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Improper coloring of sparse graphs with a given girth, I: (0,1)-colorings of triangle-free graphs

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ABSTRACT

A graph G is $(0, 1)$-*colorable* if $V(G)$ can be partitioned into two sets V_0 and V_1 so that $G[V_0]$ is an independent set and $G[V_1]$ has maximum degree at most 1. The problem of verifying whether a graph is $(0, 1)$-colorable is NP-complete even in the class of planar graphs of girth 9.

The *maximum average degree*, $\mathrm{mad}(G)$, of a graph G is the maximum of $\frac{2|E(H)|}{|V(H)|}$ over all subgraphs H of G. It was proved recently that every graph G with $\mathrm{mad}(G) \leq \frac{12}{5}$ is $(0, 1)$-colorable, and this is sharp. This yields that every planar graph with girth at least 12 is $(0, 1)$-colorable.

Let $F(g)$ denote the supremum of a such that for some constant b_g every graph G with girth g and $|E(H)| \leq a|V(H)| + b_g$ for every $H \subseteq G$ is $(0, 1)$-colorable. By the above, $F(3) = 1.2$. We find the exact value for $F(4)$ and $F(5)$: $F(4) = F(5) = \frac{11}{9}$. In fact, we also find the best possible values of b_4 and b_5: every triangle-free graph G with $|E(H)| < \frac{11|V(H)|+5}{9}$ for every $H \subseteq G$ is $(0, 1)$-colorable, and there are infinitely many not $(0, 1)$-colorable graphs G with girth 5, $|E(G)| = \frac{11|V(G)|+5}{9}$ and $|E(H)| < \frac{11|V(H)|+5}{9}$ for every *proper* subgraph H of G. A corollary of our result is that every planar graph of girth 11 is $(0, 1)$-colorable. This answers a half of a question by Dorbec, Kaiser, Montassier and Raspaud. In a companion paper, we show that for every g, $F(g) \leq 1.25$ and resolve some similar problems for the so-called (j, k)-*colorings* generalizing $(0, 1)$-colorings.

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1. Introduction

A *proper k-coloring* of a graph G is a partition of $V(G)$ into k independent sets $V_1, \dots, V_k$. A $(d_1, \dots, d_k)$*-coloring* of a graph G is a partition of $V(G)$ into sets $V_1, \dots, V_k$ such that for every $1 \le i \le k$, the subgraph $G[V_i]$ of G induced by V_i has maximum degree at most d_i. If $d_1 = \cdots = d_k = 0$, then a $(d_1, \dots, d_k)$-coloring is simply a proper k-coloring. If at least one of d_i is positive, then a $(d_1, \dots, d_k)$-coloring is called *improper* or *defective*. Several papers on improper colorings of planar graphs with restrictions on girth and of sparse graphs have appeared.

In this paper and [6] we consider the simplest versions of improper colorings, the (j, k)-colorings. Even such colorings are not simple if $(j, k) \neq (0, 0)$. In particular, Esperet, Montassier, Ochem and Pinlou [4] proved that the problem of verifying whether a given planar graph of girth 9 has a (0, 1)-coloring is NP-complete. Since the problem is hard, it is natural to consider related extremal problems.

The maximum average degree, $\mathrm{mad}(G)$, of a graph G is the maximum of $\frac{2|E(H)|}{|V(H)|}$ over all subgraphs H of G. It measures sparseness of G. Kurek and Rucin'ski [7] called graphs with low maximum average degree *globally sparse*. In particular, it is an easy consequence of Euler's formula that

$$\text{if } G \text{ is a planar graph of girth } g, \text{ then } \mathrm{mad}(G) < \frac{2g}{g-2}. \tag{1}$$

We will use the following slight refinement of the notion of $\mathrm{mad}(G)$. For $a, b \in \mathbf{R}$, a graph G is (a, b)-*sparse* if $|E(H)| < a|V(H)| + b$ for all $H \subseteq G$. For example, every forest is (1, 0)-sparse, and every graph G with $\mathrm{mad}(G) < a$ is $(a/2, 0)$-sparse.

Glebov and Zambalaeva [5] proved that every planar graph G with girth at least 16 is (0, 1)-colorable. Then Borodin and Ivanova [1] proved that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is (0, 1)-colorable. By (1), this implies that every planar graph G with girth at least 14 is (0, 1)-colorable. Borodin and Kostochka [2] proved that every graph G with $\mathrm{mad}(G) < \frac{12}{5}$ is (0, 1)-colorable, and this is sharp. This implies that every planar graph G with girth at least 12 is (0, 1)-colorable. As mentioned above, Esperet et al. [4] proved that the problem of verifying whether a given planar graph of girth 9 has a (0, 1)-coloring is NP-complete. Dorbec, Kaiser, Montassier, and Raspaud [3] mention that because of these results, the remaining open question is whether all planar graphs with girth 10 or 11 are (0, 1)-colorable.

In this paper and [6], instead of considering planar graphs with given girth, we consider graphs G with given girth that are (a, b)-sparse for small a. Let $F_{j,k}(g)$ denote the supremum of positive a such that there is some (possibly negative) b with the property that every (a, b)-sparse graph G with girth g is (j, k)-colorable. The above mentioned result in [2] implies $F_{0,1}(3) = \frac{12}{5} = 1.2$. It turns out that $F_{0,1}(g)$ does not differ much from $F_{0,1}(3)$ even for large g. In the companion paper [6] we prove that for every g, $F_{0,1}(g) \le 1.25$. We also find there the exact values of $F_{j,k}(g)$ for all g in the cases $k \ge 2j+2$ for all j.

In this paper we concentrate on (0, 1)-colorings. We prove the exact result: $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$. In fact, we also find the best possible value of b.

Theorem 1.1. *If a graph G is triangle-free and $11|A| - 9|E(G[A])| \ge -4$ for all $A \subseteq V(G)$, then G is (0, 1)-colorable. On the other hand, there are infinitely many non-(0, 1)-colorable graphs G with girth 5 such that $11|V(G)| - 9|E(G)| = -5$ and $11|A| - 9|E(G[A])| \ge -4$ for all $A \subsetneq V(G)$*

Theorem 1.1 together with (1) yields the following.

Corollary 1.2. *Every planar graph with girth at least 11 is (0, 1)-colorable.*

This answers half of the question above by Dorbec et al. [3].

The structure of the paper is as follows. In the next section we introduce *potentials* of vertex subsets of graphs, restate the theorem in a more general way using the language of potentials, and construct infinitely many graphs for which the statement of Theorem 1.1 is sharp. In Section 3 we set up the proof, consider a smallest counterexample G and derive simple properties of G. In Sections 4 and 5 we describe the structure of subsets of $V(G)$ with small potential. In Sections 6 and 7 we analyze the structure of some special subgraphs of G, so-called *shovels* and (1, 1, 1)*-trees*. Then in Section 8 we describe the discharging method and in the last two sections we use it to obtain a contradiction.

2. Restatement and a construction

In (0, 1)-coloring, we always will use colors 0 and 1, where the vertices of color 0 form an independent set and the set of vertices of color 1 induces a subgraph of maximum degree at most 1.

For technical reasons, we will consider pairs $R = (G, Z)$ where G is a graph and Z is a nonempty subset of $V(G)$ such that the vertices in Z have stronger restrictions on coloring. In (0, 1)-coloring of such pairs, every $z \in Z$ must be colored with 1 and its neighbors must be colored with 0. Vertices in Z will be called *special* and vertices in $V(G) - Z$ *typical*. Since the vertices in Z could be isolated, this setting includes the original (0, 1)-coloring problem on graphs. We also will use *potentials*: Given a pair $R = (G, Z)$, we let $\rho_R(z) = 0$ for every $z \in Z$, $\rho_R(v) = 11$ for every $v \in V(G) - Z$, and $\rho_R(e) = -9$ for every $e \in E(G)$. For all $A \subseteq V(G)$, let $\rho_R(A) = \sum_{v\in A} \rho_R(v) + \sum_{e\in E(G[A])} \rho_R(e)$. We abuse the notion by writing $\rho_G(A)$ instead of $\rho_R(A)$ for a graph G without special vertices, $A \subset V(G)$, and $R = (G', Z)$ with $V(G') = V(G) \cup Z$, $E(G') = E(G)$. When the pair R is clear from the content, we will omit the subscript.

This definition is used because for all $A \subset V(G) - Z$,

$$\rho(A) \geq 0 \quad \text{if and only if} \quad \frac{2|E(G[A])|}{|A|} \leq \frac{22}{9}. \tag{2}$$

Also, by this definition,

$$\rho(A) + \rho(B) = \rho(A \cup B) + \rho(A \cap B) + 9|E_G(A - B, B - A)| \quad \text{for all } A, B \subseteq V(G). \tag{3}$$

So, instead of Theorem 1.1, we will prove the following slightly stronger theorem in the language of potentials.

Theorem 2.1. *If a pair $R = (G, Z)$ is such that G is triangle-free and $\rho(A) \geq -4$ for all $A \subseteq V(G)$, then G is (0, 1)-colorable. On the other hand, there are infinitely many non-(0, 1)-colorable pairs $R = (G, Z)$ such that G has girth 5, all vertices of Z are isolated, $\rho(V(G) - Z) = -5$ and $\rho(A) \geq -4$ for all $A \subsetneq V(G) - Z$.*

In this section, we prove the second part of the statement of Theorem 2.1, and the rest of the paper is devoted to the proof of the first part.

Let D be the graph obtained from a 10-cycle $(x_1, \ldots, x_{10})$ by adding edge x_4x_8 (see Fig. 1). Let x_1 be the *root*, $r(D)$, of D and x_6 be the *top*, $t(D)$, of D. A useful property of (0, 1)-colorings ϕ of D is the following.

(P1) *If $\phi(x_2) = \phi(x_{10}) = 0$, then $\phi(x_6) = 1$ and x_6 has a neighbor in D of color 1.*

Proof. Since $\phi(x_2) = \phi(x_{10}) = 0$, we have $\phi(x_3) = \phi(x_9) = 1$. Since $x_4x_8 \in E(G)$ and each of x_4 and x_8 has a neighbor of color 1, they cannot be colored with the same color. By symmetry, assume that $\phi(x_4) = 1$ and $\phi(x_8) = 0$. Then $\phi(x_5) = 0$ and $\phi(x_7) = 1$. This yields (P1). □

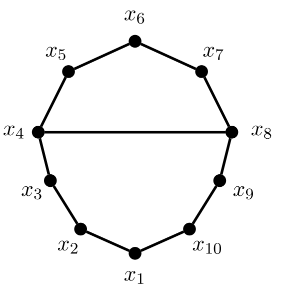

Fig. 1. $D = D_1$.

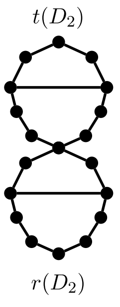

Fig. 2. D_2.

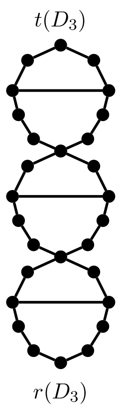

Fig. 3. D_3.

Define the sequence $D_1, D_2, \ldots$, of graphs as follows (see Figs. 2 and 3): Let $D_1 = D$. If D_i is constructed, then D_{i+1} is obtained from D_i by taking a copy D' of D disjoint from D_i, merging the root of D' with the top of D_i and choosing as the top, $t(D_{i+1})$, of D_{i+1} the top of D' and as the root, $r(D_{i+1})$, of D_{i+1} the root of D_i.

Similarly to (P1), for every i and every $(0, 1)$-coloring ϕ of D_i the following holds.

(P2) *If the neighbors of $r(D_i)$ are colored with 0, then $\phi(t(D_i)) = 1$ and $t(D_i)$ has a neighbor in D_i of color 1.*

Proof. For $i = 1$ this is exactly (P1). Suppose (P2) holds for all $i \le k$. Let ϕ be any $(0, 1)$-coloring of D_{k+1} such that the neighbors of $r(D_{k+1})$ are colored with 0. By definition, D_{k+1} is obtained by merging $t(D_k)$ with the root, $r(D')$, of a copy D' of D into a new vertex x. By the induction assumption, $\phi(t(D_k)) = 1$ and $t(D_k)$ has a neighbor of color 1 in D_k. Then both neighbors of x in D' must be colored with 0. So by (P1), $\phi(t(D')) = 1$ and $t(d') = t(D_{k+1})$ has a neighbor of color 1 in D'. □

(P3) *For every $i \ge 1$,*

(a) $\rho_{D_i}(V(D_i)) = 11$;
(b) $\rho_{D_i}(A) \ge 10$ *for every nonempty* $A \subseteq V(D_i)$,
(c) *if* $r(D_i), t(D_i) \in A$ *and* $A \ne V(D_i)$, *then* $\rho_{D_i}(A) \ge 12$.

Proof. Since D_i has $1 + 9i$ vertices and $11i$ edges, $\rho_{D_i}(V(D_i)) = 11(1 + 9i) - 9 \cdot 11i = 11$. This proves (a).

To prove (b) and (c), consider first D_1. Every proper subset A of $V(D_1)$ induces a subgraph with at most one cycle. If $D_1[A]$ is acyclic, then $|E(G[A])| \le |A| - 1$ and so $\rho_{D_1}(A) \ge 11|A| - 9(|A| - 1) = 2|A| + 9$. If $|A| \ge 2$, this is at least 13. If $D_1[A]$ has exactly one cycle C, then $|E(G[A])| \le |A|$ and so $\rho_{D_1}(A) \ge 11|A| - 9|A| = 2|A|$. Since $|C| \ge 5$, $2|A| \ge 10$. Moreover, if $|A| = 5$ and $G[A]$ has a cycle, then $A = \{x_4, x_5, x_6, x_7, x_8\}$ and so $x_1 = r(D_1) \notin A$. Otherwise, $|A| \ge 6$ and so $\rho_{D_1}(A) \ge 12$.

Suppose now that the claim holds for D_{i-1} and consider D_i for $i \ge 2$ as a copy of D_{i-1} to the top of which we merged the root of a copy D' of D. Let $x = t(D_{i-1}) = r(D')$. Let A be any nonempty proper subset of $V(D_i)$. Let $A' = A \cap V(D')$ and $A'' = A \cap V(D_{i-1})$. By induction, the claim holds for A' and A''. Thus if $A'' \subseteq \{x\}$ or $A' \subseteq \{x\}$, then (b) holds and (c) does not apply. So let $A'' - x \ne \emptyset$ and $A' - x \ne \emptyset$. If $x \notin A$, then $\rho_{D_i}(A) = \rho_{D'}(A') + \rho_{D_{i-1}}(A'') \ge 10 + 10 = 20$. Suppose $x \in A$. Then $\rho_{D_i}(A) = \rho_{D'}(A') + \rho_{D_{i-1}}(A'') - 11$ by (3). As $r(D') \in A'$ and $|A'| \ge 2$, by the argument in the previous paragraph for D_1, we know that either $\rho_{D'}(A') \ge 12$ or $A' = D'$ and $\rho_{D'}(A') = 11$. Therefore, $\rho_{D_i}(A) \ge \rho_{D_{i-1}}(A'')$ and hence (b) follows. Assume $r(D_i) = r(D_{i-1}) \in A$, $t(D_i) = t(D') \in A$ and $A \ne D_i$. If $A'' \ne D_{i-1}$, then $\rho_{D_i}(A) \ge \rho_{D_{i-1}}(A'') \ge 12$ (as $t(D') \in A''$). If $A'' = D_{i-1}$, then $A' \ne D'$, and $\rho_{D'}(A') \ge 12$ and $\rho_{D_i}(A) \ge \rho_{D_{i-1}}(A'') + 1 \ge 12$. □

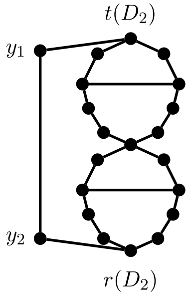

Fig. 4. $\widetilde{D}_2$.

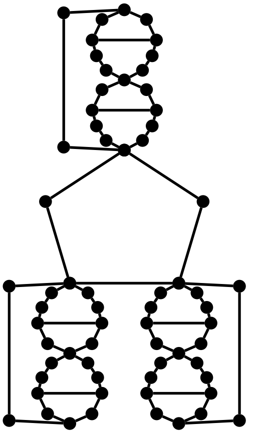

Fig. 5. G_2.

Let $\widetilde{D}_i$ be obtained from D_i by adding path $(t(D_i), y_1, y_2, r(D_i))$, where y_1 and y_2 are new vertices (see Fig. 4).

(P4) *For every $i \geq 1$, and every nonempty $A \subseteq V(\widetilde{D}_i)$, $\rho_{\widetilde{D}_i}(A) \geq 6$. Moreover, if $1 \leq |A| < |V(\widetilde{D}_i)|$, then $\rho_{\widetilde{D}_i}(A) \geq 7$.*

Proof. Let $A \subseteq V(\widetilde{D}_i)$. If $\{y_1, y_2, r(D_i), t(D_i)\} \not\subseteq A$, then $\rho(A) \geq \rho(A - y_1 - y_2)$. Since $A - y_1 - y_2 \subseteq V(D_i)$, we get $\rho(A) \geq \rho(A - y_1 - y_2) \geq 10$. Suppose $\{y_1, y_2, r(D_i), t(D_i)\} \subseteq A$. Then by (P3), if $A = V(\widetilde{D}_i)$, then $\rho(A - y_1 - y_2) = 11$ and otherwise $\rho(A - y_1 - y_2) \geq 12$. Hence $\rho(A) = \rho(A - y_1 - y_2) + 11 \cdot 2 - 9 \cdot 3 \geq 11 + 22 - 27 = 6$, with equality only if $A = V(\widetilde{D}_i)$. □

We construct graph G_i from 3 copies, F_1, F_2 and F_3, of $\widetilde{D}_i$ and two vertices, u_1 and u_2, by adding 5 edges: $r(F_1)u_1$, $u_1 r(F_2)$, $r(F_2)u_2$, $u_2 r(F_3)$ and $r(F_3)r(F_1)$ (see G_2 in Fig. 5).

(P5) *For every $i \geq 1$, $\rho_{G_i}(V(G_i)) = -5$, and for every $A \subsetneq V(G_i)$, $\rho_{G_i}(A) \geq -4$.*

Proof. Let $A \subsetneq V(G_i)$, $|A| \geq 2$. For $j \in \{1, 2, 3\}$, let $A_j = A \cap V(F_j)$. If $A_j = \emptyset$ for two indices j_1, j_2, then $\rho_{G_i}(A) \geq \rho_{F_{j_3}}(A_{j_3}) \geq 6$. If $A_j = \emptyset$ for exactly one index j_1, then $\rho_{G_i}(A) \geq \rho_{F_{j_3}}(A_{j_3}) + \rho_{F_{j_2}}(A_{j_2}) - 9 \geq 3$. If $A_j \neq \emptyset$ for all j, then

$$\rho_{G_i}(A) \geq \sum_{j=1}^{3} \rho_{F_j}(A_j) + 11 \cdot 2 - 9 \cdot 5 = \sum_{j=1}^{3} \rho_{F_j}(A_j) - 23,$$

with equality only in the case when $\{u_1, u_2, r(F_1), r(F_2), r(F_3)\} \subseteq A$. By (P4), for $j \in \{1, 2, 3\}$, $\rho_{F_j}(A_j) \geq 6$ with equality only in the case when $A_j = V(F_j)$. Thus $\rho_{G_i}(A) \geq 6 + 6 + 6 - 23 = -5$ with equality only if $A = V(G_i)$. □

(P6) *For every $i \geq 1$, G_i is not $(0, 1)$-colorable.*

Proof. Suppose f is a $(0, 1)$-coloring of G_i. Then some two consecutive vertices of the cycle $C = (r(F_1), u_1, r(F_2), u_2, r(F_3))$ are colored with 1. Thus there exists $j \in \{1, 2, 3\}$ such that $f(r(F_j)) = 1$ and $r(F_j)$ has a neighbor in C of color 1. Then all neighbors of $r(F_j)$ in F_j are colored with 0. So by (P2), $f(t(F_j)) = 1$ and $t(F_j)$ has a neighbor of color 1 in the copy of D_i contained in F_j. But then both internal vertices y_1 and y_2 of the path $(r(F_j), y_1, y_2, t(F_j))$ in the corresponding copy of $\widetilde{D}_i$ must be colored 0, a contradiction. □

By (P5) and (P6), the family $\{G_i\}_{i=1}^{\infty}$ proves the second part of the statement of Theorem 2.1.

3. Simple properties of counterexamples

Assume that $R = (G, Z)$ is a counterexample to the first part of Theorem 2.1 with the fewest vertices, and among such pairs with the largest sum of degrees of the vertices in Z. Let $n = |V(G)|$. In the rest of the paper we will show that such R does not exist. In the current section we derive some basic properties of R.

Recall that $R = (G, Z)$ where G is a triangle-free graph with $\rho(A) \geq -4$ for all $A \subset V(G)$.

Claim 3.1. $|Z| = 1$, *i.e. R has exactly one special vertex.*

Proof. By assumption, R contains at least one special vertex. Suppose there are more than one special vertices $z_1, \ldots, z_k$ in R. If R contains an edge $z_i z_j$, $\rho_R(\{z_i, z_j\}) < -4$, a contradiction. So Z forms an independent set. We merge these vertices to get a new special vertex z' whose neighborhood is $\bigcup_{i=1}^{k} N_G(z_i)$. This gives us a smaller graph G' and the pair $R' = (G', \{z'\})$.

If G' has a triangle (u, v, w), then by construction, $z' \in \{u, v, w\}$, say $z' = u$, and there are $z_i, z_j \in Z$ such that the path (z_i, v, w, z_j) is in G. But then $\rho_R(\{z_i, v, w, z_j\}) \leq 3(-9) + 2 \cdot 0 + 2 \cdot 11 = -5 < -4$, a contradiction. Thus G' is a triangle-free graph smaller than G. If $A \subset V(G')$ and $z' \notin A$, then $\rho_{R'}(A) = \rho_R(A) \geq -4$. If $z' \in A$, then $\rho_{R'}(A) = \rho_R(A \cup \{z_1, \ldots, z_k\} - z') \geq -4$. By the minimality of G, G' has a $(0, 1)$-coloring ϕ'. Let $\phi(v) = \phi'(v)$ for each typical vertex $v \in V(G)$, and $\phi(u) = 1$ for each $u \in Z$. By construction, ϕ is a $(0, 1)$-coloring of G, a contradiction. □

From now on, we assume that $Z = \{z\}$.

Claim 3.2. *If $v \in V(G) - z$, then $d(v) \geq 2$.*

Proof. Suppose $N(v) \subseteq \{w\}$. Let $G' = G - v$ and $R' = (G', Z)$. By the minimality of G, G' has a $(0, 1)$-coloring ϕ. Then letting $\phi(v) := 1 - \phi(w)$ we obtain a $(0, 1)$-coloring of G, a contradiction. □

Claim 3.3. *Suppose $A \subset V(G)$. If one of the following holds,*
(1) $v \in V(G) - A$ has more than one neighbor in A;
(2) $G - A$ has edge uw such that both u and w have neighbors in A;
(3) $z \notin A$ and the distance from z to A is at most two;
then there is a set C with $A \subset C$, $\rho(C) \leq \rho(A) - 5$, and $|C| \leq |A| + 2$.

Proof. In cases (1), (2) and (3), we take $C = A \cup \{v\}$, $A \cup \{v, w\}$, and $A \cup V(P)$ for a shortest path P from z to A, respectively. Then $\rho(C) \leq \rho(A) - 5$, $|C| \leq |A| + 2$. □

Lemma 3.4. *For every $\emptyset \neq A \subsetneq V(G)$, $\rho(A) \geq 0$. Moreover, $A = \{z\}$ is the only nonempty proper subset of $V(G)$ with $\rho(A) = 0$.*

Proof. Suppose that the lemma fails and choose the smallest $A \subsetneq V(G)$ such that $|A| \geq 2$ and $\rho(A) \leq 0$. If A is independent, then $\rho(A) \geq 11(|A| - 1) > 0$. So, by minimality we may assume that $G[A]$ is connected. If $|A| = 2$, then $\rho(A) \geq 11 - 9 = 2$, thus $|A| \geq 3$.

Since $A \subsetneq V(G)$, there is a $(0, 1)$-coloring ϕ' of $G[A]$. Let $A_0 = \{v \in A : \phi'(v) = 0\}$, $A_1 = \{v \in A : \phi'(v) = 1\}$.

Let the new graph $G' = G'(\phi')$ have $V(G') = (V(G) - A) \cup \{z', w_0\}$ and

$$\begin{aligned} E(G') = {} & E(G[V(G) - (A - z)]) \cup \{aw_0 : \exists b \in A_0, ab \in E(G)\} \\ & \cup \{az' : \exists b \in A_1, ab \in E(G)\} \cup \{w_0 z'\}. \end{aligned}$$

Let $R' = (G', Z')$ where $Z' = (Z - A) + z'$. Since $|A| \geq 3$, G' is smaller than G.

If G' contains a triangle, then by Claim 3.3, G' has a set of potential $\rho(A) - 5 \leq -5$ in G by Claim 3.3, a contradiction. Thus G' is triangle-free.

Suppose G' has a $(0, 1)$-coloring ϕ''. Then by the definition of z', $\phi''(z') = 1$ and $\phi''(w_0) = 0$. Let $\phi(v) = \phi''(v)$ for $v \in V(G) - A$, and $\phi(v) = \phi'(v)$ if $v \in A$. Then ϕ is a $(0, 1)$-coloring of G since every neighbor of $v \in A_1$ in $G - A$ is a neighbor of z' in G', and so is colored with 0, and every neighbor of $w \in A_0$ in $G - A$ is a neighbor of w_0 in G', and so is colored with 1. This contradicts the choice of G. So, G' has no $(0, 1)$-coloring. By the minimality of G, there is $B \subset V(G')$ such that

$$\rho_{R'}(B) \leq -5. \tag{4}$$

Choose a largest $B \subset V(G')$ satisfying (4). Since adding z' to a set does not increase its potential, $z' \in B$. If $w_0 \in B$, then

$$\begin{aligned} \rho_R((B - w_0 - z') \cup A) &= \rho_R(B - w_0 - z') + \rho_R(A) - 9|E_G[B - w_0 - z', A]| \\ &\leq \rho_R(B - w_0 - z') + 0 - 9|E_{G'}(B - w_0 - z', \{w_0, z'\})| \\ &< \rho_R(B - w_0 - z') + \rho_{R'}(\{w_0, z'\}) - 9|E_{G'}[B - w_0 - z', \{w_0, z'\}]| \\ &= \rho_{R'}(B). \end{aligned}$$

By (4), this is at most -5, a contradiction. If $w_0 \notin B$, then

$$\begin{aligned} \rho_R((B - z') \cup A) = {} & \rho_R(B - z') - 9|E[A, B - z']| + \rho(A) \leq \rho_{R'}(B - z') \\ & - 9|E_{G'}[z', B - z']| = \rho_{R'}(B), \end{aligned}$$

again, a contradiction to (4). □

Claim 3.5. *Let A be a subset of $V(G)$ with $4 \leq |A| \leq n - 4$, and $\rho(A) \leq 4$. No vertex $v \in V(G) - A$ has more than one neighbor in A, and $G - A$ has no edge uw such that both u and w have neighbors in A. Furthermore, if $z \notin A$, then the distance from z to A is at least 3.*

Proof. If not, we get a set C with $\rho(C) \le 4-5 < 0$ and $|C| \le |A|+2 < n$ by Claim 3.3. It is a contradiction to Lemma 3.4. □

For a subset A of $V(G)$, we define $A^* = \{v \in A : v$ has neighbors in $V(G)-A\}$.

Claim 3.6. *Assume we have a set $A \subset V(G)$ with $\rho(A) \le 4$ and ϕ is a $(0,1)$-coloring of A. Then the following statements hold.*
(1) *If $\rho(A) \le 2$, then $\phi(u) = 1$ for each $u \in A^*$.*
(2) *Color class 0 has at most one neighbor in $G - A$.*
(3) *Each typical vertex of color 1 in A^* has a neighbor in A of color 1 .*

Proof. We let $A_i = \{v \in A : \phi(v) = i\}$ for $i = 0, 1$.

Case 1. (1) does not hold. Let R' be the pair (G', Z') with $V(G') = V(G) - A \cup \{z', w_0\}$ and $E(G') = E(G[V(G)-A]) \cup \{aw_0 : \exists b \in A_0, ab \in E(G)\} \cup \{az' : \exists b \in A_1, ab \in E(G)\} \cup \{w_0z'\}$ and $Z' = Z \cup \{z'\}$.

Case 2. (2) does not hold. Then A_0 has at least two neighbors in $V(G)-A$, let x be a vertex in $N(A_0) \cap (V(G)-A)$. Let R' be the pair (G', Z') with $V(G') = V(G) - A \cup \{w_0, w_0', z'\}$, $E(G') = E(G[V(G)-A]) \cup \{uz' : \exists v \in A_1 \text{with } uv \in E(G)\} \cup \{xw_0, z'w_0, z'w_0'\} \cup \{uw_0' : u \ne x, \exists v \in A_0 \text{with } uv \in E(G)\}$ and $Z' = Z \cup \{z'\}$.

Case 3. (3) does not hold. Then $A_1 \cap N(V(G)-A)$ has a vertex x such that x has no neighbor $y \in A$ with $\phi(y) = 1$. Let R' be the pair (G', Z') with $V(G') = V(G) - A \cup \{w_0, w_1, z'\}$, $E(G') = E(G[V(G)-A]) \cup \{uz' : \exists v \in A_1 \text{with } uv \in E(G), u \ne x\} \cup \{w_1w_0, z'w_0\} \cup \{uw_0 : \exists v \in A_0 \text{with } uv \in E(G)\} \cup \{yw_1 : yx \in E(G)\}$ and $Z' = Z \cup \{z'\}$.

In Cases 1, 2, and 3, let P be the path with vertices $\{z', w_0\}$,$\{w_0', z', w_0\}$, and $\{z', w_0, w_1\}$, respectively. Note that $\rho_{R'}(V(P)) \ge \rho_R(A)$ for each case.

The set A is not independent, since otherwise $|A| = 1$ and $\rho(A) \ge 11$. In each case, both end vertices of P have a neighbor outside P in G'. By Claim 3.5, G' is triangle-free. Also, R' is smaller than R, since $|A| \ge 4$ and $|P| \le 3$. If we have a $(0,1)$-coloring ϕ' of R', then $\phi''(v) = \phi(v)$ for $v \in A$ and $\phi''(v) = \phi'(v)$ for $v \in V(G)-A$ yields a $(0,1)$-coloring ϕ'' of G, a contradiction. Thus R' has no such coloring. Then by the minimality of G, there is $B \subset V(G')$ with $\rho_{R'}(B) \le -5$. Here, $B \ne V(G')$ since $\rho_{R'}(V(G')) = \rho_R(V(G)) - \rho_R(A) + \rho_{R'}(V(P)) \ge \rho_R(V(G)) \ge -4$. Choose such set B with the smallest potential and among those with the largest size. Then $z' \in B$ since adding z' does not increase the potential.

If $\rho_R(A) \le \rho_{R'}(B \cap V(P))$, then

$$\begin{aligned}\rho_R(B - V(P) \cup A) &= \rho_R(B - V(P)) + \rho_R(A) - 9|E_G[B - V(P), A]| \\ &\le \rho_R(B - V(P)) + \rho_{R'}(B \cap V(P)) - 9|E_{G'}[B - V(P), B \cap V(P)]| \\ &= \rho_{R'}(B) \le -5,\end{aligned}$$

a contradiction.

If $\rho_R(A) > \rho_{R'}(B \cap V(P))$, then $B \cap V(P)$ must not contain an end vertex y of $V(P)$. (If $B \cap V(P)$ misses only the middle point, $\rho_{R'}(B \cap V(P)) \ge 11 \ge 4 \ge \rho_R(A)$.) Then

$$\begin{aligned}\rho_R((B - V(P)) \cup A) &= \rho_R(B - V(P)) + \rho_R(A) - 9|E_G[B - V(P), A]| \\ &\le \rho_R(B - V(P)) + 4 - 9|E_{G'}[B - V(P), A]| \le \rho_{R'}(B) + 4 \le -1.\end{aligned}$$

Since $A \ne \emptyset$, $(B - V(P)) \cup A$ must be $V(G)$. Then $\rho_R'(B \cup V(P)) \le \rho_{R'}(B) + \max\{11 - 2\cdot 9, 11\cdot 2 - 3\cdot 9\} < \rho_{R'}(B)$ since y has a neighbor in $V(G) - A$. It is a contradiction to the minimality of B. Therefore (1), (2), and (3) hold. □

We say that a path P is a *k-path* if P has $k+2$ vertices, and all internal vertices are typical and have degree 2 in G, and each of the end vertices of P either has degree at least 3 or is special. A vertex v is a *k-vertex* if $d(v) = k$. For $k \ge 2$, a k-vertex x is *$(a_1, \ldots, a_k)$-vertex* if x is typical and x is an end of distinct a_i-paths for $i = 1, 2, \ldots, k$.

Claim 3.7. *G has no alternating cycle $C = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$ such that all x_i are $(1,1,1)$-vertices (and thus all y_j are 2-vertices).*

Proof. Suppose G has such a cycle $C = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$. For $i = 1, \ldots, k$, let u_i be the neighbor of x_i outside of C and w_i be the neighbor of u_i distinct from x_i (it could be some x_j for $j \neq i$). Recall that by the definition of (1, 1, 1)-vertices, all x_i, y_i, u_i are typical. Consider $G' = G - \{x_1, y_1, u_1, \ldots, x_k, y_k, u_k\}$ and $R' = (G', \{z\})$. Then G' is a proper subgraph of G, so it has a (0, 1)-coloring ϕ'. Let $\phi(v) = \phi'(v)$ if $v \in V(G')$, and for $i = 1, \ldots, k$, let $\phi(x_i) = 1$, $\phi(y_i) = 0$ and $\phi(u_i) = 1 - \phi(w_i)$. Then ϕ is a (0, 1)-coloring for G, a contradiction. □

Let V_0 be the set of all (1, 1, 1)-vertices and their neighbors. By Claim 3.7, $G[V_0]$ is a forest. Let $T_1, \ldots, T_d$ be the components of $G[V_0]$. Each component T of $G[V_0]$ will be called a (1, 1, 1)-*tree*, and *the size*, $\|T\|$, of a (1, 1, 1)-tree T is the number of (1, 1, 1)-vertices in T.

Lemma 3.8. *Let T be a (1, 1, 1)-tree in G and ϕ be a (0, 1)-coloring of $G - T$. Then exactly one vertex of T is adjacent to a vertex v of color 1 in ϕ, and either $v = z$ or v has a neighbor in $G - T$ of color 1 .*

Proof. We use induction on the size of T. If the size of T is 1, suppose that x is the 3-vertex in T, $N(x) = \{y_1, y_2, y_3\}$, and $u_i \in N(y_i) - x$ for $i = 1, 2, 3$. For $i = 1, 2, 3$, let $\phi(y_i) = 1 - \phi(u_i)$. This coloring cannot be extended to x only if exactly one vertex in $N(x)$, say y_1 is colored with 0. Moreover, if in this case $u_1 \neq z$ and u_1 has no neighbor of color 1 in $V(G) - V(T)$, then we can recolor y_1 with 1 and color x with 0. But if $u_1 = z$ or u_1 has a neighbor of color 1 in $V(G) - V(T)$, then ϕ does not extend to T.

Suppose now that the lemma holds for (1, 1, 1)-trees of size at most $i - 1$ and let T be a (1, 1, 1)-tree of size i and ϕ be a (0, 1)-coloring of $G - T$. A *quasi-leaf* of T is a 3-vertex adjacent to two leaves of T. Suppose T has a quasi-leaf x_1 such that the two leaves of T, y_1 and y_2, adjacent to x_1 are both adjacent to vertices in $G - T$ of color 0. Then in any extension of ϕ to $\{y_1, y_2, x\}$ we must have $\phi(y_1) = \phi(y_2) = 1$ and so $\phi(x_1) = 0$. By the choice of i, the new ϕ extends to the tree $T' = T - x_1 - y_1 - y_2$ if and only if exactly one vertex T' is adjacent to a vertex v of color 1 in ϕ, and either $v = z$ or v has a neighbor in $G - T'$ of color 1. But then by construction, this also holds for T.

Thus we may assume that for every quasi-leaf x of T, some leaf y of T adjacent to x has a neighbor in T of color 1. Since every (1, 1, 1)-tree of size at least two has at least two quasi-leaves, we may assume that x and x' are quasi-leaves and y_1 and y_2 are the leaves of T adjacent to x and u_1 and u_2 are their other neighbors. Let $\phi(x) = 1$ and for $i = 1, 2$ let $\phi(y_i) = 1 - \phi(u_i)$. Since at least one of u_1 and u_2 has color 1, at least one of y_1 and y_2 gets color 0, and so the new partial coloring ϕ is a partial (0, 1)-coloring. Then at least two leaves of the tree $T' = T - x - y_1 - y_2$ have neighbors of color 1, and so by the choice of i, the new ϕ extends to T'. Thus the original ϕ extends to T. □

In the next two sections we describe all sets of low potential.

4. Sets of potential at most 2

The goal of the section is to prove the following.

Lemma 4.1. *If $A \subset V(G)$, $2 \leq |A| \leq n - 4$ and $\rho(A) \leq 2$, then A consists of one typical and one special vertex with an edge connecting them.*

It is straightforward to check the lemma in the case $|A| \in \{2, 3\}$. Suppose now that $\rho(A) \leq 2$ and $4 \leq |A| \leq n - 4$. Recall that A^* is the set of vertices in A that have neighbors in $V(G) - A$.

Claim 4.2. *If $u \in A^*$, then $u = z$.*

Proof. Assume $u \in A^*$ is typical. Consider the graph G' with $V(G') = A \cup \{w_1, w_2, w_3\}$ and $E(G') = E(G[A]) \cup \{uw_1, w_1w_2, w_2w_3\}$. Let $R' = (G', Z \cap A + w_3)$.

Since $|A| \leq n - 4$, $|V(G')| < n$. By Lemma 3.4, since $u \neq z$, $\rho(A) \geq 1$. Then $\rho_{R'}(B) \geq -4$ for every $B \subseteq V(G')$. By construction, G' is triangle-free. So by the minimality of G, G' has a (0, 1)-coloring ϕ'. Since w_3 is special, $\phi'(w_3) = 1$ and $\phi'(w_2) = 0$. So $\phi'(w_1) = 1$. Now if $\phi'(u) = 0$, this contradicts to Claim 3.6 and if $\phi'(u) = 1$ then because of w_1, u has no neighbors of color 1 in A, a contradiction to Claim 3.6. □

Proof of Lemma 4.1. By Claim 4.2, $A^* \subseteq \{z\}$. Since $4 \le |A| \le n-4$, we can $(0,1)$-color $G[A]$ and $G[(V(G)-A) \cup A^*]$ by induction hypothesis, and combine the colorings to get a $(0,1)$-coloring for G since either $A^* = \emptyset$ or $A^* = \{z\}$, and z is always colored with 1. This is a contradiction. □

5. Sets of potential at most 4

The goal of this section is to show the following lemma.

Lemma 5.1. *If $A \subseteq V(G)$ and $4 \le |A| \le n-4$, then $\rho(A) \ge 5$. Moreover, if $|A| \le 3$ and $\rho(A) \le 4$, then $z \in A$ and $G[A]$ is connected.*

Suppose that the lemma fails and choose a smallest $A \subseteq V(G)$ with $4 \le |A| \le n-4$ for which it fails. If $B \subset V(G)$, $1 \le |B| \le 3$ and $z \notin B$, then $\rho(B) \ge 11$ since G has no triangle. Thus if $G[A]$ is not connected, then by this and the minimality of A, the vertex set C of some component of $G[A]$ has potential at least 5. Thus in this case, $\rho(A-C) \le 4-5 = -1$, a contradiction to Lemma 3.4. So,

$$G[A] \text{ is connected.} \tag{5}$$

Claim 5.2. *If $u \in A^*$ is typical, then* (a) *there exists a coloring ϕ_u of $G[A]$ such that $\phi_u(u) = 0$ and $\phi_u(v) = 1$ for all $v \in A^* - u$, and* (b) *u has exactly one neighbor in $G - A$.*

Proof. Assume $u \in A^*$ is typical. Consider the graph G' with $V(G') = A \cup \{w_1, w_2, w_3\}$ and $E(G') = E(G[A]) \cup \{uw_1, w_1w_2, w_2w_3\}$. Let $R' = (G', Z \cap A + w_3)$.

Since $|A| \le n-4$, $|V(G')| < n$. By Lemma 3.4, since $u \ne z$, $\rho(A) \ge 1$. Then $\rho_{R'}(B) \ge -4$ for every $B \subseteq V(G')$. By construction, G' is triangle-free. So by the minimality of G, G' has a $(0,1)$-coloring ϕ'. Since w_3 is special, $\phi'(w_3) = 1$ and $\phi'(w_2) = 0$. So $\phi'(w_1) = 1$. If $\phi'(u) = 1$ then because of w_1, u has no neighbors of color 1 in A, a contradiction to Claim 3.6. Thus $\phi'(u) = 0$. Now Claim 3.6 yields (b), and together with Claim 3.5, also yields (a). □

Claim 5.3. *Set A^* contains a typical vertex.*

Proof. If not, then $A^* = \emptyset$ or $A^* = \{z\}$. Since $4 \le |A| \le n-3$, both $G[V(G) - (A - A^*)]$ and $G[A]$ are smaller than G, and so by the minimality of G have $(0,1)$-colorings, say ϕ and ϕ'. Then $\phi \cup \phi'$ is a $(0,1)$-coloring of G, since if $A^* \ne \emptyset$, then $A^* = \{z\}$ and $\phi(z) = \phi'(z) = 1$. □

Claim 5.4. *If $z \in A^*$, then z has no neighbor in A^*.*

Proof. Suppose z has a neighbor v in A^*. Then $vz \in E(G[A^*])$. If $A^* = \{v, z\}$, then we can color $G[A]$ and $G[V(G) - A \cup A^*]$ and combine the colorings, since v, z have the consistent colors in both colorings. So, there is $u \in A^* - z - v$. Then by Claim 5.2(a), there is a $(0,1)$-coloring ϕ_u of $G[A]$ such that $\phi_u(v) = \phi_u(z) = 1$, a contradiction to the fact that $z \in Z$. □

Proof of Lemma 5.1. Assume $4 \le |A| \le n-4$ and $\rho(A) \le 4$. Suppose $A^* - \{z\} = \{u_1, \dots, u_d\}$ and for $i = 1, \dots, d$, x_i is the unique neighbor of u_i in $G - A$. Let $X = \{x_1, \dots, x_d\}$. If $x_i = z$ for some i, then $\rho(A + x_i) = \rho(A) - 9 \le -5$, a contradiction. Furthermore, if x_i is adjacent to z, then $\rho(A + x_i + z) \le \rho(A) - 2 \cdot 9 + 11 \le -3$ and $A + x_i + z \ne V(G)$, a contradiction to Lemma 3.4. Hence

$$z \notin X \cup N(X) - A^*. \tag{6}$$

Case 1: $A^* = \{u_1\}$. Let G' be the graph with $V(G') = V(G) - A \cup \{w_0, z'\}$ and $E(G) = E(G[V(G) - A]) \cup \{u_1w_0, w_0z'\}$. Let $R' = (G', (Z - A) + z')$. If G' has a $(0,1)$-coloring ϕ, then $\phi(z') = 1$, $\phi(w_0) = 0$, and hence $\phi(u_1) = 1$. So, together with the coloring ϕ_{u_1} (see Claim 5.2(a)) of $G[A]$, ϕ will form a $(0,1)$-coloring of G, a contradiction. Thus G' has no $(0,1)$-coloring and is smaller than G, since $|A| \ge 4$. Hence there is $B \subset V(G')$ with $\rho_{R'}(B) \le -5$. Choose such B with the smallest potential and among those—with the largest size. Then $z' \in B$. If $w_0 \notin B$, then z' is isolated in $G'[B]$ and $\rho_R(B - z') = \rho_{R'}(B - z') = \rho_{R'}(B) \le -5$, a contradiction. So $w_0 \in B$ and hence $u_1 \in B$. Since $B - z' - w_0 \subset V(G)$, we have $\rho_R(B - z' - w_0) = \rho_{R'}(B - z' - w_0) = \rho_{R'}(B) - 11 + 2 \cdot 9 \le 2$. Since

$|A| \geq 4$, $|B - z' - w_0| \leq n - 4$. So by Lemmas 3.4 and 4.1, $|B - z' - w_0| \leq 2$ and $z \in B - z' - w_0$ is adjacent to $x_1 \in B - z' - w_0$, a contradiction to (6).

Case 2: $A^* = \{u_1, z\}$. Consider G' with $V(G') = A + w_0$, and $E(G') = E(G[A]) \cup \{zw_0, w_0u\}$ $E(G') = E(G[A]) \cup \{zw_0, w_0u_1\}$. Let $R' = (G', \{z\})$. If G' has a $(0, 1)$-coloring ϕ, then $\phi(w_0) = 0$ and thus $\phi(u_1) = 1$. Recall that by Claim 5.2(a), there is a $(0, 1)$-coloring ϕ_{u_1} of $G[A]$ such that $\phi_{u_1}(u_1) = 0$. By the minimality of G, $G - (A - z)$ has a $(0, 1)$-coloring f. If $f(x_1) = 0$, then $f \cup \phi$ is a $(0, 1)$-coloring of G; otherwise, $f \cup \phi_{u_1}$ is a $(0, 1)$-coloring of G, a contradiction.

Thus G' has no $(0, 1)$-coloring. Then there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Since $V(G') - w_0 \subset A \subset V(G)$, $w_0 \in B$. Hence $z, u_1 \in B$ and $\rho_R(B - w_0) \leq \rho_{R'}(B) - 11 + 2 \cdot 9 \leq 2$. Since $u_1 z \notin E(G)$ by Claim 5.4, this contradicts Lemma 4.1.

Case 3: $d \geq 2$. By Claim 3.5, all x_i are distinct and not adjacent to each other. Let G' be the graph with $V(G') = V(G) - (A - z) \cup \{w_0, w_1, z'\}$ and

$$E(G') = E(G[V(G) - (A - z)]) \cup \{zx_i : 3 \leq i \leq d\} \cup \{x_1x_2, z'w_0, w_0w_1, w_1x_1\}.$$

Observe that G' is triangle free. Otherwise x_1, x_2 have a common neighbor w. Then $\rho_R(A \cup \{w, x_1, x_2\}) \leq \rho_R(A) - 3 \leq 1$, contrary to Lemma 4.1. Let $R' = (G', \{z, z'\})$.

Suppose G' has a $(0, 1)$-coloring ϕ. Then $\phi(z) = \phi(z') = 1, \phi(x_3) = \cdots = \phi(x_d) = \phi(w_0) = 0$ and hence $\phi(w_1) = 1$. If $\phi(x_2) = \phi(x_1)$, then since $x_1x_2 \in E(G')$, their common color is 1. But x_1 already has neighbor w_1 with $\phi(w_1) = 1$. Thus $\phi(x_2) \neq \phi(x_1)$. Then by Claim 5.2(a), either $\phi \cup \phi_{u_1}$ or $\phi \cup \phi_{u_2}$ is a $(0, 1)$-coloring of G, a contradiction. So G' has no $(0, 1)$-coloring and is smaller than G, since $|A| \geq 4$. Hence there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. Then $z \in B$. Since $G' - x_1 - z - w_0 - w_1 \subset G$ and $G'[B]$ has no isolated or pendant vertices, apart from z, z', we have $x_1 \in B$. Then $\{w_0, w_1\} \subset B$. Let $X' = \{x_1, \ldots, x_d\} \cap B$. Then $|E_G[A - z, B - w_0 - w_1 - z]| = |X'|$. So

$$\rho_R(A \cup (B - w_0 - w_1 - z)) \leq \rho_R(A) + \rho_R(B - w_0 - w_1 - z) - 9(|X'| + |E_G[\{z\} \cap A^*, B - w_0 - w_1 - z]|). \quad (7)$$

Because $x_1 \in X'$, if $x_2 \notin X'$, then $|E_{G'}[z, B - w_0 - w_1 - z]| = |E_G[\{z\} \cap A^*, B - w_0 - w_1 - z]| + |X'| - 1$, and so the last expression in (7) is at most

$$4 + \rho_{R'}(B - z - w_0 - w_1) - 9(|E_{G'}[z, B - w_0 - w_1 - z]| + 1) = 4 + \rho_{R'}(B - w_0 - w_1) - 9$$
$$\leq 4 + (\rho_{R'}(B) - 2 \cdot 11 + 3 \cdot 9) - 9 = \rho_{R'}(B) \leq -5,$$

a contradiction.

If $x_2 \in B$, then on the one hand, G' has an extra edge x_1x_2, and on the other hand,

$$|E_{G'}[z, B - w_0 - w_1 - z]| = |E_G[\{z\} \cap A, B - w_0 - w_1 - z]| + |X'| - 2,$$

since $x_2 \in X'$. These edges cancel in the calculations, and we again get $\rho_R(A \cup (B - w_0 - w_1 - z)) \leq -5$, a contradiction.

To prove the "Moreover" part of the lemma, observe first that if $z \notin A$ and $1 \leq |A| \leq 3$, then $\rho(A) \geq 6$. Also, if $G[A]$ is disconnected and $1 \leq |A| \leq 3$, then $|E(G[A])| \leq |A| - 2$, and thus $\rho(A) \geq 11(|A| - 1) - 9(|A| - 2) = 2|A| + 7 \geq 9$. □

6. Shovels in G

Claim 6.1. *If G has a subgraph $P_4 = x_1x_2x_3x_4$ with $d(x_2) = d(x_3) = 2$, then $z \in \{x_1, x_2, x_3, x_4\}$.*

Proof. Assume $z \notin \{x_1, x_2, x_3, x_4\}$. By Claim 3.2, $d(x_4) \geq 2$. If $d(x_4) = 2$ and x_5 is the neighbor of x_4 distinct from x_3, let $G_0 = G - x_2 - x_3 - x_4$. By the choice of G, G_0 has a $(0, 1)$-coloring ϕ. We can extend it to G by letting $\phi(x_2) = 1 - \phi(x_1)$, $\phi(x_4) = 1 - \phi(x_5)$, and letting $\phi(x_3) = 0$ if $\phi(x_2) = \phi(x_4) = 1$ and $\phi(x_3) = 1$ otherwise. Since this contradicts the definition of G, $d(x_4) \geq 3$.

Let $G' = G - x_3x_4 + x_3z$ and $R' = (G', Z)$. Then by our definition, R' is smaller than R, since $|V(G)| = |V(G')|$ and $d_{G'}(z) > d_G(z)$. If G' has a triangle T, then $x_3z \in E(T)$. Since $N_{G'}(x_3) = \{x_2, z\}$, we need $V(T) = \{x_2, x_3, z\}$, but $x_1 \neq z$ by assumption. So G' is triangle-free.

Suppose G' has a $(0, 1)$-coloring ϕ'. If $\phi'(x_4) = 1$, then ϕ' is a $(0, 1)$-coloring of G, a contradiction. If $\phi'(x_4) = 0$, then recolor x_2 with $1 - \phi'(x_1)$ and x_3 with 1. Again, we get a $(0, 1)$-coloring of G. So, G' has no $(0, 1)$-coloring. Then by the minimality of G there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. By the choice, $z \in B$. If $x_3 \notin B$ or $x_4 \in B$, then $|E(G'[B])| \leq |E(G[B])|$ and $\rho_R(B) \leq -5$. Let $x_3 \in B$ and $x_4 \notin B$. Then by the minimality of $\rho_{R'}(B)$, $x_2 \in B$ and hence $x_1 \in B$. So by definition, $\rho_R(B) = \rho_{R'}(B) + 9 \leq 4$. Since $\{x_1, x_2, x_3, z\} \subseteq B$, by Lemma 5.1, $|B| \geq n - 3$. By the minimality of $\rho_{R'}(B)$, x_4 has at most one neighbor in $B - x_3$. So since $d(x_4) \geq 3$, x_4 has a neighbor y in $V(G) - B$ and $|B| \leq n - 2$. If $|B| = n - 2$ then by Claim 3.2, each of y and x_4 has a neighbor in B, and so $\rho_{R'}(B + y + x_4) \leq \rho_{R'}(B) + 2 \cdot 11 - 3 \cdot 9 < \rho_{R'}(B)$, a contradiction. So, we may assume that $V(G') - B = \{x_4, y, x\}$. Since G' has no triangles, by Claim 3.2, $\rho_{R'}(V(G')) \leq \rho_{R'}(B) + 3 \cdot 11 - 4 \cdot 9 < \rho_{R'}(B)$, a contradiction to the choice of B. □

Claim 6.2. *Suppose G contains a path $P = (x_1, x_2, x_3, z)$ such that $d(x_2) = d(x_3) = 2$. Then either $d(x_1) \geq 4$ or $d(x_1) = 3$ and x_1 belongs to a 5-cycle not containing x_2.*

Proof. By Claim 3.2, $d(x_1) \geq 2$. Suppose $d(x_1) = 2$ and $x_0 \neq x_2$ is a neighbor of x_1. If $x_0 = z$, then $\rho(\{x_1, x_2, x_3, z\}) = 3 \cdot 11 - 4 \cdot 9 = -3$, a contradiction to Lemma 3.4. So $x_0 \neq z$. This contradicts Claim 6.1 for the path (x_0, x_1, x_2, x_3). So $d(x_1) \geq 3$. If $d(x_1) = 3$, let x', x'' be the two neighbors of x_1 distinct from x_2.

Consider the graph G' with $V(G') = V(G) - x_1 - x_2 - x_3 - x' - x'' + x$ and

$$E(G') = E(G[V(G) - x_1 - x_2 - x_3 - x' - x'']) \cup \{xy : y \text{ is adjacent to } x' \text{ or } x'' \text{ in } G\}.$$

Let $R' = (G', Z)$. If x_1 does not belong to a 5-cycle not containing x_2, then G' is triangle-free. If G' has a $(0, 1)$-coloring ϕ, then we extend it to G by letting $\phi(x') = \phi(x'') = \phi(x)$, $\phi(x_1) = 1 - \phi(x)$, $\phi(x_2) = 1$, and $\phi(x_3) = 0$. So, G' has no $(0, 1)$-coloring. Then by the minimality of G there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size.

If $x \notin B$, then $\rho_R(B) = \rho_{R'}(B) \leq -5$. If $x \in B$, then $\rho_R(B - x + x' + x'' + x_1) = \rho_{R'}(B) + 2 \cdot 11 - 2 \cdot 9 \leq -5 + 4 = -1$, contradicting Lemma 3.4, since $x_2 \notin B - x + x' + x'' + x_1$. □

We need several new definitions. If G contains vertices x_1, x_2, x_3, z and a 5-cycle C_5 containing x_1 with $d(x_1) = 3$ as in Claim 6.1, we call the subgraph induced by $V(C_5) \cup \{x_1, x_2, x_3, z\}$ an *intact shovel*. The path x_1, x_2, x_3, z is the *handle*, the cycle C_5 is the *head* and x_1 is the *joint* of this shovel. By definition, vertices x_1, x_2, x_3 in the shovel are not adjacent to any vertex outside the shovel.

A path x_1, x_2, x_3, z in G such that x_1 has degree $t \geq 4$ will be called a *broken shovel of degree t*. Below when we speak of shovels, we have in mind both, broken and intact shovels (see Figs. 6 and 7).

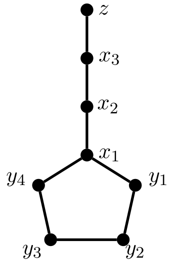

Fig. 6. An intact shovel.

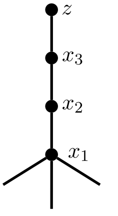

Fig. 7. A broken shovel of degree 4.

Claim 6.3. *For every shovel S, $G[V(S)] = S$.*

Proof. Suppose G has an edge e not in S that connects two vertices in $V(S)$. If S is broken, then $\rho(V(S)) \leq 3 \cdot 11 - 4 \cdot 9 = -3$, and if S is intact, then $\rho(V(S)) \leq 7 \cdot 11 - 9 \cdot 9 = -4$. In both cases, we contradict Lemma 3.4, unless $V(S) = V(G)$ and $E(G) = E(S) + e$. Suppose the latter holds and the handle of S is x_1, x_2, x_3, z. If S is broken, then $d(v_1) = 2$ since $V(S) = V(G)$, a contradiction. If S is intact and its head is $C_5 = (x_1, y_1, y_2, y_3, y_4)$, then since G has no triangles, at least one end of e is outside of C_5 and so should be z. Let the other end be v. By degree restrictions on S, $v \notin \{x_2, x_1\}$.

If $v \in \{y_1, y_4\}$, then it belongs to a 5-cycle C containing z and $\rho(V(C)) = 4 \cdot 11 - 5 \cdot 9 = -1$, a contradiction to Lemma 3.4. Otherwise, we may assume $v = y_2$. In this case, we color x_3, x_1 and y_2 with 0, and the rest with 1. □

Claim 6.4. *For any distinct shovels S and S', $V(S) \cap V(S') = \{z\}$.*

Proof. Let $S' \neq S$. Let the handle of S be x_1, x_2, x_3, z and the handle of S' be x'_1, x'_2, x'_3, z. Suppose $V(S) \cap V(S') = S_0 \neq \{z\}$.

Case 1: $x_3 \in S_0$. Since $z \in S'$, by Claim 6.3, $x_3 = x'_3$. This yields that $x_2 = x'_2$ and $x_1 = x'_1$. Then since $S \neq S'$, we may assume that S is intact, and hence $d(x_1) = 3$. So S' also is intact. Let the head of S be $(x_1, y_1, \ldots, y_4)$ and the head of S' be $(x_1, y'_1, \ldots, y'_4)$. Since $d(x_1) = 3$, we have $y'_1 = y_1$ and $y'_4 = y_4$. If $\{y_2, y_3\} \cap \{y'_2, y'_3\} = \emptyset$, then $\rho(V(S) \cup V(S')) \leq 9 \cdot 11 - 11 \cdot 9 = 0$. So by Lemma 3.4, $V(G) = V(S) \cup V(S')$ and $E(G) = E(S) \cup E(S')$. This contradicts Claim 6.1 for the path y_1, y_2, y_3, y_4. Finally, if $y'_2 = y_2$, then $\rho(V(S) \cup V(S')) \leq 8 \cdot 11 - 10 \cdot 9 = -2$. Again, by Lemma 3.4, $V(G) = V(S) \cup V(S')$ and $E(G) = E(S) \cup E(S')$. Then we color z, x_2, y_1, y_2, y_4 with 1 and the rest with 0.

Case 2: $x_3 \notin S_0$. By symmetry, we may assume $x'_3 \notin S_0$. Then also $x_2, x'_2 \notin S_0$.

Case 2.1: $x_1 \in S_0$. If S' is broken, this means that $x_1 = x'_1$ and $\rho(V(S') + x_2 + x_3) = 5 \cdot 11 - 6 \cdot 9 = 1$. Let $U = V(G) - S' - x_2 - x_3$. By Lemma 4.1, $|U| \leq 3$. If there is a vertex $u \in Y$ which has at least 2 neighbors in $S' + x_2 + x_3$, then $\rho(S' + x_2 + x_3 + u) \leq -6$, a contradiction. As $d(x_1) \geq 4$, x_1 has at least two neighbors in U. As G has no vertex of degree 1, we conclude that $U + x$ induces a 4-cycle (x, u_1, u_2, u_3). Also $d(u_2) = 2$, otherwise we get a contradiction by $\rho(V(G)) \leq 8 \cdot 11 - 11 \cdot 9 < -4$. Thus G is bipartite, and hence has a proper $(0, 1)$-coloring.

So, S' is intact. Let the head of S' be $(x'_1, y'_1, \ldots, y'_4)$. Then $\rho(V(S') + x_2 + x_3) \leq \rho(V(S')) + 2 \cdot 11 - 3 \cdot 9 = 0$, and by Lemma 3.4, $V(G) = V(S') + x_2 + x_3$ and $E(G) = E(S') \cup \{zx_3, x_3x_2, x_2x_1\}$. Since $d(x'_1) = 3$, by symmetry, we may assume that $x_1 \in \{y'_1, y'_2\}$. In both cases, we color x_3, x'_3, x'_1, y'_2 with 0 and the rest with 1.

Case 2.2: $x_1 \notin S_0$. By symmetry, we may assume $x'_1 \notin S_0$. In particular, both S and S' are intact. Since by Claim 6.3, $G[S - x_1]$ is acyclic, $G[S_0]$ is acyclic. Also, z is isolated in $G[S_0]$ and thus $\rho(S_0) \geq 11|S_0 - z| - 9(|S_0 - z| - 1) = 9 + 2|S_0 - z|$. So,

$$\rho(V(S) \cup V(S')) \leq \rho(S) + \rho(S') - \rho(S_0) \leq 5 + 5 - 9 - 2|S_0 - z| = 1 - 2|S_0 - z|. \tag{8}$$

Hence $|S_0 - z| \leq 2$ and by Lemma 3.4, $V(G) = V(S) \cup V(S')$ and $E(G) = E(S) \cup E(S')$. By Claim 6.1, each of the sets $\{y'_1, y'_2\}$ and $\{y'_3, y'_4\}$ has a vertex common with S. Thus $|S_0 - z| = 2$. Also, if $G[S_0 - z]$ has no edge, then we would have an extra -9 in (8), a contradiction. So $S_0 - z = \{y'_2, y'_3\} = \{y_2, y_3\}$. Then we can color $x_3, x'_3, y_1, y_4, y'_1, y'_4$ with color 0 and the rest with 1. □

Claim 6.5. *Let S be an intact shovel with the handle (z, x_3, x_2, x_1) and the head $C = (x_1, y_1, y_2, y_3, y_4)$. Then at least 3 edges connect C with $V(G - S)$.*

Proof. Let E' be the set of edges connecting $V(C)$ with $V(G - S)$. Suppose to the contrary that $|E'| \leq 2$. Then since $z \notin V(C)$, by Claim 6.1, $|E'| = 2$ and the ends in C of the edges in E' are distinct. So we can suppose $E' = \{y_iu, y_jv\}$, where $u, v \in V(G - S)$.

By Claim 6.1 and symmetry, we may assume that $j = 3$ and either $i = 1$ or $i = 2$. Let G' be obtained from $G - (V(S) - z)$ by adding a vertex z' and edge uz'. Let $R' = (G', \{z, z'\})$. Since edge uz' is a cut edge, G' is a triangle-free graph smaller than G. If G' has no $(0, 1)$-coloring, then by the minimality of G, there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. By the choice of G, $u, z' \in B$. Then the set $B - z'$ contains u, is disjoint from $S - z$ and $\rho_R(B - z') \leq -5 + 9 = 4$. So denoting $B' = (B - z') \cup V(S)$ and taking into account edge y_iu, we have

$$\rho_R(B') \leq \rho_R(B - z') + \rho_R(S) - 9 = 0. \tag{9}$$

By Lemma 3.4 and (9), $B' = V(G)$. But then $v \in B - z'$, and edge y_jv was not count in (9). So instead of (9), we have $\rho_R(B') \leq \rho_R(B - z') + \rho_R(S) - 18 = -9$, a contradiction.

Thus G' has a $(0, 1)$-coloring ϕ. In particular, $\phi(u) = 0$. We extend ϕ from $G' - z'$ to the whole G as follows. First, we let $\phi(x_3) = 0$, $\phi(x_2) = 1$ and $\phi(y_3) = 1 - \phi(v)$.

Case 1: $u = v$. Since G has no triangles, $i = 1$. So we extend ϕ to G by letting $\phi(y_4) = \phi(y_1) = 1$ and $\phi(x_1) = \phi(y_2) = 0$.

Case 2: $u \neq v$ and $i = 1$. Let $\phi(x_1) = 0$, $\phi(y_4) = \phi(y_1) = 1$ and $\phi(y_2) = \phi(v)$.

Case 3: $u \neq v$ and $i = 2$. If $\phi(y_3) = 1$, then let $\phi(x_1) = \phi(y_2) = 1$, and $\phi(y_1) = \phi(y_4) = 0$, otherwise let $\phi(x_1) = 0$, and $\phi(y_1) = \phi(y_4) = \phi(y_2) = 1$. □

7. On (1, 1, 1)-trees in *G*

The goal of this section is to prove that all (1, 1, 1)-trees in G are small: each of them has at most two (1, 1, 1)-vertices.

Claim 7.1. *For each* (1, 1, 1)*-tree* T*, if a vertex* $x \in V(G) - V(T)$ *is adjacent to more than one vertex in* T*, then* $zx \in E(G)$*.*

Proof. Suppose some $x \in V(G)-V(T)$ is adjacent to distinct $y, y' \in V(T)$. By the definition of (1, 1, 1)-trees, $|V(T)| \geq 4$ and $z \notin V(T)$. If $x = z$, then let $G_0 = G - V(T)$ and $R_0 = (G_0, Z)$. Since $G_0 \subset G$, it has a (0, 1)-coloring ϕ_0, and $\phi_0(z) = 1$. Then by Lemma 3.8, we can extend ϕ_0 to G, a contradiction.

Assume $x \neq z$. Let G' be obtained from $G - V(T)$ by adding vertices w and z' and edges xw and wz'. Let $R' = (G', \{z, z'\})$. Since new edges xw and wz' are cut edges, G' is a triangle-free graph. Since $|V(T)| \geq 4$, $|V(G')| < n$. If G' has a (0, 1)-coloring ϕ, then $\phi(w) = 0$ and hence $\phi(x) = 1$. So by Lemma 3.8, we can extend ϕ to G, a contradiction. Then by the minimality of G, there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. Then $z' \in B$. If $x \notin B$, then $w \notin B$ and $\rho(B - z') = \rho_{R'}(B) \leq -5$, a contradiction. So, $x \in B$ and thus $w \in B$. Then $\rho(B - z' - w) \leq \rho_{R'}(B) - 11 + 18 \leq 2$. Since $|B| \leq n - |V(T)| \leq n - 4$ and $x \neq z$, by Lemma 4.1, $B - z' - w = \{x, z\}$ and $xz \in E(G)$, as claimed. □

Lemma 7.2. *If a graph* G' *is obtained from* G *by replacing a* (1, 1, 1)— *tree* T *with another* (1, 1, 1)*-tree* T' *with the same set of vertices and the same set of leaves, then the pair* $R' = (G', Z)$ *is still a minimum counterexample to the theorem.*

Proof. Assume w is a common neighbor of two (1, 1, 1)-vertices v_1 and v_2 in a (1, 1, 1)—tree T. For $i = 1, 2$, let w_i and w_i' be the neighbors of v_i distinct from w and u_i be the neighbor of w_i distinct from v_i. By the definition of (1, 1, 1)-vertices, $d(w) = d(w_1) = d(w_2) = d(w_1') = d(w_2') = 2$. By Claim 3.7, $w_2 \neq w_1$. If $u_1 = u_2$, then $u_1 \notin T$ and by Claim 7.1, $zu_1 \in E(G)$. Then for $W = \{z, u_1, w_1, w_2, v_1, v_2, w\}$, we have $\rho(W) \leq 6 \cdot 11 - 7 \cdot 9 = 3$. So by Lemma 5.1, $n \leq 10$. If w_1' has a neighbor in $W - v_1$, then $\rho(W + w_1') \leq \rho(W) + 11 - 18 \leq -4$, contradicting Lemma 3.4, since $w_2' \notin W + w_1'$. Similarly, w_2' has no neighbor in $W - v_2$. Thus, since $n \leq 10$, there is a unique $y \in V(G) - W - w_1' - w_2'$, and $yw_1', yw_2' \in E(G)$. By Claim 6.1, y has a neighbor in W. So, $\rho(V(G)) \leq \rho(W) + 3 \cdot 11 - 5 \cdot 9 \leq -9$, a contradiction. Thus $u_1 \neq u_2$. Also, since v_1 and v_2 are adjacent only to 2-vertices in T, and T has no cycles, $u_1v_2, u_2\, v_1 \notin E(G)$.

Let G' be obtained from G by deleting edges w_1u_1, w_2u_2 and adding edges w_1u_2, w_2u_1. Let $R' = (G', Z)$. By construction, $|V(G')| = n$ and the degrees of all vertices in G' are the same as in G. Since $u_1v_2, u_2v_1 \notin E(G)$, G' has no triangles. If $\rho_{R'}(B) \leq -4$ for all $B \subseteq V(G')$, then the lemma holds for the (1, 1, 1)-tree T' obtained from T by deleting edges w_1u_1, w_2u_2 and adding edges w_1u_2, w_2u_1. So consider $B \subset V(G')$ with the smallest potential in R' and assume $\rho_{R'}(B) \leq -5$. Let $U = \{u_1, u_2, w_1, w_2\}$. If $|B \cap U| \leq 1$ or $|B \cap U| \geq 3$, then $\rho_R(B) = \rho_{R'}(B) \leq -5$, a contradiction. Furthermore, if $\rho_{R'}(B) < \rho_R(B)$, then either $B \cap U = \{w_1, u_2\}$ or $B \cap U = \{w_2, u_1\}$. By symmetry, let $B \cap U = \{w_1, u_2\}$. Then $\rho_R(B) = \rho_{R'}(B) + 9 \leq 4$ and by Lemma 5.1, either $|B| \leq 3$ or $|B| \geq n - 3$. Since $N_{G'}(w_1) = \{v_1, u_2\}$, by the minimality of $\rho_{R'}(B)$, $v_1 \in B$. For the same reason at least one of w, w_1' is in B. So $|B| \geq 4$ and thus $|B| \geq n - 3$. If $v_2 \notin B$, then $w \notin B$, a contradiction to $|B| \geq n - 3$, since the case is that $u_1, w_2 \notin B$. Thus $v_2 \in B$. Then $\rho(B + w_2) \leq \rho(B) + 11 - 18 \leq -3$, a contradiction to Lemma 3.4, since $u_1 \notin B + w_2$. This proves the lemma for our special T'.

By a sequence of switching pairs of edges as above, we can change T into any (1, 1, 1)-tree with the same set of vertices and the same set of leaves. □

Claim 7.3. *For every* $(1, 1, 1)$*-tree* T*, each vertex* $x \in V(G) - V(T)$ *is adjacent to at most one vertex in* T.

Proof. Suppose that T is a $(1, 1, 1)$-tree in G and $x \in V(G) - V(T)$ is adjacent to y_1 and y_2 in T. By Claim 7.1, $zx \in E(G)$. By Lemma 7.2, if we replace T with a $(1, 1, 1)$-tree T' with the same set of vertices and the same set of leaves such that y_1 and y_2 have a common neighbor v in T', then the pair $R' = (G', Z)$ also is a minimum counterexample to the theorem. But $\rho_{R'}(\{z, x, y_1, y_2, v\}) = -1$, a contradiction to Lemma 3.4. □

Recall that for a $(1, 1, 1)$-tree T, $\|V(T)\|$ is its size, i.e., the number of $(1, 1, 1)$-vertices in T. Let $L(T)$ denote the set of leaves of T. Then

$$|V(T)| = 3\|V(T)\| + 1 \quad \text{and} \\ |L(T)| = |N(T) - V(T)| = \|V(T)\| + 2 \text{ for each } (1, 1, 1)\text{-tree } T. \tag{10}$$

Claim 7.4. *If* x *is adjacent to* T *and* $xz \in E(G)$*, then* $\|V(T)\| = 1$.

Proof. Suppose $\|V(T)\| \geq 2$ and $x \in V(G) - V(T)$ is adjacent to z and to $y \in V(T)$. Then $d(y) = 2$ and the neighbor of y in T is some $(1, 1, 1)$-vertex v. By Lemma 7.2, we can assume that $N(v) = \{y, y', y''\}$, the other neighbor x' of y' is in $V(G) - V(T)$ and the other neighbor x'' of y'' is in T. Let G' be obtained from $G - \{y, v, y'\}$ by adding edge $y''x'$ and T' be the tree induced in G' by $V(T) - \{y, v, y'\}$. Let $R' = (G', Z)$. If R' has a $(0, 1)$-coloring ϕ, then $\phi(x) = 0$ and the number of vertices of color 1 in $N_G(T) - V(T) - x$ is not 1. But then by Lemma 3.8, we can extend the coloring of $G' - V(T')$ to a $(0, 1)$-coloring of G.

So, by the minimality of R, there exists $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Since $\rho_R(B) \geq -4$, $y'', x' \in B$. Then $\rho_R(B+v+y') = \rho_{R'}(B)+9+2(11)-3(9) \leq -5+4 = -1$. Since $y \notin B+v+y'$, this contradicts Lemma 3.4. □

Claim 7.5. *Suppose* T *is a* $(1, 1, 1)$*-tree,* $V' = \{x_1, \dots, x_t\} = N(T) - V(T)$ *and* $t \geq 5$*. Then*
(a) *Each* $x \in V'$ *is at distance at least* 3 *from* z*;*
(b) V' *is an independent set;*
(c) *Each* $w \in V(G)$ *is adjacent to at most* 2 *vertices in* V'*;*
(d) *Each* $x \in V'$ *is at distance* 2 *in* $G - V(T)$ *from at most one* $x' \in V'$.

Proof. Since $t \geq 5$, by (10), $\|V(T)\| \geq 3$ and so $|V(T)| \geq 10$.

(a) Since $\|V(T)\| \geq 3$, by Claim 7.4, $z \notin N(V')$. So, if (a) does not hold, then there is $x \in V'$ such that either $x = z$ or $dist(x, z) = 2$. In both cases there is $x' \in V' - x$ at distance at least two from z. Let G' be obtained from $G - V(T)$ by adding a new vertex w adjacent to x' and z. Let $R' = (G', Z)$. If R' has a $(0, 1)$-coloring ϕ, then $\phi(x) = \phi(x') = 1$. Thus by Lemma 3.8, we can extend ϕ to $V(T)$ to obtain a $(0, 1)$-coloring of R. So R' has no $(0, 1)$-coloring, and is smaller than R, since $|V(T)| \geq 10$. Then by the minimality of G there is a set $B \subset V(G')$ with $\rho_{G'}(B) \leq -5$. In order to have $\rho_{G'}(B) \leq -5$, we need $w \in B$ and both its neighbors, z and x', be in B. Then $\rho_R(B - w) \leq -5 + 2 \cdot 9 - 11 \leq 2$. Hence by Lemma 4.1, either $B - w = \{z, x'\}$ or $|B - w| \geq n - 3$. The former is impossible, since $zx' \notin E(G)$ and the latter does not hold, since $|V(T)| \geq 10$. This proves (a).

(b) Assume $x, x', x'' \in V'$ and $xx'' \in E(G)$. As in the proof of (a), let G' be obtained from $G - V(T)$ by adding a new vertex w adjacent to x' and z. Let $R' = (G', Z)$. If R' has a $(0, 1)$-coloring ϕ, then $\phi(x') = 1$ and either $\phi(x) = 1$ or $\phi(x'') = 1$. So by Lemma 3.8, we can extend ϕ to $V(T)$. Then we repeat the rest of the argument of the proof of (a) word by word and come to a contradiction.

(c) Assume that some $x_1, x_2, x_3 \in V'$ are adjacent to the same vertex w. By Lemma 7.2, we may assume that x_1, x_2, x_3 and some 6 vertices in T form a tree T' with 9 vertices and 8 edges. If there is a path (w, w', z) from w to z, then $\rho_R(\{z, w', w\} \cup V(T')) = 11 \cdot 11 - 9 \cdot 13 = 4$. However, this set has more than 3 and fewer than $|V(G)| - 3$ vertices. Thus we get a contradiction to Lemma 5.1. Together with (a) this yields that

$$\text{the distance from } w \text{ to } z \text{ is at least 3.} \tag{11}$$

Consider G' obtained from $G - V(T)$ by adding an edge wz. Let $R' = (G', Z)$. By (11), G' is triangle-free. If R' has a $(0, 1)$-coloring ϕ, then $\phi(w) = 0$ and hence $\phi(x_1) = \phi(x_2) = 1$. Thus by Lemma 3.8, ϕ

can be extended to a (0, 1)-coloring for G, a contradiction. Since G' is smaller than G, there is a set $B \subset V(G')$ with $\rho_{R'}(B) \le -5$. Then $w, z \in B$, so $\rho_G(B) = -5 + 9 = 4$. By Lemma 5.1, either $|B| \le 3$ or $|B| \ge n - 3$. However, B misses at least 7 vertices in T. So, $|B| \le 3$ and $w, z \in B$. By (11), $G[B]$ is disconnected. This contradicts the last statement of Lemma 5.1.

(d) Assume that $x_1, x_2, x_3 \in V'$ are such that x_3 is at distance two from x_1 and from x_2. Then by (c) there are distinct u_1, u_2 such that for $i = 1, 2$, u_i is the common neighbor of x_3 and x_i. Consider the graph G' obtained from $G - V(T)$ by adding a new vertex w and edges zw and wx_3. Let $R' = (G', Z)$. Again, if R' has a (0, 1)-coloring ϕ, then $\phi(x_3) = 1$ and hence one of u_1, u_2, say u_1 has color 0. But then $\phi(x_1) = 1$, and by Lemma 3.8, ϕ extends to a (0, 1)-coloring for G. So again, there is $B \subset V(G')$ with $\rho_{R'}(B) \le -5$. Then $w \in B$ and thus $z, x_3 \in B$. Therefore, $\rho_R(B - w) \le -5 + 2(9) - 11 \le 2$. By Lemma 4.1, either $B = \{z, x_3\}$ or $|B| \ge n - 3$. But by Claim 7.4, $x_3 z \notin E(G)$, and at least 7 vertices of T are not in B. This contradiction proves (d). □

We need a bit more notation. For a pair $R' = (G', Z')$ and a set $S \subseteq V(G')$, let $m_{R'}(S)$ be the minimal potential that may have subsets of $V(G')$ containing S. If we have two sets B_1 and B_2 containing S with $\rho_{R'}(B_1) = \rho_{R'}(B_2) = m_{R'}(S)$, then by the submodularity of $\rho_{R'}$, also $\rho_{R'}(B_1 \cap B_2) = \rho_{R'}(B_1 \cup B_2) = m_{R'}(S)$. Thus the family

$$\mathcal{M}_{R'}(S) = \{B \subseteq V(G') : B \supseteq S, \rho_{R'}(B) = m_{R'}(S)\}$$

contains a unique inclusion minimal member $M_{R'}(S)$ contained in every other member of $\mathcal{M}_{R'}(S)$.

Claim 7.6. *Let T be a (1, 1, 1)-tree with $\|V(T)\| \ge 3$. Let $V' = N(T) - V(T) = \{x_1, \ldots, x_t\}$. Let $G' = G - V(T)$ and $R' = (G', Z)$. Then for any distinct $x_1, x_2, x_3 \in V'$,* (a) $m_{R'}(\{x_1\}) = 5$, $m_{R'}(\{x_1, x_2\}) = 8$, $m_{R'}(\{x_1, x_2, x_3\}) = 11$, *and* (b) $M_{R'}(\{x_1, x_2\}) \subset M_{R'}(\{x_1, x_2, x_3\})$.

Proof. Order the vertices of V' as needed: $x_1, \ldots, x_t$. For $i = 1, \ldots, t$, let y_i be the neighbor of x_i in T and v_i be the neighbor of y_i in T. By Claim 7.5(b,d) and the symmetry between x_2, x_3, we may assume that the distance between x_1 and x_2 in G' is at least 3. By Lemma 7.2, we may assume that $v_2 = v_1$, $v_{t-1} = v_t$ and that for $j = 2, \ldots, t-2$, vertices v_j and v_{j+1} have a common neighbor u_j in T. First, we prove a half of (a):

$$\text{for any } 1 \le j \le 3, \quad m_{R'}(\{x_1, \ldots, x_j\}) \ge 3j + 2. \tag{12}$$

Indeed, suppose that $m_{R'}(\{x_1\}) \le 4$ and $B = M_{R'}(\{x_1\})$. By Lemma 5.1, either $|B| \ge n - 3$ or $|B| \le 3$. The former cannot hold, since $|V(T)| \ge 10$, so $|B| \le 3$. Then again by Lemma 5.1, $z \in B$ and $G[B]$ is connected. This contradicts Claim 7.5(a). So Case $j = 1$ of (12) is proved. Suppose now that $m_{R'}(\{x_1, x_2, x_3\}) \le 10$ and $B_1 = M_{R'}(\{x_1, x_2, x_3\})$. Let $B' = B_1 \cup \{y_1, y_2, y_3, v_2, u_2, v_3\}$. Then

$$\rho_R(B') \le \rho_R(B_1) + 11(6) - 9(8) = 10 + 66 - 72 = 4. \tag{13}$$

As above, by Lemma 5.1, since $|B'| > 4$, we have $|B'| \ge n - 3$. Since $t \ge 5$, $u_4, u_3, v_4, y_4, y_5 \notin B'$, implying $|B'| < n - 3$, a contradiction. This proves Case $j = 3$ of (12). Case $j = 2$ is very similar but simpler, so we leave it for the reader.

Let G'' be obtained from G' by adding a vertex w and edges x_1x_2, x_3w, and wz. Let $R'' = (G'', Z)$. By Claim 7.5 and the choice of x_2 and x_1, G'' has no triangles. If R'' has a (0, 1)-coloring ϕ, then $\phi(x_3) = 1$ and at least one of x_1, x_2 also has color 1. So by Lemma 3.8, ϕ can be extended to a (0, 1)-coloring of R. Thus by the minimality of G, there is a set $B \subset V(G'')$ with $\rho_{R''}(B) \le -5$. By the choice of G, B contains $\{x_1, x_2\}$ or $\{w, x_3\}$, or both. If B does not contain w, then it contains $\{x_1, x_2\}$ and $\rho_R(B) \le \rho_{R''}(B) + 9 \le 4$. In this case, by Lemma 5.1 either $|B| \le 3$ or $|B| \ge n - 3$. The latter cannot hold, since $|V(T)| \ge 10$, so $|B| \le 3$. Then by the last part of Lemma 5.1, $z \in B$ and $G[\{x_1, x_2, z\}]$ is connected. This contradicts Claim 7.5(a). Thus $w \in B$ and hence $\{w, x_3, z\} \subseteq B$. Let $B_1 = B - w$.

If B does not contain $\{x_1, x_2\}$, then $\rho_R(B_1) \le \rho_{R''}(B) + 7 \le 2$. In this case, by Lemma 4.1 either $B_1 = \{x_3, z\}$ and $zx_3 \in E(G)$ or $|B_1| \ge n - 3$. The former contradicts Claim 7.5(a) and the latter cannot hold, since $|V(T)| \ge 10$. Thus

$$B \supseteq \{x_1, x_2, x_3, w, z\} \quad \text{and} \quad \rho_R(B_1) \le -5 + 9 + 7 = 11. \tag{14}$$

Together with (12), this yields

$$\rho_R(B_1) = m_{R'}(\{x_1, x_2, x_3\}) = 11. \tag{15}$$

Let $M_1 = M_{R'}(\{x_1, x_2, x_3\})$. By the comment above the claim, $M_1 \subseteq B_1$. We now want to prove

$$M_1 \cap V' = \{x_1, x_2, x_3\}. \tag{16}$$

Indeed, suppose that $|V' \cap M_1| \geq 4$, say $\{x_1, \ldots, x_4\} \subseteq M_1$. Let $B' = M_1 \cup \{y_1, \ldots, y_4, v_2, u_2, v_3, u_3, v_4\}$. Then

$$\rho_R(B') \leq \rho_R(M_1) + 11(9) - 9(12) = 11 + 99 - 108 = 2. \tag{17}$$

By Lemma 4.1, since $|B'| > 4$, we have $|B'| \geq n - 3$. If $t \geq 6$, then $y_5, u_5, v_5, y_6 \not\in B'$, a contradiction. So, $t = 5$ and $y_5 \not\in B'$. If $x_5 \in B'$, then $\rho_R(B' + y_5) \leq \rho(B') + 11 - 9(2) \leq -5$, a contradiction. Now, let $t = 5$ and $x_5 \not\in B'$. Recall that $d(x_5) \geq 3$. If $V(G) - B' = \{x_5, y_5\}$, then by (17),

$$\rho_R(V(G)) \leq \rho_R(B') + 11(2) - 9(4) \leq 2 + 22 - 36 = -12,$$

a contradiction to the choice of G. Similarly, if $V(G) - B' = \{x_5, y_5, u'\}$ for some u', then since G has no triangles, at least 5 edges are incident with x_5, y_5 or u'. In this case, $\rho_R(V(G)) \leq \rho_R(B') + 11(3) - 9(5) \leq 2 + 33 - 45 = -10$, a contradiction again. This proves (16).

Similarly to M_1, let $M_2 = M_{R'}(\{x_1, x_2, x_4\})$. Next, we prove

$$\rho_R(M_1 \cup M_2) \geq 14. \tag{18}$$

Indeed, if $\rho_R(M_1 \cup M_2) \leq 13$ then similarly to the preceding paragraph, we let $B'' = M_1 \cup M_2 \cup \{y_1, \ldots, y_4, v_2, u_2, v_3, u_3, v_4\}$ and observe that $\rho_R(B'') \leq \rho_R(M_1 \cup M_2) + 11(9) - 9(12) = 13 + 99 - 108 = 4$. By Lemma 5.1, since $|B''| > 4$, we have $|B''| \geq n - 3$. Now we proceed as in the end of the preceding paragraph, excluding the case $x_5 \in B'$, since $x_5 \not\in M_1 \cup M_2$ by (16). Thus (18) holds.

By (3) and (18), $\rho_R(M_1 \cap M_2) \leq \rho_R(M_1) + \rho_R(M_2) - \rho_R(M_1 \cup M_2) \leq 11 + 11 - 14 = 8$. Together with (12), this proves $m_{R'}(\{x_1, x_2\}) = 8$ and part (b) of the claim. Moreover, since $M_1 \cap M_2 \cap V' = \{x_1, x_2\}$, we conclude

$$M_{R'}(\{x_1, x_2\}) \cap V' = \{x_1, x_2\}. \tag{19}$$

Now, let $D_1 = M_{R'}(\{x_1, x_2\})$ and $D_2 = M_{R'}(\{x_1, x_3\})$. Again by (3) and (18),

$$\rho_R(D_1 \cap D_2) \leq \rho_R(D_1) + \rho_R(D_2) - \rho_R(D_1 \cup D_2) \leq 8 + 8 - m_{R'}(\{x_1, x_2, x_3\}) = 16 - 11 = 5.$$

Together with (12), this completes the proof of (a) and thus of all claims. □

Now we are ready to prove the main result of the section.

Lemma 7.7. *Each* $(1, 1, 1)$-*tree* T *in* G *has* $\|V(T)\| \leq 2$.

Proof. Assume $\|V(T)\| \geq 3$. Let $V' = N(T) - V(T) = \{x_1, \ldots, x_t\}$. Since $\|V(T)\| \geq 3$, $t \geq 5$. Let $G' = G - V(T)$ and $R' = (G', Z)$. For brevity, we will use M_i for $M_{R'}(\{x_i\})$, $M_{i,j}$ for $M_{R'}(\{x_i, x_j\})$, and $M_{i,j,k}$ for $M_{R'}(\{x_i, x_j, x_k\})$. First, we prove

$$\text{for any distinct } i, j, k, \quad M_{i,j} \cup M_{j,k} = M_{i,j,k}. \tag{20}$$

Indeed, by Claim 7.6(b), $M_{i,j} \cup M_{j,k} \subseteq M_{i,j,k}$. If the containment is strict, then by the minimality of $M_{i,j,k}$, $\rho_{R'}(M_{i,j} \cup M_{j,k}) \geq 12$. Then by Claim 7.6(a) and the submodularity of $\rho_{R'}$,

$$8 + 8 = \rho_{R'}(M_{i,j}) + \rho_{R'}(M_{j,k}) \geq \rho_{R'}(M_{i,j} \cup M(j, k)) + \rho_{R'}(M_{i,j} \cap M_{j,k}) \geq 12 + m_{R'}(\{x_j\}) = 17.$$

This contradiction proves (20).

For brevity, denote $F_i(j, k) = M_{i,j} \cap M_{i,k}$ and $A(i, j, k) = M_{i,j} \cap M_{i,k} \cap M_{j,k}$. By (20) and (3), $\rho_{R'}(F_i(j, k)) \leq \rho_{R'}(M_{i,j}) + \rho_{R'}(M_{i,k}) - \rho_{R'}(M_{i,j,k}) = 8 + 8 - 11 = 5$. Together with Claim 7.6(a), this gives $\rho_{R'}(F_i(j, k)) = 5$. Similarly, $\rho_{R'}(F_j(i, k)) = \rho_{R'}(F_k(j, i)) = 5$. By definition and (20),

$$F_i(j, k) \cup F_j(i, k) = M_{i,j} \cap (M_{i,k} \cup M_{j,k}) = M_{i,j} \cap M_{i,j,k} = M_{i,j}.$$

Since $A(i,j,k) = F_i(j,k) \cap F_j(i,k)$, by (3),

$$\begin{aligned}\rho_{R'}(A(i,j,k)) &= \rho_{R'}(F_i(j,k)) + \rho_{R'}(F_j(i,k)) - \rho_{R'}(M_{i,j}) \\ &\quad - 9|E_{G'}(F_i(j,k) - F_j(i,k)), F_j(i,k) - F_i(j,k)| \\ &= (5+5-8) - 9|E_{G'}(F_i(j,k) - F_j(i,k)), F_j(i,k) - F_i(j,k)|.\end{aligned}$$

Since $\rho_{R'}(A(i,j,k)) \geq -4$, we conclude that

$$|E_{G'}(F_i(j,k) - F_j(i,k)), F_j(i,k) - F_i(j,k)| = 0 \tag{21}$$

and $\rho_{R'}(A(i,j,k)) = 2$. Then by Lemma 4.1, there is $w = w_{i,j,k} \in V(G)$ such that $A(i,j,k) = \{w, z\}$ and $wz \in E(G)$. We now claim that

$$\text{all } w_{i,j,k} \text{ coincide.} \tag{22}$$

Indeed, if (22) fails, then there are distinct i, j, k, m such that $w' = w_{i,j,m} \neq w = w_{i,j,k}$. By (21), $A(i,j,k)$ and $A(i,j,m)$ are separating sets in $G[M_{i,j}]$. It follows that w and w' are cut vertices in $G[M_{i,j}] - z$, and each of them separates x_j from x_i. By the symmetry between $F_i(j,k)$ and $F_j(i,k)$, we may assume that $w' \in F_j(i,k)$. Then $w \in F_i(j,m)$ and $F_j(i,k) \cup F_i(j,m) = M_{i,j}$. So, by (3),

$$\rho_{R'}(F_j(i,k) \cap F_i(j,m)) \leq \rho_{R'}(F_j(i,k)) + \rho_{R'}(F_i(j,m)) - \rho_{R'}(M_{i,j}) = 5+5-8 = 2.$$

Since $w, w' \in F_j(i,k) \cap F_i(j,m)$, this contradicts Lemma 4.1. Thus (22) holds. So we denote by w_0 the vertex that is $w_{i,j,k}$ for all i, j, k.

Let $F_1 = F_1(2,3)$, $F_2 = F_2(1,3)$, and for $j = 3, \ldots, t$, let $F_j = F_j(1,2)$. Let $F_0 = \bigcup_{j=1}^{t} F_j$. By (22), $F_i \cap F_j = \{w_0, z\}$ for all $1 \leq i < j \leq t$. By (21), $E_G(F_i - F_j, F_j - F_i) = \emptyset$ for all $1 \leq i < j \leq t$. Then by induction on j we have $\rho_R(\bigcup_{i=1}^{j} F_i) = 3j + 2$ and thus $\rho_R(F_0) = 3t + 2$. Now we claim

$$F_0 = V(G) - V(T). \tag{23}$$

Indeed, by (10)

$$\begin{aligned}\rho_R(F_0 \cup V(T)) &= \rho_R(F_0) + 11(3(t-2)+1) - 9(3(t-2)+t) \\ &= (3t+2) + (33t-55) - (36t-54) = 1.\end{aligned}$$

Let $W = V(G) - V(T) - F_0$. Since $\rho_R(F_0 \cup V(T)) = 1$, by Lemma 4.1, $|W| \leq 3$. Recall that $z \in F_0$. If $W = \{v\}$, then by Claim 3.2, $d_G(v) \geq 2$, and so $\rho_R(V(G)) \leq \rho_R(F_0 \cup V(T)) + 11 - 9(2) = -6$, a contradiction. If $W = \{v_1, v_2\}$ and at least four edges are incident with v_1 or v_2, then $\rho_R(V(G)) \leq \rho_R(F_0 \cup V(T)) + 11(2) - 9(4) = -13$. If $W = \{v_1, v_2\}$ and at most 3 edges are incident with v_1 or v_2, then $d_G(v_1) = d_G(v_2) = 2$ and $v_1v_2 \in E(G)$. Then by Claim 6.1, we may assume that $zv_1 \in E(G)$. Let y be the neighbor of v_2 distinct from v_1. Since $v_2 \notin V'$, $y \in F_0$, and thus there is $j \geq 1$ such that $y \in F_j$. But then $\rho_{R'}(F_j \cup W) = \rho_{R'}(F_j) + 11(2) - 9(3) = 0$, a contradiction. Suppose now that $W = \{v_1, v_2, v_3\}$. If at least 5 edges are incident with W, then $\rho_R(V(G)) \leq \rho_R(F_0 \cup V(T)) + 11(3) - 9(5) = -11$. Since G has no triangles, to have only 4 edges incident with 3 vertices of degree at least two, we need $|E(G[W])| = 2$ and each of them have degree exactly 2. This contradicts Claim 6.2. Thus, $W = \emptyset$, as claimed.

For $j = 1, \ldots, t$, let H_j be obtained from $G[F_j]$ by adding a new vertex u_j and edges u_jz and x_ju_j. Let $R_j = (H_j, Z)$. Since $m_{R'}(\{x_j\}) = \rho_{R'}(F_j) = 5$, the potential of every subset of $V(H_j)$ containing both, x_j and u_j, is at least $5 + 11 - 9(2) = -2$. The potential of any set not containing u_i is at least -4 by the choice of G. If a set $U \subset V(H_j)$ contains u_j but does not contain x_j, then $\rho_{R_j}(U) > \rho_{R_j}(U - u_j) \geq -4$ again. So by the minimality of G. Each of R_j has a $(0,1)$-coloring ϕ_j. By construction, $\phi_j(x_j) = \phi_j(z) = 1$ and $\phi_j(u_j) = 0$. Since $F_j - \{w, z\}$ has no neighbors in $F_i - \{w, z\}$ for distinct i and j, $\phi = \bigcup_{j=1}^{t} \phi_j$ is a $(0,1)$-coloring of $G - V(T)$. Furthermore, $\phi(x_j) = 1$ for every j. Then by Lemma 3.8, ϕ can be extended to a $(0,1)$-coloring of G, a contradiction. □

8. Discharging

Consider the following vertex subsets of G:

$$V_0 = \bigcup_{T \text{ is a } (1,1,1)\text{-tree in } G} V(T),$$

$$V_1 = \bigcup_{S \text{ is a shovel}} V(S) - z,$$

$$V_2 = \{v \in V(G) - V_0 - V_1 : d_G(v) = 2\},$$

$$V_3 = V(G) - V_0 - V_1 - V_2.$$

By definition, $V(G) = V_0 \cup V_1 \cup V_2 \cup V_3$. For $i = 0, 1, 2, 3$, let $G_i = G[V_i]$.

We use discharging as follows. An *item* is a vertex in $V_2 \cup V_3$ or a shovel S or a component T of G_0. Initially, every vertex or edge x has the charge $ch(x)$ equal to its ρ-value. At the end of discharging, every item U will have a charge $ch^*(U)$ so that

$$\text{the sum of } ch^*(U) \text{ over all items is equal to } \sum_{x \in V(G) \cup E(G)} \rho(x). \tag{24}$$

We use the following rules.

Rule 1: If an edge e not in a shovel is incident to exactly one 2-vertex distinct from z, then e receives 5.75 from this 2-vertex and 3.25 from the other vertex. (Recall that by Claims 6.1 and 6.2, two adjacent 2-vertices can be only in a shovel.)

Rule 2: If an edge e not in a shovel is incident neither to z nor to any 2-vertex, then e receives 4.5 from each of the ends.

Rule 3: If an edge e not in a shovel is incident to z, then it gets 3.25 from z and 5.75 from the other end.

Rule 4: Every edge in a shovel gets 9 from the shovel.

Rule 5: For every $(1, 1, 1)$-tree T, let $ch^*(T)$ be the sum of the resulting charges of its vertices.

Rule 6: For every shovel S, let $ch^*(S)$ be the sum of the resulting charges of its vertices.

Claim 8.1. *After the discharging by Rules* 1–6 *above, we have*

(a) $ch^*(e) = 0$ *for every* $e \in E(G)$*;*
(b) $ch^*(v) \le 0$ *for every* $v \in V_2 \cup V_3$*; moreover, if* $v \in V_3$ *and* $d_G(v) \ge 4$ *or* $d_{G_3}(v) \ge 2$*, then* $ch^*(v) \le -1.25$*;*
(c) $ch^*(v) \le -0.5$ *for* $d_G(v) = 2$*,* v *not in a shovel.*
(d) $ch^*(T) \le 0$ *for every* $(1, 1, 1)$*-tree* T*; moreover, if* T *contains only one* $(1, 1, 1)$*-vertex, then* $ch^*(T) \le -1/4$*;*
(e) $ch^*(S) \le -3.75$ *for every broken shovel* S*;*
(f) $ch^*(S) \le -4.75$ *for every intact shovel* S

Proof. Part (a) immediately follows from Rules 1–4.

Let $v \in V_3$. If $v = z$ and is adjacent to exactly x vertices not in shovels, then by Rule 3 it gives out charge $3.25x$. Thus (b) holds for $v = z$. Suppose $v \in V_3 - z$. Then $d(v) \ge 3$. Moreover, if $d(v) = 3$ the v has a neighbor w that either is z or has degree at least 3. If $d(v) \ge 4$, then it gives away by Rule 1 at least $3.25d(v)$, so $ch^*(v) \le 11 - 3.25d(v) \le -1.25$. If $d(v) = 3$, then by Rules 1–3, $ch^*(v) \le \max\{11 - 4.5 - 2(3.25), 11 - 2(4.5) - (3.25)\} = 0$. Moreover, if $d_{G_3}(v) \ge 2$, then $ch^*(v) \le 11 - 2(4.5) - 3.25 = -1.25$. This proves (b).

If $d_G(v) = 2$, and v is not in a shovel, then by Rule 1, it gives 5.75 to two incident edges. Thus $ch^*(v) = 11 - 2(5.75) = -0.5$. This proves (c).

Suppose a $(1, 1, 1)$-tree T contains k $(1, 1, 1)$-vertices for $k = 1$ or 2. Then T has exactly $2k + 1$ vertices of degree two. Each 2-vertex in T has charge -0.5 and each $(1, 1, 1)$-vertex has charge $11 - 3(3.25) = 1.25$. So, $ch^*(T) = 1.25k - 0.5(2k + 1) \le 0$. Moreover, if $k = 1$, then $ch^*(T) \le 0.5 - 3(0.25) \le -0.25$. This proves (d).

A broken shovel S contains three typical vertices and three edges. By Rules 1 and 2, the joint w sends out at least $(d(w) - 1)(3.25)$. So, by Rule 6, $ch^*(S) \leq 3(11) - 3(9) - (d(w) - 1)3.25 \leq 6 - 3.25(3) = -3.75$. This proves (e).

An intact shovel S contains 7 typical vertices and 8 edges. Also by Claim 6.5, S has at least three neighbors outside of it. So its total charge is at most $7(11) - 8(9) - 3(3.25) = -4.75$. □

Since the discharging preserves the total charge, it must be at least -4. Thus by Claim 8.1, if some set of items will have the total new charge less than -4, then we get a contradiction.

We will use the following immediate corollary of Lemma 3.8.

Claim 8.2. *If $G - V_0$ has a $(0, 1)$-coloring ϕ such that at most one vertex in $V_3 \cap N_G(V_0)$ is colored with 0, then ϕ can be extended to V_0. Moreover, if every component of G_0 has at least two $(1, 1, 1)$-vertex, and at most two vertices in $V_3 \cap N_G(V_0)$ have color 0, then ϕ also can be extended to V_0.*

Claim 8.3. *G has no shovels.*

Proof. Suppose S is a shovel in G.

Case 1: S is a broken shovel with the handle (v_0, v_1, v_2, z) and $d(v_0) \geq 4$. If $d(v_0) \geq 5$, then $ch^*(S) \leq 3(11) - 3(9) - 4(3.25) < -4$, a contradiction. If $d(v_0) = 4$ and v_0 has a neighbor of degree at last 3, then by Rules 6, 1 and 2, $ch^*(S) \leq 3(11) - 3(9) - 4.5 - 2(3.25) = -5 < -4$, a contradiction. Assume $d(v_0) = 4$ and all neighbors of v_0 are 2-vertices. Then $ch^*(S) \leq 6 - 3(3.25) = -3.75$.

If G_3 contains a P_3, then by Claim 8.1(b), the middle vertex w of this P_3 satisfies $ch^*(w) \leq -1.25$. So, $ch^*(S) + ch^*(w) < -4$, a contradiction. Thus G_3 does not contain P_3. Also by Claim 8.1(d), G has no shovels apart from S. We color all vertices in V_3 with color 1, and all vertices in V_2 with color 0, and color S so that v_0 is colored by 1. Then by Claim 8.2 we can extend the coloring to V_0, a contradiction.

Case 2: S is intact. Immediately follows from Claim 8.1(f). □

Claim 8.4. *The special vertex z is isolated in G.*

Proof. By Claim 8.3, $V_1 = \emptyset$. Thus by Rules 3 and 5, if $d(z) \geq 2$, then $ch^*(z) \leq -3.25 \cdot 2 = -6.5$, a contradiction.

Let $d(z) = 1$ and u be the unique neighbor of z. If $d(u) \geq 3$, then by Rules 3, $ch^*(z) = -3.25$ and $ch^*(u) \leq 11 - 5.75 - 2 \cdot 3.25 = -1.25$, a contradiction. So, let $d(u) = 2$. Then by Rule 3, $ch^*(z) \leq -3.25$. Hence $\Delta(G_3) \leq 1$, since each $w \in V_3$ of degree at least 2 in G_3 has $ch^*(w) \leq -1.25$. Also, z forms a component in G_3. Then we color V_3 with 1 and V_2 with 0. By Claim 8.2 the obtained coloring extends to V_0. □

Since z is isolated, from now on we consider coloring of $G - z$.

9. Structure of G_3

For $W \subseteq V_2 \cup V_3$, a coloring ϕ' is *standard on W* if $\phi'(v) = 1$ for $v \in V_3 \cap W$, and $\phi'(v) = 0$ for $v \in V_2 \cap W$.

Claim 9.1. *G_3 has no cycles, and no P_k with $k \geq 5$.*

Proof. Suppose G_3 contains C_k. Since G is triangle-free, $k \geq 4$. By (b) of Claim 8.1, $ch^*(C_k) \leq -1.25k < -4$, a contradiction. So G_3 is acyclic.

Suppose now that G_3 contains P_k with $k \geq 5$. Then by Claim 8.1(b), $k \leq 5$.

In order to have $ch^*(P_5) \geq -4$, every vertex in the P_5 has degree three and is adjacent to a vertex of degree two.

Also, $\Delta(G_3 - P_5) \leq 1$. Otherwise, $ch^*(P_5) + ch^*(P_3) \leq -1.25 \cdot 3 - 1.25 < -4$, a contradiction.

Let $P_5 = (v_1, \ldots, v_5)$ and x'_3 be the neighbor of v_3 outside of P_3.

If $x'_3 \in V_1$, then $ch^*(x'_3) = -0.5$ and $ch^*(P_5) + ch^*(x'_3) \leq -1.25 * 3 - 0.5 < -4$, a contradiction. So $x'_3 \notin V_1$, then we color all vertices in $G_3 - P_5$ with color 1, all vertices in V_1 with 0 and the vertices in P_5 as follows: 1, 1, 0, 1, 1. By Claim 8.2 this coloring extends to V_0. □

Claim 9.2. *G_3 does not contain $K_{1,3}$.*

Proof. Assume V_3 contains a subset X with $G[X] = K_{1,3}$ with center x.

If $d(x) \geq 4$, then $ch^*(x) \leq 11 - 3 \cdot 4.5 - 3.25 = -5.75$, a contradiction. So, $d(x) = 3$ and $ch^*(x) = -2.5$. If $\Delta(G_3 - X) \leq 1$, then we use the standard coloring on $V_2 \cup V_3 - x$ and color x with 0. By Claim 8.2 this coloring extends to V_0. So we may assume that for some $y \in V_3 - X$, $d_{G_3}(y) \geq 2$. Then $ch^*(y) \leq -1.25$. Thus in order the total charge to be at least -4, we need that every vertex in $V_3 - X - y$ has degree at most 1 in G_3, and $V_2 = \emptyset$. Then we use the standard coloring on $V_2 \cup V_3 - x - y$ and color x and y with 0. Since x has no neighbors in V_0, by Lemma 3.8, the coloring extends to V_0. □

So each component of G_3 is a K_1, K_2, P_3, or P_4. Note that if a vertex x is a component in G_3, then it is not a $(1, 1, 1)$-vertex. So in this case $d(x) \geq 4$, and $ch^*(x) \leq -2$.

Claim 9.3. *G_3 does not contain P_4.*

Proof. Suppose G_3 contains $P_4 = (v_1, v_2, v_3, v_4)$. Let $W = \{v_1, v_2, v_3, v_4\}$. Then $ch^*(v_2) + ch^*(v_3) \leq -1.25 + (-1.25) = -2.5$. Moreover, if $d(v_2) \geq 4$ or $d(v_3) \geq 4$, then $ch^*(v_2) + ch^*(v_3) \leq -1.25 + (-4.5) < -4$, a contradiction. So, $d(v_2) = d(v_3) = 3$.

If G_3 contains another 4-path $P_4' = (y_1, y_2, y_3, y_4)$ disjoint from W, we get the potential less than $2 * (-2.5) < -4$, a contradiction. Thus this P_4 is the only P_4 in G_3.

If $G_3 - W$ contains two disjoint 3-paths $P_3 = (y_1, y_2, y_3)$ and $P_3' = (u_1, u_2, u_3)$, then $ch^*(P_4) + ch^*(y_2) + ch^*(u_2) \leq -2.5 - 1.25 - 1.25 < -4$, a contradiction. Thus $G_3 - W$ contains at most one P_3.

Case 1: $G_3 - W$ contains exactly one 3-path $P_3 = (y_1, y_2, y_3)$.

If V_2 is not empty, it contains v'. Then $ch^*(\{v_2, v_3, y_2, v'\}) \leq -1.25 - 1.25 - 1.25 - 0.5 = -4.25 < -4$, a contradiction. So V_2 is empty.

Then we use the standard coloring on $V_2 \cup V_3 - v_2 - y_2$ and color v_2 and y_2 with 0. By the choice of v_2 and the case for y_2, this coloring is a $(0, 1)$-coloring for $G[V_2 \cup V_3]$.

If it extends to V_0, then we are done. Suppose not. Then some $(1, 1, 1)$-tree T of size 1 is adjacent to v_2 and y_2. Then $ch^*(\{v_2, v_3, y_2\}) + ch^*(T) \leq -3.75 - 0.25 = -4$. If there is another $(1, 1, 1)$-tree T' of size 1, then $ch^*(\{v_2, v_3, y_2\}) + ch^*(T) + ch^*(T') \leq -3.75 - 0.25 - 0.25 < -4$, a contradiction. Thus T is the only $(1, 1, 1)$-tree of size 1. Since T is adjacent to exactly 3 vertices, we may assume that y_3 is not adjacent to T. We recolor y_2 with 1 and y_3 with 0, and can extend the new coloring to V_0.

Case 2: $\Delta(G_3 - W) \leq 1$.

If some $v \in \{v_2, v_3\}$ has no neighbors in V_2, then we use the standard coloring on $V_2 \cup V_3 - v$ and color v with 0. By Claim 8.2 this coloring extends to V_0. So we may assume that for $j \in \{2, 3\}$, v_j has a neighbor $v_j' \in V_2$. Since G is triangle-free, $v_3' \neq v_2'$. Therefore, $ch^*(\{v_2, v_3, v_2', v_3'\}) \leq -1.25 - 1.25 - 0.5 - 0.5 = -3.5$. So, all vertices in $V_3 - v_2 - v_3$ are $(1, 1, 0)$-vertices and $|V_2| \leq 3$.

Case 2.1: $|V_2| = 3$, i.e. there is $w \in V_1 - v_2' - v_3'$.

Then $ch^*(w) = -0.5$ and there are no $(1, 1, 1)$-trees of size 1. For $j = 2, 3$, let $N(v_j') = \{v_j, v_j''\}$.

Suppose first that $v_3'' = v_2''$. Then we color $V_3 + v_2' + v_3' - v_2'' - v_2 - v_4$ with 1 and v_2, v_2'', v_4 and w with 0. Since v_2 has no neighbors in V_0, if this coloring is a $(0, 1)$-coloring of $G[V_2 \cup V_3]$, then it extends to V_0. So, this is not the case. The only possibility for it is that $wv_4 \in E(G)$. By the symmetric argument, $wv_1 \in E(G)$. Then we use the standard coloring on $V_2 \cup V_3 - v_1 - v_4 - w$ and color v_1 and v_4 with 0 and w with 1. The obtained coloring is a $(0, 1)$-coloring of $G[V_2 \cup V_3]$ and extends to V_0.

Suppose now that $v_3'' \neq v_2''$. We use the standard coloring on $V_2 \cup V_3 - v_2 - v_2' - v_2''$ and color v_2 and v_2'' with 0 and v_2' with 1. The only possibility that this is not a $(0, 1)$-coloring of $G[V_2 \cup V_3]$ is that $wv_2'' \in E(G)$. By the symmetric argument, $wv_3'' \in E(G)$. If $v_2'' \neq v_4$ and $v_3'' \neq v_1$, then we use the standard coloring on $V_2 \cup V_3 - v_1 - v_4$ and color v_1 and v_4 with 0. Let $v_2'' = v_4$. If $v_3'' \neq v_1$, then we use the standard coloring on $V_2 \cup V_3 - v_3 - v_3' - v_3'' - w$ and color v_3 and v_3'' with 0 and v_3' and w with 1. In both cases, the coloring extends to V_0. Finally, if also $v_3'' = v_1$, then $V(G) = W \cup \{v_2', v_3', w\}$. In this case, we color v_1, v_3 and v_2' with 0 and the rest with 1.

Case 2.2: $V_2 = \{v_2', v_3'\}$.

If $v_2'' = v_4$, then we use the standard coloring on $V_2 \cup V_3 - v_2 - v_2' - v_2''$ and color v_2 and v_2'' with 0 and v_2' with 1. Since G is triangle-free, $v_4 \neq v_3''$ and so this is a $(0, 1)$-coloring of $G[V_2 \cup V_3]$. Since v_2 has no neighbors in V_0, the coloring extends to V_0. Thus $v_2'' \neq v_4$.

If $v_3'' = v_2''$, then we use the standard coloring on $V_2 \cup V_3 - v_2 - v_2' - v_2''$ and color v_2 and w_2'' with 0 and v_2' with 1. If $v_3'' \neq v_2''$, then let w be the neighbor of v_2'' in G_3, use the standard coloring on $V_2 \cup V_3 - v_2 - v_2' - w$ and color v_2 and w with 0 and v_2' with 1. In both cases, this is a (0, 1)-coloring of $G[V_2 \cup V_3]$. Since v_2 has no neighbors in V_0, the coloring extends to V_0. □

10. Proof of Theorem 2.1

By the claims in the previous section, each component of G_3 is isomorphic to K_1, K_2 or P_3. If $\Delta(G_3) \leq 1$, then the standard coloring of $V_2 \cup V_3$ is a (0, 1)-coloring and extends to V_0. So, let (x_1, x_2, x_3) be a path in G_3 and $X = \{x_1, x_2, x_3\}$.

Case 1: $\Delta(G_3 - X) \leq 1$.

We use the standard coloring on $V_2 \cup V_3 - x_2$ and color x_2 with 0. If this is a (0, 1)-coloring of $G[V_2 \cup V_3]$, then by Claim 8.2 it extends to V_0. So, it is not. Then x_2 has a neighbor $w_1 \in V_2$. Let y_1 be the other neighbor of w_1. Since $ch^*(x_2) + ch^*(w_1) = -1.75$, every vertex $u \in V_3 - x_2$ is either a (1, 1, 0)-vertex as $d_{G_3}(u) \leq 1$ or a (1, 1, 1, 1)-vertex (otherwise $ch^*(u) \leq -3.25$ and so $ch^*(\{x_2, w_1, u\}) \leq -1.25 - 0.5 - 3.25 < -4$).

We recolor w_1 with 1. If y_1 is a (1, 1, 1, 1)-vertex, this yields a (0, 1)-coloring of $G[V_2 \cup V_3]$ which extends to V_0. So, let y_1 be a (1, 1, 0)-vertex with neighbors $w_1, w_2 \in V_2 \cup V_0$ and $y_1' \in V_3$. We now recolor y_1 with 0. If $w_2 \in V_0$, then this yields a (0, 1)-coloring of $G[V_2 \cup V_3]$. So suppose $w_2 \in V_2$ and the other neighbor of w_2 is y_2. Recolor w_2 with 1. If $y_2 \in \{x_1, x_3\}$ or is a (1, 1, 1, 1)-vertex, then this again yields a (0, 1)-coloring of $G[V_2 \cup V_3]$. So let y_2 be a (1, 1, 0)-vertex with neighbors $w_2, w_3 \in V_2 \cup V_0$ and $y_2' \in V_3$. We now recolor y_2 with 0. Again, if $w_3 \notin V_2$, then we are done; otherwise we again consider the other neighbor y_3 of w_3 and so on. Since $ch^*(w_i) = -0.5$ for $i \geq 2$, in order to have $ch^*(V_2 \cup V_3) \geq 4$, the recoloring will stop after at most 4 steps.

Case 2: There are exactly two 3-paths in G_3, (x_1, x_2, x_3) and (y_1, y_2, y_3).

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Similarly to Case 1, every vertex in $V_3 - x_2 - y_2$ is either a (1, 1, 0)-vertex or a (1, 1, 1, 1)-vertex. Also $d(x_2) = d(y_2) = 3$, since otherwise $ch^*(x_2) + ch^*(y_2) \leq -1.25 - 4.5 < -4$. We use the standard coloring on $V_2 \cup V_3 - x_2 - y_2$ and color x_2 and y_2 with 0.

If this is a (0, 1)-coloring of $G[V_2 \cup V_3]$ and extends to V_0, then we are done. Thus one of the two subcases below holds.

Case 2.1: Our coloring is not a (0, 1)-coloring of $G[V_2 \cup V_3]$.

Case 2.1.1: Both x_2 and y_2 have neighbors in V_2.

Let $x_2w_1, y_2u_1 \in E(G)$, $N(w_1) = \{x_2, q_1\}$ and $N(u_1) = \{y_2, v_1\}$. Recolor w_1 and u_1 with 1. If $q_1 \in Y$ and $v_1 \in X$ (in particular, if $w_1 = u_1$), then this gives a (0, 1)-coloring of $G[V_2 \cup V_3]$. Since x_2 has no neighbors in V_0, it extends to V_0, a contradiction. So by symmetry we may assume that $q_1 \notin Y$. Then $ch^*(\{x_2, y_2, w_1, u_1\}) \leq -1.25 - 1.25 - 0.5 - 0.5 = -3.5$. It follows that G has no (1, 1, 1, 1)-vertices and $|V_2| \leq 3$. So, q_1 is a (1, 1, 0)-vertex. Let $N(q_1) = \{w_1, q_1', w_2\}$ where $q_1' \in V_3$.

Suppose first that $u_1 \in \{x_1, x_3, q_1, q_1'\}$. Choose $q \in \{q_1, q_1'\}$ so that $q = u_1$ if $w_1 \in \{q_1, q_1'\}$, and otherwise q is not adjacent to $V_2 - w_1$, if possible. Recolor q with 0. If the new coloring is a (0, 1)-coloring of $G[V_2 \cup V_3]$, then it extends to V_0, since x_2 and y_2 have no neighbors in V_0. So it is not. If $q \neq w_1$, then by the choice of q, each of q_1, q_1' has a neighbor (say, r_1 and r_1') in $V_2 - w_1 - u_1$. Since G has no triangles, $r_1 \neq r_1'$, and so $ch^*(\{x_2, y_2, w_1, u_1, r_1, r_1'\}) \leq -1.25 - 1.25 - 0.5 - 0.5 - 0.5 - 0.5 < -4$. Thus $q = w_1$. Since the coloring is not a (0, 1)-coloring of $G[V_2 \cup V_3]$, $q = q_1'$ and q_1' has another neighbor $w_2 \in V_2$. Again, $w_2, q_1 \notin E(G)$, because G is triangle-free. Since $|V_2| \leq 3$, we know that $V_2 = \{w_1, u_1, w_2\}$ and each (1, 1, 1)-tree has size at least 2. So, after recoloring q_1' back with 1 and coloring q_1 with 0 we obtain a (0, 1)-coloring of $G[V_2 \cup V_3]$, which extends to V_0.

We may assume now that $u_1 \notin \{x_1, x_3, q_1, q_1'\}$. Let u_1' be the neighbor of u_1 in V_3. Since $|V_2| \leq 3$, we may assume that either $V_2 = \{w_1, u_1\}$ or $V_2 = \{w_1, u_1, r\}$. Then we can choose some $u \in \{u_1, u_1'\}$ and $q \in \{q_1, q_1'\}$ so that (a) either both or neither of u and q are adjacent to r (if exists), and (b) there is no (1, 1, 1)-tree of size 1 adjacent to both, q and u. Recoloring q and u with 0 and r, if it is adjacent to both q and u, with 1 yields a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0, a contradiction.

By the symmetry between x_2 and y_2, the remaining subcases are the following.

Case 2.1.2: Only x_2 has a neighbor $w_1 \in V_2$, and y_2 is adjacent to a (1, 1, 1)-tree T of size 1.

Let $N(w_1) = \{x_2, q_1\}$. Since $ch^*(\{x_2, y_2, w_1\}) + ch^*(T) \leq -1.25 - 1.25 - 0.5 - 0.25 = -3.25$, as in Case 1, every vertex $u \in V_3 - x_2 - y_2$ is a (1, 1, 0)-vertex. Moreover, $|V_2| \leq 2$, and we may assume that $V_2 = \{w_1\}$ or $V_2 = \{w_1, w_2\}$. Recolor w_1 with 1. If q_1 belongs to Y this gives a (0, 1)-coloring of $G[V_2 \cup V_3]$. Since x_2 has no neighbors in V_0, it extends to V_0. So we may assume that $q_1 \notin Y$. Let q'_1 be the neighbor of q_1 in V_3. If there is $q \in \{q_1, q'_1\}$ such that q has neighbors neither in $V_2 - w_1$ nor in T, then recoloring q with 0 yields a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0. So, each of q_1 and q'_1 has a neighbor in $V_2 - w_1$ or in T. Similarly, if there is $y \in \{y_1, y_3\}$ not adjacent to V_2 and not adjacent to T, then we recolor y_2 with 1 and q_1 and y with 0. Since we know the neighbors of x_2 and q_1, if the obtained coloring is a (0, 1)-coloring of $G[V_2 \cup V_3]$, then it extends to V_0. So, each of y_1 and y_3 has a neighbor V_2 or in T. If some $x \in \{x_1, x_3\}$ is adjacent neither to T nor to V_2, then we can use the standard coloring on $V_2 \cup V_3 - x - y_2$ and color x and y_2 with 0. This will be a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0. Thus every vertex in $Q = \{x_1, x_3, y_1, y_3, q_1, q'_1\}$ has a neighbor in T or in $V_2 - w_1$. It is impossible, since $|V_2| \leq 2$ and T has at most two neighbors in Q.

Case 2.1.3: Only x_2 has a neighbor $w_1 \in V_2$, and y_2 is not adjacent to any (1, 1, 1)-tree of size 1.

Let $N(w_1) = \{x_2, q_1\}$. Since $ch^*(\{x_2, y_2, w_1\}) \leq -1.25 - 1.25 - 0.5 = -3$, $|V_2| \leq 3$. Recolor w_1 with 1. If q_1 belongs to Y or is a (1, 1, 1, 1)-vertex, this gives a (0, 1)-coloring of $G[V_2 \cup V_3]$. Since x_2 has no neighbors in V_0, it extends to V_0. So we may assume that $q_1 \notin Y$ and is a (1, 1, 0)-vertex. Let q'_1 be the neighbor of q_1 in V_3. If there is $q \in \{q_1, q'_1\}$ such that q does not have neighbors in $V_2 - w_1$, then recoloring q with 0 yields a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0. So, each of q_1 and q'_1 has a neighbor in $V_2 - w_1$. If some $x \in \{x_1, x_3\}$ is not adjacent to V_2, then we can use the standard coloring on $V_2 \cup V_3 - x - y_2$ and color x and y_2 with 0. This will be a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0. Thus every vertex in $Q = \{x_1, x_3, q_1, q'_1\}$ has a neighbor in $V_2 - w_1$. It follows that $|V_2| = 3$ and each $w \in V_2 - w_1$ is adjacent only to vertices in Q. Since no vertex is adjacent to both q_1 and q'_1 (because G is triangle-free), there are $x \in \{x_1, x_3\}$ and $w \in V_2 - w_1$ such that $N(w) = \{x, q_1\}$. Then we use the standard coloring on $V_2 \cup V_3 - x_2 - y_2 - q - w_1$ and color x_2 and y_2 with 0 and w_1. Since we know the neighbors of all vertices in V_2, this will be a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0.

Case 2.2: Our coloring is a (0, 1)-coloring of $G[V_2 \cup V_3]$ but does not extend to V_0.

Then by Lemma 3.8, there exists a (1, 1, 1)-tree T of size 1 adjacent to both, x_2 and y_2, and such that the third neighbor of T in V_3, say q, has a neighbor of color 1. So, q is a (1, 1, 0)-vertex not in $X \cup Y$. By Rules 5, $ch^*(T) \leq -0.25$. So, $|V_2| \leq 2$. If some $x \in \{x_1, x_3\}$ is not adjacent to V_2, then we use the standard coloring on $V_2 \cup V_3 - x - y_2$ and color x and y_2 with 0. Since $q \notin X$, this will be a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0. So, both x_1 and x_3 are adjacent to V_2. Symmetrically, both y_1 and y_3 are adjacent to V_2. Then q is not adjacent to V_2. So we use the standard coloring on $V_2 \cup V_3 - x_2 - y_2 - q$ and color x_2, q and y_2 with 0. By the case, this is a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0.

Case 3: There are exactly three 3-paths in G_3, (x_1, x_2, x_3), (y_1, y_2, y_3), and (u_1, u_2, u_3).

Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $U = \{u_1, u_2, u_3\}$. Similarly to Case 1, every vertex in $V_3 - x_2 - y_2 - u_2$ is either a (1, 1, 0)-vertex. Also $d(x_2) = d(y_2) = d(u_2) = 3$ and $|V_2| \leq 2$.

If V_2 is not empty, $w \in V_2$, so $ch^*\{x_2, y_2, u_2, w\} \leq -1.25 - 1.25 - 1.25 - 0.5 < -4$, so $V_2 = \emptyset$.

Thus every vertex in $V_3 - x_2 - y_2 - u_2$ is a (1, 1, 0)-vertex.

We use the standard coloring on $V_2 \cup V_3 - x_2 - y_2 - u_2$ and color x_2, y_2 and u_2 with 0. Since this coloring is a (0, 1)-coloring of $G[V_2 \cup V_3]$, so it does not extend to V_0.

Thus there is a (1, 1, 1)-tree T adjacent to j vertices of $\{x_2, y_2, u_2\}$ with $j = 1$ or 2. And no (1, 1, 1)-tree of size 1 distinct from T is adjacent to $\{x_2, y_2, u_2\}$. Let w be the neighbor of T outside of $\{x_2, y_2, u_2\}$. Then recoloring w with 0 yields a (0, 1)-coloring of $G[V_2 \cup V_3]$ that extends to V_0.

Since there are at most three P_3s in G_3, this finishes the proof. □

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