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Planar 4-critical graphs with four triangles

Oleg V. Borodin [a,b], Zdeněk Dvořák [c], Alexandr V. Kostochka [d,a], Bernard Lidický [d], Matthew Yancey [d]

[a] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
[b] Novosibirsk State University, Novosibirsk 630090, Russia
[c] Computer Science Institute of Charles University, Prague, Czech Republic
[d] University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA

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ABSTRACT

By the Grünbaum–Aksenov Theorem (extending Grötzsch's Theorem) every planar graph with at most three triangles is 3-colorable. However, there are infinitely many planar 4-critical graphs with exactly four triangles. We describe all such graphs. This answers a question of Erdős from 1990.

1. Introduction

The classical Grötzsch's Theorem [17] says that every planar triangle-free graph is 3-colorable. The following refinement of it is known as the Grünbaum–Aksenov Theorem (the original proof of Grünbaum [18] was incorrect, and Aksenov [1] fixed the proof).

Theorem 1 ([1,7,18]). *Let G be a planar graph containing at most three triangles. Then G is* 3*-colorable.*

The example of the complete 4-vertex graph K_4 shows that "three" in Theorem 1 cannot be replaced by "four". But maybe there are not many plane 4-critical graphs with exactly four triangles (Pl_4-*graphs*, for short)?

It turned out that there are many. Havel [20] presented a Pl_4-graph H_1 (see Fig. 1) in which the four triangles had no common vertices. He used the *quasi-edge* $H_0 = H_0(u, v)$ (on the left of Fig. 1), that is, a graph in each 3-coloring of which the vertices u and v must have distinct colors. The graph

E-mail addresses: brdnoleg@math.nsc.ru (O.V. Borodin), rakdver@iuuk.mff.cuni.cz (Z. Dvořák), kostochk@math.uiuc.edu (A.V. Kostochka), lidicky@illinois.edu, bernard@alderan.cz (B. Lidický), yancey1@illinois.edu (M. Yancey).

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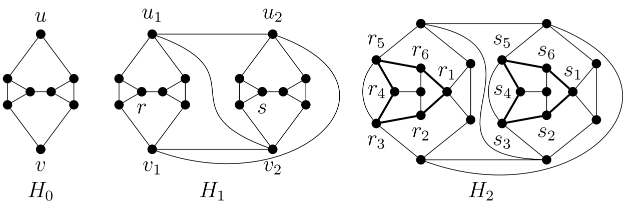

Fig. 1. A quasi-edge H_0 and Pl_4-graphs H_1 and H_2.

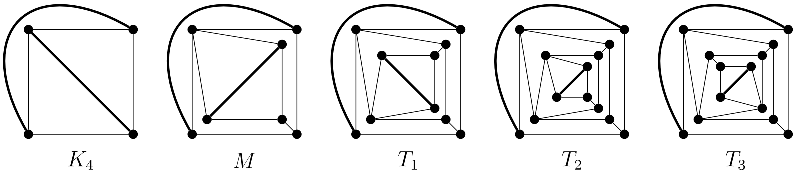

Fig. 2. Some members of $\mathcal{TW}$.

H_1 is obtained from K_4 by replacing the edges v_1u_1 and v_2u_2 with copies of the quasi-edge H_0. Then Sachs [23] in 1972 asked whether it is true that in every non-3-colorable planar graph G with exactly four triangles and no separating triangles these triangles can be partitioned into two pairs so that in each pair the distance between the triangles is less than two.

Aksenov and Mel'nikov [5,6] answered the question in the negative by constructing a Pl_4-graph H_2 (see Fig. 1) in which the distance between any two of the four triangles was at least two. Moreover, they constructed two infinite series of Pl_4-graphs. Aksenov [2] was studying Pl_4-graphs in the seventies. Nielsen and Toft [22] constructed another infinite series of Pl_4-graphs, this time with no 5-faces. According to Steinberg [24], Erdős in 1990 asked for a description of Pl_4-graphs again. Borodin [7] mentioned that he knew 15 infinite families of Pl_4-graphs. In his survey [8], he mentioned the problem of describing Pl_4-graphs among unsolved problems on 3-coloring of plane graphs.

First, we give a description of the Pl_4-graphs without 4-faces, which we call $\mathrm{Pl}_{4,4f}$*-graphs*. Thomas and Walls [25] constructed an infinite family $\mathcal{TW}$ of $\mathrm{Pl}_{4,4f}$-graphs; the first five graphs in $\mathcal{TW}$ are depicted in Fig. 2 (note that T_2 and T_3 are isomorphic graphs, but their drawings are different). If an edge e of a graph belongs to exactly two triangles, we say that e is a *diamond* edge. Each graph in $\mathcal{TW}$ contains two disjoint diamond edges, drawn in bold in Fig. 2. We define the class $\mathcal{TW}$ in terms of DHGO-compositions.

A *DHGO-composition* $O(G_1, G_2)$ of graphs G_1 and G_2 is a graph obtained as follows: delete some edge xy from G_1, split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree, and identify x with z_1 and y with z_2. Note that DHGO-composition could be found in paper by Dirac [10] and has roots in [9]. It was also used by Gallai [15] and Hajós [19]. Ore used it for a composition of complete graphs.

If xy is a diamond edge of G_1 and G_2 is K_4, then we say that $O(G_1, G_2)$ is a *diamond expansion* of G_1. The class $\mathcal{TW}$ consists of all graphs that can be obtained from K_4 by diamond expansions.

Note that H_1 is a $\mathrm{Pl}_{4,4f}$-graph but is not in $\mathcal{TW}$. A graph is *k-Ore* if it is obtained from a set of copies of K_k by a sequence of DHGO-compositions. It was observed in [21] that every k-Ore graph is k-critical. A partial case of Theorem 6 in [21] is the following.

Theorem 2 (*[21]*). *Let G be an n-vertex* 4*-critical graph. Then* $|E(G)| \geq \frac{5n-2}{3}$. *Moreover,* $|E(G)| = \frac{5n-2}{3}$ *if and only if G is a* 4*-Ore graph.*

We will see below that every $\mathrm{Pl}_{4,4f}$-graph is a 4-Ore graph. On the other hand, our first result says the following.

Theorem 3. *Every* 4*-Ore graph has at least four triangles. Moreover, a* 4*-Ore graph G has exactly four triangles if and only if G is a* $\mathrm{Pl}_{4,4f}$*-graph.*

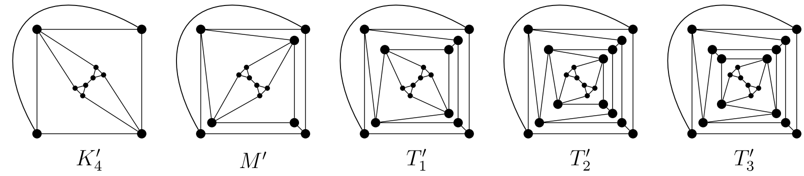

Fig. 3. Some members of $\mathcal{TW}_1$.

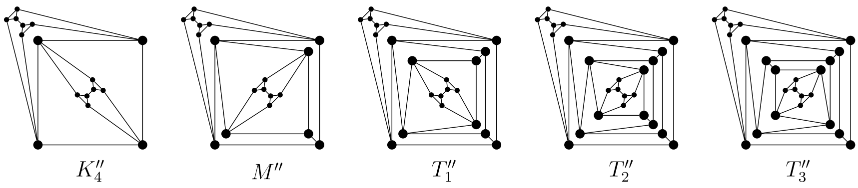

Fig. 4. Some members of $\mathcal{TW}_2$.

This reduces a "topological" problem on plane graphs to a problem on abstract graphs. And it turns out that $\mathrm{Pl}_{4,4f}$-graphs do not differ much from the Thomas–Walls graphs. Let $\mathcal{TW}_1$ denote the graphs obtained from a graph in $\mathcal{TW}$ by replacing a diamond edge by the Havel's quasi-edge H_0 (see Fig. 3). Let $\mathcal{TW}_2$ denote the graphs obtained from a graph in $\mathcal{TW}_1$ by replacing a diamond edge by the Havel's quasi-edge H_0 (see Fig. 4). Note that H_0 contains no diamond edges, and thus each graph in $\mathcal{TW}_2$ can be obtained from a graph in $\mathcal{TW}$ by replacing two vertex-disjoint diamond edges by the Havel's quasi-edge H_0.

Theorem 4. *The class of* $\mathrm{Pl}_{4,4f}$*-graphs is equal to* $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$.

The Pl_4-graphs may have arbitrarily many 4-faces. Let G be a plane graph, let $C_P = xz'yx'zy'$ be a 6-cycle in G and let Δ be the closed disk bounded by C_P. Let P be the subgraph of G consisting of the vertices and edges drawn in Δ. If all neighbors of the vertices x', y' and z' belong to P and all faces of G contained in Δ have length 4, then we say that P is a *patch*. In other words, a patch P is a subgraph of G which is a quadrangulation of the interior of a 6-cycle (x, x', y, y', z, z') such that x', y' and z' have no neighbors outside of P. The patch P is *critical* if x', y' and z' have degree at least 3 and every 4-cycle in P bounds a face.

We will see that if G_0 is a Pl_4-graph and a vertex $v \in V(G_0)$ has exactly 3 neighbors x, y and z, then the graph G_v obtained from $G_0 - v$ by inserting a critical patch P with boundary $C_P = xz'yx'zy'$ (where x', y' and z' are new vertices) is again a Pl_4-graph. For example, the graph H_2 in Fig. 1 is obtained from the graph H_1 by replacing the vertices r and s with patches bounded by the cycles $r_1 \dots r_6$ and $s_1 \dots s_6$, respectively. This gives a way to construct from every $\mathrm{Pl}_{4,4f}$-graph an infinite family of Pl_4-graphs. Note that the infinite series of Pl_4-graphs by Nielsen and Toft [22] are graphs that are obtained from K_4 by replacing one vertex by a patch P. Our main result is that every Pl_4-graph can be obtained this way.

Theorem 5. *A plane* 4*-critical graph has exactly four triangles if and only if it is obtained from a* $\mathrm{Pl}_{4,4f}$*-graph by replacing several (possibly zero) non-adjacent* 3*-vertices with critical patches.*

Thus, even though there are infinitely many Pl_4-graphs, we know the structure of all of them. This fully answers Erdős's question from 1990. In particular, the result yields that Sachs had the right intuition in 1972: his question has positive answer if we replace "less than two" with "at most two". Also, Aksenov and Mel'nikov [6] conjectured, in particular, that H_1 is the unique smallest Pl_4-graph with the minimum distance 1 between triangles and H_2 is the unique smallest Pl_4-graph with the minimum distance 2 between triangles. Our description confirms this.

Havel [20] asked the following question: Does there exist a constant C such that every planar graph with the minimal distance between triangles at least C is 3-colorable? The graph H_2 shows that $C \geq 3$, and a further example of Aksenov and Mel'nikov [6] shows that $C \geq 4$, which is the best known lower bound. The existence of (large) C was recently proved by Dvořák, Král' and Thomas [11]. It is conjectured by Borodin and Raspaud that $C = 4$ is sufficient, and our result confirms this conjecture for graphs with four triangles. For more details, see a recent survey of Borodin [8].

The proof of Theorem 5 can be converted to a polynomial-time algorithm to find a 3-coloring of a planar graph with four triangles or to decide that no such coloring exists. However, a more general algorithm of Dvořák, Král' and Thomas [12] can also be used, and thus we do not provide further details.

The structure of the paper is as follows. In the next section we study the structure of 4-Ore graphs. In Section 3 we prove Theorems 3 and 4. In the last section, we describe all Pl_4-graphs by proving Theorem 5.

2. The structure of 4-Ore graphs

In this section, we study 4-Ore graphs with few triangles. The following claim is simply a reformulation of the definition of a k-Ore graph.

Claim 6. *Every k-Ore graph $G \neq K_k$ has a separating set $\{x, y\}$ and two vertex subsets A and B such that*

- *$A \cap B = \{x, y\}$, $A \cup B = V(G)$ and no edge of G connects $A \setminus \{x, y\}$ with $B \setminus \{x, y\}$,*
- *x and y are non-adjacent in G and have no common neighbor in B,*
- *the graph G' obtained from $G[A]$ by adding the edge xy is a k-Ore graph, and*
- *the graph G'' obtained from $G[B]$ by identifying x with y into a new vertex $x * y$ is a k-Ore graph.*

Our first goal is to obtain a similar decomposition when we restrict ourselves to 4-Ore graphs with 4 triangles (Theorem 11 below).

Claim 7. *Every edge in each 4-Ore graph is contained in at most 2 triangles.*

Proof. We prove the claim by induction on the order of a graph. Let G be a 4-Ore graph and let uv be any edge of G and assume that the claim holds for all graphs with less than $|V(G)|$ vertices.

Each edge of K_4 is contained in exactly two triangles, and thus we can assume that $G \neq K_4$. Let $\{x, y\}$, A, B, G' and G'' be as in Claim 6. Since u is adjacent to v, either $\{u, v\} \subset A$ or $\{u, v\} \subset B$. Let $G_0 \in \{G[A], G[B]\}$ be the graph containing the edge uv, and let $G'_0 \in \{G', G''\}$ be the corresponding 4-Ore graph. Let $u'v'$ be the edge of G'_0 corresponding to uv. Every triangle of G containing uv maps to a triangle in G'_0 containing $u'v'$. Since $|V(G'_0)| < |V(G)|$, the edge $u'v'$ is contained in at most two triangles in G'_0 by induction, and thus uv is contained in at most two triangles in G. □

For a graph G, let $t(G)$ denote the number of triangles in G.

Claim 8. *If G is a 4-Ore graph, then $t(G) \geq 4$. For every vertex $z \in V(G)$, every graph G_z obtained from G by splitting z satisfies $t(G_z) \geq 2$. Furthermore, if $G \neq K_4$, then $t(G - z) \geq 2$, and if $G = K_4$, then $t(G - z) = 1$.*

Proof. Let G be a 4-Ore graph. We proceed by induction and assume that the claim holds for all graphs with less than $|V(G)|$ vertices. Since the claim holds for K_4, we can assume that $G \neq K_4$. Let $\{x, y\}$, A, B, G' and G'' be as in Claim 6. By induction hypothesis and Claim 7, $G[A] = G' - xy$ has at least $t(G') - 2 \geq 2$ triangles. Furthermore, $G[B]$ is obtained from G'' by splitting a vertex, and thus it has at least two triangles by induction. It follows that G has at least four triangles.

Consider now a vertex z of G. If $z \notin \{x, y\}$, then $G - z$ contains $G[A]$ or $G[B]$ as a subgraph, and thus $t(G - z) \geq 2$. If $z \in \{x, y\}$, say $z = x$, then $G - z$ contains $G' - x$ and $G'' - x * y$ as vertex-disjoint subgraphs, and by induction each of them has a triangle; hence, $t(G - z) \geq 2$. Finally, any graph G_z obtained from G by splitting z contains $G - z$ as a subgraph, and thus $t(G_z) \geq t(G - z) \geq 2$. □

Claim 9. *If G is a 4-Ore graph with $t(G) \geq 5$, then $t(G - u - v) \geq 1$ for each $u, v \in V(G)$.*

Proof. We proceed by induction and assume that the claim holds for all graphs with less than $|V(G)|$ vertices. Note that $G \neq K_4$, since $t(G) \geq 5$. Let $\{x, y\}$, A, B, G' and G'' be as in Claim 6. If $u, v \in A$, then $G'' - x * y$ is a subgraph of $G - u - v$ and $t(G - u - v) \geq t(G'' - x * y) \geq 1$ by Claim 8. Hence, by symmetry we can assume that $u \in B \setminus \{x, y\}$. Suppose that $v \in B$. We can assume that $v \neq x$, and thus $G' - y$ is a subgraph of $G - u - v$. Again, Claim 8 implies that $t(G - u - v) \geq t(G' - y) \geq 1$.

Finally, consider the case that $u \in B \setminus \{x, y\}$ and $v \in A \setminus \{x, y\}$. Since $t(G) \geq 5$, either $G[A]$ or $G[B]$ contains at least three triangles. In the former case, let $G_0 = G[A]$, $G'_0 = G'$, $r = v$ and $s = x$. In the latter case, let $G_0 = G[B]$, $G'_0 = G''$, $r = u$ and $s = x * y$. If r is contained in at most two triangles in G_0, then $t(G - u - v) \geq t(G_0 - r) \geq t(G_0) - 2 \geq 1$; hence, assume that r is contained in at least 3 triangles in G_0, and thus also in G'_0. Note that $G'_0 \neq K_4$, since $t(G_0) \geq 3$. By Claim 8, we have $t(G'_0) \geq t(G'_0 - r) + 3 \geq 5$. Note that $G'_0 - r - s$ is a subgraph of $G - u - v$, and thus $t(G - u - v) \geq t(G'_0 - r - s) \geq 1$ by induction. □

Claim 10. *If G is a 4-Ore graph with $t(G) \geq 5$, then $t(G - v) \geq 3$ for each $v \in V(G)$.*

Proof. Note that $G \neq K_4$, since $t(G) \geq 5$. Let $\{x, y\}$, A, B, G' and G'' be as in Claim 6. Since $t(G) \geq 5$, either $G[A]$ or $G[B]$ contains at least three triangles. In the former case, let $G_0 = G[A]$, $G_1 = G[B]$, $G'_0 = G'$ and $s = x$. In the latter case, let $G_0 = G[B]$, $G_1 = G[A]$, $G'_0 = G''$ and $s = x * y$. Since G_0 has at least three triangles, it follows that $G'_0 \neq K_4$. If $v \notin V(G_0)$, then $t(G - v) \geq t(G_0) \geq 3$. Therefore, assume that $v \in V(G_0)$. If $v \in \{x, y\}$, then $G' - v$ and $G'' - x * y$ are vertex-disjoint subgraphs of $G - v$, and since at least one of G' and G'' is not equal to K_4, we have $t(G - v) \geq t(G' - v) + t(G'' - x * y) \geq 3$ by Claim 8.

Finally, suppose that $v \notin \{x, y\}$, and thus G_1 is a subgraph of $G - v$. By Claim 8, there are at least two triangles in G_1. We claim that $G_0 - v$ contains a triangle. This is clear if v belongs to at most two triangles in G_0, since $t(G_0) \geq 3$. Otherwise, v belongs to at least three triangles in G_0, and thus also in G'_0. By Claim 8, we have $t(G'_0) \geq t(G'_0 - v) + 3 \geq 5$. We conclude that $t(G_0 - v) \geq t(G'_0 - v - s) \geq 1$ by Claim 9. Therefore, $t(G - v) = t(G_0 - v) + t(G_1) \geq 3$. □

A 4, 4-*Ore graph* is a 4-Ore graph with exactly 4 triangles. The main result of this section is the following.

Theorem 11. *Suppose G is a 4, 4-Ore graph distinct from K_4. Let $\{x, y\}$, A, B, G' and G'' be as in Claim 6. Then both G' and G'' are 4, 4-Ore graphs, xy is a diamond edge of G', and $t(G[B]) = 2$. Furthermore, if $G'' \neq K_4$, then $x * y$ belongs to exactly two triangles of G''.*

Proof. Note that $t(G[A]) \geq 2$ and $t(G[B]) \geq 2$ by Claims 7 and 8, and since $t(G[A]) + t(G[B]) = t(G) = 4$, it follows that $t(G[A]) = t(G[B]) = 2$. If $t(G') \geq 5$, then we would have $t(G[A]) \geq t(G' - x) \geq 3$ by Claim 10. If $t(G'') \geq 5$, then we would have $t(G[B]) \geq t(G'' - x * y) \geq 3$ by Claim 10. It follows that $t(G') \leq 4$ and $t(G'') \leq 4$, and by Claim 8, we conclude that $t(G') = t(G'') = 4$, i.e., both G' and G'' are 4, 4-Ore graphs.

Since xy belongs to $t(G') - t(G[A]) = 2$ triangles in G', it is a diamond edge. Similarly, $x * y$ belongs to at least $t(G'') - t(G[B]) = 2$ triangles in G''. Furthermore, if $G'' \neq K_4$ and $x * y$ belonged to at least three triangles, then we would have $t(G'') \geq t(G'' - x * y) + 3 \geq 5$ by Claim 8, which is a contradiction. □

3. A description of 4, 4-Ore graphs and $Pl_{4,4f}$-graphs

With Theorem 11, it is easy to characterize all 4, 4-Ore graphs. The *Moser spindle* is the Ore composition of two K_4's, depicted in Fig. 2 as M.

Lemma 12. *Every 4, 4-Ore graph belongs to $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$.*

Proof. Let G be a 4, 4-Ore graph. We proceed by induction and assume that every 4, 4-Ore graph with less than $|V(G)|$ vertices belongs to $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$. Note that $K_4 \in \mathcal{TW}$, and thus we can assume that $G \neq K_4$. Let $\{x, y\}$, A, B, G' and G'' be as in Theorem 11. By induction hypothesis, we have $G', G'' \in \mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$. Since G' has a diamond edge xy and G'' has a vertex $x * y$ belonging to at least two triangles, we conclude that $G', G'' \notin \mathcal{TW}_2$. Since $G[B]$ has exactly two triangles, an inspection of the graphs in $\mathcal{TW} \cup \mathcal{TW}_1$ (see Figs. 2 and 3) shows that either

- G'' is the Moser spindle, $x * y$ is its vertex of degree four, $G[B]$ is the Havel's quasiedge H_0 and x and y are its vertices of degree two, or
- x has degree two in $G[B]$, y has degree one in $G[B]$ and $x * y$ is incident with a diamond edge $(x * y)z$ of G'', where z is the neighbor of y in $G[B]$.

In the former case, G is obtained from G_1 by replacing a diamond edge with the Havel's quasiedge H_0, and thus $G \in \mathcal{TW}_1 \cup \mathcal{TW}_2$. In the latter case, observe that the described DHGO-composition of graphs from $\mathcal{TW} \cup \mathcal{TW}_1$ results in a graph from $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$. □

We can now describe $\mathrm{Pl}_{4,4f}$-graphs.

Proof of Theorems 3 and 4. By Claim 8, every 4-Ore graph has at least four triangles. Therefore, it suffices to prove that the following classes are equal to each other:

- 4, 4-Ore graphs,
- $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$, and
- $\mathrm{Pl}_{4,4f}$-graphs.

By Lemma 12, every 4, 4-Ore graph belongs to $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$. Every graph in $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$ is planar, 4-critical (since it is 4-Ore) and has four triangles, and thus it is a $\mathrm{Pl}_{4,4f}$-graph. Therefore, it suffices to show that every $\mathrm{Pl}_{4,4f}$-graph G is 4, 4-Ore. Since G has four triangles, we only need to prove that G is 4-Ore. Consider a plane drawing of G without 4-faces. Let e, n and s be the number of edges, vertices and faces of the drawing of G, respectively. Note that G has at most four triangular faces and all other faces of G have length at least 5. Therefore, G has at least $\frac{1}{2}(5(s-4)+3\cdot 4)$ edges. Note that G is connected (since it is 4-critical), and thus $s = e + 2 - n$ by Euler's formula. It follows that $e \leq (5n-2)/3$, and G is 4-Ore by Theorem 2. □

4. A description of Pl_4-graphs

We are going to characterize planar 4-critical graphs with 4 triangles. To deal with short separating cycles, we use the notion of criticality with respect to a subgraph. Let G be a graph and C be its (not necessarily induced) proper subgraph. We say that G is *C-critical (for 3-coloring)* if for every proper subgraph $H \subset G$ such that $C \subseteq H$, there exists a 3-coloring of C that extends to a 3-coloring of H, but not to a 3-coloring of G.

Notice that 4-critical graphs are exactly C-critical graphs with $C = \emptyset$. Furthermore, it is easy to see that if F is a 4-critical graph and $F = G \cup G'$, where $C = G \cap G'$, then either $G = C$ or G is C-critical. We mainly use the following reformulation.

Lemma 13 (*Dvořák et al. [13]*)**.** *Let G be a plane graph and let Λ be a connected open region of the plane whose boundary is equal to a cycle C of G, such that Λ is not a face of G. Let H be the subgraph of G drawn in the closure of Λ. If G is 4-critical, then H is C-critical.*

This is useful in connection with the following result of Gimbel and Thomassen [16], which was also obtained independently by Aksenov et al. [3].

Theorem 14 (*Gimbel and Thomassen [16]*)**.** *Let G be a plane triangle-free graph with the outer face bounded by a cycle C of length at most 6. If G is C-critical, then C is a 6-cycle and all internal faces of G have length four.*

Furthermore, they also exactly characterized the colorings of C that do not extend to G.

Theorem 15 (*Gimbel and Thomassen [16]*)**.** *Let G be a plane graph with the outer face bounded by a cycle $C = c_1 \dots c_6$ of length 6, such that all other faces of G have length 4. A 3-coloring φ of $G[V(C)]$ does not extend to a 3-coloring of G if and only if $\varphi(c_1) = \varphi(c_4)$, $\varphi(c_2) = \varphi(c_5)$ and $\varphi(c_3) = \varphi(c_6)$.*

An analogous result for a 7-cycle was obtained by Aksenov et al. [4]. Their paper is in Russian. For a shorter proof in English see [14].

Theorem 16 (*Aksenov et al. [4]*)**.** *Let G be a plane triangle-free graph with the outer face bounded by a cycle $C = c_1 \dots c_7$ of length 7. The graph G is C-critical and $\varphi : V(C) \to \{1, 2, 3\}$ is a 3-coloring of C that does not extend to a 3-coloring of G if and only if G contains no separating cycles of length at most five and one of the following propositions is satisfied up to relabeling of vertices (see Fig. 5 for an illustration).*

(a) *The graph G consists of C and the edge c_1c_5, and $\varphi(c_1) = \varphi(c_5)$.*
(b) *The graph G contains a vertex v adjacent to c_1 and c_4, the cycle $c_1c_2c_3c_4v$ bounds a 5-face and every face drawn inside the 6-cycle $vc_4c_5c_6c_7c_1$ has length four; furthermore, $\varphi(c_4) = \varphi(c_7)$ and $\varphi(c_5) = \varphi(c_1)$.*

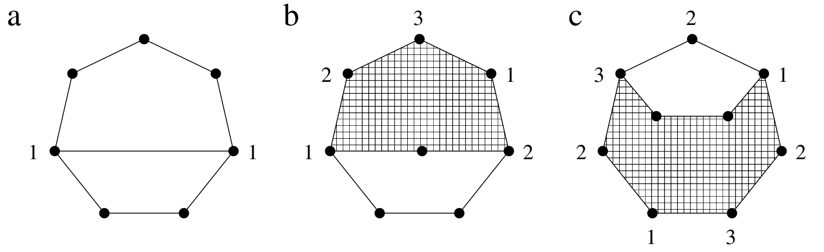

Fig. 5. Critical graphs with a precolored 7-face.

(c) *The graph G contains a path c_1uvc_3 with $u, v \notin V(C)$, the cycle $c_1c_2c_3vu$ bounds a 5-face and every face drawn inside the 8-cycle $uvc_3c_4c_5c_6c_7c_1$ has length four; furthermore, $\varphi(c_3) = \varphi(c_6)$, $\varphi(c_2) = \varphi(c_4) = \varphi(c_7)$ and $\varphi(c_1) = \varphi(c_5)$.*

By inspection of the cases (a), (b) and (c) of Theorem 16 we obtain the following fact.

Fact 17. *Let G be a plane triangle-free graph with the outer face bounded by a cycle C of length 7. Suppose that G is C-critical and that φ is a precoloring of C that does not extend to a 3-coloring of G. Let x, y and z be consecutive vertices of C. If $\varphi(x) = \varphi(z)$, then y is incident with a 5-face of G.*

We also use the following result dealing with graphs with a triangle.

Theorem 18 (*Aksenov [1]*)**.** *Let G be a plane graph with the outer face bounded by a cycle $C = c_1c_2 \dots$ of length at most 5. If G is C-critical and contains exactly one triangle T distinct from C, then C is a 5-cycle, all internal faces of G other than T have length exactly four and T shares at least one edge with C. Furthermore, if C and T share only one edge c_1c_2 and φ is a 3-coloring of C that does not extend to a 3-coloring of G, then $\varphi(c_1) = \varphi(c_3)$ and $\varphi(c_2) = \varphi(c_5)$.*

Recall that the notion of a patch was defined in the introduction. First, we need to argue that replacing vertices of degree 3 by critical patches preserves criticality and the number of triangles.

Lemma 19. *Let G be a 4-critical graph, let v be a vertex of G of degree 3 and let x, y and z be the neighbors of v in G. Let G′ be a graph obtained from $G - v$ by inserting a patch P with boundary $C_P = xz'yx'zy'$, where x', y' and z' are new vertices. Then G′ is 4-critical if and only if the patch P is critical. Furthermore, if P is critical, then $t(G') = t(G)$.*

Proof. Clearly, every 4-critical graph has minimum degree at least 3. Furthermore, by Lemma 13 and Theorem 14, if Λ is an open disk bounded by a 4-cycle in a 4-critical plane graph and no triangle is contained in Λ, then Λ is a face. Therefore, if G′ is 4-critical, then P is a critical patch.

Suppose now conversely that P is a critical patch. Consider an edge $e \in E(G')$. If $e \notin E(P)$, then e is an edge of G not incident with v. Since G is 4-critical, there exists a 3-coloring φ of $G - e$. The vertex v is properly colored, and thus we can by symmetry assume that $\varphi(x) = \varphi(y)$. By Theorem 15, φ extends to a 3-coloring of P, showing that $G' - e$ is 3-colorable.

On the other hand, consider the case that $e \in E(P)$. Since G is 4-critical, there exists a 3-coloring ψ of $G - v$ such that $\psi(x) = 1$, $\psi(y) = 2$ and $\psi(z) = 3$. Suppose that φ does not extend to a 3-coloring of $P - e$. If $e \notin E(C_P)$, this implies that $P - e$ contains a C_P-critical subgraph P'. By Theorem 14, all faces of P' distinct from C_P have length 4. However, e is drawn inside one of the faces of P', and thus P would contain a 4-cycle not bounding a face, contrary to the assumption that P is a critical patch.

Finally, suppose that $e \in E(C_P)$, say $e = xz'$. Note that all faces of P have even length, and thus P is bipartite. Since z' has degree at least 3 and every 4-cycle in P bounds a face, we conclude that $xz'y$ is the only path of length at most two between x and y in P. Let P_1 be obtained from $P - e$ by adding the edge xy, and note that P_1 is triangle-free. Let $C_1 = xyx'zy'$ be the 5-cycle bounding the outer face of P_1. Since φ does not extend to a 3-coloring of $P - e$, it also does not extend to a 3-coloring of P_1, and thus P_1 contains a C_1-critical subgraph. This contradicts Theorem 14.

We conclude that $G' - e$ is 3-colorable for every $e \in E(G')$. Since G′ does not contain isolated vertices, this implies that every proper subgraph of G′ is 3-colorable. Suppose that G′ has a proper 3-coloring θ.

Since G is not 3-colorable, θ cannot be extended to v, and thus we can assume that $\theta(x) = 1, \theta(y) = 2$ and $\theta(z) = 3$. However, that implies that $\theta(x) = \theta(x'), \theta(y) = \theta(y')$ and $\theta(z) = \theta(z')$. Since θ is a 3-coloring of P, this contradicts Theorem 15. Therefore, G' is not 3-colorable, and thus it is 4-critical.

Now we establish a bijection f between triangles in G and G'. If a triangle T in G does not contain v, then T also appears in G', and we set $f(T) = T$. If T contains v, say $T = vxy$, then we set $f(T) = z'xy$. Since f is injective, it suffices to show that it is surjective. Suppose that there exists a triangle $T' \subset G'$ that is not in the image of f. Then T' contains an edge of P. Since P is bipartite, T' contains an edge outside of P, and since x, y and z are non-adjacent in the patch P, we conclude that T' intersects P in a path of length two, say xwy, and x and y are adjacent in G. Since $f(vxy) = xz'y$ and T' is not in the image of f, we conclude that $w \neq z'$. Since P is a critical patch, the 4-cycle $xz'yw$ bounds a face. However, this implies that z' has degree two, which is a contradiction. Therefore, f is indeed a bijection, and thus $t(G') = t(G)$. □

By Lemma 19, the graphs described in Theorem 5 are indeed 4-critical and have exactly 4 triangles. If G is obtained from a $\mathrm{Pl}_{4,4f}$-graph by replacing non-adjacent vertices of degree 3 with (not necessarily critical) patches, then we say that G is an *expanded* $\mathrm{Pl}_{4,4f}$*-graph*. By Lemma 19, it remains to show that every Pl_4-graph is an expanded $\mathrm{Pl}_{4,4f}$-graph. First we state several properties of expanded $\mathrm{Pl}_{4,4f}$-graphs.

Claim 20. *If P is a patch in an expanded $\mathrm{Pl}_{4,4f}$-graph G, then no vertex of P is incident with exactly one edge that does not belong to P.*

Proof. Let G_0 be the $\mathrm{Pl}_{4,4f}$-graph from which G is obtained by replacing vertices with patches. Let v be the vertex of G_0 that is replaced by P and let x, y and z be the neighbors of v. All vertices of $V(P) \setminus \{x, y, z\}$ only have neighbors in P. Each of x, y and z has degree at least 3 in G_0, and thus each of them is incident with at least two edges that are not incident with v. So, each of x, y and z is incident with at least two edges of G that do not belong to P. □

Claim 21. *Every face of an expanded $\mathrm{Pl}_{4,4f}$-graph G has length 3, 4 or 5, and every 4-face of G belongs to a patch.*

Proof. Let G_0 be the $\mathrm{Pl}_{4,4f}$-graph from which G is obtained by replacing vertices with patches. By Theorem 4, G_0 belongs to $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$, and thus each face of G_0 has length 3 or 5. Furthermore, observe that replacing a 3-vertex v by a patch transforms each face of G_0 incident with v into a face of G of the same length, whose boundary shares a path of length two with the boundary cycle of P. Therefore, every face of G which is not contained in a patch has length 3 or 5. □

Furthermore, we will use the following simple property of critical graphs.

Claim 22. *If G is a 4-critical graph and T is a triangle of G, then $G - V(T)$ is connected. In particular, if G is a plane graph, then every triangle in G bounds a face.*

Proof. If $G - V(T)$ is not connected, then there exist proper subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = T$. Since G is 4-critical, both G_1 and G_2 are 3-colorable and by permuting the colors if necessary, we can assume that their 3-colorings match on T. Together, they would give a 3-coloring of G, which is a contradiction. □

A *stretching* of a plane graph G at a vertex $w \in V(G)$ is a graph G_2 obtained from G by the following procedure. Let $e_1, \ldots, e_k$ be the edges incident with w as drawn in the clockwise order around it. Choose $m < k$ and let G' be obtained from G by removing w, adding two new vertices w_1 and w_2 and adding edges between w_1 and the endpoints of $e_1, \ldots, e_m$ distinct from w, and between w_2 and the endpoints of $e_{m+1}, \ldots, e_k$ distinct from w. Let G_1 be obtained from G' by either adding a new vertex z adjacent to w_1 and w_2, or by adding an edge between w_2 and the endpoint z of e_1 distinct from w. Finally, for each face f of G_1 incident with zw_2, if $|f| = 6$, then replace f by a quadrangulation, and if $|f| = 7$, then replace it by a graph satisfying (a), (b) or (c) of Theorem 16, resulting in the graph G_2. We call G_1 the *intermediate graph* of the stretching, and the faces of G_1 incident with zw_2 are called *special*.

Lemma 23. *Every Pl_4-graph is an expanded $\mathrm{Pl}_{4,4f}$-graph.*

Proof. Assume that G has the fewest vertices among the Pl_4-graphs that are not expanded $Pl_{4,4f}$-graphs. Then G does not contain any patches, since replacing a patch by a 3-vertex would give a smaller counterexample. We are going to need the following stronger claim.

(1) *Let C be a 6-cycle in G and let Δ be the open region of the plane bounded by C, such that all faces in Δ have length four. Then Δ contains at most one vertex, and if it contains a vertex, then each vertex of C is incident with an edge that is not drawn in the closure of Δ.*

Proof. Let $C = v_1v_2v_3v_4v_5v_6$ and suppose that Δ contains at least one vertex, and if it contains only one, then all edges incident with v_6 are drawn in the closure of Δ. Let G' be the graph obtained from G by removing the vertices in Δ (which does not include C) and adding a vertex v adjacent to v_1, v_3 and v_5. By Theorem 15, G' is not 3-colorable, since every 3-coloring of C in which v_1, v_3 and v_5 do not have pairwise distinct colors can be extended to the subgraph of G drawn in the closure of Δ. Therefore, G' has a 4-critical subgraph G''. Note that $|V(G'')| < |V(G)|$, since if Δ contains only one vertex of G, then v_6 has degree 2 in G', and thus $v_6 \notin V(G'')$.

Since both G and G'' are 4-critical, G'' is not a proper subgraph of G. Hence v and all edges incident to it belong to G''. Triangles in G'' that are not in G can only be created by adding v to an edge, say v_1v_3; in this case, $v_1v_2v_3$ is a triangle in G which bounds a face (by Claim 22), and thus v_2 has degree two in G' and does not belong to G'', which means that creating a new triangle in G'' destroys another triangle. Thus $t(G'') \leq t(G) = 4$. Since G'' is not 3-colorable, it contains exactly four triangles. By the minimality of G, the graph G'' is an expanded $Pl_{4,4f}$-graph and by Claim 21, each face of G'' has length at most 5. Let G_1 be the graph obtained from $G'' - v$ by adding the subgraph H of G contained in the closure of Δ. Then G_1 is a subgraph of G.

If at least two faces of G'' incident with v have length 4, then v is a vertex of a patch P'' of G'' and all neighbors of v belong to P''. We claim that $P = (P'' - v) + H$ is a patch; this is clear if v is incident with three 4-faces in G''. If v is incident with exactly two 4-faces, then let f be the face incident with v of length other than four. Note that at least one edge e of C does not belong to G'' and in G, it is drawn in the region of the plane corresponding to f. By symmetry, we can assume that $e = v_1v_6$. Since all faces of G'' have length at most 5, if $v_6 \in V(G'')$, then v_1 and v_6 would be joined by a path Q of length two in G''. The path Q together with the edge v_1v_6 would form a triangle in G which does not correspond to any triangle in G'', and thus G'' would only have at most three triangles, which is a contradiction. Therefore, $v_6 \notin V(G'')$, and thus all neighbors of v_6 in G_1 belong to P, showing that P is a patch. Consequently, G_1 is an expanded $Pl_{4,4f}$-graph.

Hence, we can assume that at most one face of G'' incident with v has length four. If exactly one face incident with v had length 4, then v would belong to a patch P and it would be incident with exactly one edge not belonging to P, contradicting Claim 20. Therefore, no face incident with v has length four, and as in the previous paragraph, we conclude that H is a patch in G_1 that replaces the vertex v of degree 3 of G''. Again, it follows that G_1 is an expanded $Pl_{4,4f}$-graph.

Since G_1 is not 3-colorable and G is 4-critical, $G = G_1$ and G is an expanded $Pl_{4,4f}$-graph. This is a contradiction. □

(2) *G does not contain separating 4-cycles.*

Proof. Suppose that $C = v_1v_2v_3v_4$ is a separating 4-cycle in G. Let Λ_1 and Λ_2 be the two connected regions obtained from the plane by removing C. For $i \in \{1, 2\}$, let G_i be the subgraph of G drawn in the closure of Λ_i. By Lemma 13, G_i is C-critical, and by Theorem 18, G_i contains at least two triangles. Since $t(G) = 4$, we conclude that $t(G_1) = t(G_2) = 2$.

For $i, j \in \{1, 2\}$, let $G_{i,j}$ be the graph obtained from G_i by adding the edge v_jv_{j+2}. Note that $C + v_jv_{j+2}$ has a unique 3-coloring up to permutation of colors. Therefore, any 3-colorings of $G_{1,j}$ and $G_{2,j}$ could be combined to a 3-coloring of G. We conclude that at least one of $G_{1,j}$ and $G_{2,j}$ is not 3-colorable. By symmetry, we can assume that $G_{1,1}$ is not 3-colorable.

Let $G'_{1,1}$ be a 4-critical subgraph of $G_{1,1}$. By Claim 22, each triangle in $G'_{1,1}$ bounds a face, and thus v_1v_3 belongs to at most two triangles of $G'_{1,1}$. Since $t(G_1) = 2$, it follows that $t(G'_{1,1}) \leq 4$. Since $G'_{1,1}$ is 4-critical, it follows that $t(G'_{1,1}) = 4$, i.e., $G'_{1,1}$ is a Pl_4-graph, and that v_1v_3 belongs to two triangles of $G'_{1,1}$.

Let $v_1v_2'v_3$ and $v_1v_4'v_3$ be the triangles incident with v_1v_3 labeled so that for $k \in \{2, 4\}$, v_k is either equal to v_k' or it is drawn in $G_{1,1}$ in the region of the plane corresponding to the face of $G_{1,1}'$ bounded by $v_1v_k'v_2$.

Since C is a separating cycle, $|V(G_{1,1}')| < |V(G)|$, and by the minimality of G, it follows that $G_{1,1}'$ is an expanded $\mathrm{Pl}_{4,4f}$-graph. By Claim 21, every face of $G_{1,1}'$ has length at most five. Consider a face f of $G_{1,1}'$ not incident with v_1v_3. Since the subgraph of G drawn in the closure of f contains no triangles, by Lemma 13 and Theorem 14, f is a face of G as well. Furthermore, if $v_k \neq v_k'$ for some $k \in \{2, 4\}$, then Lemma 13 and Theorem 14 imply that $v_1v_k'v_3v_k$ is a face of G and v_k has degree two in $G_{1,1}$. It follows that $V(G_{1,1}) \setminus V(G_{1,1}') \subseteq \{v_2, v_4\}$ and each vertex $v \in V(G_{1,1}) \setminus V(G_{1,1}')$ is adjacent in $G_{1,1}$ only to v_1 and v_3.

Suppose that $G_{1,1}'$ has a 4-face. In that case, it contains a patch P. Let x, y and z be the vertices of the boundary cycle K of P that have neighbors only in P. Note that $v_1v_3 \notin E(K)$, since both faces incident with v_1v_3 are triangles. Furthermore, at most two of x, y and z are incident with C, since they form an independent set. Therefore, we can assume that all edges incident with x in G belong to P. This contradicts (1). We conclude that $G_{1,1}'$ has no 4-faces, and thus it belongs to $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$ by Theorem 4.

Consequently, both vertices of the diamond edge v_1v_3 of $G_{1,1}'$ have degree exactly 3 in $G_{1,1}'$. Since $G_{1,1}'$ is 4-critical, there exists a 3-coloring φ of $G_{1,1}' - \{v_1, v_3\}$ and $\varphi(v_2') \neq \varphi(v_4')$. Set $\varphi(v_k) = \varphi(v_k')$ for $k \in \{2, 4\}$ and note that φ is a proper 3-coloring of $G_{1,2}$. Since G is not 3-colorable, it follows that $G_{2,2}$ is not 3-colorable. By an argument symmetrical to the one for $G_{1,1}$, we conclude that $G_{2,2}$ contains a $\mathrm{Pl}_{4,4f}$-graph $G_{2,2}'$ as a subgraph such that $V(G_{2,2}) \setminus V(G_{2,2}') \subseteq \{v_1, v_3\}$ and each vertex $v \in V(G_{2,2}) \setminus V(G_{2,2}')$ is only adjacent to v_2 and v_4 in $G_{2,2}$.

Suppose that $G_{1,1}' \neq G_{1,1}$, and thus say $v_2 \in V(G_{1,1}) \setminus V(G_{1,1}')$ is adjacent in G_1 only to v_1 and v_3. Since v_2v_4 is a diamond edge of $G_{2,2}'$, vertex v_2 has degree 3 in $G_{2,2}'$. If $G_{2,2}' = G_{2,2}$, this would imply that v_1 and v_3 are the only neighbors of v_2 in G_2, and thus v_2 would have degree two in G, contrary to the assumption that G is 4-critical. Hence, we can assume that say $v_1 \in V(G_{2,2}) \setminus V(G_{2,2}')$ is adjacent in G_2 only to v_2 and v_4. Since $G_{1,1}'$ and $G_{2,2}'$ are 4-critical, there exist 3-colorings φ_1 of $G_{1,1}' - v_1v_3$ and φ_2 of $G_{2,2}' - v_2v_4$ such that $\varphi_1(v_1) = \varphi_1(v_3) = 1$, $\varphi_1(v_2') = 2$, $\varphi_1(v_4') = 3$, $\varphi_2(v_2) = \varphi_2(v_4) = 3$, $\varphi_2(v_1') = 2$ and $\varphi_2(v_3') = 1$. Then each vertex of G is colored by φ_1 or φ_2 and if a vertex (v_3 or v_4) belongs to both $G_{1,1}'$ and $G_{2,2}'$, then it is assigned the same color by φ_1 and φ_2. Thus the union of φ_1 and φ_2 gives a 3-coloring of G, which is a contradiction.

Therefore, $G_{1,1}' = G_{1,1}$, and by symmetry, $G_{2,2}' = G_{2,2}$. It follows that G has no 4-face, and thus G is a $\mathrm{Pl}_{4,4f}$-graph. This is a contradiction. □

(3) *If K is a separating 5-cycle in G and Δ is a region of the plane bounded by K such that Δ contains at most one triangle, then Δ contains exactly one vertex, which has three neighbors in K.*

Proof. By Theorem 14 and the criticality of G, Δ contains a triangle T, as Δ is not a face of G. Since K is separating, Theorem 18 implies that T shares exactly one edge with K. Let $\{z\} = V(T) \setminus V(K)$. Note that $T \cup K$ contains a 6-cycle, bounding an open region $\Delta_0 \subset \Delta$. Since all edges incident with z are drawn in the closure of Δ_0, (1) implies that Δ_0 contains no vertices, and thus z is the only vertex in Δ. □

(4) *The following configuration does not appear in G: a path $z_1z_2z_3$ and a vertex z of degree 3 adjacent to z_1, z_2 and z_3.*

Proof. Since G does not contain separating triangles, if z_1 is adjacent to z_3, then $G = K_4$. This is a contradiction, since G is not a $\mathrm{Pl}_{4,4f}$-graph.

Let G' be the graph obtained from $G - z$ by identifying z_1 with z_3 to a new vertex u. Since G is not 3-colorable, G' also is not 3-colorable. Let G'' be a 4-critical subgraph of G'. Note that G'' has at least four triangles, and since the triangles zz_1z_2 and zz_2z_3 disappear during the construction of G', we conclude that G'' contains at least two triangles ux_1x_2 and uy_1y_2 such that $z_1x_1x_2z_3$ and $z_1y_1y_2z_3$ are (not necessarily disjoint) paths between z_1 and z_3 in $G - \{z, z_2\}$. Since G'' contains no separating triangles by Claim 22, we conclude that G'' contains exactly two such triangles, and thus $t(G'') = 4$. Let Δ_1 be the open disk bounded by $z_1zz_3x_2x_1$ in G corresponding to the face ux_1x_2 of G', and let Δ_2 be the open disk bounded by $z_1z_2z_3y_2y_1$ in G corresponding to the face uy_1y_2 of G'. By swapping the labels of x_i and y_i (for $i \in \{1, 2\}$) if necessary, we can assume that Δ_1 and Δ_2 are disjoint. Note that since $t(G'') = 4$, neither Δ_1 nor Δ_2 contains a triangle of G, and thus Δ_1 and Δ_2 are faces of G by (3).

By the minimality of G, it follows that G'' is an expanded $\mathrm{Pl}_{4,4f}$-graph, and all faces of G'' have length at most 5. By Lemma 13 and Theorem 14, all faces of G'' other than ux_1x_2 and uy_1y_2 are also faces of G. In particular, since G does not contain patches, G'' also does not contain patches, and thus G'' has no 4-faces. It follows that G has no 4-faces. Therefore, G is a $\mathrm{Pl}_{4,4f}$-graph, which is a contradiction. □

(5) *The following configuration does not appear in G: a triangle $T = z_1z_2z_3$ such that all vertices of T have degree* 3 *and z_3 is adjacent to a vertex x_3 distinct from z_1 and z_2 that has degree* 3 *and belongs to a triangle.*

Proof. Let x_1 and x_2 be the neighbors of z_1 and z_2, respectively, outside of T. By (4), we have $x_1 \neq x_2 \neq x_3 \neq x_1$. Furthermore, the vertices x_1, x_2 and x_3 form an independent set in G, as otherwise every 3-coloring of $G - V(T)$ would extend to G, contrary to the 4-criticality of G. Let G' be the graph obtained from $G - V(T) - x_3$ by adding the edge x_1x_2. Note that every 3-coloring of G' extends to a 3-coloring of G, and thus G' is not 3-colorable. Let G'' be a 4-critical subgraph of G. Note that G'' contains at least four triangles, and since T as well as the triangle incident with x_3 disappear during the construction of G', it follows that x_1x_2 belongs to at least two triangles in G''. By Claim 22, x_1x_2 belongs to exactly two triangles, each of them bounding a face of G''. Therefore, G contains a 4-cycle x_1ux_2v separating two of its triangles from T. By (2), it follows that u is adjacent to v. By (3), either $x_1z_1z_2x_2u$ or $x_1z_1z_2x_2v$ bounds a face, and thus u or v has degree 3. This contradicts (4). □

Let $v_1v_2v_3v_4$ be a 4-face in G, if possible chosen so that it contains two adjacent vertices that are only incident with 4-faces. Since G contains no separating triangles, v_1v_3 and v_2v_4 are not edges. Suppose that v_1 and v_3 are joined by a path $v_1x_1x_2v_3 \subset G - \{v_2, v_4\}$ of length 3, and that v_2 and v_4 are joined by a path $v_2y_1y_2v_4 \subset G - \{v_1, v_3\}$ of length 3. By symmetry and planarity, we can assume that $x_1 = y_1$. If both v_1 and v_2 have degree 3, then every 3-coloring of $G - \{v_1, v_2\}$ extends to G, contrary to the assumption that G is 4-critical. By symmetry, v_1 has degree at least 4, and since $v_4y_2y_1v_1$ is not a separating 4-cycle, v_1 is adjacent to y_2. By (3) applied to the 5-cycle $v_4v_1v_2y_1y_2$, we have $x_2 = y_2$. But then $V(G) = \{v_1, v_2, v_3, v_4, x_1, x_2\}$ and G is 3-colorable.

Therefore, we can by symmetry assume that v_1 and v_3 are not joined by a path of length 3 in $G - \{v_2, v_4\}$. Let G' be the graph obtained from G by identifying v_1 and v_3 to a single vertex w. Clearly, G' has exactly the four triangles that originally belonged to G (possibly with v_1 or v_3 relabeled to w). Since every 3-coloring of G' gives a 3-coloring of G, we conclude that G' is not 3-colorable. Let G'' be a 4-critical subgraph of G'. By the minimality of G, G'' is an expanded $\mathrm{Pl}_{4,4f}$-graph. In particular, all faces of G'' have length at most 5. For a face f of G'', let C_f denote the corresponding cycle in G (either equal to f up to relabeling of w to v_1 or v_3, or obtained from f by replacing w by the path $v_1v_2v_3$). Since all triangles of G are faces of G'', Theorem 14 implies that if $|C_f| = |f|$, then C_f is a face of G. Furthermore, if $|C_f| = |f| + 2$, then $4 \leq |f| \leq 5$ and C_f does not bound a face of G, since v_2 has degree at least 3. Let G_f denote the subgraph of G drawn in the region of the plane bounded by C_f and corresponding to f, and note that G_f is one of the graphs described by Theorem 14 or Theorem 16. Therefore, the following holds.

(6) *The graph G is obtained by stretching from an expanded* $\mathrm{Pl}_{4,4f}$*-graph G_0 at a vertex w_0, where G and G_0 have the same triangles.*

In particular, all faces of G have length at most 5. By the choice of the 4-face $v_1v_2v_3v_4$, we can also assume that the following property is satisfied.

(7) *Let G_0 and w_0 be as in (6). Let $v_1v_2v_3$ be the path replacing w_0 in the intermediate graph of the stretching. The path $v_1v_2v_3$ is contained in the boundary of a* 4*-face in G. Furthermore, if G contains an edge whose vertices are only incident with* 4*-faces, then v_1 is only incident with* 4*-faces.*

Let G_0 and w_0 satisfying (6) be chosen so that if G_0 contains at least one patch, then (7) is satisfied, and subject to that with $|V(G_0)|$ as small as possible. Observe that G_0 does not necessarily have to be created from G by contracting a 4-face. Let G_1 be the intermediate graph of the stretching, let f_1 and h_1 be its special faces and let f_0 and h_0 be the corresponding faces of G_0.

(8) *The graph G_0 has no* 4*-faces.*

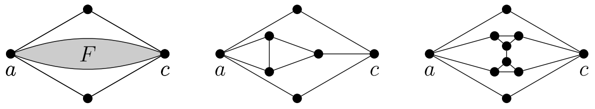

Fig. 6. Possibilities for a fragment F.

Proof. Suppose that G_0 has a 4-face, and thus it contains a patch P_0 bounded by a 6-cycle $C_0 = z_1z_2z_3z_4z_5z_6$, where z_2, z_4 and z_6 only have neighbors in P_0. Let r_2, r_4 and r_6 be the faces of G_0 incident with z_2, z_4 and z_6, respectively, whose length is not four. Since G_0 is 2-connected, these faces are pairwise distinct. If C_0 is a 6-cycle in G, then the open region Δ corresponding to P_0 contains only 4-faces and at least one vertex of G lies inside Δ. There exists $k \in \{2, 4, 6\}$ such that $f_0 \neq r_k \neq h_0$, and thus all edges incident with z_k in G are drawn in the closure of Δ. This contradicts (1); hence, C_0 contains w_0 and corresponds to an 8-cycle C_1 in G_1 containing the path $v_1v_2v_3$. Note that all faces of G drawn in the open region Λ of the plane bounded by C_1 and corresponding to P_0 have length 4. By symmetry, we can assume that $w_0 \in \{z_1, z_2\}$ and f_0 is contained in the patch P_0.

Suppose that h_0 does not share an edge with C_0; then, $w_0 = z_1$, r_2 and r_6 are faces of G and v_1 is incident with one of them. Since neither r_2 nor r_6 is a 4-face, (7) implies that G does not contain an edge whose vertices are only incident with 4-faces. Hence, $|V(P_0) \setminus V(C_0)| = 1$ and the only neighbors of z_1 in P_0 are z_2 and z_6. By (7), G contains a 4-face $v_1v_2v_3z$. If z lies inside Λ, then note that it is adjacent neither to z_2 nor to z_6, and thus z has a neighbor inside Λ. If z lies outside Λ, then similarly v_2 has a neighbor inside Λ. In both cases G contains an edge whose vertices are incident only with 4-faces, a contradiction. Therefore, h_0 shares an edge with C_0 (we can assume that $h_0 = r_2$). Since the same conclusions would have to hold for any other patch in G_0, the only patch in G_0 is P_0.

Label the vertices of C_1 so that the vertices of $V(C_0) \setminus \{w_0\}$ retain their labels and $C_1 = z_1xyz_2z_3z_4z_5z_6$. Note that all edges incident with y, z_2, z_4 and z_6 in G_1 are drawn in the closure of Λ. Also, either x is adjacent to z_1 in G_0 and $|h_1| = |h_0| = 5$, or $|h_0| = 5$ and $|h_1| = 7$ and all edges incident with x are drawn in the closure of Λ. Let $h_0 = z_1z_2z_3ab$, where b may be equal to x. Since $C = z_1baz_3z_4z_5z_6$ is a 7-cycle in G whose interior (the part of the plane bounded by C and containing Λ) does not contain any triangle, it satisfies (b) or (c) of Theorem 16. Let G_0' be the graph obtained from G_0 by replacing the patch P_0 with a 3-vertex z' adjacent to z_1, z_3 and z_5. Since G_0 has only one patch, G_0' has no patches and it is a $\mathrm{Pl}_{4,4f}$-graph. Furthermore, G is obtained from G_0' by stretching at vertex z' (split out the edge $z'z_3$, then make its new endvertex adjacent to z_5 and relabel the vertices created from z' by z_4 and z_6). Since $|V(G_0')| < |V(G_0)|$, this is a contradiction with the choice of G_0. □

Suppose that $\{a, c\}$ is a vertex-cut in G_0 and F is an induced subgraph of G_0 such that all edges of G_0 incident with $V(F) \setminus \{a, c\}$ belong to F. We say that F is a *fragment with attachments a and c* if F either isomorphic to the Havel's quasiedge H_0 and a and c are its vertices of degree two, or if F consists of a 4-cycle $au_1u_2u_3$ and edges u_1u_3 and u_2c. See Fig. 6 for an illustration. Since all triangles of G_0 belong to G and stretching does not preserve size of at least one face, G_0 is not K_4, and thus G_0 contains two fragments.

Since stretching has at most two special faces and fragment contains configurations forbidden by (4) and (5), the stretching must occur at a vertex of each of the fragments. Hence, G_0 does not have two vertex-disjoint fragments, and we conclude that G is one of the graphs M and K_4' depicted in Figs. 2 and 3. The possible intermediate graphs G_1 are drawn in Fig. 7. Thus we have the three cases below.

Case 1: $G_1 = X_1$. Let ψ be a 3-coloring of G_1 such that $\psi(v_1) = \psi(x) = \psi(v_4) = 1$, $\psi(v_2) = \psi(x_1) = \psi(x_2) = 2$ and $\psi(v_3) = \psi(x_3) = \psi(x_4) = 3$. This coloring does not extend to a 3-coloring of G, and by symmetry, we can assume that it does not extend to a 3-coloring of G_{h_0}. By Fact 17, it follows that x, x_3 and v_4 are incident with a common 5-face in G, and thus x_3 has degree 3. This contradicts (4).

Case 2: $G_1 = X_2$. Let ψ be a 3-coloring of G_1 such that $\psi(v_1) = \psi(x) = \psi(y_1) = 1$, $\psi(v_2) = \psi(x_1) = \psi(y_2) = 2$ and $\psi(v_3) = \psi(x_2) = \psi(y) = 3$. By symmetry, ψ does not extend to G_{h_0}, and by Theorem 16, x_2 is adjacent to v_3 and $xx_2v_3y_2y$ is a 5-face. However, then y_2 has degree 3 and we again obtain a contradiction with (4).

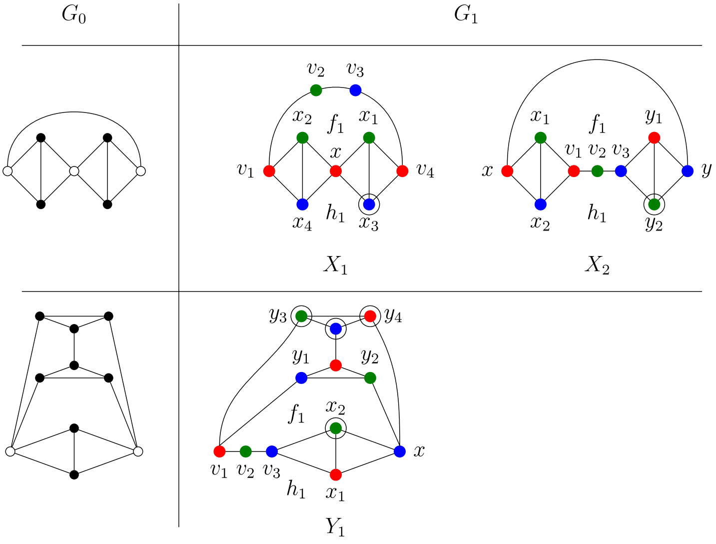

Fig. 7. Stretchable $\mathrm{Pl}_{4,4f}$-graphs.

Case 3: $G_1 = Y_1$. Let ψ be a 3-coloring of G_1 such that $\psi(v_1) = \psi(x_1) = \psi(y_4) = 1$, $\psi(v_2) = \psi(y_2) = \psi(x_2) = \psi(y_3) = 2$ and $\psi(v_3) = \psi(x) = \psi(y_1) = 3$. If ψ does not extend to a 3-coloring of G_{f_0}, then by Fact 17, G contains a 5-face incident with v_3, x_2 and x; hence, x_2 has degree 3 and contradicts (4). Since G is not 3-colorable, it follows that ψ does not extend to a 3-coloring of G_{h_0}. By Fact 17, G contains a 5-face incident with v_1, y_3 and x, and by Theorem 14, y_4 is incident with the 5-face as well. However, then both y_3 and y_4 have degree 3 in G, which contradicts (5).

This finishes the proof of Lemma 23. □

Proof of Theorem 5. By Lemma 19, every graph obtained from a $\mathrm{Pl}_{4,4f}$-graph by replacing non-adjacent 3-vertices with critical patches is 4-critical and has exactly four triangles. Conversely, if G is a Pl_4-graph, then it is an expanded $\mathrm{Pl}_{4,4f}$-graph by Lemma 23, and since G is 4-critical, all the patches in G are critical by Lemma 19. □

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