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Hypergraph Ramsey numbers: Triangles versus cliques

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ABSTRACT

A celebrated result in Ramsey Theory states that the order of magnitude of the triangle-complete graph Ramsey numbers $R(3,t)$ is $t^2/\log t$. In this paper, we consider an analogue of this problem for uniform hypergraphs. A *triangle* is a hypergraph consisting of edges e, f, g such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. For all $r \geqslant 2$, let $R(C_3, K_t^r)$ be the smallest positive integer n such that in every red–blue coloring of the edges of the complete r-uniform hypergraph K_n^r, there exists a red triangle or a blue K_t^r. We show that there exist constants $a, b_r > 0$ such that for all $t \geqslant 3$,

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \leqslant R\left(C_3, K_t^3\right) \leqslant b_3 t^{\frac{3}{2}}$$

and for $r \geqslant 4$

$$\frac{t^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}+o(1)}} \leqslant R\left(C_3, K_t^r\right) \leqslant b_r t^{\frac{3}{2}}.$$

This determines up to a logarithmic factor the order of magnitude of $R(C_3, K_t^r)$. We conjecture that $R(C_3, K_t^r) = o(t^{3/2})$ for all $r \geqslant 3$. We also study a generalization to hypergraphs of cycle-complete graph Ramsey numbers $R(C_k, K_t)$ and a connection to $r_3(N)$, the

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maximum size of a set of integers in $\{1, 2, \dots, N\}$ not containing a three-term arithmetic progression.

1. Introduction

A *hypergraph* is a pair (V, E) where V is a set whose elements are called *vertices* and E is a family of subsets of V called *edges*. If all edges have size r, then the hypergraph is referred to as an *r-graph*. Throughout this paper, C_k denotes a *loose k-cycle*, namely the hypergraph with edges $e_1, \dots, e_k$ such that $|e_i \cap e_{i+1}| = 1$ for $i = 1, \dots, k-1$, $|e_1 \cap e_k| = 1$, and $e_i \cap e_j = \emptyset$ otherwise. In particular, a *loose triangle* is a hypergraph consisting of three edges e, f, g such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. Since we consider only loose cycles and triangles, we will omit the word "loose". A hypergraph is *linear* if any pair of distinct edges of the hypergraph intersect in at most one vertex.

An *independent set* in a hypergraph is a set of vertices containing no edges of the hypergraph. Let K_t^r denote the t-vertex *complete r-graph*, i.e., the t-vertex r-graph whose edges are all r-element subsets of the vertex set. In this paper we consider the cycle versus complete hypergraph Ramsey numbers $R(C_k, K_t^r)$ – this is the minimum n such that every n-vertex r-graph contains either a cycle C_k or an independent set of t vertices. Our main effort will be on the triangle-complete hypergraph Ramsey number $R(C_3, K_t^r)$. A celebrated result of Kim [13] together with earlier bounds by Ajtai, Komlós and Szemerédi [2] shows that

$$R(C_3, K_t) = \Theta\left(\frac{t^2}{\log t}\right) \quad \text{as } t \to \infty.$$

This establishes the order of magnitude of these Ramsey numbers for graphs.

1.1. Triangle-free hypergraphs

The study of the independence number in triangle-free hypergraphs was initiated by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] and used to give a counterexample to a conjecture of Erdős on the Heilbronn problem [17] on the largest area of a triangle with vertices from n points in the unit square. Motivated also by the triangle-complete graph Ramsey numbers, in this paper we determine for $r \geqslant 3$ the order of magnitude of the triangle-complete Ramsey numbers for r-graphs up to logarithmic factors:

Theorem 1.1. *There exist constants $a, b_3 > 0$ such that for all $t \geqslant 1$,*

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \leqslant R\big(C_3, K_t^3\big) \leqslant b_3 t^{\frac{3}{2}}.$$

For each $r > 3$, there exist constants $a_r, b_r > 0$ such that for all $t \geqslant 1$,

$$\frac{t^{\frac{3}{2}}}{(\log t)^{\frac{3}{4} + \frac{a_r}{\sqrt{\log\log t}}}} \leqslant R\big(C_3, K_t^r\big) \leqslant b_r t^{\frac{3}{2}}.$$

We shall see that $b_r \leqslant (2r)^{9/2}$ for all $r \geqslant 3$. The upper bound in Theorem 1.1 is proved in Section 3. The lower bound in Theorem 1.1 comes from a construction that combines randomness and linear algebra and a construction of triangle-free hypergraphs coming from sets with no three-term arithmetic progressions, presented in Section 5. The preliminaries required to analyze this construction are presented in Section 4. Some of the ideas of the construction were recently used in [14] to study a related problem. In light of Theorem 1.1, we make the following conjecture:

Conjecture 1.1. *For all fixed* $r \geqslant 3$,

$$R\big(C_3, K_t^r\big) = o\big(t^{3/2}\big) \quad \text{as } t \to \infty.$$

We shall see in Section 2 that if H is a triangle-free hypergraph (the edges may have arbitrary size) on n vertices, then H contains an independent set of size at least $\lfloor\sqrt{n}\rfloor$. By Theorem 1.1, this is not tight for r-uniform hypergraphs for each fixed $r \geqslant 3$. It would be interesting to see if it is tight when edges whose size depends on n are allowed.

1.2. Linear triangle-free hypergraphs

We indicate a connection between independent sets in *linear* triangle-free hypergraphs and Roth's Theorem [17] on arithmetic progressions. Let $r_3(N)$ denote the largest size of a set of integers in $\{1, 2, \ldots, N\}$ containing no three-term arithmetic progressions. This problem has attracted much attention, starting with the original theorem of Roth [17] showing that $r_3(N) = o(N)$. The best current known bounds are as follows: for some constant $c > 0$,

$$\frac{N}{e^{c\sqrt{\log N}}} \leqslant r_3(N) \leqslant \frac{N}{(\log N)^{1-o(1)}}.$$

The lower bound, which comes from a construction of Behrend [4], is essentially unchanged for more than sixty years. The upper bound, due to Sanders [19] improves many earlier results which gave smaller powers of $\log N$ in the denominator. Let $RL(C_3, K_t^3)$ denote the minimum n such that every linear triangle-free 3-graph on at least n vertices contains an independent set of size t. We prove the following theorem:

Theorem 1.2. *There are constants* $\tilde{a}, \tilde{b} > 0$ *such that for all* $t \geqslant 1$

$$\frac{t^{\frac{3}{2}}}{e^{\tilde{a}\sqrt{\log t}}} \leqslant RL\big(C_3, K_t^3\big) \leqslant \frac{\tilde{b}t^{\frac{3}{2}}}{\sqrt{\log t}}.$$

Furthermore, if for some $c > 0$, $RL(C_3, K_t^3) = O(t^{3/2}(\log t)^{-3/4-c})$, *then*

$$r_3(N) = O\left(\frac{N}{(\log N)^{\frac{4c}{3}}}\right).$$

It would be interesting if one could prove that $r_3(N) = o(N)$ using Theorem 1.2 above. The bound $RL(C_3, K_t^3) = O(t^{3/2}/\sqrt{\log t})$ may also be evidence for Conjecture 1.1, that $R(C_3, K_t^3) = o(t^{3/2})$.

1.3. k-Cycle-free hypergraphs

The construction used in Theorem 1.1 extends more generally to give lower bounds on all *cycle-complete hypergraph Ramsey numbers*. The cycle C_3 is precisely a hypergraph triangle. We give for all $k, r \geqslant 3$ a construction of C_k-free r-graphs with low independence number, based on known results on the C_k-free bipartite Ramanujan graphs of Lubotzky, Phillips and Sarnak [16]. Specifically, we prove the following theorem by a suitable and fairly straightforward modification of the construction. We write $f = O^*(g)$ to denote that for some constant $c > 0$, $f(t) = O((\log t)^c g(t))$, and $f = \Omega^*(g)$ is equivalent to $g = O^*(f)$.

Theorem 1.3. *For fixed* $r, k \geqslant 3$,

$$R\big(C_k, K_t^r\big) = \Omega^*\big(t^{1+\frac{1}{3k-1}}\big) \quad \text{as } t \to \infty.$$

The key point of this theorem is that the exponent $1+1/(3k-1)$ of t is bounded away from 1 by a constant independent of r, and strictly improves for all $r, k \geqslant 5$ the lower bounds given by considering appropriate random hypergraphs, namely

$$R\bigl(C_k, K_t^r\bigr) = \Omega^*\bigl(t^{1+\frac{1}{kr-r-k}}\bigr) \quad \text{as } t \to \infty.$$

In the case $r = 2$, namely for graphs, the best available constructions for lower bounds on $r(C_k, K_t^r)$ indeed come from appropriate random graphs; in particular the C_k-free random graph process studied by Bohman and Keevash [7].

By using the known constructions of extremal bipartite graphs of girth 12, arising from generalized hexagons, we obtain the following improvement of the lower bound in Theorem 1.3 for C_5, i.e. for *loose pentagons*:

Theorem 1.4. *For fixed $r \geqslant 3$, there exists a constant $c_r > 0$ such that*

$$R\bigl(C_5, K_t^r\bigr) \geqslant c_r \left(\frac{t}{\log t}\right)^{\frac{5}{4}} \quad \text{as } t \to \infty.$$

The main part of this theorem is the exponent 5/4; we suspect that this exponent may be tight as $t \to \infty$, and perhaps even more generally, that $r(C_k, K_t^r) = \Theta^*(t^{k/(k-1)})$ for all $r, k \geqslant 3$. Our second conjecture is as follows:

Conjecture 1.2. *For all $r \geqslant 3$,*

$$R\bigl(C_5, K_t^r\bigr) = O\bigl(t^{5/4}\bigr) \quad \text{as } t \to \infty.$$

For graphs, the best current bounds are $a_2 t^{\frac{4}{3}}/\log t \leqslant R(C_5, K_t) \leqslant b_2 t^{3/2}/\sqrt{\log t}$, for some constants $a_2 > 0$ and $b_2 > 0$, where the upper bound is due to Caro, Li, Rousseau and Zhang [9] and the lower bound is from Bohman and Keevash [7].

2. Non-uniform hypergraphs

The goal of this section is to give a simple proof that any triangle-free hypergraph on n vertices has an independent set of size at least $\lfloor\sqrt{n}\rfloor$. Recall that the *chromatic number* $\chi(H)$ of a hypergraph H is the minimum k such that there is an assignment of k colors to the vertices such that no subset of vertices of the same color forms an edge of H.

Theorem 2.1. *Let H be any hypergraph on n vertices not containing a triangle and in which $|e| \geqslant 2$ for all $e \in H$. Then*

$$\alpha(H) \geqslant \lfloor\sqrt{n}\rfloor.$$

Proof. Suppose for a contradiction that $\alpha(H) < \lfloor\sqrt{n}\rfloor$. Then $\chi(H) > k := \lfloor\sqrt{n}\rfloor$. So, H contains a $(k+1)$-vertex-critical subgraph H', which means that $\chi(H') = k+1$ but $\chi(H' - v) \leqslant k$ for every $v \in V(H')$. By Corollary 3 on page 431 of [6] (see also [22] and [15]), the *strong degree* of each vertex in H' is at least k, i.e. for each $v \in V(H')$ there are k edges $e_1, e_2, \dots, e_k$ such that $e_i \cap e_j = \{v\}$ for all $1 \leqslant i < j \leqslant k$. In words, the e_is share v and nothing else. Choose a vertex v_i in each $e_i \backslash \{v\}$. Since H' has no triangles, the set $\{v_1, \dots, v_k\}$ is an independent set of H of size $k \geqslant \lfloor\sqrt{n}\rfloor$, which is a contradiction. □

This result is almost tight since $R(C_3, t) = \Theta(t^2/(\log t))$, so there are n-vertex triangle-free graphs with independence number of order $\sqrt{n \log n}$. It would be interesting to see if for hypergraphs (not necessarily uniform) where every edge has size at least three, the above lower bound on the independence number is tight.

3. Proof of Theorem 1.1: Upper bound

The aim of this section is to prove the upper bound of Theorem 1.1. For $r \geqslant 3$, let $\triangle_r$ denote the family of all triangle-free hypergraphs each of whose edges has size at least three and at most r. The upper bound on $R(C_3, K_t^r)$ in Theorem 1.1 will be derived as a direct consequence of the following more general statement about hypergraphs in $\triangle_r$:

Theorem 3.1. *For every $r \geqslant 3$ and $G \in \triangle_r$, $\alpha(G) \geqslant |V(G)|^{2/3}/(8r^3)$.*

This section is devoted to the proof of Theorem 3.1, which gives the constant $b_r = (8r^3)^{3/2} = (2r)^{9/2}$ in the upper bound in Theorem 1.1.

3.1. Expandable sets

In this section we state and prove a sequence of preliminary results needed for the proof of Theorem 3.1.

A set S of vertices of $G \in \triangle_r$ is called *expandable* if, for every $T \subseteq V(G) - S$ with $|T| \leqslant 2r$, there is an edge of G containing S and disjoint from T, otherwise S is *non-expandable*. For example, if S is an edge of G, then it is expandable, and every set $S \subset V(G)$ of size more than r is non-expandable.

Let G be an n-vertex graph in $\triangle_r$ with the smallest $\sum_{e \in E(G)} |e|$ for which Theorem 3.1 fails. Certainly G has at least one edge.

Lemma 3.2. *No three expandable sets in G form a triangle.*

Proof. If sets S_1, S_2, S_3 form a triangle, then by the definition of expandable sets, there is an edge $e_1 \supseteq S_1$ disjoint from $(S_2 \cup S_3) \setminus S_1$, there is an edge $e_2 \supseteq S_2$ disjoint from $(e_1 \cup S_3) \setminus S_2$, and there is an edge $e_3 \supseteq S_3$ disjoint from $(e_1 \cup e_2) \setminus S_3$. Now e_1, e_2, e_3 form a triangle in G, contradicting $G \in \triangle_r$. □

Lemma 3.3. *Let $S \subset V(G)$ be an expandable set and $|S| \geqslant 3$. Then no edge of G of size more than $|S|$ contains S.*

Proof. Suppose an expandable set S with $|S| \geqslant 3$ is contained in $e \in E(G)$ with $|e| \geqslant |S| + 1$. Let $V(G') = V(G)$ and $E(G') = E(G) - e + S$. By Lemma 3.2, $G' \in \triangle_r$. Since $\sum_{e \in E(G')} |e| < \sum_{e \in E(G)} |e|$, by the minimality of G, $\alpha(G') \geqslant |V(G')|^{2/3}/8r^3 = n^{2/3}/8r^3$. But every independent set in G' is also independent in G, and so $\alpha(G) \geqslant \alpha(G') \geqslant n^{2/3}/8r^3$, a contradiction to the choice of G. □

Lemma 3.4. *For every $3 \leqslant i < j \leqslant r$ no i-element subset of $V(G)$ is contained in more than $(2r)^{j-i}$ edges of size j.*

Proof. We use induction on $j - i$. If a $(j-1)$-element $S \subset V(G)$ is contained in $2r + 1$ edges of size j in G, then S is expandable, a contradiction to Lemma 3.3. Suppose now that $3 \leqslant i \leqslant j - 2$ and an i-element $S \subset V(G)$ is contained in $m \geqslant (2r)^{j-i} + 1$ edges $e_1, e_2, \ldots, e_m \in E(G)$ of size j. By Lemma 3.3, S is not expandable. This means that for some set T of $2r$ vertices of $V(G) \setminus S$, we have $(e_i \setminus S) \cap T \neq \emptyset$ for every $1 \leqslant i \leqslant m$. In other words, T intersects the part outside S of every e_i.

By the pigeonhole principle, there is an $x \in T$ such that the set $S \cup \{x\}$ is contained in at least $(2r)^{j-i-1} + 1$ edges among $e_1, e_2, \ldots, e_m$, a contradiction. □

Corollary 3.5. *For every $3 \leqslant j \leqslant k$ each 2-element non-expandable subset of $V(G)$ is contained in at most $(2k)^{j-2}$ edges of size j.*

Proof. Suppose that $S = \{x, y\}$ is a non-expandable pair of vertices in G is contained in $m \geqslant (2k)^{j-2} + 1$ edges $e_1, \ldots, e_m$ of size j. Then some $2k$ vertices $x_1, \ldots, x_{2k}$ outside S intersect all edges of G containing S, and in particular, all edges $e_1, \dots, e_m$. Then by the pigeonhole principle, for some $1 \leqslant t \leqslant 2k$, the 3-element set $S + x_t$ is contained in at least $(2k)^{j-3} + 1$ edges among $e_1, \dots, e_m$, a contradiction to Lemma 3.4. □

3.2. Proof of Theorem 3.1

In this section we complete the proof of Theorem 3.1. For $3 \leqslant i \leqslant r$, let G_i be the subgraph of G consisting of all edges of size i, that is, $E(G_i) = \{e \in E(G)\colon\ |e| = i\}$. For convenience, denote $n = |V(G)|$.

Lemma 3.6. *For every* $3 < j \leqslant r$, $|E(G_j)| \leqslant (2r)^{j-2}\binom{n}{2}$.

Proof. Let $e \in E(G_j)$ and $x, y, z \in e$. By Lemma 3.2, at least one of the pairs $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ is non-expandable and thus, by Corollary 3.5, is contained in at most $(2r)^{j-2}$ edges of G_j. Since every $e \in E(G_j)$ contains such a pair, the lemma follows. □

Lemma 3.7. $|E(G_3)| \geqslant n^{5/3}/4r^2$.

Proof. Suppose that $|E(G_3)| < n^{5/3}/4r^2$. Let $p = n^{-1/3}/4r^2$ and let W be a random subset of $V(G)$ where each $v \in V(G)$ is in W with probability p independently of all other vertices. By Lemma 3.6, for $j \geqslant 4$, the expected number of edges of size j in $G[W]$ is at most

$$\left|E(G_j)\right|p^j \leqslant (2r)^{j-2}\binom{n}{2}\left(4r^2\right)^{-j}n^{-j/3} \leqslant (2r)^{-j}n^{2/3}.$$

By assumption, the expected number of edges of size 3 in $G[W]$ is at most

$$n^{5/3}p^3/4r^2 = (2r)^{-8}n^{2/3} \leqslant (2r)^{-5}n^{2/3}.$$

So, the expectation of $|W| - |E(G[W])|$ is at least

$$pn - \sum_{j=4}^{r}(2r)^{-j}n^{2/3} - (2r)^{-5}n^{2/3} \geqslant pn - 2(2r)^{-4}n^{2/3} = \left(1 - \frac{1}{2r^2}\right)pn.$$

Thus there is a particular subset U of $V(G)$ with $|U| - |E(G[U])| \geqslant 0.9pn$. Then deleting a vertex from each edge in $G[U]$ we obtain an independent subset U' of U with $|U'| \geqslant 0.9pn$, so $\alpha(G) \geqslant n^{2/3}/5r^2 > n^{2/3}/8r^3$, a contradiction to the choice of G. □

The key part of the proof will be to produce an independent set in $H = G_3$ of size at least $n^{2/3}/8r^3$ that is *also an independent set in* G, using the preceding lemmas. By Lemma 3.7, $|E(H)| \geqslant (2r)^{-2}n^{5/3}$. Let $d = 3|E(H)|/n$ be the average degree of H, so $d \geqslant 3n^{2/3}/4r^2$. An edge $e \in H$ is called *k-light* if exactly k pairs of vertices of e have codegree in H at most r. An edge is *heavy* if it is 0-light. We see quickly that H has no heavy edges: for a heavy edge $\{x, y, z\} \in H$, since $r + 1 \geqslant 4$, we can greedily choose distinct vertices $a, b, c \notin \{x, y, z\}$ such that edges $\{a, x, y\}, \{b, y, z\}, \{c, x, z\}$ form a triangle, since each of the pairs $(x, y), (y, z), (z, x)$ has codegree at least $r + 1 \geqslant 4$. We now consider two cases.

Case 1. The number of edges in H that are 2-light or 3-light is at least $2|E(H)|/3$.

For each vertex v, let $d'(v)$ be the number of edges e of H containing v such that e is either 2-light or 3-light and v is incident to two light pairs of e. Then $\sum_v d'(v)$ counts each such e one or three times so $\sum_v d'(v) \geqslant 2|E(H)|/3$. Therefore some vertex v of H is in at least $2|E(H)|/3n = 2d/9$ edges, where two pairs of codegree (in H) at most r in each edge contain v. Let $e_1, e_2, \dots, e_m$ be such a set of edges on v with $m \geqslant 2d/9$. Then the link graph $L(v)$ consisting of pairs $e_i \backslash \{v\}$ has maximum degree at most r. It follows by Vizing's Theorem that $L(v)$ has a matching of size $\ell \geqslant m/(r + 1)$. This means that we have found edges, say $e_1, e_2, \dots, e_\ell$ sharing no vertices other than v, and such

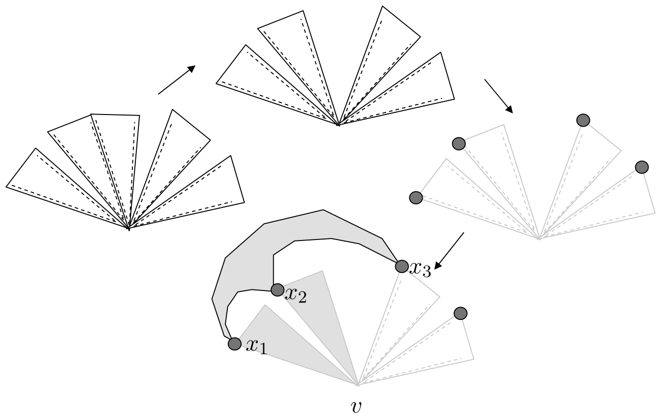

Fig. 1. Finding an independent set in Case 1.

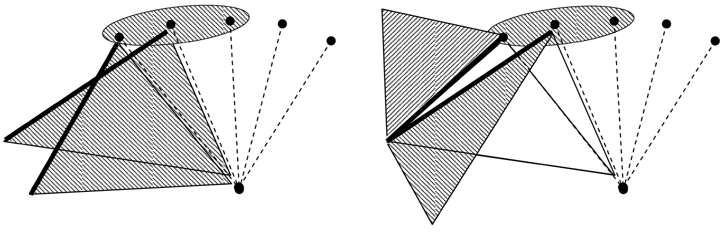

Fig. 2. Finding an independent set in Case 2.

that in each e_i the two pairs containing v have codegree at most r. Now pick $x_1, x_2, \dots, x_\ell$ where $x_i \in e_i \backslash \{v\}$ for $1 \leqslant i \leqslant \ell$. We claim that this is an independent set in the entire hypergraph G. If not, then say $e = \{x_1, \dots, x_j\} \in E(G)$. Then $\{e, e_1, e_2\}$ is a triangle in G, since e_1 and e_2 share only v, e and e_1 share only x_1, and e and e_2 share only x_2 – see Fig. 1. This independent set has size $\ell \geqslant m/(r+1) \geqslant 2d/9(r+1) \geqslant n^{2/3}/8r^3$. This completes the proof in Case 1.

Case 2. The number of 1-light edges in H is at least $|E(H)|/3$.

For each vertex v, let $d''(v)$ be the number of edges e of H containing v such that e is 1-light and v is incident to the light pair of e. Then $\sum_v d''(v)$ counts each such e exactly twice so $\sum_v d''(v) \geqslant 2|E(H)|/3$. By averaging, some v in H lies in at least $2|E(H)|/3n = 2d/9$ 1-light edges such that the pair of codegree in H at most r in each edge contains v. Then there are at least $2d/9r$ distinct vertices $x_1, x_2, \dots, x_m$ such that the codegree of (v, x_i) is at most r, and there is a 1-light edge $e_i \supset \{v, x_i\}$ for all $i \in \{1, 2, \dots, m\}$. Since each e_i is 1-light, exactly two pairs of vertices in e_i have codegree at least $1 + r$, and in particular, all e_is are distinct. We claim that $\{x_1, x_2, \dots, x_m\}$ is again an independent set in G. Suppose not, and that $\{x_1, \dots, x_j\}$ is an edge. Let $e_i \backslash \{x_i, v\} = \{y_i\}$. Note that every y_i is disjoint from $\{x_1, \dots, x_j\}$, otherwise if say $y_i = x_j$, then $\{v, x_i\}$ and $\{v, x_j\}$ both have codegree less than $1 + r$, but they lie in the edge e_i, which has only one pair of codegree less than $1 + r$ – a contradiction. So every y_i is disjoint from $\{x_1, \dots, x_j\}$. Now we claim that $y_1 = y_2 = \cdots = y_j$. If say $y_1 \neq y_2$ (left drawing in Fig. 2), consider the triples $\{v, x_1, y_1\}, \{v, x_2, y_2\}$ and the edge $\{x_1, x_2, \dots, x_j\}$. Since $y_1, y_2, x_1, \dots, x_j$ are all distinct, this is a triangle. So $y_1 = y_2 = \cdots = y_j = y$. Now consider the pairs $\{y, x_1\}, \{y, x_2\}$ (shown in black bold lines in the right drawing in Fig. 2). Since $\{y, x_1\}$ and $\{y, x_2\}$ are pairs in e_1 and e_2, respectively, and they do not contain v, by the choice of e_1 and e_2, those pairs have codegree at least $1 + r$. So we can pick $z_1 \neq z_2$ with $z_1, z_2 \notin \{x_1, \dots, x_j, y, v\}$ such that $\{x_1, y, z_1\}, \{x_2, y, z_2\}, \{x_1, x_2, \dots, x_j\}$ is a triangle – namely $z_1, z_2, x_1, \dots, x_j$ are all distinct. This shows that $\{x_1, x_2, \dots, x_m\}$ is independent, and it has size at least $2d/9r \geqslant n^{2/3}/6r^3$.

4. Generalized quadrangles and a spectral lemma

Generalized quadrangles were first constructed by Tits [21] and described as graphs by Benson [5]. Let G_q denote a *generalized quadrangle of order* q, which is a $(q+1)$-regular $(q+1)$-uniform C_2, C_3-free hypergraph on q^3+q^2+q+1 vertices. Generalized quadrangles of order q exist whenever q is a prime power.

4.1. A general spectral lemma

In this section, we employ a lemma which relates the distribution of edges in a bipartite graph to spectral properties of its adjacency matrix. This lemma is an analog of a well-known spectral lemma in graph theory (see for example [3]) which is frequently referred to as the expander mixing lemma, and is used especially in the context of (n, d, λ)-graphs and pseudorandom graphs. The lemma we give may be referred to as the expander mixing lemma for bipartite graphs, appears in a different form in [11] and in [12]. For completeness, we give the proof here and it is very similar to the proof for non-bipartite graphs in [3].

Lemma 4.1. *Let $G(U,V)$ be a d-regular bipartite graph with adjacency matrix A and let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ be the eigenvalues of A. Let $\lambda = \max\{|\lambda_i|: i \notin \{1,N\}\}$. Then for any sets $X \subseteq U$ and $Y \subseteq V$, the number $e(X,Y)$ of edges from X to Y satisfies*

$$\left| e(X,Y) - \frac{d}{|V|}|X||Y| \right| \leqslant \lambda\sqrt{|X||Y|}.$$

Proof. Let χ_X and χ_Y denote the characteristic vectors of X and Y. Let $x_1, x_2, \ldots, x_N$ be an orthonormal basis of eigenvectors of A, where x_i is the eigenvector corresponding to λ_i, and write

$$\chi_X = \sum_{i=1}^{N} s_i x_i, \qquad \chi_Y = \sum_{i=1}^{N} t_i x_i.$$

Then

$$e(X,Y) = \langle A\chi_X, \chi_Y \rangle = \lambda_1 s_1 t_1 + \lambda_N s_N t_N + \sum_{i=2}^{N-1} \lambda_i s_i t_i.$$

The values of s_1, t_1, s_N and t_N are recovered quickly from the knowledge of the first and last eigenvectors, x_1 and x_N, recalling x_1 is the constant unit vector and x_N is the unit vector which is constant on $V(G_q)$ and minus that constant on $E(G_q)$. Noting that $\|\chi_X\|^2 = |X|$ and $\|\chi_Y\|^2 = |Y|$, and using $\lambda_1 = d = -\lambda_N$, it is straightforward to see that

$$e(X,Y) = \frac{d}{|V|}|X||Y| + \sum_{i=2}^{N-1} \lambda_i s_i t_i.$$

Finally, by the Cauchy–Schwarz inequality,

$$\sum_{i=2}^{N-1} \lambda_i s_i t_i \leqslant \lambda(A)\left(\sum_{i=1}^{N} s_i^2\right)^{1/2}\left(\sum_{i=1}^{N} t_i^2\right)^{1/2}$$

and the sums are $\|\chi_X\| = \sqrt{|X|}$ and $\|\chi_Y\| = \sqrt{|Y|}$ respectively. □

This lemma will be used in the context of hypergraphs (in particular for the hypergraph $H = G_q$) in the following way: if H is a hypergraph, then the *bipartite incidence graph* of H is the bipartite graph $B(H)$ whose parts are $V(H)$ and $E(H)$, and $\{v,e\} \in E(B(H))$ if and only if $v \in e$. We denote by $A(H)$ the adjacency matrix of the bipartite incidence graph $B(H)$, and when $|V(H)| = |E(H)|$ we denote by $\lambda(H)$ the largest absolute value of the eigenvalues of $A(H)$ other than λ_1 and λ_N. Lemma 4.1 is applied to $B(H)$ to give the following hypergraph formulation:

Lemma 4.2. *Let H be a d-uniform d-regular n-vertex hypergraph and let $X \subseteq V(H)$ and $Y \subseteq E(H)$. Then*

$$\left|\sum_{e\in Y}|X\cap e| - \frac{d}{|V|}|X||Y|\right| \leqslant \lambda(H)\sqrt{|X||Y|}.$$

In particular, if $\lambda(H) \leqslant \delta\sqrt{d}$ and $|X| \geqslant 2\tau n/d$, then the number of edges $e \in E(H)$ such that $|X\cap e| \geqslant \tau$ is at least $n - 2\delta^2 n/\tau$.

Proof. For the first inequality, if H is a d-uniform d-regular hypergraph, then $B(H)$ is d-regular. Applying Lemma 4.1 gives

$$\left|e(X,Y) - \frac{d}{|V|}|X||Y|\right| \leqslant \lambda(H)\sqrt{|X||Y|}.$$

We note that

$$e(X,Y) = \sum_{e\in Y}|X\cap e|.$$

This gives the first inequality of Lemma 4.2. Applying this inequality with $\lambda(H) \leqslant \delta\sqrt{d}$, we obtain for any $Z \subseteq E(H)$,

$$\left|\sum_{e\in Z}|X\cap e| - \frac{d}{n}|X||Z|\right| \leqslant \delta\sqrt{d|X||Z|}.$$

Now let $Y = \{e \in E(H)\colon\ |X\cap e| \geqslant \tau\}$ and $Z = E(H)\backslash Y$. Suppose for a contradiction that $|Z| > 2\delta^2 n/\tau$. By the definition of Z,

$$\sum_{e\in Z}|X\cap e| < \tau|Z|.$$

By the preceding inequality,

$$\tau|Z| > \sum_{e\in Z}|X\cap e| > \frac{d}{n}|X||Z| - \delta\sqrt{d|X||Z|}.$$

Since $|X| \geqslant 2\tau n/d$, we get

$$\tau|Z| < \delta\sqrt{2\tau n|Z|}.$$

This contradicts $|Z| > 2\delta^2 n/\tau$. □

We remark that for fixed $|X|$, $d|X|/|V|$ is exactly the expected value of $|X\cap e|$ when X is a random set whose elements are chosen from $V(H)$ independently with probability $|X|/|V|$.

4.2. *Spectral properties of $A(G_q)$*

In order to apply Lemma 4.2 to G_q, we determine $\lambda(G_q)$. Since G_q is $(q+1)$-uniform and $(q+1)$-regular, the bipartite incidence graph $B(G_q)$ is $(q+1)$-regular. Since $B(G_q)$ is connected, this implies $q+1$ and $-(q+1)$ are eigenvalues of $A = A(G_q)$ with multiplicity 1. By the definition of a generalized quadrangle, for every vertices x and y in distinct partite sets of $B(G_q)$, there exists exactly one x,y-path of length 3. Since each entry $a^3_{i,j}$ of A^3 is the number of i,j-walks of length 3, we have

$$A^3 = J + qA$$

where J is the block matrix

$$J = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

and K is the square all 1 matrix with appropriate dimensions. If $\lambda \notin \{-(q+1), q+1\}$ is an eigenvalue of A, then an eigenvector x for λ is orthogonal to the constant unit vector and so $Kx = 0$. It follows that $\lambda^3 = q\lambda$ and therefore $\lambda \in \{-\sqrt{q}, 0, \sqrt{q}\}$. Since the eigenvalues of $A(G_q)$ other than $-(q+1)$ and $(q+1)$ are not all zero, $\lambda(G_q) = \sqrt{q}$. A more complete analysis of these eigenvalues and their multiplicities was achieved by Haemers [11]. Since we have $\lambda(G_q) = \sqrt{q}$, Lemma 4.2 gives the following:

Corollary 4.3. *Let $X \subseteq V(G_q)$ where $|X| \geqslant 2\tau n/(q+1)$, and let Y be the set of $e \in E(G_q)$ such that $|X \cap e| \geqslant \tau$. Then*

$$|Y| \geqslant n - \frac{2n}{\tau}.$$

Proof. Since $\lambda(G_q) = \sqrt{q}$, applying Lemma 4.2 with $\delta = 1$ and $d = q+1$ gives the result. □

5. Proof of Theorem 1.1: Lower bound

Based on the generalized quadrangle G_q, we now specify the construction of a triangle-free n-vertex hypergraph H_q with independence number $O(n^{2/3}\sqrt{\log n})$, which gives the lower bound in Theorem 1.1 for $r = 3$. Let $\tau = \lfloor 4 \log q \rfloor$. The idea is to place randomly a carefully chosen triangle-free 3-graph F_q on $q+1$ vertices into each of the edges of G_q, independently for each edge of G_q, to form a new hypergraph H_q with $n = q^3 + q^2 + q + 1$ vertices. We then use the spectral result in the form of Corollary 4.3 to deduce that a set of $2\tau n/q$ vertices of G_q must intersect almost all edges of G_q in roughly τ vertices. Together with some basic probability, we use this to deduce that the expected number of independent sets of size $2\tau n/q$ in H_q is $o(1)$, and therefore some H_q has independence number $2\tau n/q$, as required. A similar idea will be used in the lower bound in Theorems 1.2 and 1.4, and the appropriate modifications to F_q will be made in Section 5.3 to obtain the lower bound in Theorem 1.1 for $r > 3$.

5.1. The hypergraph F_q

Throughout this section, $\tau = \lfloor 8\sqrt{\log q} \rfloor$. To describe H_q, we use the auxiliary hypergraph F_q with vertex set $[q+1]$, defined as follows. Let $V = \{v_{ij}\colon 1 \leqslant i, j \leqslant \tau\}$ be a τ^2-element subset of $[q+1]$ and let $S_1, \dots, S_\tau, T_1, \dots, T_\tau$ be a partition of $[q+1] - V$ into sets whose sizes differ by at most one. Let $S = \bigcup_{i=1}^{\tau} S_i$ and $T = \bigcup_{j=1}^{\tau} T_j$. The edge set of F_q is the set of all triples $\{v_{ij}, a, b\}$ such that $a \in S_i$ and $b \in T_j$. Note that F_q is actually 3-partite, with parts V, S and T. Then H_q is constructed by taking independently for each $e \in G_q$ a random bijection π_e from $V(F_q)$ to e and letting a triple in e be an edge if its pre-image is an edge in F_q.

Lemma 5.1. *H_q is triangle-free.*

Proof. Since G_q is linear and triangle-free, it is sufficient to verify that F_q is triangle-free. Suppose F_q has a triangle. Since F_q is 3-partite, some vertex in V belongs to two of the edges of the triangle. Let this vertex be $v_{ij} \in V$, and these two edges be $\{v_{ij}, s_i, t_j\}$ and $\{v_{ij}, s'_i, t'_j\}$. Now the third edge must be either $\{v, s_i, t'_j\}$ or $\{v, s'_i, t_j\}$ for some $v \in V$. By definition of F_q, this implies that $v = v_{ij}$, a contradiction. □

Next we bound from above the probability that a set of τ vertices of $e \in E(G_q)$ is an independent set in H_q.

Lemma 5.2. *Let I be a τ-element subset of $e \in E(G_q)$. Then as $q \to \infty$, the probability that I is independent in H_q is at most $1 - \frac{\tau^3 - o(\tau^3)}{4eq}$.*

Proof. Let N be the number of τ-sets X of $V(F_q) = [q+1]$ that are not independent in F_q. A lower bound for N is obtained by picking an element $v_{ij} \in V$, an element $s \in S_i$, an element $t \in T_j$ and $\tau - 3$ elements in $[q+1] - (V \cup S_i \cup T_j)$. As $q \to \infty$, this gives

$$\begin{aligned}
N &\geqslant \sum_{v_{ij} \in V} |S_i||T_j| \binom{q+1-|S_i|-|T_j|-|V|}{\tau-3} \\
&\geqslant \tau^2 \cdot \left\lfloor \frac{q}{2\tau} \right\rfloor^2 \binom{q+1-2\lceil \frac{q+1-\tau^2}{2\tau} \rceil - \tau^2}{\tau-3} \\
&\geqslant \tau^2 \cdot \left\lfloor \frac{q}{2\tau} \right\rfloor^2 \binom{(1-1/\tau)(q+1)-\tau^2}{\tau-3} \\
&= \big(1-o(1)\big)\tau^2 \left(\frac{q}{2\tau}\right)^2 \binom{(1-1/\tau)q}{\tau-3} \\
&= \big(1-o(1)\big)\frac{q^2}{4}(1-1/\tau)^{\tau-3}\frac{q^{\tau-3}}{(\tau-3)!} \\
&= \big(1-o(1)\big)\frac{\tau^3}{4eq}\frac{(q+1)^\tau}{\tau!} \\
&= \big(1-o(1)\big)\frac{\tau^3}{4eq}\binom{q+1}{\tau}.
\end{aligned}$$

Now the probability that $I \subset e$ is not independent in H_q is

$$\frac{|\{\pi_e \colon I \text{ is not independent under } \pi_e\}|}{(q+1)!} = \frac{N\tau!(q+1-\tau)!}{(q+1)!} = \frac{N}{\binom{q+1}{\tau}}.$$

The lower bound on N now gives the desired result. □

5.2. Independence number of H_q

If $n = q^3 + q^2 + q + 1$ for some prime power q, then we show that with positive probability, H_q has no independent set of size more than $2\tau n/q$ if n is large enough and $\tau = \lfloor 8\sqrt{\log q} \rfloor$. Note that $2\tau n/q < 16n^{2/3}\sqrt{\log n}$ if n is large enough. If n is not of this form, pick the smallest prime power q such that $n \leqslant q^3 + q^2 + q + 1$, and remove $q^3 + q^2 + q + 1 - n$ vertices from H_q. The new hypergraph H'_q has $\alpha(H'_q) \leqslant \alpha(H_q)$. Since it is well-known that there exists a prime q: $n^{1/3} \leqslant q \leqslant 2n^{1/3}$, H'_q has no independent set of size more than $2\tau n/q = O(n^{2/3}\sqrt{\log n})$, as required to finish the proof of the lower bound in Theorem 1.1.

Suppose that $X \subset V(H_q) = V(G_q)$ is an independent set of size $\lceil 2\tau n/q \rceil$ in H_q. By Corollary 4.3, at least $n - 2n/\tau$ of the edges of G_q contain at least τ vertices of I. Let $Y = Y(X)$ be this set of edges. For each $e \in Y$, $X \cap e$ is an independent set in the random hypergraph F_q on e. Let B_e be the event that $X \cap e$ is independent in F_q. By Lemma 5.2,

$$P(B_e) \leqslant 1 - \frac{\tau^3}{11q}$$

provided q is large enough. The events B_e are independent over $e \in Y$, and therefore the expected number of independent sets of size $2\tau n/q$ in H_q is at most

$$\sum_{X:\ |X|=\lceil 2\tau n/q\rceil}\prod_{e\in Y}P(B_e)\leqslant\left(1-\frac{\tau^3-o(\tau^3)}{4eq}\right)^{n-2n/\tau}\binom{n}{\lceil 2\tau n/q\rceil}$$
$$\leqslant\exp\left(-\frac{n(\tau^3-o(\tau^3))}{4eq}+\frac{2\tau n}{q}\log\frac{n}{q}\right). \tag{1}$$

Since $\tau=\lfloor 8\sqrt{\log q}\rfloor$ and $n=q^3+q^2+q+1$, as $q\to\infty$, we have

$$-\frac{n(\tau^3-o(\tau^3))}{4eq}+\frac{2\tau n}{q}\log\frac{n}{q}\leqslant\frac{n\tau}{q}\left[-\frac{\tau^2-o(\tau^2)}{4e}+2\log q^2\right]$$
$$\leqslant\frac{n\tau}{q}\left[-\frac{(1-o(1))64\log q}{4e}+4\log q\right].$$

Thus the quantity in (1) decays to zero. Therefore with high probability, H_q has no independent set of size more than $2\tau n/q<16n^{2/3}\sqrt{\log n}$ if n is large enough. This proves the lower bound in Theorem 1.1 for $r=3$. We next turn to the case $r>3$.

5.3. The hypergraph $H_{q,r}$

In this section we prove the lower bound in Theorem 1.1 for $r>3$. Take $H_{q,r}$ to consist of randomly placed copies of a carefully chosen hypergraph $F_{q,r}$ on $q+1$ vertices in the edges of G_q. The hypergraph $F_{q,r}$ takes the role of the hypergraph F_q in the preceding section. To describe $F_{q,r}$, we first review a known construction of linear r-graphs based on a construction of dense sets without three-term arithmetic progressions.

5.4. Description of $F_{q,r}$

Erdős, Frankl and Rödl [10] showed that for every $r\geqslant 3$ there is a constant $c_r>0$ such that for each $N\in\mathbb{N}$ there exists a linear triangle-free r-partite r-graph $J(N,r)$ with N vertices in each part and at least $N^2/\exp(c_r\sqrt{\log N})$ edges. Their construction is based on and generalizes the construction of Ruzsa and Szemerédi [18] for $r=3$ of a dense linear triangle-free 3-graph. The Ruzsa–Szemerédi construction is in turn derived from Behrend's construction [4] of relatively dense sets of integers with no three-term arithmetic progressions. Using the Erdős–Frankl–Rödl construction, we describe a triangle-free (but not linear) r-graph $F_{q,r}$ on $q+1$ vertices for each $r>3$. This is the key in the description of $H_{q,r}$ for the proof of Theorem 1.1.

Fix $r>3$, and let $C_r>0$ be a constant depending on r, to be chosen later. Let J be the Erdős–Frankl–Rödl hypergraph $J(\tau,r-1)$, where

$$\tau=\left\lceil(\log q)^{1/2}\exp\left(-C_r\sqrt{\log\log q}\right)\right\rceil=(\log q)^{1/2-o(1)}.$$

For convenience let $m=|E(J)|$ and let $V_1,\dots,V_{r-1}$ be the parts of J. To define $V(F_{q,r})$, associate pairwise disjoint sets S_v to the vertices $v\in V(J)$, and let W be a set of m vertices disjoint from all the sets S_v and indexed by the edges of J, namely $W=\{v_e:\ e\in E(J)\}$. Then let

$$V(F_{q,r})=W\cup\bigcup_{v\in V(J)}S_v$$

where the S_v are as equal in size as possible subject to

$$q+1=m+\bigcup_{v\in V(J)}S_v.$$

This ensures that $F_{q,r}$ has exactly $q+1$ vertices. The edges of $F_{q,r}$ are defined as follows. For every $e=\{v_1,\dots,v_{r-1}\}\in J(\tau,r-1)$ with $v_i\in V_i$, recall that $v_e\in W$, and let

$$F_e=\left\{v_e\cup\{x_1,\dots,x_{r-1}\}:\ x_i\in S_{v_i},\ i=1,\dots,r-1\right\}.$$

Then

$$E(F_{q,r}) = \bigcup_{e \in J(\tau, r-1)} F_e.$$

Loosely speaking, the edges $e \in E(J)$ are being replaced with complete $(r-1)$-partite $(r-1)$-graphs $K(e)$ with parts of size roughly $q/(r-1)\tau$, and then we form the edges of $F_{q,r}$ by enlarging each of the edges of $K(e)$ with the new vertex v_e. It is straightforward to check (by both the linearity of J and the fact that J is triangle-free) that $F_{q,r}$ is triangle-free (although it is not linear). The key lemma about $F_{q,r}$ is now as follows:

Lemma 5.3. *Let $r \geqslant 3$ and I be a τ-element subset of $e \in E(G_q)$. Then for some $d_r > 0$, the probability that I is independent in $H_{q,r}$ is at most*

$$1 - \frac{\tau^{3-d_r/\sqrt{\log \tau}}}{q}.$$

Proof. Let N be the number of τ-element subsets of $V(F_{q,r}) = [q+1]$ that are not independent in $F_{q,r}$. Since every τ-element set obtained by picking an element $w_e \in W$, an element from each set S_v such that $v \in e$, and then $\tau - r$ elements in $[q+1] \setminus (W \cup \bigcup_{v \in e} S_v)$ is not independent, we have

$$N \geqslant \sum_{w_e \in W} \left(\prod_{v \in e} |S_v|\right) \binom{q + 1 - \sum_{v \in e} |S_v| - |W|}{\tau - r}.$$

Since all S_v have almost the same cardinality, as $q \to \infty$ the right-hand side is at least

$$\big(m + o(m)\big) \cdot \left(\frac{q}{\tau}\right)^{r-1} \cdot \left(\frac{q + 1 - q/\tau}{\tau - r}\right)^{\tau - r} \geqslant \big(m + o(m)\big) \cdot \frac{\tau}{qer^r} \binom{q+1}{\tau}.$$

So we can choose $d_r > 0$ depending only on r such that the last expression is at least

$$\frac{\tau^{3-d_r/\sqrt{\log \tau}}}{q} \binom{q+1}{\tau}.$$

This bound proves the lemma. $\square$

The rest of the proof for $H_{q,r}$ carries through as for H_q, except at the end, the expected number of independent sets of size $2\tau n/q$ in $H_{q,r}$ is now by Lemma 5.3 at most

$$\left(1 - \frac{\tau^{3-d_r/\sqrt{\log \tau}}}{q}\right)^{n - 2n/\tau} \binom{n}{2\tau n/q} < \exp\left(-\frac{\tau^{3-d_r/\sqrt{\log \tau}} n}{2q} + \frac{2\tau n \log n}{q}\right).$$

We have chosen τ to ensure

$$\tau^{3-d_r/\sqrt{\log \tau}} > 6\tau \log n.$$

This ensures that the expected number of independent sets of size $2\tau n/q$ in $H_{q,r}$, for large enough n, is less than

$$\exp\left(-\frac{\tau n \log n}{q}\right) < \exp\big(-n^{2/3} \log n\big) < 1.$$

We conclude that with positive probability, for large enough n and a large enough constant C_r,

$$\alpha(H_{q,r}) \leqslant 2\tau n/q \leqslant 2n^{2/3} (\log n)^{1/2 + C_r/\sqrt{\log \log n}}.$$

This gives the lower bound on Ramsey numbers in Theorem 1.1. $\square$

6. Proof of Theorem 1.2

To prove the upper bound in Theorem 1.2, it is sufficient to show that every n-vertex linear triangle-free 3-graph has an independent set of size $\Omega(n^{2/3}(\log n)^{1/3})$. Let H be such a 3-graph. By the main theorem in [1],

$$\alpha(H) = \Omega\left(\frac{n\sqrt{\log d}}{\sqrt{d}}\right)$$

where d is the average degree of H. The union of all pairs $e\backslash\{v\}$ for edges e containing a vertex v of degree at least d in H is an independent set of $2d$ vertices in H, since H is linear and triangle-free. Therefore

$$\alpha(H) = \Omega\left(\min_d \max\left\{d, \frac{n\sqrt{\log d}}{\sqrt{d}}\right\}\right) = \Omega\left(n^{2/3}(\log n)^{1/3}\right).$$

This completes the proof of the upper bound in Theorem 1.2.

6.1. *Proof of Theorem 1.2: Lower bound*

Based on the hypergraph G_q, for $n = q^3 + q^2 + q + 1$ and q a prime power, we construct an n-vertex linear triangle-free 3-graph H_q^* with $\alpha(H_q^*) \leqslant n^{2/3}\exp(A\sqrt{\log n})$ for some $A > 0$. If n is not of that form, then as in the proof of Theorem 1.1 we use the distribution of primes and a large subhypergraph of H_q^* to obtain the same result with perhaps a slightly larger implicit constant. Let $N = \lfloor (q+1)/3 \rfloor$ and let $F_q^* = J(N,3)$, where $J(N,3)$ is defined in Section 5.4. Then $|E(F_q^*)| = |E(J)| = \Omega(qr_3(q))$. The main lemma we require counts independent sets of size τ in F_q^*.

Lemma 6.1. *As $q \to \infty$ the number of independent sets of size τ in F_q^* is at most*

$$\left(1 - \Omega\left(\frac{\tau^3 r_3(q)}{q^2}\right)\right)\binom{q+1}{\tau}.$$

Proof. Let N be the number of non-independent sets of size τ in F_q^*. It is sufficient to show

$$N = \Omega\left(\frac{\tau^3 r_3(q)}{q^2}\right)\binom{q+1}{\tau}.$$

Since $M := |E(F_q^*)| = \Omega(qr_3(q))$, by inclusion–exclusion,

$$\begin{aligned} N &\geqslant M \cdot \binom{q-2}{\tau-3} - \binom{M}{2}\binom{q-4}{\tau-5} \\ &= M \cdot \binom{q+1}{\tau}\frac{\tau(\tau-1)(\tau-2)}{(q+1)q(q-1)}\left(1 - \frac{(M-1)(\tau-3)(\tau-4)}{2(q-2)(q-3)}\right) \\ &= \Omega\left(\frac{\tau^3 r_3(q)}{q^2}\right)\binom{q+1}{\tau}. \end{aligned}$$

This is the required bound on N. □

As before, we construct H_q^* by placing a randomly permuted copy of F_q^* in each edge of G_q. The expected number of independent sets of size $\lceil 2\tau n/q \rceil$ in H_q^* is then at most

$$\left(1 - O\left(\frac{\tau^3 r_3(q)}{q^2}\right)\right)^{n-2n/\tau}\binom{n}{\lceil 2\tau n/q \rceil}$$

using Lemma 6.1 and Corollary 4.3 as in the proof of Theorem 1.1. Choose τ to satisfy

$$\frac{4\tau n\log n}{q}<\frac{n\tau^3 r_3(q)}{q^2}$$

which ensures that the expected number of independent sets is $o(1)$. It is sufficient to take

$$\tau^2=\bigl(1+o(1)\bigr)\frac{4q\log n}{r_3(q)}.$$

Then with high probability

$$\alpha\bigl(H_q^*\bigr)<\frac{2\tau n}{q}<\frac{8n\sqrt{q\log n}}{q\sqrt{r_3(q)}}.$$

To obtain from this the lower bound on $RL(C_3,K_t^3)$, let $n=RL(C_3,K_t^3)$ so that

$$\frac{8n\sqrt{q\log n}}{q\sqrt{r_3(q)}}>t.$$

Since $r_3(q)>q/\exp(c\sqrt{\log q})$ for some $c>0$, this gives the lower bound on $RL(C_3,K_t^r)$ in Theorem 1.2.

Finally, we connect a bound on Ramsey numbers to $r_3(N)$. According to the above proof, if $n=RL(C_3,K_t^3)=O(t^{3/2}(\log t)^{-3/4-c})$, then

$$\frac{n\sqrt{q\log n}}{q\sqrt{r_3(q)}}=\Omega(t).$$

Put $N=q$. Recalling $n=N^3+o(N^3)$,

$$r_3(N)=O\left(\frac{N^5\log N}{t^2}\right).$$

The definition of n in terms of t gives

$$t=\Omega\bigl(n^{2/3}(\log n)^{1/2+2c/3}\bigr)=\Omega\bigl(N^2(\log N)^{1/2+2c/3}\bigr).$$

Therefore

$$r_3(N)=O\left(\frac{N}{(\log N)^{4c/3}}\right).$$

This completes the proof of Theorem 1.2. □

7. Proof of Theorem 1.3

For Theorem 1.3, which states that

$$R\bigl(C_k,K_t^r\bigr)=\Omega^*\bigl(t^{1+\frac{1}{3k-1}}\bigr),$$

we let $G_{k,q}$ be an n-vertex $(q+1)$-uniform $(q+1)$-regular hypergraph with no cycles of length at most k, such that q is a maximum relative to n and such that $\lambda(G_{k,q})\leqslant 2\sqrt{q}$.

A construction of hypergraphs $G_{k,q}$ for primes $q\equiv 1 \bmod 4$ can be obtained from the construction of Ramanujan graphs of Lubotzsky, Phillips and Sarnak [16]. These $G_{k,q}$ are constructed from the following bipartite graphs of [16]: Let p,q be primes congruent to 1 modulo 4 with $p>16$. If $(\frac{p}{q})=-1$, then $B_{p,q}$ is a bipartite $(q+1)$-regular graph with $p(p^2-1)$ vertices in each part and no cycle of length less than $4\log_q(p/4)$. If $(\frac{p}{q})=1$, then $B_{p,q}$ is a bipartite $(q+1)$-regular graph with $p(p^2-1)/2$ vertices in each part and no cycle of length less than $2\log_q p$. In both cases $B_{p,q}$ has no cycle of length less than $2\log_q p$ since $p>16$, and the second largest eigenvalue in absolute value except the first and last is at most $2\sqrt{q}$.

So, given $k \geqslant 4$, we first choose a prime $q \equiv 1 \bmod 4$, then choose a smallest prime $p \equiv 1 \bmod 4$ with $p > q^k$. By the previous paragraph, for $n \in \{\frac{1}{2}p(p^2-1), p(p^2-1)\}$, there exists a $2n$-vertex bipartite $(q+1)$-regular graph $B_{p,q}$ of girth greater than $2k$. This $B_{p,q}$ is the bipartite incidence graph of a C_k-free $(q+1)$-graph $G_{k,q}$ on n vertices. And if we choose the smallest possible p, then $n < (1+o(1))q^{3k}$. Furthermore, it follows that $\lambda(G_{k,q}) \leqslant 2\sqrt{q}$.

Let $F_{k,q,r}$ denote the r-graph consisting of a vertex-disjoint union of $\tau = \lfloor 4\log q \rfloor$ stars of size $\lfloor q/\tau \rfloor$ on q vertices. In each edge of $G_{k,q}$, put a randomly permuted copy of $F_{k,q,r}$ to get the r-graph $H_{k,q,r}$. Corollary 4.3 shows that if X is a set of at least $2\tau n/q$ vertices of $H_{k,q,r}$, then at least $n - 8n/\tau$ edges of $G_{k,q}$ contain at least τ vertices of X. The expected number of independent sets in $H_{k,q,r}$ of size $2\tau n/q$ is at most

$$\left(1 - \frac{\tau^2}{10q}\right)^{n-8n/\tau} \binom{n}{2\tau n/q} < \exp\left(-\frac{\tau^2 n}{20q} + \frac{2\tau n \log n}{q}\right)$$

provided q is large enough. The choice of τ ensures this decays to zero. Therefore with positive probability,

$$\alpha(H_{k,q,r}) = O\left(\frac{\tau n}{q}\right) = O\left(n^{1-1/3k} \log n\right),$$

as long as $q > c_k n^{1/3k}$ for some constant c_k depending only on k.

Now suppose we are given $k \geqslant 4$ and an integer n not of the form required to construct $B_{p,q}$ and hence $G_{k,q}$ and $H_{k,q,r}$. For such an n, we will choose p, q so that the construction above is possible on n' vertices with $n < n' < 8n$, and then restrict the resulting $H_{k,q,r}$ (which has n' vertices) to a subhypergraph with only n vertices. The resulting n-vertex r-graph would again have independence number $O(n^{1-1/3k} \log n)$.

Given $k \geqslant 4$ and a sufficiently large n, choose a prime $q \equiv 1 \bmod 4$ such that

$$\frac{1}{2}(2n)^{1/3k} < q < (2n)^{1/3k}.$$

Such a q exists by the prime number theorem in arithmetic progressions. Next choose a prime $p \equiv 1 \bmod 4$ such that

$$(3n)^{1/3} < p < 2n^{1/3}.$$

Again, by the prime number theorem in arithmetic progressions, we can find such a p because n is sufficiently large. Now set $n' = p(p^2-1)/2$ or $p(p^2-1)$ depending on whether $(\frac{p}{q})$ is 1 or -1, and construct $H_{k,q,r}$ as described above. The resulting $(q+1)$-graph $H_{k,q,r}$ contains no C_k as $q < (2n)^{1/3k} < (3n)^{1/3k} < p^{1/k}$. Finally, observe that

$$n' > p^3/2 - p/2 > 3n/2 - n^{1/3} > n$$

and $n' < p^3 < 8n$. Moreover, $q > c_k n^{1/3k}$ so the above bound on the independence number holds as $n \to \infty$.

This shows that for any $r \geqslant 3$ and $k \geqslant 4$,

$$R\left(C_k, K_t^r\right) = \Omega^*\left(t^{1+\frac{1}{3k-1}}\right).$$

7.1. *Proof of Theorem 1.5*

The specialization of the above arguments to $k = 5$ comes from the existence of generalized hexagons (see [8] or [20]). The generalized hexagons G_q exist for prime powers q and can be viewed as $(q+1)$-uniform $(q+1)$-regular hypergraphs G_q on $q^5+q^4+q^3+q^2+q+1$ vertices containing no cycles of length at most five, and moreover the associated matrix $A(G_q)$ has $\lambda(G_q) = \sqrt{q}$ once more. Using the hypergraph $F_{k,q,r}$ in each edge of the hypergraph G_q as before gives the result: we obtain a hypergraph $H_{5,q,r}$ with

$$\alpha(H_{5,q,r}) = O\left(n^{4/5} \log n\right)$$

from which the lower bound on Ramsey numbers $R(C_5, K_t^r) = \Omega(t^{5/4}(\log t)^{-5/4})$ for all $r \geqslant 3$ follows.

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