $\alpha \leq \frac{4}{3}$. Then $1 - 2(\alpha - 1) \geq \frac{1}{3}$. Let S' be a subset of S with $\frac{1}{k}|S'| = 1 - 2(\alpha - 1)$ such that $\frac{1}{k}|S' \cap S_1| \geq \frac{1}{3}$. Observe that the normalized number of edges between A and S' is at most

$$\frac{1}{k^2}|A| \cdot |S'| - \frac{1}{2}(\alpha - 1) \cdot \frac{1}{3},$$

by the definition of S_1. Hence, using Lemma 4′ for $G[A \cup (S \setminus S')]$, we get that

$$\frac{1}{k^2}(|E(G)| - |E(G-A)|) \leq \frac{1}{6}\left((3\alpha - 2)^2 + 4(3\alpha - 2) - 2\right) + \alpha(3 - 2\alpha) - \frac{1}{6}(\alpha - 1)$$
$$= \frac{1}{6}(-3\alpha^2 + 17\alpha - 5).$$

Case 1: $\beta \leq \frac{3}{2}$. Adding Matula's estimate for $G[B \cup S]$ and subtracting $\frac{19}{12}(\alpha + \beta)$, we get

$$\frac{1}{k^2}|E(G)| - \frac{19}{12}(\alpha + \beta) \leq \frac{1}{6}(-3\alpha^2 + 17\alpha - 5) + \frac{1}{6}\left((\beta + 1)^2 + 4(\beta + 1) - 2\right) - \frac{19}{12}(\alpha + \beta)$$
$$= \frac{1}{12}(-6\alpha^2 + 15\alpha + 2\beta^2 - 7\beta - 4).$$

Since $\alpha \leq \beta \leq \frac{3}{2}$, the maximum is attained when $\beta = \alpha$, in which case the last expression turns into $-\frac{1}{3}(\alpha - 1)^2 \leq 0$.

Case 2: $\beta > \frac{3}{2}$. Then, instead of using Matula's bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that

$$\frac{1}{6}(-3\alpha^2 + 17\alpha - 5) - \frac{19}{12}\alpha = -\frac{1}{12}(6\alpha^2 - 15\alpha + 10) \leq 0. \tag{4}$$

Since the discriminant of the quadratic $6\alpha^2 - 15\alpha + 10$ is negative, (4) holds for all α, and we are done. ∎

Since Lemmas 9 and 10 contradict each other, we conclude that such graph G does not exist. This completes the proof of the theorem.

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References

[1] W. Mader, Existenz *n*-fach zusammenhängender Teilgraphen in Graphen genügend großen Kantendichte, Abh. Math. Sem. Univ. Hamburg 37 (1972) 86–97.
[2] W. Mader, Connectivity and edge-connectivity in finite graphs, in: B. Bollobás (Ed.), Surveys in Combinatorics, Cambridge University Press, London, 1979, pp. 66–95.
[3] D.W. Matula, Ramsey theory for graph connectivity, J. Graph Theory 7 (1983) 95–105.
[4] R. Yuster, A note on graphs without *k*-connected subgraphs, Ars Combin. 67 (2003) 231–235.