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A list version of graph packing

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ABSTRACT

We consider the following generalization of graph packing. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order n and $G_3 = (V_1 \cup V_2, E_3)$ a bipartite graph. A bijection f from V_1 onto V_2 is a *list packing* of the triple (G_1, G_2, G_3) if $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and for all $v \in V_1$, $vf(v) \notin E_3$. We extend the classical results of Sauer and Spencer and Bollobás and Eldridge on packing of graphs with small sizes or maximum degrees to the setting of list packing. In particular, we extend the well-known Bollobás–Eldridge Theorem, proving that if $\Delta(G_1) \le n-2$, $\Delta(G_2) \le n-2$, $\Delta(G_3) \le n-1$, and $|E_1| + |E_2| + |E_3| \le 2n-3$, then either (G_1, G_2, G_3) packs or is one of 7 possible exceptions.

1. Introduction

The notion of graph packing is a well-known concept in graph theory and combinatorics. Two graphs on n vertices are said to *pack* if there is an edge-disjoint placement of the graphs onto the same set of vertices. In 1978, two seminal papers, [6] and [1], on extremal problems on graph packing appeared in the same journal. In particular, Sauer and Spencer [6] proved sufficient conditions for packing two graphs with bounded product of maximum degrees.

Theorem 1 ([6]). *Let G_1 and G_2 be two graphs of order n. If $2\Delta(G_1)\Delta(G_2) < n$, then G_1 and G_2 pack.*

This result is sharp and later Kaul and Kostochka [5] characterized all graphs for which Theorem 1 is sharp.

Theorem 2 ([5]). *Let G_1 and G_2 be two graphs of order n and $2\Delta(G_1)\Delta(G_2) \le n$. Then G_1 and G_2 do not pack if and only if one of G_1 and G_2 is a perfect matching and the other is either $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.*

Bollobás and Eldridge [1] and, independently, Sauer and Spencer gave sufficient conditions for packing two graphs with given total number of edges.

Theorem 3 ([1,6]). *Let G_1 and G_2 be two graphs of order n. If $|E(G_1)| + |E(G_2)| \le \frac{3}{2}n - 2$, then G_1 and G_2 pack.*

This result is best possible, since $G_1 = K_{1,n-1}$ and $G_2 = \frac{n}{2}K_2$ do not pack. Bollobás and Eldridge [1] proved the stronger result that the bound of Theorem 3 can be significantly strengthened when $\Delta(G_1) < n-1$ and $\Delta(G_2) < n-1$.

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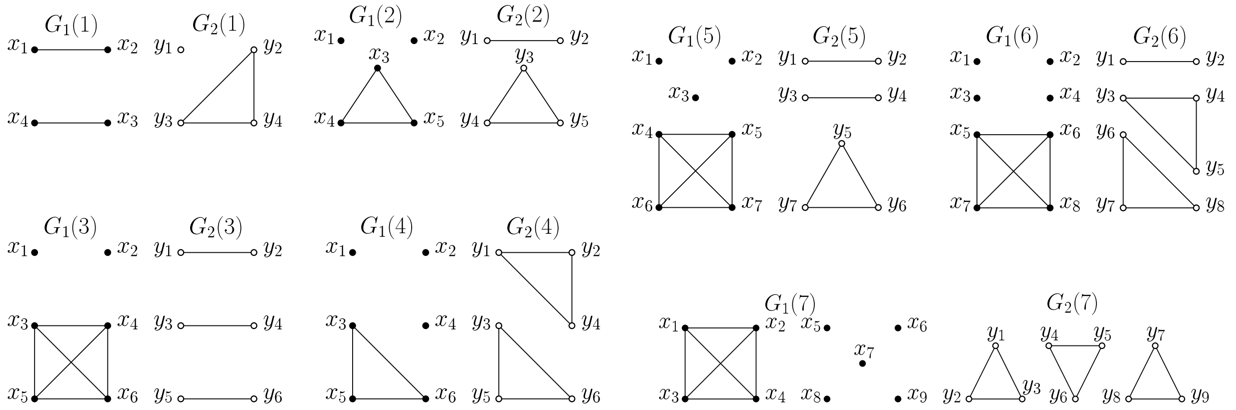

Fig. 1. Bad pairs in Theorems 4 and 6.

Theorem 4 ([1]). *Let G_1 and G_2 be two graphs of order n. If $\Delta(G_1), \Delta(G_2) \leq n-2$, $|E(G_1)| + |E(G_2)| \leq 2n-3$, and $\{G_1, G_2\}$ is not one of the following pairs: $\{2K_2, K_1 \cup K_3\}$, $\{\overline{K}_2 \cup K_3, K_2 \cup K_3\}$, $\{3K_2, \overline{K}_2 \cup K_4\}$, $\{\overline{K}_3 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K}_3 \cup K_4\}$, $\{\overline{K}_4 \cup K_4, K_2 \cup 2K_3\}$, $\{\overline{K}_5 \cup K_4, 3K_3\}$ (Fig. 1), then G_1 and G_2 pack.*

This result is also sharp, since the graphs $G_1 = C_n$ and $G_2 = K_{1,n-2} \cup K_1$ satisfy the maximum degree conditions, have $2n-2$ edges, and do not pack. There are other extremal examples.

Variants of the packing problem have been studied and, in particular, restrictions of permissible packings arise both within proofs and are posed as independent questions. The notion of a bipartite packing was introduced by Catlin [2] and was later studied by Hajnal and Szegedy [4]. This variation of traditional packing involves two bipartite graphs $G_1 = (X_1 \cup Y_1, E_1)$ and $G_2 = (X_2 \cup Y_2, E_2)$ where permissible packings send X_1 onto X_2 and Y_1 onto Y_2. The problem of fixed-point-free embeddings, studied by Schuster in 1978, considers a different restriction to the original packing problem [7]. In this case, $G_1 = G$ is packed with $G_2 = G$ under the additional restraint that no vertex of G_1 is mapped to its copy in G_2. In [9], Schuster's result is used to prove a necessary condition for packing two graphs with given maximum and average degrees.

In this paper, we introduce the language of list packing in order to model such problems. A *list packing* of the graph triple (G_1, G_2, G_3) with $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_1 \cup V_2, E_3)$ is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and for each $u \in V_1$, $uf(u) \notin E_3$. Note that both G_1 and G_2 are graphs on n vertices so that G_3 has $2n$ vertices, and one can think of the edge set E_3 as a list of restrictions that must be avoided when packing G_1 and G_2.

This notion is closely related to Vizing's concept of list coloring [8]. Suppose we wish to color a graph G with the colors $\{1, \ldots, k\}$. A *list assignment* L is a function on the vertex set $V(G)$ that returns a set of colors $L(v) \subseteq \{1, \ldots, k\}$ permissible for v. A *list coloring*, more specifically an *L-coloring*, is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$. The problem of list coloring G can be stated within the framework of list packing. A proper L-coloring of a graph G is equivalent to a list packing where $G_1 = G$ along with an appropriate number of isolated vertices, G_2 is a disjoint union of K_n's each representing a color, and E_3 consists of all edges going between a vertex $v \in V_1$ and the copies of K_n corresponding to colors *not* in $L(v)$. Note the list $L(v)$ denotes permissible colors in a list coloring while $N_3(v)$ specifies forbidden vertices in a list packing.

Similarly, the variations of packing discussed above can be modeled using this framework. A bipartite packing is a packing of the triple (G_1, G_2, G_3) where E_3 consists of all edges between X_i and Y_{3-i} for $i = 1, 2$. A fixed-point-free embedding is a packing of the triple (G, G, G_3) where $E_3 = \{(v, v) : v \in V(G)\}$. Further, several important theorems on the ordinary packing can be stated in terms of list packing. The results of this paper prove natural generalizations of Theorems 1–4. In particular, we extend Theorems 1 and 2 as follows.

Theorem 5. *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If*

$$\Delta(G_1)\Delta(G_2) + \Delta(G_3) \leq n/2, \tag{1}$$

then $\mathbf{G}$ does not pack if and only if $\Delta(G_3) = 0$ and one of G_1 or G_2 is a perfect matching and the other is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$. Consequently, if $\Delta(G_1)\Delta(G_2) + \Delta(G_3) < n/2$, then $\mathbf{G}$ packs.

The main result of this paper is the following list version of Theorem 4.

Theorem 6. *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If $\Delta(G_1), \Delta(G_2) \leq n-2$, $\Delta(G_3) \leq n-1$, $|E_1| + |E_2| + |E_3| \leq 2n-3$ and the pair $\{G_1, G_2\}$ is none of the 7 pairs in Fig. 1, then $\mathbf{G}$ packs.*

Theorem 6 is sharp and has more sharpness examples than Theorem 4. First, the condition $\Delta(G_3) \leq n-1$ cannot be removed, since a vertex in V_1 adjacent to all vertices in V_2 cannot be placed at all (Fig. 2(a)). The restriction on $|E_1|+|E_2|+|E_3|$ is also sharp, as there are several examples of graphs with $|E_3| > 0$ and edge sum equal to $2n-2$ that do not pack.

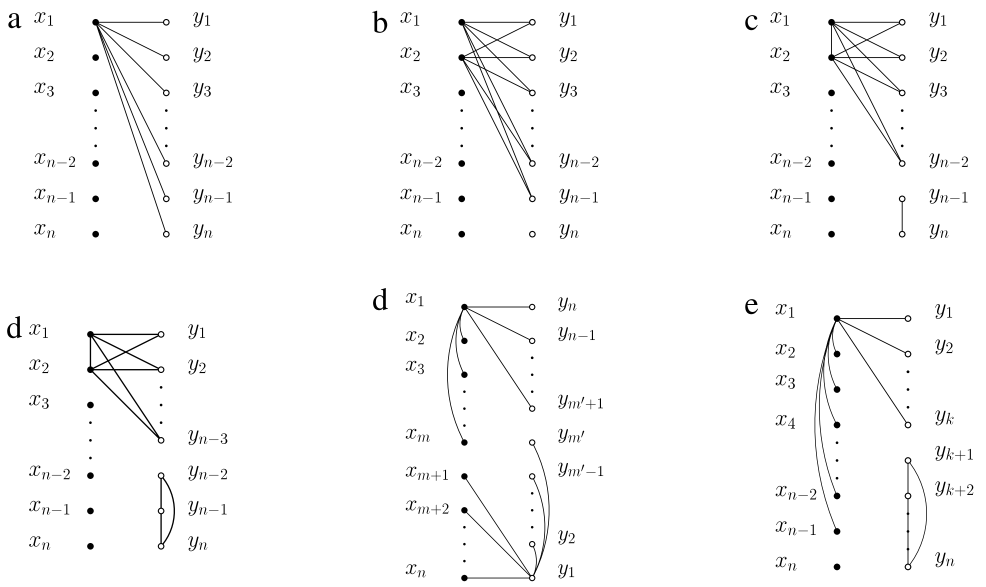

Fig. 2. Sharpness examples for Theorem 6. In each of the above figures, the left column of vertices corresponds to V_1 and the right column corresponds to V_2.

For example, let G_1 and G_2 be independent sets and $x_1, x_2 \in V_1$ be adjacent to the same $n-1$ vertices in V_2 (Fig. 2(b)). In this case (G_1, G_2, G_3) does not pack. If E_1 consists of a single edge x_1x_2, E_2 consists of a single edge $y_{n-1}y_n$, and E_3 consists of all edges between $\{x_1, x_2\}$ and $V_2 - y_{n-1} - y_n$ (Fig. 2(c)), then (G_1, G_2, G_3) also does not pack. Similarly, if E_1 contains a single edge, G_2 contains a triangle and $(n-3)$ isolated vertices, and G_3 consists of all edges between non-isolated vertices in G_1 and isolated vertices in G_2, then (G_1, G_2, G_3) does not pack (Fig. 2(d)).

Alternatively, consider $G_1 = K_{1,m-1} \cup \overline{K}_{n-m}$, $G_2 = K_{1,m'-1} \cup \overline{K}_{n-m'}$ (for any choice of m, m' such that $m-1 \neq n-m'$), and E_3 consisting of all edges between the center of the star in G_1 and isolated vertices in G_2 as well as between the center of the star in G_2 and isolated vertices in G_1 (Fig. 2(e)). Indeed, since $m-1 \neq n-m'$, mapping the center of the star in G_1 to the center of the star in G_2 will create a conflict. Then, since the center of the star in G_1 must be mapped to a leaf in G_2 and a leaf in G_1 must be mapped to the center of the star in G_2, (G_1, G_2, G_3) does not pack. Finally, consider $G_1 = K_{1,n-1} \cup K_1$, $G_2 = C_k \cup \overline{K}_{n-k}$ (for any choice of k), and let E_3 consist of all possible edges between the center of the star in G_1 and isolated vertices in G_2 (Fig. 2(f)). In this case, (G_1, G_2, G_3) does not pack since the center of the star in G_1 is adjacent to $n-2$ vertices in G_1, but must be mapped to a vertex in the cycle in G_2.

The notion of list packing arose while the authors were working on a conjecture of Żak [9] on packing n-vertex graphs with given sizes and maximum degrees. In this situation, list packing provides a stronger inductive assumption that facilitates a proof. In [3], we heavily use Theorems 5 and 6 of this paper to get an approximate solution to Żak's conjecture.

The paper is organized as follows. In the next paragraph, we introduce some notation. In Section 2, we prove Theorem 5. Section 3 contains some preliminary results, including an extension of Theorem 3 that will be used as a base case in the proof of Theorem 6. In Section 4 we prove the main result by induction on the size of the vertex set.

1.1. Notation

A *graph triple* $\mathbf{G} = (G_1, G_2, G_3)$ of order n consists of a pair of n-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets together with a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$. For $i \in \{1, 2, 3\}$, let $e_i = |E_i|$. Let $V = V(\mathbf{G}) = V_1 \cup V_2$. An edge in $E_1 \cup E_2$ is a *white* edge, while an edge in E_3 is a *yellow* edge. The edge set of $\mathbf{G}$ is $E(\mathbf{G}) = E_1 \cup E_2 \cup E_3$.

Let $i \in \{1, 2\}$ and $v \in V_i$. Then the *white neighborhood* of v, denoted $N_i(v)$, is the set of neighbors of v in G_i, and $d_i(v) = |N_i(v)|$. A vertex in $N_i(v)$ is a *white neighbor* of v. For convenience, we say that $N_{3-i}(v) = \emptyset$ (and hence $d_{3-i}(v) = 0$). The *yellow neighborhood* of v, denoted $N_3(v)$, is the set of neighbors of v in G_3 and $d_3(v) = |N_3(v)|$. A vertex in $N_3(v)$ is a *yellow neighbor* of v. Furthermore, the *neighborhood* of v, denoted $N(v)$, is the disjoint union $N_i(v) \cup N_3(v)$ and vertices in the neighborhood are *neighbors*. The *degree* of v is $d_i(v) + d_3(v)$ and is denoted $d(v)$. For $i \in \{1, 2, 3\}$, define $\Delta_i = \Delta_i(\mathbf{G})$ to be $\max_{v \in V} d_i(v)$.

If $W \subseteq V$ with $|W \cap V_1| = |W \cap V_2|$, then the triple induced by W is $\mathbf{G}[W] = (G_1[W], G_2[W], G_3[W])$, where $G_i[W]$ is the subgraph of G_i induced by the set W. Similarly, the triple $\mathbf{G} - W$ is $(G_1 - W, G_2 - W, G_3 - W)$. The *underlying graph* $\underline{\mathbf{G}}$ of a triple $\mathbf{G}$ is the graph with vertex set $V(\mathbf{G})$ and edge set $E(\mathbf{G})$.

Finally, we say the graph triple $\mathbf{G}$ *packs* if the triple has a list packing.

2. Proof of Theorem 5

($\Leftarrow$) Suppose G_1 is a perfect matching and G_2 contains $K_{\frac{n}{2}+1}$ or $\frac{n}{2}$ is odd and $G_2 = K_{\frac{n}{2},\frac{n}{2}}$. In the first case, for any mapping $f : V_1 \to V_2$, some edge of G_1 will be mapped to an edge in the clique $K_{\frac{n}{2}+1}$. In the second case, since $\frac{n}{2}$ odd, some edge of a perfect matching on V_2 has one endpoint in each partite set of G_2. Thus, $\mathbf{G} = (G_1, G_2, G_3)$ cannot pack.

($\Rightarrow$) Assume that a graph triple $\mathbf{G}$ is a counterexample with the minimum $|E_3|$. By Theorem 1, $E_3 \neq \emptyset$. By the minimality of $|E_3|$, we may assume that there is a mapping f which has a conflict at only one edge $vw \in E_3$, i.e., $f(v) = w$. For each $a \in V_1 - v$, define the mapping f_a by $f_a(v) = f(a)$, $f_a(a) = w$ and $f_a(x) = f(x)$ for all $x \in V_1 - a - v$. We claim that there is a mapping f_a that satisfies:

(i) $f_a(N_1(a)) \cap N_2(w) = \emptyset$,
(ii) $f_a(N_1(v)) \cap N_2(f(a)) = \emptyset$,
(iii) $f_a(a) \notin N_3(v)$, and
(iv) $w \notin N_3(a)$.

Indeed, $V_1 - v$ has at most $\Delta_1\Delta_2$ vertices that may violate (i), at most $\Delta_1\Delta_2$ vertices that may violate (ii), at most $\Delta_3 - 1$ vertices that may violate (iii) and at most $\Delta_3 - 1$ vertices that may violate (iv). Since $\mathbf{G}$ does not pack, $n - 1 = |V_1 - v| \leq (\Delta_3 - 1) + (\Delta_3 - 1) + 2\Delta_1\Delta_2$. But this inequality yields $n + 1 \leq 2(\Delta_3 + \Delta_1\Delta_2)$, contradicting (1).

Thus some f_a satisfies (i)–(iv). Then under f_a there is no conflict along edge vw and no new conflicts are introduced. Since the only conflict in f was along vw, f_a is a packing, a contradiction to the choice of $\mathbf{G}$. □

3. Preliminary facts

The following lemma is an extension of Theorem 3.

Lemma 7. *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If $\Delta_3 \leq n - 1$ and $e_1 + e_2 + e_3 \leq \lfloor \frac{3}{2}n \rfloor - 2$, then the triple $\mathbf{G} = (G_1, G_2, G_3)$ packs.*

Proof. It is enough to prove the lemma in the case

$$e_1 + e_2 + e_3 = \left\lfloor \frac{3}{2}n \right\rfloor - 2. \tag{2}$$

The proof will proceed by induction on n. If $n = 1$, then $\mathbf{G}$ contains no edges and it packs. If $n = 2$, then $e_1 + e_2 + e_3 = 1$ and $\mathbf{G}$ also packs. If $e_3 = 0$, then the result holds by Theorem 3. If $e_1 = 0$ or $e_2 = 0$, then the problem reduces to finding a perfect matching in $K_{n,n} - E_3$. By the König–Egerváry Theorem, if $K_{n,n} - E_3$ has no perfect matching, then it has a vertex cover C with $|C| = n - 1$. This means that $G_3 - C$ is a complete bipartite graph with $n + 1$ vertices, say $G_3 - C = K_{k,n+1-k}$. Since $\Delta_3 \leq n - 1$, we have $2 \leq k \leq n - 1$ and so $|E(G_3 - C)| = k(n + 1 - k) \geq 2n - 2$, contradicting (2). Therefore, $e_1, e_2, e_3 > 0$ and so $n \geq 4$.

We now claim that

$$\Delta_3 \leq n - 2. \tag{3}$$

Otherwise, by symmetry, we may assume that $d_3(v) = n - 1$ for some $v \in V_1$. Let $V_2 - N_3(v) = \{y\}$. Then at most $n/2 - 1$ edges in $\mathbf{G}$ are not adjacent to v. In particular, there is $u \in V_2$ that has no neighbors in $(V_1 \cup V_2) - v$. If $u = y$, then we pack $\mathbf{G} - v - y$ by induction and extend this packing by assigning v to y.

If $uv \in E_3$ and there is $w \in V_1 - v$ with $d(w) \geq 1$, then consider $\mathbf{G}' = \mathbf{G} - w - u$. The total number of edges decreases by at least 2, and v is incident with exactly $n - 2$ yellow edges. So, since $\mathbf{G}'$ contains at most $\lfloor \frac{3}{2}n \rfloor - 4$ edges, $\Delta_3(\mathbf{G}') = n - 2$. Thus $\mathbf{G}'$ packs by induction, and we can extend the packing by sending w to u. Finally, if $uv \in E_3$ and G_1 has no edges, it is enough to find an ordering $(v_1, \ldots, v_n)$ of V_1 and an ordering $(y_1, \ldots, y_n)$ of V_2 such that $v_iy_i \notin E_3$ for all i. We order V_1 so that $v_1 = v$ and $d_3(v_{i+1}) \leq d_3(v_i)$ for all $1 \leq i \leq n - 1$ and find a nonneighbor y_i for v_i greedily one by one for $i = 1, \ldots, n$. This is possible, since $G_3 - v_1$ has at most $n/2 - 1$ edges and so for $i \geq 2$, v_i has at most $\frac{n/2-1}{i-1}$ neighbors in $V_2 - \{y_1, \ldots, y_{i-1}\}$. This proves (3).

We now proceed in three cases.

Case 1: *There exists an $i \in \{1, 2\}$ and a vertex $x \in V_i$ such that $d_i(x) = 0$ and $d_3(x) > 0$.* By symmetry, we may assume $i = 1$. If there exists $y \in V_2 - N_3(x)$ with $d_3(x) + d(y) \geq 2$, then by (3) the triple $\mathbf{G} - x - y$ satisfies the lemma. By induction, $\mathbf{G} - x - y$ has a packing and this packing can be extended to $\mathbf{G}$ by assigning x to y. Otherwise, we may assume $d(y) = 0$ for every $y \in V_2 - N_3(x)$ and $d_3(x) = 1$. Let $N_3(x) = \{z\}$. Since $\Delta_3 \leq n - 1$, there exists a vertex $w \in V_1 - N_3(z)$ that can be mapped to z. As $d(y) = 0$ for each $y \in V_2 - z$, any bijection from $V_1 - w$ onto $V_2 - z$ is a packing of $\mathbf{G} - w - z$. This packing extends to a packing of $\mathbf{G}$ by assigning w to z.

Case 2: *There exists an $i \in \{1, 2\}$ and a vertex $x \in V_i$ such that $d_i(x) = d_3(x) = 0$.* Again, we may assume $i = 1$. Similarly to Case 1, if there exists $y \in V_2$ with $d(y) \geq 2$, then the triple $\mathbf{G} - x - y$ satisfies the lemma. By induction $\mathbf{G} - x - y$ has

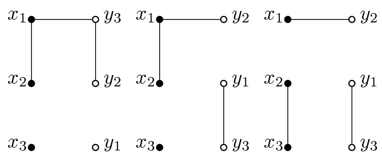

Fig. 3. Graph triples of order $n = 3$ and $e_1 = e_2 = e_3 = 1$. In each case, the function $f : V_1 \to V_2$ defined by $f(x_i) = y_i$ for each $i \in \{1, 2, 3\}$ is a packing.

a packing and this packing can be extended to $\mathbf{G}$ by assigning x to y. So we may assume that $d(y) \leq 1$ for all $y \in V_2$. Then since $e_3 > 0$, there is $y \in V_2$ with $d_3(y) = 1$ and $d_2(y) = 0$. But this means we now have Case 1.

Case 3: *For $i \in \{1, 2\}$, $d_i(v) \geq 1$ for every $v \in V_i$.* By (2), there is $x \in V_1 \cup V_2$ with $d(x) \leq 1$. By symmetry, we may assume $x \in V_1$. By the case assumption, $d_1(x) = 1$, and so $d_3(x) = 0$. Let $N_1(x) = \{z\}$. Since $e_3 > 0$, there is $y \in V_2$ incident with a yellow edge. Let $\mathbf{G}''$ be obtained from the triple $\mathbf{G} - x - y$ by joining z with an edge to each vertex in $N_2(y)$. Note that we have deleted $1 + d_2(y) + d_3(y)$ edges and added only $d_2(y)$ edges. Since $d_3(y) \geq 1$, $|E(\mathbf{G}'')| \leq \left\lfloor \frac{3}{2}(n-1) \right\rfloor - 2$.

For $i \in \{1, 2\}$, $d_i(v) \geq 1$ for each $v \in V_i$, so $e_1, e_2 \geq \frac{n}{2}$. Every vertex in $\mathbf{G}''$ is incident to at most Δ_3 yellow edges present in $\mathbf{G}$ and at most $d_2(y) \leq \Delta_2$ newly added yellow edges. Hence, each vertex in $\mathbf{G}''$ is incident to at most $e_2 + e_3 \leq (\frac{3}{2}n - 2) - e_1 \leq n - 2$ yellow edges. Thus the triple $\mathbf{G}''$ satisfies the conditions of Lemma 7 and, by induction, $\mathbf{G}''$ packs. Due to the added yellow edges, z was sent to a vertex in $V_2 - N_2(y)$. Therefore, this packing extends to a packing of $\mathbf{G}$ by mapping x to y. $\square$

Lemma 7 along with the following corollary will serve as a base case for a proof of Theorem 6.

Corollary 8. *Suppose $\mathbf{G}$ is a graph triple (G_1, G_2, G_3) of order $n \geq 1$. If $e_1 + e_2 + e_3 \leq n$, then either:*

(1) *$\mathbf{G}$ has a packing, or*
(2) *$e_1 = e_2 = 0$ and for some $i \in \{1, 2\}$, there exists $v \in V_i$ adjacent to all vertices in V_{3-i}, or*
(3) *$n = 2$, $e_3 = 0$ and $G_1 \cong G_2 \cong K_2$.*

Proof. For $n \geq 4$, the result follows from Lemma 7. If $n = 1$, then either there are no edges and so $\mathbf{G}$ packs, or there is a single edge in E_3, and (2) holds.

If $n = 2$ and $e_1 + e_2 = 2$, then (3) holds. If $n = 2$, $e_3 = 1$ and $e_1 + e_2 \leq 1$, then $\mathbf{G}$ has a packing. Finally, if $n = 2$ and $e_3 = 2$, then either $\mathbf{G}$ has a packing or (2) holds.

The last case is $n = 3$. If $e_3 = 0$, then in the worst case, $e_1 + e_2 = 3$. In this case, either $\{G_1, G_2\} \cong \{K_{1,2}, K_2 \cup K_1\}$ or $\{G_1, G_2\} \cong \{K_3, \overline{K}_3\}$ and so $\mathbf{G} = (G_1, G_2, \overline{K}_6)$ packs in all cases. Suppose now $e_1 = 0$. Then similarly to the proof of Lemma 7, $\mathbf{G}$ packs if $K_{3,3} - E_3$ has a perfect matching. If $K_{3,3} - E_3$ has no such matching, then by the König–Egerváry Theorem, G_3 has a complete bipartite subgraph with 4 vertices. Since $e_3 \leq 3$, the only possibility is that $G_3 = K_{1,3}$, i.e. (2) holds. Thus, $e_1, e_2, e_3 \geq 1$, which means $e_1 = e_2 = e_3 = 1$. Up to isomorphism, there are only 3 cases, and Fig. 3 shows a packing in each case. $\square$

4. Proof of Theorem 6

Let $\mathbf{G} = (G_1, G_2, G_3)$ of order n be a counterexample to Theorem 6 with the smallest order. By Corollary 8, $n \geq 4$. Also, by Theorem 4, we may assume $E_3 \neq \emptyset$.

Lemma 9. $\Delta_3 \leq n - 2$.

Proof. Suppose that there exist $v \in V_1$ and $w \in V_2$ such that $N_3(v) = V_2 - w$. Since $|E(\mathbf{G} - v - w)| \leq (2n-3) - (n-1) = n-2$, the triple $\mathbf{G} - v - w$ packs by Corollary 8. If $d_1(v) = 0$ or $d_2(w) = 0$, then additionally placing v on w is a packing of $\mathbf{G}$. So assume $d_1(v) \geq 1$ and $d_2(w) \geq 1$.

Let $\mathbf{G}' = (G'_1, G'_2, G'_3)$ be obtained from $\mathbf{G}$ by deleting, in G_3, all $n - 1$ edges connecting v with V_2 and all edges (maybe zero) connecting w with V_1. We now show that after mapping v to w, there are enough isolated vertices in either $V_2 - w$ or $V_1 - v$ to complete the packing.

First suppose v and w are in different components of the underlying graph $\underline{\mathbf{G}'}$. Define X and Y to be the vertex sets of the component of $\underline{\mathbf{G}'}$ containing v and w, respectively (possibly $X = Y$). Define $Z = X \cup Y$ and let $z = |Z|$. For $i \in \{1, 2\}$, let $Z_i = Z \cap V_i$, with size z_i. The graph $\underline{\mathbf{G}'} - Z$ has $2n - z$ vertices and at most $2n - 3 - (n-1) - (z-2)$ edges. So $\underline{\mathbf{G}'} - Z$ has at least $(2n - z) - (n - z) = n$ components, and at least z of them have no edges, i.e. are singletons. At least z_1 of the singletons are in V_2 or at least z_2 of them are in V_1. Suppose the former holds. In particular, there is a set $S \subseteq V_2 - w$ of singletons with $|S| = d_1(v)$.

Consider the triple $\mathbf{G}'' = \mathbf{G} - v - w - N_1(v) - S$. The triple $\mathbf{G}''$ has order $n - d_1(v) - 1$ and $|E(\mathbf{G}'')| \leq 2n - 3 - (n-1) - d_1(v) - d_2(w) = n - 2 - d_1(v) - d_2(w)$. The number of edges in $\mathbf{G}''$ is strictly less than the order of $\mathbf{G}''$, so by Corollary 8, $\mathbf{G}''$ packs. This packing, together with the placement of v and $N_1(v)$, gives a packing of $\mathbf{G}$, a contradiction. $\square$

Lemma 10. $\Delta_1, \Delta_2 \leq n - 3$.

Proof. Suppose $\Delta_1 = n-2$, the other case is symmetric. Let $v, v' \in V_1$ and $N_1(v) = V_1 - v - v'$.

Case 1: *There is $w \in V_2 - N(v)$ with no neighbors in V_2.* Consider the triple $\mathbf{G}' = \mathbf{G} - v - w$. Since $d(v) \geq n-2$, $|E(\mathbf{G}')| \leq (2n-3)-(n-2) = n-1$. By Lemma 9, $\Delta_3(\mathbf{G}') \leq n-2$, so $\mathbf{G}$ packs by Corollary 8. This packing can be extended to a packing of $\mathbf{G}$ by sending v to w.

Case 2: *Every $w \in V_2 - N(v)$ has a white neighbor.* Let W' be the set of vertices in V_2 reachable from V_1 in the underlying graph $\underline{\mathbf{G}}$, and let $W = V_2 - W'$. Since $\underline{\mathbf{G}} - W$ has at least $(n-2) + |W'|$ edges, $|W'| \leq n-1$. So $W \neq \emptyset$ and if $d_1(v') = a$, then

$$|E(\underline{\mathbf{G}}[W])| \leq (2n-3) - (n-2) - a - |W'| = |W| - 1 - a. \tag{4}$$

Let W_1 be the vertex set of a smallest tree component in $G_2[W]$. By the case assumption, every vertex in $G_2[W]$ has positive degree. Since there are no yellow edges incident to W, the degree of each vertex in $G_2[W]$ is equal to its degree in $\mathbf{G}$. Let $y \in W_1$ be a vertex of degree 1 in $G_2[W]$ and let $y' \in W_1$ be the neighbor of y. Suppose $d_2(y') = b$. Let $\mathbf{G}' = (G_1', G_2', G_3')$ be the triple obtained from $\mathbf{G} - \{v, v', y, y'\}$ by adding the $a(b-1)$ yellow edges connecting the white neighbors of v' with the (necessarily white) neighbors of y' distinct from y. The graph triple $\mathbf{G}'$ has $2(n-2)$ vertices and

$$|E(\mathbf{G}')| \leq 2n-3-(n-2)-a-b+a(b-1) = n-1-2a+b(a-1). \tag{5}$$

If $\mathbf{G}'$ packs, then because of the added edges, this packing extends to a packing of $\mathbf{G}$ by sending v to y and v' to y'. Suppose it does not.

Case 2.1: $a \leq 1$. Then by (5),

$$\begin{aligned} &|E(\mathbf{G}')| \leq n-2 \textit{ with equality only if } a=0,\, b=1, \textit{ and the only edges in} \\ &E(\mathbf{G}) - E(\mathbf{G}') \textit{ are } yy' \textit{ and the } n-2 \textit{ white edges incident with } v. \end{aligned} \tag{6}$$

By Corollary 8, $|E(\mathbf{G}')| = n-2$ and $\mathbf{G}'$ either has no white edges or has no yellow edges, since $\mathbf{G}'$ does not pack. Then (6) yields $a = 0$, $b = 1$, and $E(\mathbf{G}) - E(\mathbf{G}')$ has no yellow edges. Since $e_3 > 0$, this implies $\mathbf{G}'$ has no white edges, but this contradicts the case conditions together with $b = 1$.

Case 2.2: $a \geq 2$. By (4), $G_2[W]$ has at least $a+1$ tree components. So by the choice of W_1,

$$2 \leq b+1 \leq |W_1| \leq |W|/(a+1) \leq n/(a+1) \tag{7}$$

and thus $b \leq -1 + n/(a+1)$. Since $a \geq 2$, by (5),

$$\begin{aligned} |E(\mathbf{G}')| &\leq n-1-2a+\left(\frac{n}{a+1}-1\right)(a-1) \\ &= n-3a+n\frac{a}{a+1}-\frac{n}{a+1} \\ &\leq n+n\frac{a}{a+1}-3a-3 \\ &\leq n+n\frac{a}{a+1}-9 < 2(n-2)-3. \end{aligned}$$

Since $\mathbf{G}'$ does not pack and the last strict inequality ensures that the examples from Fig. 1 do not appear as $\mathbf{G}'$, by induction, some vertex z in $\mathbf{G}'$ has yellow degree $n-2$ or white degree at least $n-3$. But since we deleted at least $n-2+a+b \geq n+1$ edges out of $2n-3$ in $\mathbf{G}$, the number of edges in $E(\mathbf{G}') \cap E(\mathbf{G})$ (and thus the total number of white edges in $E(\mathbf{G}')$) is at most $n-4$. It follows that the vertex z has yellow degree $n-2$ in $\mathbf{G}'$ and is incident to an added yellow edge. Since all added yellow edges connect W_1 with V_1, $z \in W_1 \cup V_1$.

If $z \in W_1$, then by the definition of W, all $n-2$ yellow edges incident to z are in $E(\mathbf{G}') - E(\mathbf{G})$. By the construction of $\mathbf{G}'$, this yields $a \geq n-2$, which contradicts (7) since $n \geq 4$. Thus $z \in V_1$ and is adjacent to each vertex in $V(G_2')$. But by the definition of W and $\mathbf{G}'$, the vertices in $W - W_1$ are not incident with yellow edges in $\mathbf{G}'$. This is a contradiction, since $W - W_1 \neq \emptyset$ by (4). $\square$

Lemma 11. *Every vertex of* $\mathbf{G}$ *has a white neighbor.*

Proof. Suppose $v \in V$ has no white neighbor. Without loss of generality, assume $v \in V_1$.

Case 1: $d_3(v) = 0$.

Case 1.1: Some $w \in V_2$ has degree at least 2. Then $\mathbf{G} - v - w$ contains at most $2(n-1)-3$ edges. By Lemmas 9 and 10, $\mathbf{G} - v - w$ satisfies the conditions of Theorem 6 for $n' = n-1$. Since any packing of $\mathbf{G} - v - w$ can be extended to a packing of $\mathbf{G}$ by sending v to w, it does not pack. So by the minimality of $\mathbf{G}$, $\mathbf{G} - v - w$ is one of the examples in Fig. 1. In particular, $G_3 - v - w$ has no yellow edges. This means all yellow edges in $\mathbf{G}$ are incident to w. Since each of the examples in Fig. 1 has exactly $2(n-1)-3$ edges, $d(w) = 2$.

If both edges adjacent to w are yellow, since every graph in Fig. 1 contains 3 vertices of positive degree, there is some $v' \in V_1 - N(w)$ with $d(v') \geq 1$. Then $\mathbf{G} - v' - w$ contains fewer than $2(n-1)-3$ edges and no yellow edges. By Theorem 4, $\mathbf{G} - v' - w$ packs and this packing can be extended to a packing of $\mathbf{G}$ by sending v' to w.

Since $e_3 > 0$, the remaining possibility is that w has exactly one neighbor $w' \in V_2$ and one neighbor in V_1. As above, we can choose some $v' \in V_1 - N_1(w)$ with positive degree. Create a new graph triple $\mathbf{G}'$ from $\mathbf{G}$ by removing v' and w and adding yellow edges from w' to $N_1(v')$. This triple $\mathbf{G}'$ has exactly $2(n-1)-3$ edges, and all yellow edges in $\mathbf{G}'$ are incident to w', since w was the only vertex in $\mathbf{G}$ incident to yellow edges. So $\Delta_3(\mathbf{G}') = d_1(v') \le n-3$ by Lemma 10. Additionally, no white edges were added, so again by Lemma 10, $\Delta_1(\mathbf{G}')$, $\Delta_2(\mathbf{G}') \le n-3$. Thus, $\mathbf{G}'$ satisfies the conditions of Theorem 6 and has at least one yellow edge. Hence $\mathbf{G}'$ is not one of the examples from Fig. 1. By the minimality of $\mathbf{G}$, the triple $\mathbf{G}'$ packs, and this packing can be extended to a packing of $\mathbf{G}$ by sending v' to w.

Case 1.2: $d(w) \le 1$ for each $w \in V_2$. If there exists $w \in V_2$ such that $d(w) = 0$, then in view of Case 1.1, each $u \in V_1$ has degree at most 1, and $\mathbf{G}$ packs by Corollary 8.

Thus, $d(w) = 1$ for each $w \in V_2$. Since $e_3 > 0$, there exists $w \in V_2$ such that $d_3(w) = d(w) = 1$. Let $N_3(w) = \{v'\}$. Fix $u \in V_1 - v'$ with $d(u)$ maximum. If $d(u) = 0$, then

$$\sum_{v \in V_1 \cup V_2} d(v) \le d_3(v') + n \le \Delta_3(\mathbf{G}) + n \le 2n-1.$$

In particular, $|E(\mathbf{G})| < n$. Corollary 8 and the strict inequality imply that $\mathbf{G}$ packs. So suppose $d(u) \ge 1$. Since $d(w) = 1$ and $d(u) \ge 1$, $|E(\mathbf{G} - u - w)| \le 2(n-1) - 3$. By Lemmas 9 and 10, $\mathbf{G} - u - w$ satisfies the conditions of Theorem 6. If $d(u) = 1$, then v' is the only vertex in $V_1 \cup V_2$ with degree at least 2, and hence $\mathbf{G} - u - v$ is not one of the examples from Fig. 1. Similarly, if $d(u) \ge 2$, then $|E(\mathbf{G} - u - w)| \le 2(n-1) - 4$ and again $\mathbf{G} - u - w$ is not one of the examples from Fig. 1. Therefore, there is a packing of $\mathbf{G} - u - w$, and sending u to w extends this packing to a packing of $\mathbf{G}$.

Case 2: $d_3(v) \ge 1$. Among the vertices in $V_2 - N_3(v)$ with maximum degree, let w be a vertex that minimizes $d_3(w)$. By Case 1, $d(w) \ge 1$. Consider the triple $\mathbf{G}' := \mathbf{G} - v - w$. Since $d(v) + d(w) \ge 2$ and $vw \notin E(\mathbf{G})$, by Lemmas 9 and 10, $\mathbf{G}'$ satisfies the conditions of Theorem 6 for $n' = n - 1$. If $\mathbf{G}'$ packs, then the packing extends to a packing of $\mathbf{G}$ by sending v to w. Therefore by Theorem 6, $d(v) = d(w) = 1$, and $\mathbf{G}'$ is an example from Fig. 1.

However, by the choice of w and the fact that $d(w) = 1$, all vertices in $V_2 - N_3(v)$ have degree at most 1 in $\mathbf{G}$ and, hence, in $G_2 - w$. By inspection, $G_2 - w$ is either $G_1(1)$ or $G_2(3)$ in Fig. 1, as every other graph in Fig. 1 has at least two vertices with degree at least 2. Since each of $G_2(1)$ and $G_1(3)$ has an isolated vertex, and by Case 1, $\mathbf{G}$ has no isolated vertices in V_1, we have removed an incident yellow edge when deleting w. It follows that $d_3(w) = d(w) = 1$. Each of $G_1(1)$ and $G_2(3)$ has at least 4 vertices incident to exactly one white edge. Since $d_2(w) = 0$, in the process of removing v and w from $\mathbf{G}$, we have removed only one edge incident to $V_2 - w$. Thus, G_2 contains a vertex with degree 1 incident to a white edge, contradicting our choice of w. □

Proof of Theorem 6. Let $\mathbf{G}$ be our minimum counterexample. Since $e_3 > 0$, $\mathbf{G}$ has a yellow edge xy with $x \in V_1$ and $y \in V_2$. Since $|E(\mathbf{G})| \le 2n-3 < 2n$, there are vertices of degree at most 1. We may assume that $v \in V_1$ and $d(v) \le 1$. By Lemma 11, v has a white neighbor, v'. Since $d(v) = 1$, $v \ne x$. We obtain the triple $\mathbf{G}'$ from $\mathbf{G}$ by removing v and y, and adding a yellow edge from v' to each vertex in $N_2(y)$. Then, $|E(\mathbf{G}')| \le |E(\mathbf{G})| - 2 \le 2(n-1) - 3$. The triple $\mathbf{G}'$ has at least one yellow edge (connecting v' with a white neighbor of y), so it is not an example from Fig. 1. Since we have not added any white edges, by Lemma 10, $\Delta_1(\mathbf{G}')$, $\Delta_2(G') \le n-3$. If $\Delta_3(G') \le n-2$, then $\mathbf{G}'$ satisfies the conditions of Theorem 6 and so there exists a packing of $\mathbf{G}'$. This packing extends to a packing of $\mathbf{G}$ by sending v to y.

Thus, $\Delta_3(\mathbf{G}') = n-1$. By Lemma 11, $e_1 + e_2 \ge n$, so $\Delta_3 \le e_3 \le n-3$. Since v' is the only vertex whose degree in $\mathbf{G}'$ exceeds the degree in $\mathbf{G}$ by at least 2, it is the only vertex with yellow degree $n-1$ in $\mathbf{G}'$. In particular, by construction this implies that in $\mathbf{G}$, every vertex in $V_2 - y$ is either in $N_3(v')$ or in $N_2(y)$.

Since the underlying graph $\underline{\mathbf{G}}$ of $\mathbf{G}$ contains $2n$ vertices and at most $2n-3$ edges, it contains at least 3 tree components. Consider a tree component T that contains neither v' nor y. Since every vertex in $V_2 - y$ is adjacent to y or v',

$$T \text{ contains only vertices in } V_1 \text{ that do not have neighbors in } V_2. \tag{8}$$

By Lemma 11, T is not a single vertex. Let $u \in V_1$ be a leaf vertex, so $d(u) = 1$, and let $u' \in V_1$ be its neighbor.

Consider the triple $\mathbf{G}''$ formed from $\mathbf{G} - u - y$ by adding a yellow edge from u' to each vertex in $N_2(y)$. As with $\mathbf{G}'$, $|E(\mathbf{G}'')| \le |E(\mathbf{G})| - 2 \le 2(n-1) - 3$ and $\mathbf{G}''$ contains a yellow edge, so it is not an example from Fig. 1. No white edges have been added, so by Lemma 10, $\Delta_1(\mathbf{G}'')$, $\Delta_2(\mathbf{G}'') \le n-2$. By (8), u' is incident to exactly $d_2(y) \le \Delta_2 \le n-3$ yellow edges and every other vertex in $\mathbf{G}''$ is incident to at most $\Delta_3 + 1 \le n-2$ yellow edges. So $\mathbf{G}''$ satisfies the conditions of Theorem 6. Therefore, there exists a packing of $\mathbf{G}''$, and this packing extends to a packing of $\mathbf{G}$ by sending u to y. □

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