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Turán problems and shadows I: Paths and cycles

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ABSTRACT

A *k-path* is a hypergraph $P_k = \{e_1, e_2, \ldots, e_k\}$ such that $|e_i \cap e_j| = 1$ if $|j-i| = 1$ and $e_i \cap e_j = \emptyset$ otherwise. A *k-cycle* is a hypergraph $C_k = \{e_1, e_2, \ldots, e_k\}$ obtained from a $(k-1)$-path $\{e_1, e_2, \ldots, e_{k-1}\}$ by adding an edge e_k that shares one vertex with e_1, another vertex with e_{k-1} and is disjoint from the other edges.

Let $\mathrm{ex}_r(n, G)$ be the maximum number of edges in an r-graph with n vertices not containing a given r-graph G. We prove that for fixed $r \geq 3$, $k \geq 4$ and $(k,r) \neq (4,3)$, for large enough n:

$$\mathrm{ex}_r(n, P_k) = \mathrm{ex}_r(n, C_k) = \binom{n}{r} - \binom{n - \lfloor \frac{k-1}{2} \rfloor}{r} + \begin{cases} 0 & \text{if } k \text{ is odd} \\ \binom{n-\lfloor \frac{k-1}{2} \rfloor -2}{r-2} & \text{if } k \text{ is even} \end{cases}$$

and we characterize all the extremal r-graphs. We also solve the case $(k,r) = (4,3)$, which needs a special treatment. The case $k = 3$ was settled by Frankl and Füredi.

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This work is the next step in a long line of research beginning with conjectures of Erdős and Sós from the early 1970s. In particular, we extend the work (and settle a conjecture) of Füredi, Jiang and Seiver who solved this problem for P_k when $r \geq 4$ and of Füredi and Jiang who solved it for C_k when $r \geq 5$. They used the delta system method, while we use a novel approach which involves random sampling from the shadow of an r-graph.

1. Introduction

An r-uniform hypergraph, or simply *r-graph*, is a family of r-element subsets of a finite set. Given a set $\mathcal{F}$ of r-graphs, an *$\mathcal{F}$-free r-graph* is an r-graph containing none of the members of $\mathcal{F}$. Let the *Turán number of $\mathcal{F}$*, $\mathrm{ex}_r(n, \mathcal{F})$, denote the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. When $\mathcal{F} = \{F\}$ we write $\mathrm{ex}_r(n, F)$. An n-vertex $\mathcal{F}$-free r-graph H is *extremal for $\mathcal{F}$* if $|H| = \mathrm{ex}_r(n, \mathcal{F})$.

In this paper we promote the idea of determining $\mathrm{ex}_r(n, \mathcal{F})$ for certain classes $\mathcal{F}$ by randomly sampling from the shadow of an $\mathcal{F}$-free r-graph H and using Hall-type combinatorial lemmas to determine the structure of the shadow and hence the structure of H. This paper focuses solely on paths and cycles. Our next paper will consider more general structures.

1.1. Definitions of paths and cycles

There are several natural generalizations to hypergraphs of paths and cycles in graphs. A *Berge k-cycle* is a hypergraph consisting of k distinct edges $e_0, \dots, e_{k-1}$ such that there exist k distinct vertices $v_0, v_1 \dots, v_{k-1}$ with $v_i \in e_{i-1} \cap e_i$ for all $i = 0, 1, \dots, k-1$ (indices count modulo k). Let $\mathcal{BC}_k$ denote the family of all Berge k-cycles. A *minimal k-cycle* is a Berge cycle $\{e_0, e_1, \dots, e_{k-1}\}$ such that $e_i \cap e_j \neq \emptyset$ if and only if $|j-i| = 1$ or $\{i, j\} = \{0, k-1\}$, and no vertex belongs to all edges. Let $\mathcal{C}_k$ denote the family of minimal k-cycles. Furthermore, a *linear k-cycle* is the member $C_k \in \mathcal{C}_k$ such that $|e_i \cap e_{i+1}| = 1$ for all $i = 0, 1, \dots, k-1$.

Every Berge (respectively, minimal and linear) *k-path* is obtained from a Berge (respectively, minimal and linear) $(k+1)$-cycle by deleting one edge. The family of Berge (respectively, minimal) k-paths is denoted by $\mathcal{BP}_k$ (respectively, $\mathcal{P}_k$). The linear k-path is denoted by P_k. The most restricted structures above are linear k-cycles and k-paths. We will refer to these simply as *k-cycles* and *k-paths*. In this paper, we study the extremal functions for k-paths and k-cycles and minimal k-paths and k-cycles.

1.2. The extremal function for k-cycles and k-paths

The extremal problem for P_k has been studied extensively. In the case of graphs, the Erdős–Gallai Theorem [9] shows $\mathrm{ex}(n, P_k) \leq \frac{k-1}{2}n$ and this is tight whenever $k|n$.

Frankl [11] solved the simplest case for r-graphs, namely $\mathrm{ex}_r(n, P_2)$, answering a question of Erdős and Sós. As far as exact results are concerned, it appears that even the next smallest case $\mathrm{ex}_r(n, P_3)$ was not determined until very recently. Füredi, Jiang and Seiver [14] determined $\mathrm{ex}_r(n, P_k)$ precisely for all $r \geq 4$, $k \geq 3$ and n large while also characterizing the extremal examples. They conjectured a similar result for $r = 3$. In this paper, we prove their conjecture and determine the extremal structures for large n.

The extremal problem for r-graphs for C_3 is also well-researched [6,12], indeed, the case $r = 2$ is precisely Mantel's theorem from 1907. Frankl and Füredi [12] showed that the unique extremal r-graph on $[n]$ not containing C_3 consists of all edges containing some $x \in [n]$, for large enough n. For $r = 3$, Csákány and Kahn [6] proved the same result for all $n \geq 6$. More recently, Füredi and Jiang [15] determined the extremal function for C_k for all $k \geq 3$, $r \geq 5$ and large n; their results substantially extend earlier results of Erdős and settled a conjecture of the last two authors for $r \geq 5$. They used the delta system method.

Our main result extends the Füredi–Jiang Theorem to the case of $r = 3, 4$. To describe the result, we need some notation. Let $[n] := \{1, 2, \dots, n\}$, and for $L \subset [n]$ let $S^r_L(n)$ denote the r-graph on $[n]$ consisting of all r-element subsets of $[n]$ intersecting L.

Theorem 1.1. *Let $r \geq 3$, $k \geq 4$, and $\ell = \lfloor \frac{k-1}{2} \rfloor$. For sufficiently large n,*

$$\mathrm{ex}_r(n, P_k) = \binom{n}{r} - \binom{n-\ell}{r} + \begin{cases} 0 & \text{if } k \text{ is odd} \\ \binom{n-\ell-2}{r-2} & \text{if } k \text{ is even} \end{cases}$$

with equality only for $S^r_L(n)$ if k is odd and $S^r_L(n) \cup F$ where F is extremal for $\{P_2, 2P_1\}$ on $n - \ell$ vertices. The same result holds for k-cycles except the case $(k, r) = (4, 3)$, in which case

$$\mathrm{ex}_3(n, C_4) = \binom{n}{r} - \binom{n-1}{r} + \max\left\{n - 3, 4\left\lfloor \frac{n-1}{4} \right\rfloor\right\}$$

with equality only for 3-graphs of the form $S^3_L(n) \cup F$ where F is extremal for P_2 on $n-1$ vertices.

Remarks. (1) By the Erdős–Ko–Rado Theorem [10], $\mathrm{ex}_r(n - \ell, \{P_2, 2P_1\}) = \binom{n-\ell-2}{r-2}$ for sufficiently large n, and a result of Erdős and Sós (see [11]) gives $\mathrm{ex}_3(n - 1, P_2) = \max\{n - 3, 4\lfloor \frac{n-1}{4} \rfloor\}$. These results account for the lower order terms in the expressions for $\mathrm{ex}_r(n, P_k)$ and $\mathrm{ex}_r(n, C_k)$ in Theorem 1.1.

(2) The proof of Theorem 1.1 restricted to the case of k-paths is substantially simpler than the proof for k-cycles.

(3) It was recently shown by Bushaw and Kettle [3] that the Turán problem for disjoint k-paths can be easily solved once we know the extremal function for a single k-path. As we have now solved the k-paths problem for all $r \geq 3$, the corresponding extremal questions for disjoint k-paths are also completely solved (for large n). A similar situation likely holds for disjoint k-cycles, as recently observed by Gu, Li and Shi [16].

1.3. The extremal function for minimal k-cycles and minimal k-paths

The related problems of determining $\mathrm{ex}(n, \mathcal{P}_k)$ and $\mathrm{ex}_r(n, \mathcal{C}_k)$ have also received considerable attention, indeed the case of $\mathcal{P}_2$ is the celebrated Erdős–Ko–Rado theorem. The last two authors [24] proved that $\mathrm{ex}(n, \mathcal{P}_3) = \binom{n-1}{r-1}$ for all $r \geq 3$ and $n \geq 2r$. The case of $\mathcal{C}_3$ goes back to Chvátal [4] in 1973, and in [23] the last two authors proved that $\mathrm{ex}_r(n, \mathcal{C}_3) = \binom{n-1}{r-1}$ for all $r \geq 3$ and $n \geq 3r/2$ thereby settling an old conjecture of Erdős [7]. They also proved some bounds for all k, r and conjectured that both of these extremal functions are asymptotic to $\ell\binom{n}{r-1}$. Füredi, Jiang and Seiver [14] proved the conjecture in strong form and determined $\mathrm{ex}(n, \mathcal{P}_k)$ for all $k, r \geq 3$ and n large. Füredi and Jiang [15] later determined $\mathrm{ex}(n, \mathcal{C}_k)$ exactly for all $k \geq 3$, $r \geq 4$ and n large. Our second theorem determines $\mathrm{ex}_r(n, \mathcal{C}_k)$ as well as the extremal $\mathcal{C}_k$-free r-graphs for all $r \geq 3$ and n large.

Theorem 1.2. *Let $r \geq 3$, $k \geq 5$, and $\ell = \lfloor \frac{k-1}{2} \rfloor$. Then for sufficiently large n,*

$$\mathrm{ex}_r(n, \mathcal{C}_k) = \binom{n}{r} - \binom{n-\ell}{r} + \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even} \end{cases}$$

with equality only for r-graphs of the form $S_L^r(n)$ with $|L| = \ell$ if k is odd, and $S_L^r(n)$ plus an edge when k is even. Also for each $r \geq 3$,

$$\mathrm{ex}_r(n, \mathcal{C}_4) = \binom{n}{r} - \binom{n-1}{r} + \left\lfloor \frac{n-1}{r} \right\rfloor$$

with equality only for r-graphs of the form $S_L^r(n) \cup F$ where F comprises $\lfloor \frac{n-1}{r} \rfloor$ disjoint edges.

The proof is very similar to that of Theorem 1.1 and some steps are easier, so we only indicate the differences in the proofs. The reader may observe that the approach also yields a proof for minimal paths that is substantially shorter than that in [14]. Furthermore, we believe our methods with some additional refinements give polynomial bounds on n relative to r and k above which Theorem 1.1 and Theorem 1.2 hold.

1.4. The extremal problem for Berge k-paths and k-cycles

Interesting results on the Turán-type problems for Berge k-paths and Berge k-cycles, were obtained by Bollobás and Győri [1] and in a series of papers by Győri, Katona and Lemons, in particular, in [17–19]. The bounds differ from those in Theorems 1.1 and 1.2. In particular, they are linear in n for $\mathrm{ex}_r(n, \mathcal{BP}_k)$. We do not study $\mathrm{ex}_r(n, \mathcal{BC}_k)$ in this paper. But if we forbid the family of Berge k-cycles or Berge k-paths in which no vertex belongs to at least 3 edges, then the answer is the same as in Theorem 1.2, apart from $k = 4$: the proof of the upper bound simply applies here, and the construction of $S_L^r(n)$ if k is odd and $S_L^r(n)$ plus one edge if k is even also applies. We remark that Turán-type problems for Berge cycles with other additional restrictions have been extensively studied in the literature. Very recently, Jiang and Collier-Cartaino [5] showed that a 2-linear r-graph on n vertices with no $2k$-cycle has $O(n^{1+1/k})$ edges, generalizing the Even Cycle Theorem of Bondy and Simonovits [2]. As another instance, for the minimal 4-cycle $C = \{e, f, g, h\}$ with $e \cup f = g \cup h$ and $e \cap f = g \cap h = \emptyset$, Erdős [8] conjectured $\mathrm{ex}_r(n, C) = O(n^{r-1})$, and this was proved by Füredi [13] (see also [13,22,25]). It seems likely that in this case the extremal C-free r-graphs for $r > 3$ are those in Theorem 1.2 for $k = 4$, and Füredi [13] conjectured $\mathrm{ex}_r(n, C) \sim \binom{n-1}{r-1}$.

1.5. Organization

We prove Theorem 1.1 in four steps in Section 6; first we give an asymptotic version, then a stability version followed by the proof of the exact result for cycles and the exact result for paths. The heart of the proof and all the new ideas lie in the proof of stability. The method of proof for the exact results in Sections 6.3 and 7 is not novel and the specific approach we take was used in [21].

Theorem 1.2 is proved in Section 7. In Sections 3–5 we prepare the background for passing from cycles and paths in the shadow of an r-graph to cycles and paths in the r-graph itself.

2. Notation and terminology

2.1. General notation

Edges of an r-graph H sometimes will be written as unordered lists, for instance, xyz represents $\{x, y, z\}$. For $X \subset V(H)$, let $H - X = \{e \in H : e \cap X = \emptyset\}$. The *codegree* of a set $S = x_1x_2 \dots x_s$ of vertices of H is $d_H(S) = |\{e \in H : S \subset e\}|$; when $s = r - 1$, the *neighborhood in H of S* is $N_H(S) = \{x : S \cup \{x\} \in H\}$, so that $|N_H(S)| = d_H(S)$. For vertices x, y in a hypergraph, an *x, y-path* is a path $P = e_0e_1 \dots e_k$ where $x \in e_0 - e_1$ and $y \in e_k - e_{k-1}$. Throughout the rest of the paper, when we say k-cycle (path) we mean linear k-cycle (path).

2.2. Shadows in hypergraphs

Now we state the crucial definitions involving shadows in hypergraphs. Let ∂H denote the $(r - 1)$-graph of sets contained in some edge of H — this is the *shadow* of H. The edges of ∂H will be called the *sub-edges* of H. If $G \subset \partial H$ and $F \subset H$ is obtained from G by adding distinct vertices of $V(H) - V(G)$ to each edge of G, then we say that G *expands* to F.

For $2 \le s < r$, let $\partial^1 H := \partial H$ and $\partial^s H = \partial^{s-1}\partial H$. The strategy to prove Theorem 1.1 is to find a cycle in the shadow of an r-graph that can be expanded to a cycle in the r-graph itself.

Definition 2.1. Let H be an r-graph. For $G \subset \partial H$ and $e \in G$, the *list* of e is

$$L_G(e) = N_H(e) - V(G).$$

The elements of $L_G(e)$ are called *colors*. We let $L_G = \bigcup_{e \in G} L_G(e)$ and

$$\hat{G} = \{e \cup \{x\} : e \in G, x \in L_G(e)\}.$$

Note that all these definitions are relative to the fixed host hypergraph H and the fixed subgraph G of ∂H. A key idea is that if C is a k-cycle or k-path in ∂H and the family $\{L_C(e) : e \in C\}$ has a system of distinct representatives, then $\hat{C}$ contains a k-cycle or k-path, and so H contains a k-cycle or k-path.

3. Full, superfull and linear hypergraphs

3.1. Full subgraphs

An r-graph H is *d-full* if every sub-edge of H has codegree at least d. Thus H is d-full exactly when the minimum non-zero codegree in H is at least d.

The following lemma extends the well-known fact that any graph G has a subgraph of minimum degree at least $d+1$ with at least $|G| - d|V(G)|$ edges.

Lemma 3.1. *For $r \geq 2$, $d \geq 1$, every n-vertex r-graph H has a $(d+1)$-full subgraph F with*

$$|F| \geq |H| - d|\partial H|.$$

Proof. A *d-sparse sequence* S is a maximal sequence $e_1, e_2, \dots, e_m \in \partial H$ such that $d_H(e_1) \leq d$, and for all $i > 1$, e_i is contained in at most d edges of H which contain none of $e_1, e_2, \dots, e_{i-1}$. The r-graph F obtained by deleting all edges of H containing at least one member of a d-sparse sequence S is $(d+1)$-full. Since S has length at most $|\partial H|$, we have $|F| \geq |H| - d|\partial H|$. □

Lemma 3.2. *Let $r \geq 3$, $k \geq 3$ and let H be a non-empty rk-full r-graph. Then $C_k, P_{k-1} \subset H$.*

Proof. Consider the graph $F = \partial^{r-2} H$. Every edge of H yields a K_r in F, so F contains a 3-cycle C_3. As H is rk-full, each edge of F is in at least rk triangles in F. We claim that F contains a k-cycle: we start from C_3, and for $i = 3, \dots, k-1$, obtain an $(i+1)$-cycle C_{i+1} from i-cycle C_i by using one of the at least $rk - i + 2$ triangles containing an edge of C_i and no other vertices of C_i. Let a k-cycle C_k in F have edges $f_1, \dots, f_k$. Choose in H edges $e_1 = f_1 \cup g_1, \dots, e_k = f_k \cup g_k$ so that to maximize the size of $Y = \bigcup_{i=1}^k e_i$. Suppose $C = \{e_1, \dots, e_k\}$ is not a k-cycle in H. Then there are distinct i, j such that

$g_i \cap g_j \neq \emptyset$. Pick $v \in g_i \cap g_j$. Let $Z = \{z \in V(H) : (f_i \cup g_i \cup \{z\}) - \{v\} \in H\}$. Since H is rk-full, $|Z| \geq rk$. As C is not a k-cycle, $|Y| < rk$ and so there exists $z \in Z - Y$. Replacing e_i with $e = (f_i \cup g_i \cup \{z\}) - \{v\}$, we enlarge Y, a contradiction. So H contains C_k and thus P_{k-1}. □

3.2. Superfull subgraphs

Recall that $\ell = \lfloor (k-1)/2 \rfloor$.

Definition 3.3. An ℓ-full r-graph H is ℓ-superfull if for every edge e of H at most one sub-edge of e has codegree less or equal to rk.

Lemma 3.4. *Let $k, r \geq 3$, and let H be an ℓ-superfull r-graph such that H contains a minimal k-cycle (respectively, a minimal k-path). Then H contains a k-cycle (respectively, a k-path).*

Proof. The proofs for paths and cycles are similar, so we only do the case of cycles. Let $C \subset H$ be a minimal k-cycle with maximum $|V(C)|$. If C is not a k-cycle, then we find consecutive edges $f, g \in C$ with $|f \cap g| \geq 2$. Let $x, y \in f \cap g$. Since H is ℓ-superfull, we may assume $d_H(f - \{x\}) \geq rk$. Since $|V(C)| < rk$, we find $z \notin V(C)$ such that $h = f \cup \{z\} - \{x\} \in H$. Then $C' = C \cup \{h\} - \{f\}$ has more vertices than C, a contradiction. □

Lemma 3.5. *Let $r \geq 3$, $k \geq 4$ and let H be an ℓ-superfull r-graph containing a set W of at least rk vertices such that every $(r-1)$-subset of W has codegree exactly ℓ. Let G be the set of all $(r-1)$-subsets of W. If H has no k-cycle or no k-path, then for some set L of ℓ vertices of $H - W$, $L_G(e) = L$ for every $(r-1)$-set $e \subset W$.*

Proof. If $e \cup \{x\} \in H$ for some $x \in W$, then all $(r-1)$-subsets of $e \cup \{x\}$ have codegree exactly ℓ, contradicting the fact that H is ℓ-superfull. Thus, $N_H(e) \cap W = \emptyset$ for all $e \in G$.

Suppose that $L_G(f) \neq L_G(e)$ for some $e, f \in G$. Then there are $e_1, e_2 \in G$ such that $|e_1 \cap e_2| = 1$ and $L_G(e_2) \neq L_G(e_1)$, since from $|W| \geq rk \geq 4r$, for every two distinct $e, f \in G$, there is $g \in G$ sharing exactly one vertex with each of e and f. In particular,

$$\left|L_G(e_1) \cup L_G(e_2)\right| \geq \ell + 1. \tag{1}$$

Case 1: $\ell \geq 2$ and H has no k-cycle. Let $e_3, \dots, e_{\ell+1} \in G$ be such that $C = \{e_1, e_2, \dots, e_{\ell+1}\}$ is an $(\ell+1)$-cycle. By (1), the family $\{L_G(e_i) : 1 \leq i \leq \ell+1\}$ has a system of distinct representatives $\{v_i \in L_G(e_i) : 1 \leq i \leq \ell+1\}$. As observed above, $v_i \notin W$ for all i.

Let $e_i \cap e_{i+1} = \{w_{i+1}\}$ and $X_i = e_i \cup \{v_i\} - \{w_i, w_{i+1}\}$, with subscripts modulo $\ell+1$. Then each of $X_i \cup \{w_i\}$ and $X_i \cup \{w_{i+1}\}$ has codegree at least rk in H, since H is ℓ-superfull and e_i has codegree exactly ℓ. Thus for each $1 \leq i \leq \ell$, we can select edges

$f_i, g_i \in H$ with $X_i \cup \{w_i\} \subset f_i$ and $X_i \cup \{w_{i+1}\} \subset g_i$ forming a minimal $(2\ell+2)$-cycle in H if k is even. We let $f_{\ell+1} = g_{\ell+1} = e_{\ell+1}$ to obtain a minimal $(2\ell+1)$-cycle if k is odd. In both cases, H contains a minimal k-cycle, and so by Lemma 3.4, H contains a k-cycle.

Case 2: $\ell = 1$ and H has no 4-cycle. Let e_3 be a sub-edge such that $\{e_1, e_2, e_3\}$ is a 3-cycle. For $i = 1, 2, 3$, let $L_G(e_i) = \{v_i\}$ and $e_i \cap e_{i+1} = \{w_i\}$. Note again that $v_i \notin W$. By symmetry, we may assume that $v_1 \notin \{v_2, v_3\}$. Since H is ℓ-superfull and e_1 has codegree exactly ℓ, the sub-edges $e' = e_1 - w_1 + v_1$ and $e'' = e_1 - w_3 + v_1$ have codegrees at least $3r$. So we can select edges $g_1 \supset e'$ and $g_2 \supset e''$, so that $\{e_2, e_3, g_1, g_2\}$ is a minimal 4-cycle in H. Applying Lemma 3.4, we conclude that H contains a 4-cycle.

Case 3: H has no k-path. We repeat Case 1, except we use an $(\ell+1)$-path instead of C. □

3.3. Linear hypergraphs

In the last two sections we showed how to pass from cycles and paths in the shadow of full and superfull subgraphs of an r-graph H to cycles and paths in H itself. Here we consider the case that all sub-edges have bounded codegrees. The following fact is due to Erdős (see Theorem 1 in [7]):

Proposition 3.6. *(See Erdős [7].) For $r, t \geq 2$ there exists $n_0 = n_0(r, t)$ such that for all $n > n_0$, every n-vertex r-graph H with $|H| > n^{r - t^{1-r}}$ contains the complete r-partite r-graph $K^r_{t,\dots,t}$.*

Definition 3.7. An n-vertex r-graph H is (t, c)-*sparse* if every t-set of vertices lies in at most c edges of H. If $c = 1$, then H is t-linear.

The famous Ruzsa–Szemerédi $(6, 3)$-Theorem [26] shows that any linear 3-graph on n vertices and $\Omega(n^2)$ edges contains C_3. The following generalization was proved for $r = 3$ by Sárkőzy and Selkow [27] using the Regularity Lemma. We avoid the use of regularity for $r > 3$ and for these r the proof below actually gives a bound of $O(n^{r-1-\delta})$ for some $\delta > 0$.

Proposition 3.8. *Fix $c > 0$ and $r, k \geq 3$. Let H be an n-vertex $(r-1, c)$-sparse r-graph not containing P_k or not containing C_k. Then $|H| = o(n^{r-1})$.*

Proof. It suffices to prove the result for C_k since $P_k \subset C_{k+1}$. In view of the Ruzsa–Szemerédi $(6, 3)$-Theorem [26], we consider only $r \geq 4$. Consider the graph with vertex set H in which two vertices are adjacent if the intersection of the corresponding edges of H has size $r - 1$. Since H is $(r-1, c)$-sparse, this graph has maximum degree less than rc, so it contains an independent set H_0 of size at least $|H|/rc$. This means that H_0 is an $(r-1)$-linear r-graph.

Assume that $\epsilon > 0$, n is sufficiently large, and $|H_0| > \epsilon n^{r-1}$. A standard averaging argument shows that there is an r-partite subgraph of H_0 with at least $(r!/r^r)|H_0|$ edges. Let $X_1, \ldots, X_r$ be the r parts and consider the edge-colored $(r-1)$-partite $(r-1)$-graph $H' \subset \partial H_0$ with parts $X_1, \ldots, X_{r-1}$ where the color of the edge $\{x_1, \ldots, x_{r-1}\}$, with $x_i \in X_i$ for $i \in [r-1]$ is the unique $x_r \in X_r$ such that $\{x_1, \ldots, x_r\} \in H_0$. Such x_r is unique as H_0 is $(r-1)$-linear. We will find a *rainbow* C_k in H' — in other words a k-cycle in H' whose colors are all unique. Since $|H'| > (\epsilon r!/r^r)n^{r-1}$ and n is large, by Proposition 3.6, there is a complete $(r-1)$-partite $(r-1)$-graph $K = K_{k,k,\ldots,k,s} \subset H'$ where $s = k^{2r-3} + 1$ that has the same $(r-1)$-partition as H'. Since H_0 is $(r-1)$-linear, every color class S_c in H' is $(r-2)$-linear. Now construct a hypergraph H^* with vertex set X_r (these are the colors of H') and s edges, where the ith edge consists of the set of colors on edges incident to the ith vertex of K in the part of size s. Note that H^* need not be uniform, but its edges have size at most k^{r-2}.

Pick a color c (recall that c is a vertex of H^*). The number of edges of H^* (these correspond to vertices of K in X_{r-1}) containing c is at most k^{r-2} since S_c is $(r-2)$-linear. So H^* has maximum degree at most k^{r-2}, edges of size at most k^{r-2}, and size s. Therefore H^* has a matching M of size $s' = \lceil s/k^{2r-4} \rceil > k$ (by the greedy algorithm). This means that K contains the complete $(r-1)$-partite $(r-1)$-graph $K' = K_{k,k,\ldots,k,s'}$ with partite sets $X'_1, \ldots, X'_{r-1}$, $|X'_1| = \ldots = |X'_{r-2}| = k$, and $|X'_{r-1}| = s'$ (here X'_{r-1} corresponds to M) such that

$$\text{no two edges } e, e' \text{ with the same color are incident to different vertices in } X'_{r-1}. \tag{2}$$

Let $x \in X'_{r-1}$. We claim that

$$\text{there is a pair } \{e_1, e_2\} \text{ of edges in } K' \text{ of different colors such that } e_1 \cap e_2 = \{x\}. \tag{3}$$

Indeed consider two edges $e = \{x_1, \ldots, x_{r-2}, x\}$ and $e' = \{x_1, \ldots, x_{r-3}, x'_{r-2}, x\}$ of K' that differ only in $(r-2)$th coordinate. Since H_0 is an $(r-1)$-linear, they have different colors. Then for any edge $e'' \in K'$ that shares only x with $e \cup e'$, either $\{e, e''\}$ or $\{e', e''\}$ satisfies (3).

Consider a k-cycle $C' = \{e_1, \ldots, e_k\}$ in K' such that e_1 and e_2 satisfy (3) and for every $i \neq 1$, the vertex $v_i \in e_i \cap e_{i+1}$ is not in X'_{r-1}. By (2) and (3), C' is a rainbow k-cycle in K' and we expand it to a k-cycle in H. □

4. Cycles and paths from shadows

We now present the key lemmas which show how to expand k-paths and k-cycles in ∂H to paths and cycles in H itself. Throughout this section, $r, k \geq 3$ and $\ell = \lfloor \frac{k-1}{2} \rfloor$.

4.1. Paths

Lemma 4.1. *Let $k \geq 3$, let H be an r-graph and let $P = \{e_0, e_1, \dots, e_{2^{2\ell+1}-1}\}$ be a $2^{2\ell+1}$-path in ∂H. If $|L_P(e)| \geq \ell + 1$ for all $e \in P$, then $\hat{P}$ contains a k-path whose first edge contains e_0.*

Proof. As $\lfloor (k-1)/2 \rfloor = \lfloor (k-2)/2 \rfloor$ for k even, it is enough to consider even $k \geq 4$. First we prove the lemma for $k = 4$, and then apply an inductive proof. The case $k = 4$ is split into two cases:

Case 1: $L_P(e_0) \cap L_P(e_i) \neq \emptyset$ for some $i > 1$.
Let $\alpha \in L_P(e_0) \cap L_P(e_i)$ and let $e_i, f, g, h \in P$ form a path vertex-disjoint from e_0 — this exists since P has eight edges. Define $L'(e) = L_P(e) - \{\alpha\}$ for $e \in P$. If we find distinct $\beta \in L'(f)$ and $\gamma \in L'(g)$, then $\{e_0 \cup \{\alpha\}, e_i \cup \{\alpha\}, f \cup \{\beta\}, g \cup \{\gamma\}\}$ is a 4-path. Otherwise, $L_P(f) = L_P(g) = \{\alpha, \alpha'\}$ for some α'. The same argument with f in place of e_i shows $L_P(g) = L_P(h) = \{\alpha, \alpha'\}$, in which case the required 4-path is $\{e_0 \cup \{\alpha\}, e_i \cup \{\alpha\}, f \cup \{\alpha'\}, h \cup \{\alpha'\}\}$.

Case 2: $L_P(e_0) \cap L_P(e_i) = \emptyset$ for all $i > 1$.
Let $L_P(e_0) = \{\alpha, \beta\}$. If $L_P(e_0) \cap L_P(e_1) \neq \emptyset$, say, $\beta \in L_P(e_1)$, then by the case, we may pick distinct $\gamma \in L_P(e_2)$ and $\delta \in L_P(e_3)$ so that $\{e_0 \cup \{\alpha\}, e_1 \cup \{\beta\}, e_2 \cup \{\gamma\}, e_3 \cup \{\delta\}\}$ is a 4-path, as required. Suppose $L_P(e_0) \cap L_P(e_1) = \emptyset$. If there is $\gamma \in L_P(e_1) \cap L_P(e_3)$, then choose any $\lambda \in L_P(e_4) - \gamma$, and the edges $e_0 \cup \{\alpha\}, e_1 \cup \{\gamma\}, e_3 \cup \{\gamma\}, e_4 \cup \{\lambda\}$ form a 4-path. Otherwise, as $|L_P(e_i)| \geq 2$ for $i \geq 1$, we can choose all distinct $\alpha_1 \in L_P(e_1)$, $\alpha_2 \in L_P(e_2)$, $\alpha_3 \in L_P(e_3)$, and the edges in the set $\{e_i\{\alpha_i\} : i = 1, 2, 3\}$ together with $e_0 \cup \{\alpha\}$ form a 4-path.

Now suppose $k \geq 6$. If for some $i > 1$ we have $\beta \in L_P(e_0) \cap L_P(e_i)$, let $P' = \{e_{i+1}, e_{i+2}, \dots, e_{i+2^{k-3}}\}$ if $i \leq 2^{k-3} + 1$ and $P' = \{e_{i-1}, e_{i-2}, \dots, e_{i-2^{k-3}}\}$ if $i > 2^{k-3} + 1$ (note that $i - 2^{k-3} \geq 2$). Let $e_0' = e_{i+1}$ if $i \leq 2^{k-3} + 1$ and $e_0' = e_{i-1}$ if $i > 2^{k-3} + 1$. Let us remove β from all lists of edges of P'. Then P' is a 2^{k-3}-path all of whose lists have size at least ℓ. So by induction on k, $\hat{P} - \beta$ has a $(k-2)$-path $\{f_2, f_3, \dots, f_{k-1}\}$ where $e_0' \subset f_2$. Set $f_0 = e_0 \cup \{\beta\}$, $f_1 = e_i \cup \{\beta\}$. Then $\{f_0, f_1, \dots, f_k\}$ is the required k-path. So we may assume for all $i > 1$, $L_P(e_0) \cap L_P(e_i) = \emptyset$. If we find $\gamma \in L_P(e_1) - L_P(e_0)$, then remove γ from all lists $L_P(e_i)$ where $i \geq 2$. Let $\hat{P}' = \hat{P} - L_P(e_0) - \{\gamma\}$ if γ exists and $\hat{P}' = \hat{P} - L_P(e_0)$ otherwise (in this case $L_P(e_1) \subset L_P(e_0)$). By induction, $\hat{P}'$ contains a $(k-2)$-path $\{f_2, f_3, \dots, f_{k-1}\}$ with $e_2 \subset f_2$ as the lists sizes have reduced by at most one. Set $f_0 = e_0 \cup \{\alpha\}$, $f_1 = e_1 \cup \{\beta\}$ with $\alpha \neq \beta$, $\alpha \in L_P(e_0)$ and $\beta \in L_P(e_1) \cup \{\gamma\}$ (if γ exists we may choose $\beta = \gamma$); this works since $|L_P(e)| \geq 2$ for $e \in P$. Now $\{f_0, f_1, \dots, f_{k-1}\} \subset \hat{P}$ is a k-path. □

4.2. Cycles

To extend Lemma 4.1 to k-cycles, we need the following technical definition.

Definition 4.2. Let H be an r-graph where $r \geq 3$. Let $\Psi_t(H)$ be the set of complete $(r-1)$-partite $(r-1)$-graphs $G \subset \partial H$ with parts of size t and $|L_G(e)| > \ell$ for all $e \in G$, and if $r = 3$ and k is odd, then in addition for $xy \in G$, there is $xy\alpha \in \hat{G}$ such that

(a) $\min\{d_H(x\alpha), d_H(y\alpha)\} \geq 2$ and
(b) $\max\{d_H(x\alpha), d_H(y\alpha)\} \geq 3k + 1$.

The additional technical conditions for $r = 3$ and k odd will become apparent in the proof of Case 2 of Lemma 4.4 below. We also will use the following simple fact.

Lemma 4.3. *Let $p \geq 1$ and $q \in \{2p, 2p+1\}$, and let $S_1, S_2, \dots, S_q$ be sets such that*

$$S_i \cap S_j = \emptyset \quad \text{for } i \leq p \text{ and } j \geq p+2, \quad \text{and} \quad |S_i| \geq p+1 \quad \text{for } i \leq p \tag{4}$$

and $|S_i| \geq p$ for $i \geq p+1$. Then $\{S_1, S_2, \dots, S_q\}$ has a system of distinct representatives, unless $q = 2p+1$ and all S_j for $j > p$ are all equal and of size p.

Proof. If we find an SDR for $\{S_{p+1}, \dots, S_q\}$, then by (4) we can greedily extend it to an SDR for $\{S_1, \dots, S_q\}$. So suppose we cannot. Since $|S_i| \geq p$ for every i, this means $q - p > p$ (i.e., $q = 2p+1$) and all $S_{p+1}, \dots, S_{2p+1}$ are the same. □

Lemma 4.4. *Let $r \geq 3, k \geq 4$. Then there exists a $t_0 = t_0(r, k)$ such that for all $t > t_0$ and for all C_k-free r-graphs H, $\Psi_t(H) = \emptyset$.*

Proof. Suppose $G \in \Psi_t(H)$. Let M be a set of $s = 2^{k-2}(r-1)$ pairwise disjoint edges of G. Suppose first that $\alpha \in L_G(e)$ for all $e \in M$. Let $F \subset G$ be a complete $(r-1)$-partite subgraph of G with parts of size 2^{k-2} and

- $V(F) \subset V(M)$,
- $\forall f \in F$, $e \in M$, $|f \cap e| \leq 1$
- $\forall f \in F$ there exist $r-1$ distinct $e \in M$ with $|f \cap e| = 1$.

We will show that $\hat{F}$ contains a $(k-2)$-path avoiding α. For $k \geq 5$, F contains a 2^{k-2}-path, so by Lemma 4.1, $\hat{F}$ contains a $(k-2)$-path that does not use α on its lists. If $k = 4$ and F has lists of size 1 after removing α, we cannot use Lemma 4.1 to find a $(k-2)$-path as $k - 2 < 3$. To find a 2-path in $\hat{F}$ in this case, consider any 3-path $\{f_1, f_2, f_3\}$ in F. Suppose $\beta_i \in L_G(f_i) - \alpha$ for $i = 1, 2, 3$. If $\beta_1 = \beta_3$, then $\{f_1 \cup \beta_1, f_3 \cup \beta_1\}$ is a 2-path; otherwise either $\{f_1 \cup \beta_1, f_2 \cup \beta_2\}$ or $\{f_2 \cup \beta_2, f_3 \cup \beta_3\}$ is a 2-path. For all $k \geq 4$ we have found $x, y \in V(F) \subset V(M)$ and an xy-path $\hat{P} \subset \hat{F} - \{\alpha\}$ of length $k-2$. Picking edges $e, f \in M$ with $x \in e$ and $y \in f$, $\hat{P} \cup \{e \cup \{\alpha\}, f \cup \{\alpha\}\}$ is a k-cycle in $\hat{G}$, a contradiction. We conclude that

$$\text{no color appears in the lists of } s \text{ pairwise disjoint edges of } G. \tag{5}$$

For every $e \in G$, fix a subset $L'_G(e)$ of $L_G(e)$ with $|L'_G(e)| = \ell + 1$. Let $m = \lfloor t/(s+2) \rfloor$. For $i \in [m]$, let $F_i \subset G$ be vertex-disjoint complete $(r-1)$-partite graphs with parts of size $s+2$, and $L'_i = \bigcup\{L'_G(e) : e \in F_i\}$. Then $|L'_1| \le (\ell+1)|F_1| \le (s+2)^r$. For each color $\alpha \in L'_1$, by (5), there are at most s different i for which $\alpha \in L'_i \cap L'_1$. So $L'_i \cap L'_1 \ne \emptyset$ for at most $(s+2)^{r+1}$ values $i \in [m]$. Choose t so that $m > (s+2)^{r+1}$. Then for some $i > 1$, $L'_i \cap L'_1 = \emptyset$, say for $i = 2$. Let $F = F_1 \cup F_2$ and let X, Y be two parts of F. Select $e \in G$ with $e \cap V(F_1) = \{x\} \subset X$ and $e \cap V(F_2) = \{y\} \subset Y$.

Case 1: $r > 3$, or $r = 3$ and k is even. Let $e \cup \{\alpha\} \in \hat{G}$. By the symmetry between L'_1 and L'_2 we may suppose $\alpha \notin L'_1$. Let $q = k-1$ and $p = \ell$. If p is odd, then let $U = V(F_1) \cap X$ and $V = V(F_2) \cap Y$, while if p is even, then let $U = V(F_1) \cap Y$ and $V = V(F_2) \cap X$.

Let f be any edge $f \in G$ with $|f \cap U| = 1 = |f \cap V|$ and $|f \cap V(F)| = 2$. Since U and V are subsets of different parts in F and $r > 3$, or $r = 3$ and k is even, there is a q-path $Q = \{f_1, f_2, \ldots, f_q\}$ from x to y in G with $f_i \subset F_1$ for $i \le p$, $f_{p+1} = f$, and $f_i \subset F_2$ for $i > p+1$. Indeed, the case $r > 3$ is trivial as we have space to easily obtain Q while the case $r - 1 = 2$ and k even relies on the careful placement of the edge f based on the parity of p. If Q expands to a q-path $\hat{Q} \subset \hat{G} - \alpha$, then $\hat{Q} \cup \{e \cup \{\alpha\}\}$ is a k-cycle in $\hat{G}$, a contradiction. Therefore

$$Q \text{ does not expand to a } q\text{-path in } \hat{G} - \alpha. \tag{6}$$

Now let $S_i = L'_G(f_i) - \alpha$ for $1 \le i \le q$. Since $L'_1 \cap L'_2 = \emptyset$, we have $S_i \cap S_j = \emptyset$ for $i \le p$ and $j > p+1$, and since $\alpha \notin L'_1$, $|S_i| > p$ for $i \le p$, and $|S_i| \ge |L'_G(f_i)| - 1 \ge p$ for $i > p$. By (6), the family $\{S_1, S_2, \ldots, S_q\}$ has no system of distinct representatives. By Lemma 4.3, all S_i for $i > p$ are identical of size $p = \ell$, and since $|L'_g(f_i)| = \ell + 1$, we have $\alpha \in L'_G(f)$. Since f was any edge with $|f \cap U| = 1 = |f \cap V|$ and $|f \cap V(F)| = 2$, G is complete $(r-1)$-partite, t is large, and $|U|, |V| \ge s$, we have s disjoint edges of G whose lists all contain α, contradicting (5). This finishes Case 1.

Case 2: $r = 3$ and k is odd. Let $q = k-2$ and $p = \ell - 1$, so $q = 2p+1$. Since $G \in \Psi_t(H)$, some $xy\alpha \in \hat{G}$ satisfies (a) and (b) in Definition 4.2. Again, since $L'_1 \cap L'_2 = \emptyset$, we may suppose $\alpha \notin L'_1$. By symmetry we may assume $d_H(x\alpha) > 3k$ and $d_H(y\alpha) > 1$. Choose an edge $y\alpha\beta \in H$ with $\beta \ne x$. Note that possibly $\beta \in V(G)$. For $i = 1, 2$, let $X_i = X \cap V(F_i) - \{x, \beta\}$ and $Y_i = Y \cap V(F_i) - \{y, \beta\}$. Let $f \in G$ be such that

$$\begin{aligned} |f \cap X_1| = 1 = |f \cap Y_2| \quad & \text{if } q \equiv 1 \pmod 4, \\ |f \cap X_2| = 1 = |f \cap Y_1| \quad & \text{if } q \equiv 3 \pmod 4. \end{aligned}$$

Since q is odd, there is a q-path $Q = \{f_1, f_2, \ldots, f_q\}$ from x to y in G with $f_i \subset F_1$ for $i \le p$, $f_{p+1} = f$, and $f_i \subset F_2$ for $i > p+1$. If Q expands to a q-path $\hat{Q} \subset \hat{G} - \alpha - \beta$, then select $\gamma \in V(H) - V(\hat{Q}) - \alpha - \beta$ so that $x\alpha\gamma \in H$ — this is possible since $d_H(x\alpha) > 3k$ — and then $\hat{Q} \cup \{x\alpha\gamma, y\alpha\beta\}$ is a k-cycle in $\hat{G}$. So

$$Q \text{ does not expand to a } q\text{-path in } \hat{G} - \alpha - \beta. \tag{7}$$

Let $S_i = L'_G(f_i) - \alpha - \beta$. Since $L'_1 \cap L'_2 = \emptyset$, we have $S_i \cap S_j = \emptyset$ for $i \le p$ and $j > p+1$, and since $\alpha \notin L'_1$, $|S_i| = |L'_G(f_i) - \beta| \ge \ell > p$ for $i \le p$, and $|S_i| \ge |L'_G(f_i)| - 2 \ge p$ for $i > p$. By (7), the family $\{S_1, S_2, \dots, S_q\}$ has no system of distinct representatives. By Lemma 4.3, all S_i for $i > p$ are identical, and $|S_i| = p$ for all i, which forces $\alpha \in S_i$, in particular, $\alpha \in L'_G(f)$. Since f was an arbitrary edge joining X_1 to Y_2 or joining X_2 to Y_1 and $|X_i|, |Y_i| \ge s$ for $i = 1, 2$, this contradicts (5). $\square$

5. Random sampling

We use a random sampling technique and Lemmas 4.4 and 4.1 to find k-cycles and k-paths in an r-graph H when H has many sub-edges of codegree at least $\ell + 1$.

Lemma 5.1. *Let $\delta > 0$, $r \ge 3$ and $k \ge 4$. Let H be an r-graph, and $E \subset \partial H$ with $|E| > \delta n^{r-1}$. Suppose that $d_H(f) \ge \ell + 1$ for every $f \in E$ and, if $r = 3$ and k is odd, then in addition, for every $f = xy \in E$ there is $e_f = xy\alpha \in H$ such that $\min\{d_H(x\alpha), d_H(y\alpha)\} \ge 2$ and $\max\{d_H(x\alpha), d_H(y\alpha)\} \ge 3k+1$. Then for large enough n, H contains P_k and C_k.*

Proof. By Lemmas 4.4 and 4.1, it is enough to prove that $\Psi_t(H) \ne \emptyset$ for a large enough t.

Let $m = \ell + 1$ and T be a random subset of $V(H)$ obtained by picking each vertex independently with probability $p = 1/2$. Let

$$F = \{f \in E : f \subset T, |N_H(f) - T| \ge m, e_f - f \not\subset T\}.$$

For $f \in E$ and any choice of edges $e_1, e_2, \dots, e_m \in H$ containing f such that $e_1 = e_f$, the probability that $f \subset T$ and $e_i - f \not\subset T$ for $i \in [m]$ is exactly $p^{r-1}(1-p)^m$. Therefore

$$\mathbb{E}(|F|) \ge |E| p^{r-1}(1-p)^m \ge \delta 2^{-m-r+1} n^{r-1}.$$

So there is a $T \subset V(H)$ with $|F| \ge \delta 2^{-m-r+1} n^{r-1}$. If n is large enough, Proposition 3.6 gives a complete $(r-1)$-partite $G \subset F$ with parts of size t. Since $|L_G(f)| \ge |N_H(f) - T| \ge m$ for $f \in G$, $G \in \Psi_t(H)$ for $r \ge 4$ and for even k when $r = 3$. Suppose $r = 3$ and k is odd. Then since for every $f \in G$, $e_f \in \hat{G}$, again $G \in \Psi_t(H)$. $\square$

6. Proof of Theorem 1.1

6.1. Part I: asymptotics

Theorem 6.1. *Let $r \ge 3$, $k \ge 4$.*

(a) If H is an n-vertex $(\ell+1)$-full r-graph and $C_k \not\subset H$ or $P_k \not\subset H$, then $|H| = o(n^{r-1})$.
(b) $\mathrm{ex}_r(n, P_k) \sim \mathrm{ex}_r(n, C_k) \sim \ell\binom{n}{r-1}$.

Proof. To prove (a), we first show

$$|\partial H| = o\big(n^{r-1}\big). \tag{8}$$

Suppose that $|\partial H| > \delta n^{r-1}$ where $\delta > 0$, and n is large. If $r > 3$ or $r = 3$ and k is even, then by Lemma 5.1 with $E = \partial H$, if n is large enough, then H contains a k-cycle and a k-path, a contradiction.

For $r = 3$ and k odd, let H^* be the set of edges of H containing no pair of codegree at least $3k$. Then H^* is $(2, 3k)$-sparse, so by Proposition 3.8, $|H^*| = o(n^2)$. Let $F = \partial H - \partial H^*$ so that for every $f \in F$, there is an edge $e \in H$ containing f and containing a pair f' with $d_H(f') > 3k$ (possibly, $f' = f$). Then $|F| \geq |\partial H| - |\partial H^*| \geq \delta n^2 - o(n^2) > (\delta/2)n^2$ if n is large enough.

If all edges of H containing a pair $f \in F$ have all their sub-edges of codegree greater than $3k$, map f to itself. Otherwise, pick an edge of H containing f and containing some pair f' of codegree at most $3k$, and map f to f' (again $f = f'$ is possible). This map is at most $6k$ to one, and therefore we have a set E of $(\delta/12k)n^2$ pairs in ∂H each of codegree at least $\ell + 1$ in H and each $f \in E$ is contained in some edge $e_f \in H$ in which some pair has codegree at least $3k + 1$. Since H is $(\ell + 1)$-full, the conditions of Lemma 5.1 hold for E, and so H contains a k-cycle and a k-path, a contradiction. So we proved (8) in both cases.

Now by Lemma 3.1, H has an $r(k + 1)$-full subgraph H' with

$$|H'| \geq |H| - r(k + 1)|\partial H|.$$

By Lemma 3.2, if $H' \neq \emptyset$, then $P_k, C_k \subset H' \subset H$, which is a contradiction. We conclude $H' = \emptyset$, and so $|H| \leq r(k + 1)|\partial H| = o(n^{r-1})$, which proves (a).

Now we determine the asymptotic value of $\mathrm{ex}_r(n, C_k)$ and $\mathrm{ex}_r(n, P_k)$. The construction $S_L^r(n)$ in the statement of Theorem 1.1 shows $\mathrm{ex}_r(n, C_k), \mathrm{ex}_r(n, P_k) \geq \binom{n}{r} - \binom{n-\ell}{r} \sim \ell\binom{n}{r-1}$. Suppose H is an r-graph and $C_k \not\subset H$ or $P_k \not\subset H$. By Lemma 3.1, H has an $(\ell + 1)$-full subgraph H' with $|H'| \geq |H| - \ell|\partial H|$. By (a), $|H'| = o(n^{r-1})$. So $|H| \leq |H'| + \ell|\partial H| \leq o(n^{r-1}) + \ell\binom{n}{r-1}$. □

6.2. Part II: stability

Theorem 6.2. *Fix $r \geq 3$, $k \geq 4$ and let H be an n-vertex r-graph with $|H| \sim \ell\binom{n}{r-1}$ containing no k-cycle or no k-path. Then there exists $G^* \subset \partial H$ with $|G^*| \sim \binom{n}{r-1}$ and a set L of ℓ vertices of H such that $L_{G^*}(e) = L$ for every $e \in G^*$. In particular, $|H - L| = o(n^{r-1})$.*

Proof. Let H^* be the set of edges of H not containing any sub-edge of codegree at least $rk + 1$. Then H^* is $(r - 1, rk)$-sparse, so Proposition 3.8 implies $|H^*| = o(n^{r-1})$. Let $H' = H - H^*$, so $|H'| \sim |H|$. We construct sequences $f_1, f_2, \ldots, f_q \in \partial H'$ and

$H_0, H_1, \ldots, H_q \subset H$ with $H_0 = H'$ as follows. Suppose H_i is constructed and let $d_i(f) = d_{H_i}(f)$. A sub-edge f of H_i is of type

(i) if $d_i(f) < \ell$,
(ii) if $d_i(f) = \ell$ and some $e \in H_i$ containing f contains a sub-edge $g \neq f$ with $d_i(g) = \ell$,
(iii) if $\ell < d_i(f) < rk$.

If H_i has no sub-edges of types (i)–(iii), let $q = i$ and stop. Otherwise, let f be a sub-edge of H_i of minimum type, and $H_{i+1} = H_i - \{e \in H_i : f \subset e\}$ and $f_{i+1} = f$.

Every sub-edge $f \in \partial H_q$ has $d_q(f) \geq \ell$ (since f is not type (i)) so H_q is certainly ℓ-full. Also, no edge has more than one sub-edge of codegree less than rk, for then we have a sub-edge of type (ii) or (iii). Therefore H_q is ℓ-superfull.

Claim 1. $|\partial H_q| \sim \binom{n}{r-1}$.

Proof. Let E be the set of f_i of type (iii), and for each $f \in E$, let e_f be any edge of H' containing f. Suppose $|E| > \delta n^{r-1}$. If $r \geq 4$ or $r = 3$ and k is even, this contradicts Lemma 5.1. Let $r = 3$ and k be odd. By definition every edge of H_i containing f_i of type (iii) has each of its subedges of codegree at least $\ell \geq 2$ and $d_H(f_i) \geq \ell + 1$. Since every edge in H' contains some pair of codegree at least $3k + 1$ in H, the conditions of Lemma 5.1 are met by E. Again, by this lemma, H contains P_k and C_k, a contradiction. So, $|E| = o(n^{r-1})$. Since we have deleted q sub-edges, $|\partial H_q| \leq \binom{n}{r-1} - q$. Note that if a sub-edge of type (ii) was chosen, then H_{i+1} will have a sub-edge of type (i). So, if $\epsilon > 0$ and $q = \epsilon\binom{n}{r-1}$, then for n sufficiently large,

$$
\begin{aligned}
|H_q| &\geq |H'| - q\left(\ell - \frac{1}{2}\right) - rk|E| \geq \ell|\partial H_q| - o\left(n^{r-1}\right) + \frac{\epsilon}{2}\binom{n}{r-1} - rk|E| \\
&\geq \ell|\partial H_q| + \frac{\epsilon}{4}\binom{n}{r-1}.
\end{aligned}
$$

By Lemma 3.1, H_q has an $(\ell+1)$-full subgraph with at least $\frac{\epsilon}{4}\binom{n}{r-1}$ edges, contradicting Theorem 6.1. So $q = o(n^{r-1})$, and $\ell|\partial H_q| \leq |H_q| \leq \ell|\partial H_q| + o(n^{r-1})$, which imply $|\partial H_q| \sim \binom{n}{r-1}$. □

Let G' be the subgraph of ∂H_q formed by the sub-edges of codegree ℓ in H_q.

Claim 2. $|G'| \sim \binom{n}{r-1}$.

Proof. Let $G'' = \partial H_q - G'$. Since H_q is ℓ-superfull, the codegree of every $f \in G''$ is at least $rk \geq \ell + 1$. So if $r \geq 4$ or $r = 3$ and k is even, then by Lemma 5.1 with $E = G''$, $|G''| = o(n^{r-1})$. If $r = 3$ and k is odd, then $\ell \geq 2$ and since H_q is ℓ-superfull, the conditions of Lemma 5.1 are satisfied. So again we get $|G''| = o(n^{r-1})$, and thus $|G'| \sim \binom{n}{r-1}$ as required. □

Claim 3. *For some $G^* \subset G'$, $|G^*| \sim \binom{n}{r-1}$ and all edges of G^* have the same list in H_q.*

Proof. Let N be the number of rk-cliques in G'. Since $|G'| \sim \binom{n}{r-1}$, we easily see that $N \sim \binom{n}{rk}$. By averaging, some edge $e^* \in G'$ is contained in at least

$$\frac{N}{|G'|}\binom{rk}{r-1}$$

rk-cliques in G'. By Lemma 3.5, for every rk-clique K in G' containing e^* and each $e \in K$, the set $L := L_{H_q}(e^*)$ is disjoint from $V(K)$ and coincides with $L_{H_q}(e)$.

Let $G^* \subset G'$ be the set of edges of G' contained in a common rk-clique of G' with e^*. By the previous paragraph, $L_{H_q}(f) = L$ for all $f \in G^*$. The number of pairs (K, f) where K is an rk-clique in G' containing e^* and $f \in K$ is disjoint from e^* is at least

$$\frac{N\binom{rk}{r-1}\binom{rk-r+1}{r-1}}{|G'|}.$$

The number of rk-cliques containing both e^* and f is at most $\binom{n}{rk-2r+2}$. We conclude

$$|G^*| \geq \frac{N\binom{rk}{r-1}\binom{rk-r+1}{r-1}}{|G'|\binom{n}{rk-2r+2}}.$$

Using $|G'| \sim \binom{n}{r-1}$ and $N \sim \binom{n}{rk}$, a straightforward calculation shows $|G^*| \sim \binom{n}{r-1}$. □

6.3. Part IIIa: exact result for cycles

Fix $r \geq 3$, $k \geq 4$ and let n be large. Let H be an n-vertex r-graph containing no k-cycle and with $|H| = \binom{n}{r} - \binom{n-\ell}{r} + f(n,k,r)$, where $f(n,k,r) = 0$ if k is odd, $f(n,k,r) = \mathrm{ex}_r(n-\ell, \{P_2, 2P_1\}) = \binom{n-\ell-2}{r-2}$ if k is even and $(k,r) \neq (4,3)$ and $f(n,4,3) = \mathrm{ex}_3(n-\ell, P_2)$.

Let $\beta = 1/10$. Theorem 6.2 implies that for n sufficiently large, $\mathrm{ex}_r(n, C_k) < 2\ell\binom{n}{r-1}$ and consequently, there is a $c = c(k,r)$ such that $\mathrm{ex}_r(n, C_k) < cn^{r-1}$ for all $n \geq 1$. Choose α sufficiently small so that

$$c2^{r-1}\left(k^3r^r\right)^{r-1}\alpha^{(r-2)} < \beta/2. \tag{9}$$

Finally, choose n sufficiently large so that all inequalities involving α, k, r in the proof below are valid. By Theorem 6.2, there exists $L = \{x_1, \ldots, x_\ell\} \subset [n]$ such that $|H - L| \leq \alpha n^{r-1}$. Let $B = H - L$ be the set of edges of H that are disjoint from L so $|B| < \alpha n^{r-1}$. If k is odd, then we shall show that $B = \emptyset$. If k is even then we shall show that B is an extremal family with no P_2 and $2P_1$ unless $k = 4$, $r = 3$, in which case B is an extremal family with no P_2. This proves both the extremal result and the characterization of equality. Let

$$M = \left\{ e \in \binom{[n]}{r} - H : e \cap L \neq \emptyset \right\},$$

so that

$$|B| = |M| + f(n,k,r).$$

If $M = \emptyset$, then we are done, so we may suppose for a contradiction that $M \neq \emptyset$ and $|B| > f(n,k,r)$. Set $m := |M|$ so that $m \leq |B| < \alpha n^{r-1}$.

Claim 1. *There exist pairwise disjoint $(r-2)$-sets $Z_1, Z_2, \dots, Z_{kr} \subset V(H) - L$ such that for each $i \in [kr]$ and $j \in [\ell]$*

$$d_H\big(Z_i \cup \{x_j\}\big) \geq n - r + 1 - \frac{krm}{\binom{n-\ell}{r-2}}.$$

If $r \geq 4$ there exists an additional $(r-2)$-set Z_{kr+1} that is disjoint from Z_i for $i \in [kr-1]$ and $|Z_{kr+1} \cap Z_{kr}| = 1$.

Proof. Pick an $(r-2)$-set $T \subset V(H) - L$ uniformly at random. Let $\overline{H} = \{e \subset V(H) : |e| = r, e \notin H\}$. For $j \in [\ell]$, let

$$X_j = d_{\overline{H}}\big(T \cup \{x_j\}\big) = n - r + 1 - d_H\big(T \cup \{x_j\}\big).$$

In other words, X_j counts the number of r-sets $e \notin H$ with $T \cup \{x_j\} \subset e$. The number of r-sets $e \supset \{x_j\}$ with $e \notin H$ is at most m. For each such e, let $X_j(e)$ be the indicator for the event that $T \subset e$. Then

$$\mathbb{E}(X_j) = \sum_e \mathbb{E}\big(X_j(e)\big) \leq m \frac{\binom{r-1}{r-2}}{\binom{n-\ell}{r-2}} < \frac{rm}{\binom{n-\ell}{r-2}}.$$

By Markov's inequality,

$$\mathbb{P}\left(X_j > \frac{krm}{\binom{n-\ell}{r-2}}\right) < 1/k.$$

This implies that

$$\mathbb{P}\left(\exists j : X_j > \frac{krm}{\binom{n-\ell}{r-2}}\right) < \ell/k < 1/2.$$

In other words, the number of T for which $d_H(T \cup \{x_j\}) \geq n - r + 1 - krm/\binom{n-\ell}{r-2}$ for all j is at least $\binom{n-\ell}{r-2}/2$.

Now consider the family of all $(r-2)$-sets described above, and let $T_1, \dots, T_t$ be a maximum matching in this family. If $t < kr$, then all other sets of this family have an element within $\bigcup_i T_i$, which implies that the number of such T is less than $\binom{n-\ell}{r-2}/2$, because n is sufficiently large. This contradiction shows that $t \geq kr$.

If $r \geq 5$, then by a result of Frankl [11] that $\mathrm{ex}_{r-2}(n-\ell, P_2) = O(n^{r-4})$, we can find two sets T_1, T_2 with $|T_1 \cap T_2| = 1$ and then find the remaining $kr-1$ sets using the greedy procedure described above. If $r = 4$, then we use the fact that a graph with $\Omega(n^2)$ edges has a 2-path together with a disjoint from it matching of size $kr-1$. □

Claim 2. *Let $Z = \bigcup_i Z_i$ and $Y = V(H) - (L \cup Z)$. Then there exists a set $D \subset Y$ such that H contains all edges of the form $Z_i \cup \{x_j, y\}$, for all $i \in [kr]$, $x_j \in L$ and $y \in D$ and*

$$|D| = n - \ell kr - \left\lceil \frac{k^3 r^r m}{n^{r-2}} \right\rceil.$$

Proof. For each $i \in [kr]$ and $j \in [\ell]$, let $S_{i,j} = \{y \in Y : Z_i \cup \{x_j, y\} \notin H\}$. Claim 1 implies that $|S_{i,j}| < krm/\binom{n-\ell}{r-2}$. Let $S = \bigcup_{i,j} S_{i,j}$. Then

$$|S| < \frac{(kr\ell)krm}{\binom{n-\ell}{r-2}} < \frac{k^3 r^r m}{n^{r-2}}.$$

We may add points arbitrarily to S till $D := Y - S$ has the required size. □

Claim 3. *No two edges $e, e' \in B$ have $|e \cap e'| = 1$ and $(e - e') \cap D \neq \emptyset$ and $(e' - e) \cap D \neq \emptyset$. If $k \geq 5$ is odd, then no edge $e \in B$ has $|e \cap D| \geq 2$. If $k \geq 6$ is even and $r = 3$, then there are no two disjoint edges each with at least two points in D.*

Proof. For k even and $|e \cap e'| = 1$ suppose $u \in e - e'$, $v \in e' - e$ and $u, v \in D$. Then there is a path P of length $k-2$ in H between u and v consisting of edges $Z_i \cup \{x_j, y\}$ with $y \in D$ and such that $V(P) \cap (e \cup e') = \{u, v\}$. All vertices of L will have degree two in P. Now $P \cup \{e, e'\}$ is a k-cycle in H. For $k \geq 5$ odd and $r \geq 4$, we repeat the same argument except that we use Z_{kr-1} and Z_{kr} which have a common intersection point. Thus we use $\ell - 1$ of the x_j's in two edges and the last x_j together with Z_{kr} and $(e' - e) \cap D$. Lastly, for $k \geq 5$ odd and $r = 3$, we use a particular Z_i twice to complete the odd cycle (since $|Z_i| = 1$, this approach is valid only for $r = 3$).

For $k \geq 5$ odd, suppose $u, v \in e \cap D$. Then again there is a path P of length $k-1$ in H between u and v consisting of edges $Z_i \cup \{x_j, y\}$ with $y \in D$ such that $V(P) \cap e = \{u, v\}$, and $P \cup \{e\}$ is a k-cycle in H.

Finally, if $k \geq 6$ is even, $r = 3$, $e = uvw$, $e' = u'v'w'$ with $e \cap e' = \emptyset$, and $\{u, v, u', v'\} \subset D$, then we form a C_k as follows: If $k = 6$ we use the edges $e, x_1z_1u, x_1z_2u', e', x_2z_3v', x_2z_4v$ where $Z_i = \{z_i\}$ for all i. If $k > 6$ then instead of the edge x_2z_4v, we use an edge x_2z_4y for some $y \in D$, expand the path using the remaining x_i's and z_i's, and close the path with $x_\ell z_{2\ell} v$. We obtain a cycle of length $2\ell + 2 = k$ as desired. □

Claim 4. $m > \binom{n-3r-3k}{r-2}$.

Proof. Suppose that k is even and there are $e, e' \in B$ with $|e \cap e'| = 1$. Let $u \in e - e'$ and $v \in e' - e$ and let f be an r-set with $f \cap (e \cup e') = \{u\}$ and $|f \cap L| = 1$. If no such r-set is an edge of H, then $m \geq \binom{n-|e \cup e' \cup L|}{r-2}$ and we are done. So we may assume that there is such an $f \in H$. If $k > 4$, then let g be an r-set disjoint from f and with $g \cap (e \cup e') = \{v\}$ and $|g \cap L| = 1$. If $k = 4$, then let g be an r-set with $g \cap (e \cup e' \cup f) = \{v\} \cup (f \cap L)$. Let us argue that $g \notin H$. Indeed, if $k > 4$ and $g \in H$, then we find a path P of length $k - 2$ in H as in Claim 3 containing f and g, and $P \cup \{e, e'\}$ is a k-cycle in H. If $k = 4$, then e, e', f, g is already a 4-cycle. Since $g \notin H$ we have $g \in M$ and hence

$$m = |M| \geq \binom{n - |e \cup e' \cup f \cup L|}{r-2} > \binom{n-3r-3k}{r-2}.$$

If $r > 3$, then by Frankl's theorem [11], $|B| > f(n, k, r)$ implies that there exist $e, e' \in B$ with $|e \cap e'| = 1$. Now we are done by the preceding argument. If $r = 3$ and $k = 4$, then by definition of $f(n, 4, 3)$ we find e, e' with $|e \cap e'| = 1$ and we are again done. If $r = 3$ and $k \geq 6$ is even and we cannot find such e, e' with a singleton intersection, then there are $e, e' \in B$ with $e \cap e' = \emptyset$ (this is easy to see since if we have more than $f(n, k, 3) = n - \ell - 2$ triples on $n - \ell$ points and no singleton intersection, then we must have many disjoint complete 3-graphs on four points). Then for every i and every $u \in e \cup e'$, $d_H(x_i u) < 3k$ for otherwise we can build a k-cycle using e, e' and $k - 2$ edges each containing some x_i and at most one point of $e \cup e'$ (many of the edges will not intersect $e_1 \cup e_2$ if k is large). This immediately gives at least $n - 9 - 3k$ triples in M that contain both x_i and u and Claim 4 is proved in this case.

If k is odd, then pick any edge $e \in B$ and apply a similar argument. □

For $0 \leq i \leq r$, define $B_i^r = \{e \in B : |e \cap (Y - D)| = i\}$.

Claim 5. $|B_r^r| < \beta m$.

Proof. Recall that c satisfies $\mathrm{ex}_r(n, C_k) < cn^{r-1}$ for all $n \geq 1$. As B_r^r itself has no C_k, we can apply this weaker bound to obtain

$$|B_r^r| \leq \mathrm{ex}_r(n - |D|, C_k) < c(n - |D|)^{r-1}.$$

Since n is large, Claim 4 implies that $c2^{r-1}(\ell kr)^{r-1} < (\beta/2)m$ and Claim 2 gives

$$|B_r^r| < c\left(\ell kr + \frac{k^3 r^r m}{n^{r-2}}\right)^{r-1} < c2^{r-1}\left((\ell kr)^{r-1} + \left(\frac{k^3 r^r m}{n^{r-2}}\right)^{r-1}\right) < \frac{\beta}{2}m + c'\frac{m^{r-1}}{n^{(r-2)(r-1)}}$$

where $c' = c2^{r-1}(r^r k^3)^{r-1}$. By (9) and $m < \alpha n^{r-1}$,

$$c' \frac{m^{r-1}}{n^{(r-2)(r-1)}} = c'm\left(\frac{m}{n^{r-1}}\right)^{r-2} \leq c'm\alpha^{r-2} < \frac{\beta}{2}m$$

and the claim follows. □

Claim 6. *$|B^r_{r-1}| < \beta m$ for $r \geq 4$ and $|B^3_2| < 3m/4$.*

Proof. Partition B^r_{r-1} into $P^r \cup Q^r$, where P^r comprises those r-sets $e \in B^r_{r-1}$ with $d_{B^r_{r-1}}(e - D) = 1$. Clearly $|P^r| < \binom{|Y|-|D|}{r-1} < (\beta/2)m$ as in Claim 5.

Let us now focus on Q^r. Let F be the collection of $(r-1)$-sets $f \subset Y - D$ such that there exists $e \in B^r_{r-1}$ with $f \subset e$. We now partition the argument depending on whether $r = 3$ or $r \geq 4$

Suppose that $r = 3$. Then F is a (graph) matching for if we have vw and vw' in F, then we have (by definition of Q^3) distinct vertices y, y' and edges $vwy, vw'y'$ in B^3_2. This contradicts Claim 3. We will prove that $|Q^3| \leq 2m/3$. Suppose for contradiction that $|Q^3| > 2m/3$. Then by averaging, there is a vertex $u \in D$ with $d_{B^3_2}(u) \geq \lceil 2m/(3n) \rceil := t$. Let $v_1w_1, \ldots, v_tw_t$ be the neighbors of u in Q^3 (meaning that $uv_iw_i \in Q^3$ for all i). Note that these pairs form a matching. Given $i < j$, there are at least $2(|D| - 2)$ sets of M containing an element of $\{v_i, w_i\}$ or at least $2(|D| - 2)$ edges of M containing an element of $\{v_j, w_j\}$. Indeed, if this is not the case, then we can form a copy of C_k using uv_iw_i and uv_jw_j. Since the pairs $\{v_iw_i\}^t_{i=1}$ form a matching this implies that $|M| \geq 2(|D| - 2)(t - 1)$. Since m is large by Claim 4 and α is small this is at least $2 \times (0.9)n \times (\frac{2m}{3n} - 1) > m$, contradiction.

Next suppose that $r \geq 4$. In this case F is a collection of $(r - 1)$-sets on D that have no singleton intersection by Claim 3. We conclude by a result of Keevash–Mubayi–Wilson [20] that $|F| < \binom{n-|D|}{r-3}$ and hence that

$$|Q^r| < |F|n < \binom{n - |D|}{r - 3}n.$$

By Claim 2, there exists C depending only on k and r such that this is at most

$$Cn\left(\frac{m}{n^{r-2}}\right)^{r-3} = Cm\frac{m^{r-4}}{n^{(r-2)(r-3)-1}}.$$

Since $m < n^{r-1}$, $(r - 1)(r - 4) < (r - 2)(r - 3) - 1$ and n is large, the last expression is at most $(\beta/2)m$ and the claim follows. □

Since $|B| = m+f(n, k, r)$, Claims 5 and 6 imply that $|B^r_{r-1}|+|B^r_r| < (2\beta+3/4)m < m$ and therefore $|B^r_0 \cup \ldots \cup B^r_{r-2}| > f(n, k, r)$.

If k is odd, then $B^r_0 \cup \ldots \cup B^r_{r-2} \neq \emptyset$. If k is even and $r \geq 4$ then there are edges $e, e' \in B^r_0 \cup \ldots \cup B^r_{r-2}$ such that $|e \cap e'| = 1$. This is because for $r \geq 4$ the extremal function for P_2 is the same as the extremal function for $\{P_2, 2P_1\}$ by [11] as long as n is sufficiently large (in both cases the extremal example is obtained by taking all r-sets that intersect a specific set of two points). If $(k, r) = (4, 3)$, then by definition of $f(n, 4, 3)$ there are edges $e, e' \in B_0^3 \cup B_1^3$ such that $|e \cap e'| = 1$. Finally, if $k \geq 6$ is even, $r = 3$ and $|B_0^3 \cup B_1^3| > f(n, k, 3) = n - \ell - 2$ then we find two edges $e, e' \in B_0^3 \cup B_1^3$ with $|e \cap e'| \leq 1$. In all four cases above we contradict Claim 3. This completes the proof of Theorem 1.1. □

6.4. Part IIIb: exact result for paths

We closely follow the proof in Section 6.3 except that we replace $f(n, k, r)$ by $h(n, k, r)$, where $h(n, k, r) = 0$ if k is odd and $h(n, k, r) = ex_r(n - \ell, \{P_2, 2P_1\})$ if k is even. Claims 1, 2 and 5 follow immediately and Claim 4 follows by a very similar proof. We strengthen Claim 3 as follows.

Claim 3′. *No two edges $e, e' \in B$ have $|e \cap e'| \leq 1$, $(e - e') \cap D \neq \emptyset$ and $(e' - e) \cap D \neq \emptyset$. If k is odd, then no edge $e \in B$ has $|e \cap D| \geq 1$.*

Proof. In the first case, we may form a path using the two vertices of $e \triangle e'$ in D and 2ℓ other edges. This is a path of length $2\ell + 2 \geq k$. In the case when k is odd, we form a path of length $2\ell + 1 = k$ ending at e by the same procedure. □

If k is odd, then Claim 3′ implies that $B = B_r^r$ and Claim 5 implies the contradiction $m \leq |B| < \beta m$. Let us suppose that k is even. We now observe that Claim 6 also holds (in fact we can improve the argument when $r = 3$ to obtain $4(|D| - 1)$ instead of $2(|D| - 1)$ as it is easier to form a k-path), so $|B_0^r \cup \ldots \cup B_{r-2}^r| > h(n, k, r)$ and we find a P_2 or a $2P_1$ in this union. This contradicts Claim 3′ and completes the proof. □

7. Proof of Theorem 1.2

In this short section we show how to modify the proof of Theorem 1.1 to prove Theorem 1.2. The case of minimal paths is easier than minimal cycles, so we concentrate only on minimal cycles. We only prove the case $r = 3$ as all other cases are covered by the result of Füredi–Jiang [15] (though our proof works just as easily for all $r \geq 3$ and $k \geq 4$). We closely follow the proof of Theorem 1.1. We may assume that $k \geq 4$ is even as the case $k = 3$ is already solved in [6,12,23] and if $k \geq 5$ is odd, then we apply Theorem 1.1 directly. Since $C_k \in \mathcal{C}_k$, we immediately obtain a stability result (Theorem 6.2) for $\mathcal{C}_k$. Now we repeat the proof in Section 6.3 with $f(n, k, r)$ replaced by $f(k)$, where $f(k) = 0$ if k is odd, $f(k) = \lfloor (n-1)/r \rfloor$ if $k = 4$ and $f(k) = 1$ if $k \geq 6$ is even. The proofs of Claims 1, 2, 4, 5 and 6 remain the same or very similar and we do not repeat them. Claim 3 can be strengthened by replacing $|e \cap e'| = 1$ with $|e \cap e'| \geq 1$ since it is enough to find a minimal cycle.

Suppose that $k = 4$, $\ell = 1$ and we are trying to find a minimal 4-cycle. Then $|B_2^3| + |B_3^3| < (\beta + 3/4)m \leq (1/10 + 3/4)m < (6/7)m$ and therefore $|B_0^3| + |B_1^3| = m + f(k) -$

$|B_2^3| - |B_3^3| > f(k) + m/7$. If $|B_0^3| > f(k)$, then we find $e, e' \in B_0^3$ with $e \cap e' \neq \emptyset$ which contradicts (the strengthened) Claim 3. So we may assume that $|B_0^3| \leq f(k)$ and $|B_1^3| > m/7$. Each edge of B_1^3 has a vertex in $Y - D$, and since n is large, $|Y - D| < m/7$. Therefore there is a vertex $v \in Y - D$ with $d_{B_1^3}(v) > 1$. This again contradicts Claim 3.

Now we suppose that $k \geq 6$ is even, and $f(k) = 1$. If $|B_0^3| > f(k) = 1$, then there are two edges $e, e' \subset D$ and this contradicts Claim 3 (no matter what their intersection size). We may therefore assume that $|B_1^3| > m/7$ and this again contradicts Claim 3 as above. □

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