European Journal of Combinatorics 46 (2015) 89–94

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The minimum number of edges in a 4-critical graph that is bipartite plus 3 edges

A.V. Kostochka [a,b], B.M. Reiniger [a]

[a] Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA

[b] Institute of Mathematics, Novosibirsk, Russia

ARTICLE INFO

Article history:
Received 6 June 2014
Accepted 9 December 2014
Available online 5 January 2015

ABSTRACT

Rödl and Tuza proved that sufficiently large $(k+1)$-critical graphs cannot be made bipartite by deleting fewer than $\binom{k}{2}$ edges, and that this is sharp. Chen, Erdős, Gyárfás, and Schelp constructed infinitely many 4-critical graphs obtained from bipartite graphs by adding a matching of size 3 (and called them $(B+3)$-graphs). They conjectured that every n-vertex $(B+3)$-graph has much more than $5n/3$ edges, presented $(B+3)$-graphs with $2n-3$ edges, and suggested that perhaps $2n$ is the asymptotically best lower bound. We prove that indeed every $(B+3)$-graph has at least $2n-3$ edges.

1. Introduction

A graph G is said to be $(k+1)$-*critical* if it is $(k+1)$-chromatic but every proper subgraph G is k-colorable.

In this paper we consider $(k+1)$-critical graphs that are “nearly bipartite” in the following sense. For an integer ℓ, we say a graph is a $B+E_\ell$ *graph* if it is obtained from a bipartite graph by adding some ℓ edges. We say that a graph is a $B+M_\ell$ *graph* if it is obtained from a bipartite graph by adding a matching of size ℓ. (In [2], a $B+M_\ell$ graph is denoted by $B+\ell$.)

Disproving a conjecture by Erdős, Rödl and Tuza [7] showed that for all sufficiently large n, there are n-vertex $(k+1)$-critical $B+E_{\binom{k}{2}}$ graphs. They also showed that this is best possible: when n is large enough, there are no n-vertex $(k+1)$-critical $B+E_\ell$ graphs with $\ell < \binom{k}{2}$.

E-mail addresses: kostochk@math.uiuc.edu (A.V. Kostochka), reinige1@illinois.edu (B.M. Reiniger).

http://dx.doi.org/10.1016/j.ejc.2014.12.002
0195-6698/

Chen, Erdős, Gyárfás, and Schelp [2], strengthened Rödl and Tuza's result for $k = 3$: they showed that for all sufficiently large n, there are 4-critical n-vertex $B + M_3$ graphs.

We focus on the case $k = 3$ and ask how few edges such a graph may have. Chen et al. [2] provided for every $n \geq 7$ such a graph with $2n-3$ edges. They "suspect[ed]" that any 4-critical n-vertex $B+M_3$ graph has at least $2n$ edges asymptotically, and "dare[d] to conjecture only that they have significantly more than $5n/3$ edges". Gyárfás renewed interest to the problem in [3]. Here we prove that indeed, every such graph has at least $2n - 3$ edges. Furthermore, we prove the same result for any 4-critical $B + E_3$ graph.

Theorem 1.1. *If G is a 4-critical $B + E_3$ graph, then $|E(G)| \geq 2|V(G)| - 3$.*

We use techniques from [5,6].

For $A \subseteq V(G)$, we let $G[A]$ denote the subgraph of G induced by A. When $A \cap B = \emptyset$, we let $G[A, B]$ denote the bipartite subgraph with parts A, B consisting of all edges from A to B.

2. Proof

For $A \subseteq V(G)$, define the *potential*, $\rho_G(A)$, to be $2|A| - |E(G[A])|$.

Theorem 1.1 is equivalent to the statement

$$\rho_G(V(G)) \leq 3 \text{ for every 4-critical } B + E_3 \text{ graph } G.$$

We will frequently use the fact that, for $A, B \subseteq V(G)$,

$$\rho_G(A \cup B) + \rho_G(A \cap B) = \rho_G(A) + \rho_G(B) - |E(G[A - B, B - A])|.$$

Lemma 2.1. *Suppose $G \neq K_4$ is a 4-critical graph such that $E(G) = E(B) \cup E(S)$ where B is bipartite and $|E(S)| = 3$. Let V_1, V_2 be the bipartition of $V(B)$ with $|V_1 \cap V(S)| \geq |V_2 \cap V(S)|$. Then either*

(1) *$G[V_1 \cap V(S)] = K_3$, and $\rho_G(V(S)) = 3$; or*
(2) *$G[V_1 \cap V(S)] = P_3$, $G[V_2 \cap V(S)] = K_2$, and $\rho_G(V(S)) \leq 3$; or*
(3) *$G[V_1 \cap V(S)] = 2K_2$, $G[V_2 \cap V(S)] = K_2$, and $\rho_G(V(S)) \leq 5$.*

Proof. Observation: If there is an independent set I that intersects each edge of S, then $G-I$ is bipartite and so G is 3-colorable.

Chen et al. [2] showed that the only $B+E_2$ graph that is 4-critical is K_4. So each edge of S lies within one of V_1, V_2. If all three edges of S are in V_1, then $G[V(S)] = K_3$ by the observation. Otherwise two edges of S, say ab and cd, lie in V_1 and one edge, say xy, lies in V_2. Then $G[V_1 \cap V(S)]$ is either $2K_2$ (a, b, c, d are distinct) or P_3 (say $b = c$). By the observation, x must be adjacent to both of a, b or both of c, d; by symmetry assume $xa, xb \in E(G)$. Similarly, y must be adjacent to both a, b or to both c, d; since G does not contain a K_4, we have $yc, yd \in E(G)$. These four edges together with $E(S)$ imply the given inequalities on potential. □

For a graph G as in the hypothesis of Lemma 2.1, let

$$P(G) = \min_{V(S) \subseteq A \subseteq V(G)} \rho_G(A).$$

Suppose the theorem fails. Among all counterexamples, choose G to have the maximum $P(G)$, and subject to this, to have the minimum number of vertices. (Note that for any G, $P(G) \leq \rho_G(V(S)) \leq 5$, so the maximum exists.) For this G, let S, V_1, V_2 be as in Lemma 2.1. Let a, b, c, d, x, y be vertices not in $V(G)$, and let M be the matching with edges ab, cd, and xy.

Claim 2.2. *If $V(S) \subseteq R \subseteq V(G)$, then $\rho_G(R) \geq 4$. If furthermore $8 < |R| < |V(G)|$, then $\rho_G(R) \geq 5$.*

Proof. If $R = V(G)$, then the claim follows from G being a counterexample. So we henceforth consider $R \subsetneq V(G)$. If the first statement of the claim fails, then there is an R such that

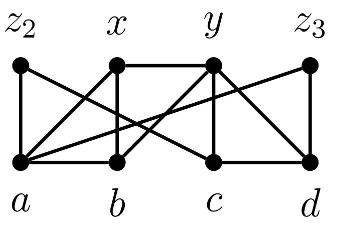

Fig. 1. The graph $G'[R']$.

(i) $\rho_G(R) = P(G) \le 3$. If the first statement of the claim holds but the second statement fails, then there is an R such that (ii) $8 < |R| < |V(G)|$ and $\rho_G(R) = 4$. Fix such an R in either case.

Take any 3-coloring ϕ of $G[R]$, and construct the graph G' as follows. Let $R' = V(M) \cup \{z_2, z_3\}$, where z_2, z_3 are new vertices. Let $V(G') = (V(G) - R) \cup R'$.

Let $E(G'[V(G) - R]) = E(G[V(G) - R])$, and

$$E(G'[R']) = \{ab, ax, bx, by, xy, cd, cy, dy, z_2a, z_2c, z_3a, z_3d\}.$$

See Fig. 1 for $G'[R']$. For each $i \in [3]$, let

$$C_i = \{v \in V(G) - R : v \text{ is adjacent to some vertex of color } i\},$$

and let $E(G'[R', V(G') - R'])$ be such that

$$N_{G'}(a) - R' = V_2 \cap C_1, \qquad N_{G'}(d) - R' = V_2 \cap C_2, \qquad N_{G'}(c) - R' = V_2 \cap C_3,$$
$$N_{G'}(y) - R' = V_1 \cap C_1, \qquad N_{G'}(z_2) - R' = V_1 \cap C_2, \qquad N_{G'}(z_3) - R' = V_1 \cap C_3,$$
$$N_{G'}(b) \subseteq R', \qquad N_{G'}(x) \subseteq R'.$$

Then G' is not 3-colorable. Indeed, if it had a proper 3-coloring ψ, then by renaming colors as necessary, we can assume that $\psi(a) = \psi(y) = 1$, $\psi(c) = \psi(z_3) = 3$, and $\psi(d) = \psi(z_2) = 2$. Then coloring G by ϕ on R and ψ on $V(G) - R$ is a proper coloring, a contradiction. Hence there exists a 4-critical subgraph $G'' \subseteq G'$; note that $E(G'') \supset E(M)$.

For any A such that $V(M) \subseteq A \subseteq V(G')$, we have

$$|E(G'[A - R', A \cap R'])| \le |E(G'[A - R', R'])| \le |E(G[A - R', R])|.$$

Also, any subset of R' containing $V(M)$ has potential equal to 4. Hence for any $A \subseteq V(G'')$ containing $V(M)$,

$$\begin{aligned}
\rho_{G''}(A) &\ge \rho_{G'}(A) \\
&= \rho_{G'}(A - R') + \rho_{G'}(A \cap R') - |E(G'[A - R', A \cap R'])| \\
&\ge \rho_G(A - R') + 4 - |E(G[A - R', R])| \\
&= (4 - \rho_G(R)) + (\rho_G(A - R') + \rho_G(R) - |E(G[A - R', R])|) \\
&= (4 - \rho_G(R)) + \rho_G((A - R') \cup R).
\end{aligned}$$

Since $\rho(R) \le 4$, we have $P(G'') \ge P(G)$. If $\rho(R) < 4$, then $P(G'') > P(G)$; if $\rho(R) = 4$ and $|R| > 8$, then $|V(G'')| < |V(G)|$. In either case, by the extremality of G, we have $\rho_{G''}(V(G'')) \le 3$. Taking $A = V(G'')$ above, we have

$$\rho_G((V(G'') - R') \cup R) \le \rho_{G''}(V(G'')) - 4 + \rho_G(R) \le \rho_G(R) - 1.$$

If R satisfies (i), then the set $(V(G'') - R') \cup R$ contradicts the minimality of $\rho_G(R)$. Hence the first statement of the claim holds. If R satisfies (ii), then the set $(V(G'') - R') \cup R$ has potential at most 3, contradicting the first statement. Hence the second statement of the claim holds as well. □

Claim 2.2 and Lemma 2.1 imply that S is a matching, and we will henceforth assume it is M, with $V_1 \cap V(M) = \{a, b, c, d\}$ and $V_2 \cap V(M) = \{x, y\}$. Furthermore we obtain that $\rho_G(V(M)) \in \{4, 5\}$,

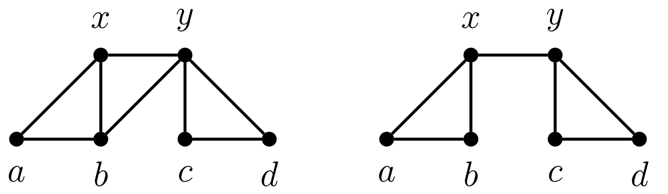

Fig. 2. The two cases for $G[V(M)]$.

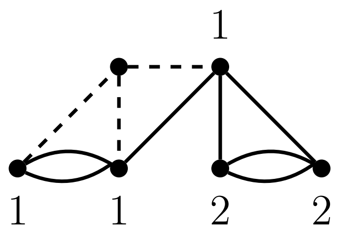

Fig. 3. $G'[V(M)]$ with f.

i.e. $|E(G[V(M)])| \in \{7, 8\}$. From the proof of Lemma 2.1 we see that

$$E(G[V(M)]) \supseteq \{ab, ax, bx, xy, cd, cy, dy\},$$

with equality or, up to symmetry, with the extra edge *by* (see Fig. 2). We will need to consider these two cases separately. First we introduce a few lemmas that the arguments will have in common.

The following lemma is an old result by Hakimi [4]. A simpler version of it was used by Alon and Tarsi in [1].

Lemma 2.3 (*Theorem 4 in [4]*). *Given a multigraph H and a function $f : V(H) \to \mathbb{N}$, one of the following holds.*

(1) *There is a subset $A \subseteq V(H)$ such that $|E(H[A])| > \sum_{v\in A} f(v)$.*
(2) *There is an orientation of H such that for every $v \in V(H)$, $d^+(v) \le f(v)$.*

A *kernel* in a digraph is an independent set S such that for every $v \in V(D) - S$, there is some $s \in S$ such that $vs \in E(D)$. A digraph is called *kernel-perfect* if every induced subdigraph has a kernel.

Lemma 2.4 (*Lemma 10 in [6]*). *Let A be an independent set in a graph H and $B = V(H) - A$. Let D be the digraph obtained from H by replacing each edge in $H[B]$ by a pair of opposite arcs and by an arbitrary orientation of $H[A, B]$. Then D is kernel-perfect.*

Lemma 2.5 (*Bondy, Boppana, and Siegel, see [1]*). *If D is a kernel-perfect digraph and L is a list assignment such that for every $v \in V(D)$, $|L(v)| \ge 1 + d^+(v)$, then D is L-colorable.*

Now we consider two cases.

Case $\rho(V(M)) = 4$:

Let G' be obtained from G by doubling the two edges *ab* and *cd* of M. Define $f : V(G' - x) \to \mathbb{N}$ by $f|_{N(x)} \equiv 1, f|_{V(G')-N[x]} \equiv 2$, and apply Lemma 2.3 to $G' - x, f$.

If Conclusion (2) of Lemma 2.3 holds, then the orientation of $G' - x$ must have antiparallel edges *ab* and *cd*. Indeed, *ab* must be antiparallel in order to have the outdegrees of a and b at most 1, then *yb* must be oriented from y, hence *cy* and *dy* must both be oriented toward y, and thus *cd* must be antiparallel (see Fig. 3). By Lemma 2.4, the orientation is kernel-perfect. We extend this orientation to one of G' by making x a sink; the result is still kernel-perfect, and now $d^+(v) \le 2$ for every $v \in V(G)$. By Lemma 2.5, G' and also G are 3-choosable, a contradiction.

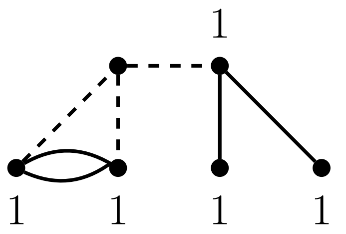

Fig. 4. $G'_x[V(M)]$ with f.

So Conclusion (1) of Lemma 2.3 holds. That is, there exists an $A \subseteq V(G')$ with $\rho_{G'}(A) \le -1 + |A \cap N(x)|$. This implies that

$$\rho_G(A) \le -1 + |A \cap N(x)| + |E(M) \cap E(G[A])|. \quad (1)$$

If $A \cap V(M) = \emptyset$, then

$$\rho_G(A + x) \le \rho_G(A) + 2 - |A \cap N(x)| \le 1$$
$$\rho_G(A + x + y) \le 1 + (2 - 1) = 2$$
$$\rho_G(A + V(M)) \le 2 + 2(4 - 3) - 1 = 3.$$

But this contradicts Claim 2.2.

So suppose $A \cap V(M) \ne \emptyset$. If $y \in A$, then we even save one, in the middle step of the previous paragraph. If exactly one of a, b (resp. c, d) is in A, then again we can even save one, in the third step above. If both $a, b \in A$ (resp. $c, d \in A$), then our bound in the first line above coming from Eq. (1) is worse by one (or two if all of $a, b, c, d \in A$). But then we save one (or two) in the third line of the computations above. So we still have a contradiction.

Case $\rho(V(M)) = 5$:

Let $G'_x = G - x - cd + ab$, and define $f : V(G'_x) \to \mathbb{N}$ by $f|_{N(x)\cup\{c,d\}} \equiv 1$, $f|_{V(G'_x)-(N[x]\cup\{c,d\})} \equiv 2$, and apply Lemma 2.3 to G'_x, f. (See Fig. 4.)

If Conclusion (2) of Lemma 2.3 holds, then the orientation must have antiparallel edges ab. Extend the orientation of G'_x to an orientation of $G - x + ab + cd$ by orienting the double edges cd antiparallel; then Lemma 2.4 implies the orientation is kernel-perfect. Extend to an orientation of $G + ab + cd$ by making x a sink; the result is still kernel-perfect, and now $d^+(v) \le 2$ for every v. By Lemma 2.5, $G + ab + cd$ and also G are 3-choosable, a contradiction.

So Conclusion (1) of Lemma 2.3 holds. That is, there exists an $A \subseteq V(G'_x)$ with $\rho_{G'_x}(A) \le -1 + |A \cap (N(x) \cup \{c, d\})|$. This implies that

$$\rho_G(A + x) \le 1 + \epsilon_1 + \epsilon_2, \quad (2)$$

where $\epsilon_1 = \begin{cases} 1 & \text{if } A \cap \{c,d\} \ne \emptyset, \\ 0 & \text{if } A \cap \{c,d\} = \emptyset \end{cases}$ and $\epsilon_2 = \begin{cases} 1 & \text{if } A \supseteq \{a,b\}, \\ 0 & \text{otherwise.} \end{cases}$

Adding to A in turn $\{y\}$, $\{c, d\}$, and $\{a, b\}$ each adds at most 1 to the potential. If A already intersects $\{y\}$, $\{c, d\}$, or $\{a, b\}$, then instead no potential is gained. Hence $A \cap V(M) = \emptyset$ or $A \cap V(M) = \{a, b\}$; otherwise $\rho(A + V(M)) \le 3$, contradicting Claim 2.2.

Similarly, we construct the graph G'_y, and find a set B such that $\rho_G(B + y) \le 1 + \epsilon_3$, where $\epsilon_3 = \begin{cases} 1 & \text{if } B \supseteq \{c,d\}, \\ 0 & \text{else.} \end{cases}$

We have $x \notin A \cup B$, $y \notin A \cup B$, and $B \cap V(M) = \emptyset$ or $B \cap V(M) = \{c, d\}$. So

$$\rho(A + x + B + y) + \rho(A \cap B) \le 2 + \epsilon_2 + \epsilon_3 - |E(G[A + x - B, B + y - A])|. \quad (3)$$

Let $C = A \cap B$. Then $C \cap V(M) = \emptyset$. If $|C| \le 2$, then $\rho(C) \ge 0$; if $|C| > 2$, then by Claim 2.2 we have

$$5 \le \rho(C + V(M)) \le \rho(C) + \rho(V(M)) - \rho(C \cap V(M)) = \rho(C) + 5,$$

so that still $\rho(C) \ge 0$. Furthermore, xy contributes to the last term of (3), and so by (3),

$$\rho(A + x + B + y) \le 1 + \epsilon_2 + \epsilon_3.$$

This implies that $\rho(A + B + V(M)) \le 3$, which contradicts Claim 2.2. $\square$

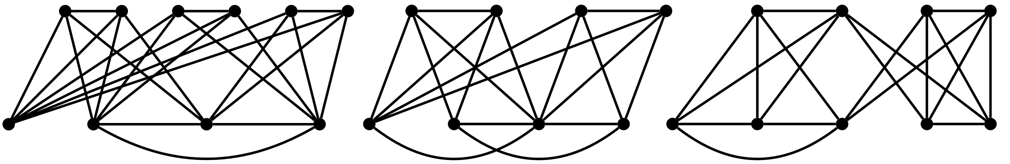

Fig. 5. Three 5-critical $B + E_6$ graphs with few edges.

3. Comments

1. The lemmas imply that if G is obtained from a bipartite graph B by adding a set S of at most 3 edges, $\rho(A) \geq 0$ for every $A \subseteq V(G)$ and $\rho(A) \geq 4$ for every $V(S) \subseteq A \subseteq V(G)$, then G is not only 3-colorable, but also 3-list-colorable.

2. When a graph is a $(k+1)$-critical $B+E_{\binom{k}{2}}$ graph, how few edges may it contain? Rödl and Tuza [7] gave examples for infinitely many n of $(k+1)$-critical n-vertex $B+E_{\binom{k}{2}}$ graphs with $(k-1)n-\binom{k}{2}+1$ edges. For $k = 3$, Chen et al. [2] gave examples for every $n \geq 7$ with only $2n - 3$ edges, and our result shows that this is sharp. For $k = 4$, Fig. 5 shows three examples of 5-critical n-vertex $B + E_6$ graphs with fewer than $3n - 5$ edges. The middle one, which is obtained from the Moser Spindle by adding a dominating vertex, was kindly shown to us by a referee. It is interesting whether for infinitely many n there exist 5-critical n-vertex $B + E_6$ graphs with fewer than $3n - 5$ edges.

Acknowledgments

The authors thank András Gyárfás for attracting attention to the problem and helpful discussions; and the anonymous referee for helpful comments, in particular, for pointing out the second of the graphs in Fig. 5. Research of the first author is supported in part by National Science Foundation grant DMS-1266016 and by grants 12-01-00631 and 12-01-00448 of the Russian Foundation for Basic Research. Research of the second author is supported in part by National Science Foundation grant DMS-0838434.

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