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Defective 2-colorings of sparse graphs

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ABSTRACT

A graph G is (j,k)-colorable if its vertices can be partitioned into subsets V_1 and V_2 such that every vertex in $G[V_1]$ has degree at most j and every vertex in $G[V_2]$ has degree at most k. We prove that if $k \geqslant 2j+2$, then every graph with maximum average degree at most $2(2-\frac{k+2}{(j+2)(k+1)})$ is (j,k)-colorable. On the other hand, we construct graphs with the maximum average degree arbitrarily close to $2(2-\frac{k+2}{(j+2)(k+1)})$ (from above) that are not (j,k)-colorable. In fact, we prove a stronger result by establishing the best possible sufficient condition for the (j,k)-colorability of a graph G in terms of the minimum, $\varphi_{j,k}(G)$, of the difference $\varphi_{j,k}(W,G)=(2-\frac{k+2}{(j+2)(k+1)})|W|-|E(G[W])|$ over all subsets W of $V(G)$. Namely, every graph G with $\varphi_{j,k}(G) > \frac{-1}{k+1}$ is (j,k)-colorable. On the other hand, we construct infinitely many non-(j,k)-colorable graphs G with $\varphi_{j,k}(G)=\frac{-1}{k+1}$.

1. Introduction

A graph G is called *improperly* $(d_1,\dots,d_k)$*-colorable*, or just $(d_1,\dots,d_k)$*-colorable*, if the vertex set of G can be partitioned into subsets $V_1,\dots,V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leqslant i \leqslant k$. This notion generalizes those of proper k-coloring (when $d_1=\dots=d_k=0$) and d-improper k-coloring (when $d_1=\dots=d_k=d \geqslant 1$).

Proper and d-improper colorings have been widely studied. As shown by Appel and Haken [1,2], every planar graph is 4-colorable, i.e. $(0,0,0,0)$-colorable. Cowen, Cowen and Woodall [8] proved that

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every planar graph is 2-improperly 3-colorable, i.e. $(2,2,2)$-colorable. This latter result was extended by Havet and Sereni [10] to sparse graphs that are not necessarily planar: for every $k \geqslant 0$, every graph G with $\mathrm{mad}(G) < \frac{4k+4}{k+2}$ is k-improperly 2-colorable, i.e. (k,k)-colorable.

Recall that for a graph G, $\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\}$ is the maximum over the average degrees of the subgraphs of G. The *girth*, $g(G)$, of a graph G is the length of a shortest cycle in G.

We will consider probably the simplest version of defective colorings, defective colorings with two colors. For nonnegative integers j and k, let $F(j,k)$ denote the supremum of x such that every graph G with $\mathrm{mad}(G) \leqslant x$ is (j,k)-colorable. It is easy to see that $F(0,0) = 2$. Indeed, since the odd cycle C_{2n-1} has $\mathrm{mad}(G) = 2$ and is not $(0,0)$-colorable, $F(0,0) \leqslant 2$. On the other hand, each graph with $\mathrm{mad}(G) < 2$ has no cycles and therefore is bipartite, i.e., $(0,0)$-colorable.

Glebov and Zambalaeva [9] proved that every planar graph G with $g(G) \geqslant 16$ is $(0,1)$-colorable. This was strengthened by Borodin and Ivanova [3] by proving that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(0,1)$-colorable, which implies that every planar graph G with $g(G) \geqslant 14$ is $(0,1)$-colorable. In [4], it was proved that $F(0,1) = \frac{12}{5}$. In particular, this implies that every planar graph G with $g(G) \geqslant 12$ is $(0,1)$-colorable.

For each integer $k \geqslant 2$, Borodin et al. [5] proved that every graph G with $\mathrm{mad}(G) < \frac{3k+4}{k+2} = 3 - \frac{2}{k+2}$ is $(0,k)$-colorable. On the other hand, for all $k \geqslant 2$ Borodin et al. [5] constructed non-$(0,k)$-colorable graphs with mad arbitrarily close to $\frac{3k+2}{k+1} = 3 - \frac{1}{k+1}$.

Recently, it was proved by Borodin et al. [7] that every graph G with $\mathrm{mad}(G) < \frac{10k+22}{3k+9}$, where $k \geqslant 2$, is $(1,k)$-colorable. On the other hand, [7] presents a construction of non-$(1,k)$-colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$.

The purpose of this paper is to prove an exact result for a wide range of j and k.

Theorem 1. *Let*

$$j \geqslant 0 \quad \text{and} \quad k \geqslant 2j+2. \tag{1}$$

Then $F(j,k) = 2(2 - \frac{k+2}{(j+2)(k+1)})$.

In particular, together with [4], Theorem 1 yields exact values for $F(0,k)$ for every k. If $j \leqslant k < 2j+2$, then we do not know the exact answer apart from the cases $j=0$ and $k \in \{0,1\}$. Furthermore, the formula for $F(0,1)$ differs from that in Theorem 1.

In fact, to derive Theorem 1, we will need a more precise statement. For a graph G and $W \subseteq V(G)$, let

$$\varphi_{j,k}(W,G) := \left(2 - \frac{k+2}{(j+2)(k+1)}\right)|W| - \left|E\bigl(G[W]\bigr)\right|. \tag{2}$$

By definition, $\mathrm{mad}(G) \leqslant 2(2 - \frac{k+2}{(j+2)(k+1)})$ if and only if $\varphi_{j,k}(W,G) \geqslant 0$ for every $W \subseteq V(G)$.

Theorem 2. *Let j and k satisfy (1). Every graph G such that*

$$\varphi_{j,k}(W,G) > -\frac{1}{k+1} \quad \textit{for every } W \subseteq V(G), \tag{3}$$

is (j,k)-colorable. Moreover, restriction (3) is sharp.

The second part of Theorem 2 means that there exist infinitely many non-(j,k)-colorable graphs G for which the non-strict version of (3) holds.

Since each planar graph G satisfies $\mathrm{mad}(G) < \frac{2g(G)}{g(G)-2}$, from Theorem 2 we easily deduce:

Corollary 1. *Let G be a planar graph and*

$$k \geqslant \max\left\{-1 + \frac{g(G)-2}{(g(G)-4)(j+2) - g(G)+2}, 2j+2\right\}.$$

Then G is (j,k)*-colorable. In particular, G is*:

1) $(0,2)$*-colorable if* $g(G) \geqslant 8$,
2) $(0,4)$*-colorable if* $g(G) \geqslant 7$,
3) $(1,4)$*-colorable if* $g(G) \geqslant 6$*, and*
4) $(2,6)$*-colorable if* $g(G) \geqslant 5$.

Borodin et al. [5] constructed a planar graph with girth 6 which is not $(0,k)$-colorable for any k, and proved that every planar graph G with $g(G) \geqslant 7$ is $(0,8)$-colorable and $g(G) \geqslant 8$ is $(0,4)$-colorable. It follows from Borodin et al. [7] that every planar graph G with $g(G) \geqslant 7$ is $(1,2)$-colorable, and with $g(G) \geqslant 6$ is $(1,5)$-colorable. Among other results, Borodin et al. [6] also proved that planar graphs with girth 5 are $(2,13)$- and $(3,7)$-colorable. Note that all these bounds are now strengthened by Corollary 1. Still, we suspect that Corollary 1 can be further improved. Also, the result by Havet and Sereni [10] yields that every planar graph G with $g(G) \geqslant 5$ (respectively, $g(G) \geqslant 6$, and $g(G) \geqslant 8$) is $(4,4)$-colorable (respectively, $(2,2)$-colorable, and $(1,1)$-colorable).

In the next section we show the sharpness of Theorem 2 and lay the ground for its proof. The proof is delivered in Section 3.

2. Preliminaries and proof of the sharpness in Theorem 2

For $i \geqslant 1$ and a graph G, an *i-flag in G* is an $(i+2)$-vertex pendant block B of G in which the non-cut vertices induce $K_{1,i}$ and the cut vertex (we will call it the *base vertex*) is adjacent to all other vertices.

For convenience, the two colors that we will use are j and k: each vertex of color j (respectively, k) in a (j,k)-coloring is adjacent to at most j (respectively, k) vertices of its color. By definition, in a (j,k)-coloring, at least one vertex of each star $K_{1,j+1}$ is colored with k. This yields the following useful observation.

Claim 1. *In any* (j,k)*-coloring of a* $(j+1)$*-flag F, at least two vertices of F are colored with color k.*

An (i,j)*-host* in a graph G is a vertex contained in $i+1$ $(j+1)$-flags as the base vertex. An (i,j)-host is *peripheral* if its degree in G is $1+(i+1)(j+2)$. In other words, $v \in V(G)$ is a peripheral (i,j)-host if it is adjacent to only one vertex apart from the vertices of the $i+1$ $(j+1)$-flags in which v is the base vertex.

Claim 1 readily implies

Claim 2. *In any* (j,k)*-coloring of a graph G, each* (k,j)*-host is colored with j.*

Let $G_0(j,k)$ be obtained from $k+1$ vertex-disjoint copies of the star $K_{1,j+1}$ by adding a new vertex v_0 adjacent to all vertices in all copies of $K_{1,j+1}$. By construction, v_0 is a (k,j)-host in $G_0(j,k)$, and hence by Claim 2 in any (j,k)-coloring of $G_0(j,k)$, v_0 must be colored with j.

Let $G_1(j,k)$ be obtained from $j+2$ copies of $G_0(j,k)$ with (k,j)-hosts $v_0, v_1, \dots, v_{j+1}$ by adding the $j+1$ edges connecting v_0 with $v_1, \dots, v_{j+1}$. Suppose that $G_1(j,k)$ has a (j,k)-coloring f. Then by Claim 2, $f(v_0) = f(v_1) = \cdots = f(v_{j+1}) = j$, and so vertex v_0 of color j has $j+1$ neighbors of the same color, a contradiction. Thus, $G_1(j,k)$ has no (j,k)-coloring.

In order to calculate the minimum of $\varphi_{j,k}(W, G_1(j,k))$ over all $W \subseteq V(G_1(j,k))$, observe the following.

Claim 3.

(a) *Adding to some* $W \subseteq V(G)$ *a vertex* $w \in V(G) - W$ *adjacent to exactly two vertices of W decreases* $\varphi_{j,k}(W,G)$ *by* $\frac{k+2}{(j+2)(k+1)}$.
(b) *Adding to some* $W \subseteq V(G)$ *a* $(j+1)$*-flag sharing with W exactly one vertex decreases* $\varphi_{j,k}(W,G)$ *by* $\frac{1}{k+1}$.

(c) *Adding to some $W \subseteq V(G)$ a peripheral (k, j)-host v together with its $k+1$ $(j+1)$-flags such that v is adjacent to a vertex in W decreases $\varphi_{j,k}(W, G)$ by $\frac{k+2}{(j+2)(k+1)}$.*

Indeed, (a) is evident. To see (b), observe that we add $j+2$ vertices and $2j+3$ edges, and so the net gain in $\varphi_{j,k}$ is

$$(j+2)\left(2-\frac{k+2}{(j+2)(k+1)}\right)-2j-3=1-\frac{k+2}{k+1}=-\frac{1}{k+1}.$$

Now (c) easily follows from the definition and (b).

We can obtain $G_1(j,k)$ from the star $K_{1,j+1}$ by consecutive adding of $(j+2)(k+1)$ $(j+1)$-flags. For the vertex set $V_0=\{v_0, v_1, \ldots, v_{j+1}\}$ inducing $K_{1,j+1}$ in $G_1(j,k)$, we have

$$\varphi_{j,k}\big(V_0, G_1(j,k)\big)=(j+2)\left(2-\frac{k+2}{(j+2)(k+1)}\right)-j-1=j+2-\frac{1}{k+1}.$$

So by Claim 3(b),

$$\varphi_{j,k}\big(V\big(G_1(j,k)\big), G_1(j,k)\big)=j+2-\frac{1}{k+1}-(j+2)(k+1)\frac{1}{k+1}=-\frac{1}{k+1}.$$

Since $G_1(j,k)$ can be obtained from every of its induced non-empty connected subgraphs by a sequence of the operations described in Claim 3, we know that $\varphi_{j,k}(W, G_1(j,k)) \geqslant -\frac{1}{k+1}$ for every $W \subseteq V(G_1(j,k))$. Thus $G_1(j,k)$ is one of the examples showing the sharpness of Theorem 2.

Let $H_1(j,k)$ be the graph obtained from $G_1(j,k)$ with host vertices $v_0, v_1, \ldots, v_{j+1}$ by deleting all vertices (apart from v_{j+1}) of one $(j+1)$-flag containing v_{j+1}. By Claim 3(b),

$$\varphi_{j,k}\big(V\big(H_1(j,k)\big), H_1(j,k)\big)=\varphi_{j,k}\big(V\big(G_1(j,k)\big), G_1(j,k)\big)+\frac{1}{k+1}=0.$$

Repeating the argument of the previous paragraph, we conclude that $\varphi_{j,k}(W, H_1(j,k)) \geqslant 0$ for every $W \subseteq V(H_1(j,k))$.

By Claims 2 and 1 and the definition of (j,k)-colorings, in each (j,k)-coloring f of $H_1(j,k)$, the following should hold:

(i) $f(v_0)=f(v_1)=\cdots=f(v_j)=j$, in particular, v_0 has j neighbors of color j;
(ii) $f(v_{j+1})=k$ and v_{j+1} has exactly k neighbors of color k.

Now we construct $G_i(j,k)$ for $i=2,3\ldots$. Suppose that $G_{i-1}(j,k)$ is constructed and let x be a peripheral (k, j)-host in it. Let y be a vertex of degree 2 in one of the $(j+1)$-flags containing x. We obtain $G_i(j,k)$ from $G_{i-1}(j,k)$ and a copy of $H_1(j,k)$ by deleting edge xy and adding an edge connecting y with the $(k-1, j)$-host v_{j+1} in $H_1(j,k)$. Again, $G_i(j,k)$ has no (j,k)-coloring and in (3) we have non-strict inequality. Thus we have infinitely many non-(j,k)-colorable graphs which satisfy the non-strict version of (3).

3. Proof of Theorem 2

3.1. Some notation

A vertex of degree d (respectively, at least d, at most d) is called a *d-vertex* (respectively, *d^+-vertex*, *d^--vertex*). By a *k-path* we mean a path with precisely k internal vertices all of which have degree 2. A *$(k_1, k_2, \ldots, k_t)$-vertex* is a vertex v of degree t that is a starting point of a k_1-, a k_2-, ..., and a k_t-path that are all distinct.

For a graph G, the vertices in $(j+1)$-flags that are not cut-vertices are called *ghosts*. The other vertices of G are *non-ghosts*. By $\widetilde{G}$ we denote the graph obtained from G by deleting ghosts. Recall that each of $H_1(j,k)$ and $G_1(j,k)$ has exactly $j+2$ non-ghosts and that $\widetilde{H}_1(j,k)=\widetilde{G}_1(j,k)=K_{1,j+1}$.

Let Z be obtained from $G_1(j,k)$ by deleting v_{j+1} and the vertices of all $(j+1)$-flags containing it. By definition, Z is isomorphic to the subgraph of $H_1(j,k)$ obtained by deleting v_{j+1} and the vertices of all $(j+1)$-flags containing it.

We say that a graph G is *smaller* than a graph G' if either $\widetilde{G}$ has fewer vertices of degree at least two than $\widetilde{G}'$, or if they have the same number of such vertices but $|V(\widetilde{G})| < |V(\widetilde{G}')|$, or both these parameters are the same, and $|E(\widetilde{G})| < |E(\widetilde{G}')|$, or all these three parameters are the same, but $|V(G)| < |V(G')|$.

For example, $H_1(j,k)$ is smaller than $G_1(j,k)$, since $\widetilde{H}_1(j,k) = \widetilde{G}_1(j,k)$ and $H_1(j,k)$ has fewer vertices. Graph Z is smaller that either of these graphs.

3.2. *Structural properties of a minimum counterexample*

Let G be a smallest counterexample to Theorem 2. Clearly, G is connected and has no pendant vertices. For shortness, $\varphi(W,H)$ will denote $\varphi_{j,k}(W,H)$, and $\varphi(W)$ will denote $\varphi(W,G)$.

Lemma 1. *G does not have (i,j)-hosts for $i \geqslant k+1$.*

Proof. Suppose to the contrary that $v \in V(G)$ is an (i,j)-host for some $i \geqslant k+1$. Delete the vertices (apart from v) of one $(j+1)$-flag F based on v. Then the new graph G' is smaller than G and hence has a (j,k)-coloring c'. By Claim 2, $c'(v) = j$. So, we may extend c' to the whole G by coloring each vertex of $F - v$ with k (recall that $k \geqslant 2j+2$). □

Lemma 2. *If $W \subseteq V(G)$ and $\emptyset \neq W \neq V(G)$ then $\varphi(W) \geqslant \frac{1}{(j+2)(k+1)}$.*

Proof. Suppose that $\emptyset \neq W \neq V(G)$, $\varphi(W) \leqslant 0$. Then there is a non-empty $W' \subseteq W$ such that $G[W']$ is connected and $\varphi(W') \leqslant 0$. We may choose a maximal W' with this property that is distinct from $V(G)$. If $G[W']$ contains a vertex w of degree at most 1 in it, then

$$\varphi(W'-v) \leqslant \varphi(W') - \left(2 - \frac{k+2}{(j+2)(k+1)}\right) + 1 \leqslant 0 - 2 + \frac{k+2}{(j+2)(k+1)} + 1 \leqslant \frac{-1}{k+1},$$

a contradiction to the choice of G. So, $\delta(G[W']) \geqslant 2$.

By Claim 3(a),

$$\text{each } w \in V(G) - W' \text{ has at most one neighbor in } W'. \tag{4}$$

By Claim 3(b), a $(j+1)$-flag of G cannot have exactly one vertex in W'. So,

$$\text{each } (j+1)\text{-flag of } G \text{ either is completely in } W' \text{ or is disjoint from } W'. \tag{5}$$

It follows that $\widetilde{G}[W']$ is a subgraph of $\widetilde{G}$. Thus, $G[W']$ is smaller than G. Since $G[W']$ is a subgraph of G, it satisfies the conditions of the theorem. So, $G[W']$ has a (j,k)-coloring c'. Construct the graph $G^* = G^*(W')$ from $G - W'$ as follows:

(a) add to $G - W'$ a copy of Z;
(b) each $w \in V(G - W')$ that is adjacent to a vertex of color j in c' is joined by an edge to v_0 in Z;
(c) to each $w \in V(G - W')$ that is adjacent to a vertex of color k in c' add $k+1$ $(j+1)$-flags based on w.

We will prove the following three facts: (I) G^* is smaller than G; (II) Condition (3) holds for G^*; and (III) any (j,k)-coloring c^* of G^* yields a (j,k)-coloring of G. These three facts together will imply the lemma.

By (4) and (5), if $w \in V(G) - W'$ belongs to $V(\widetilde{G}^*)$, then it was in $V(\widetilde{G})$. Furthermore, since the edges connecting W' with $V(G) - W'$ were not in $(j+1)$-flags, if $w \in V(G) - W'$ is a non-leaf vertex in $\widetilde{G}^*$, it was also a non-leaf in $\widetilde{G}$. Recall that Z contains exactly $j+1$ vertices of $\widetilde{G}^*$, and at most one of them is a non-leaf in $\widetilde{G}^*$. Thus, if (I) does not hold, then W' contains at most one non-leaf of $\widetilde{G}$. The only connected graph with at most one non-leaf is a star. So, the subgraph of $\widetilde{G}$ contained in W' is some star $K_{1,i}$. If (I) does not hold, then $i \leqslant j$. But then by Lemma 1, W' contains at most $(i+1)(k+1)$ $(j+1)$-flags and hence by Claim 3(b),

$$\begin{aligned}\varphi(W') &\geqslant \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(i+1) - i - (i+1)(k+1)\frac{1}{k+1} \\ &= 1 - \frac{(i+1)(k+2)}{(j+2)(k+1)} \geqslant 1 - \frac{(j+1)(k+2)}{(j+2)(k+1)}.\end{aligned}$$

Since $k > j$, the last expression is positive, a contradiction to $\varphi(W') \leqslant 0$. This proves (I).

Choose now a set U with the minimum $\varphi(U, G^*)$. If (II) does not hold, then $\varphi(U, G^*) \leqslant -\frac{1}{k+1}$. Let $U^* = U - V(Z)$, $U'' = U^* \cap V(G)$, and $U' = U \cap V(Z)$. By the minimality of $\varphi(U, G^*)$, if a $(j+1)$-flag shares a vertex with U, then it is contained in $G^*[U]$.

Suppose that exactly x edges in G connect U'' with vertices of color k in c', and exactly y edges connect U'' with vertices of color j in c'. Then by Claim 3(b), the $x(k+1)$ $(j+1)$-flags added to vertices in U'' while constructing G^* to form U^* decrease $\varphi(U'', G^*)$ by exactly x. Thus

$$\varphi(U, G^*) = \varphi(U', G^*) + \varphi(U'', G^*) - x - y = \varphi(U', Z) + \varphi(U'', G) - x - y. \tag{6}$$

Define $Y = W' \cup U''$. Similarly to (6) we have

$$\varphi(Y, G) = \varphi(W', G) + \varphi(U'', G) - x - y. \tag{7}$$

Since $\varphi(U', G) \geqslant \varphi(V(Z), Z) > 0$, comparing (6) with (7) we have $\varphi(Y, G) < \varphi(U, G^*) \leqslant -\frac{1}{k+1}$, a contradiction to (3). This proves (II).

By (I) and (II) and by the minimality of G, G^* has a (j,k)-coloring c^*. We define c by letting $c(v) := c'(v)$ for $v \in W'$ and $c(v) := c^*(v)$ for $v \in V(G) - W'$. Recall that by Claim 2, all the $j+1$ (k, j)-hosts in $V(Z)$ and the vertices in $V(G) - W'$ adjacent to vertices of color k in c' are colored with j. Furthermore, since v_0 has j neighbors in Z of color j, all its neighbors in $V(G^*) - V(Z)$ are colored with k. Thus, c is a (j,k)-coloring of G, a contradiction. □

For every vertex $w \in V(\widetilde{G})$, let $\widetilde{d}(w)$ be the degree of w in $\widetilde{G}$ and $h(w)$ denote the number of $(j+1)$-flags based on w.

Lemma 3. *For every $w \in V(\widetilde{G})$, $\widetilde{d}(w) + h(w) \geqslant k+2$. In particular, every 2-vertex in G is a ghost.*

Proof. Suppose that $w \in V(\widetilde{G})$ and $\widetilde{d}(w) + h(w) \leqslant k+1$. Denote $d := \widetilde{d}(w)$ and $h := h(w)$. Let $x_1, \dots, x_d$ be the neighbors of w in $\widetilde{G}$ and $F_1, \dots, F_h$ be the $(j+1)$-flags based on w. If $d = 0$, then $V(\widetilde{G}) = \{w\}$ and it is easy to (j,k)-color G.

Suppose now that $d = 1$. Let $G' = G - F_1 - \dots - F_h - w$. Since G' is less than G, it has a (j,k)-coloring c'. If $c'(x_1) = k$, then we color w with j and each vertex in $\bigcup_{i=1}^{h} F_i - w$ with k. If $c'(x_1) = j$, then we color w with k and in each F_i based on w we color a vertex of degree $j+1$ in $F_i - w$ with k and the remaining $j+1$ vertices with j. It says “a vertex”, since for $j = 0$, there are two such vertices in $F_i - w$.

Finally, let $d \geqslant 2$. Consider G^* obtained from $G - F_1 - \dots - F_h - w$ by adding a $(j+1)$-flag $F(s)$ based on x_s for every $s = 1, \dots, d$. As in the previous lemma, we will prove that (I) G^* is smaller than G; (II) Condition (3) holds for G^*; and (III) any (j,k)-coloring c^* of G^* yields a (j,k)-coloring of G.

Statement (I) holds, since we deleted a vertex $w \in V(\widetilde{G})$ and added only ghost vertices. Suppose that (II) fails, i.e., $\varphi(W^*, G^*) \leqslant -\frac{1}{k+1}$ for some $W^* \subseteq V(G^*)$. Let $W = W^* \cap V(G)$. If $|\{x_1, \dots, x_d\} \cap W| \leqslant 1$, then by Claim 3(b),

$$\varphi(W, G) \leqslant \varphi(W^*, G^*) + \frac{1}{k+1} \leqslant 0,$$

a contradiction to Lemma 2. Suppose that $|\{x_1,\ldots,x_d\}\cap W|=r\geqslant 2$. Then $\varphi(W,G)\leqslant\varphi(W^*,G^*)+r\frac{1}{k+1}$ and hence

$$\varphi(W+w,G)\leqslant\varphi\big(W^*,G^*\big)+r\frac{1}{k+1}+\left(2-\frac{k+2}{(j+2)(k+1)}\right)-r.$$

For $r\geqslant 2$, this is at most

$$\varphi\big(W^*,G^*\big)+\frac{2}{k+1}-\frac{k+2}{(j+2)(k+1)}\leqslant\varphi\big(W^*,G^*\big),$$

since $(k+2)\geqslant 2(j+2)$. It follows that (3) does not hold for G, a contradiction to the choice of G.

By (I) and (II) and by the minimality of G, G^* has a (j,k)-coloring c^*. If $c^*(x_1)=\cdots=c^*(x_d)=k$, then we color w with j and each vertex in $\bigcup_{i=1}^h F_i-w$ with k. Suppose not. In this case we color w with k and in each flag F_i based on w we color a vertex of degree $j+1$ in F_i-w with k and the remaining $j+1$ vertices with j. In this way, w will have at most $(d-1)+h$ neighbors of color k. Recall that by Claim 1, each x_s had a neighbor of color k in $F(s)$. Thus, we get a (j,k)-coloring of G. □

Comparing Lemmas 1 and 3, we obtain

Corollary 2. *$\widetilde{G}$ has no isolated vertices. Furthermore, each pendant vertex in $\widetilde{G}$ is a peripheral (k,j)-host.*

If $j\geqslant 1$ and a peripheral (k,j)-host x is adjacent to another peripheral (k,j)-host y, then $V(\widetilde{G})=\{x,y\}$ and we can color x and y with j and the remaining vertices of G with k, a contradiction. From this and Corollary 2 we deduce

Lemma 4. *If $j\geqslant 1$, then peripheral (k,j)-hosts in $\widetilde{G}$ are not adjacent. In particular, if $j\geqslant 1$, then $\widetilde{G}$ has a vertex of degree at least 2.*

For every $w\in V(\widetilde{G})$, let $d_1(w)$ denote the number of its neighbors that are peripheral (k,j)-hosts and $d_2(w)=\widetilde{d}(w)-d_1(w)$. We are interested in vertices w with $d_2(w)=1$.

Lemma 5. *Let $w\in V(\widetilde{G})$ with $d_2(w)=1$. Then*:

(a) $h(w)\geqslant k$;
(b) $d_1(w)\geqslant j$;
(c) $h(w)+d_1(w)\geqslant k+j+1$.

Proof. For shortness, let $h:=h(w)$, $d_1:=d_1(w)$. Let $F_1,\ldots,F_h$ be the $(j+1)$-flags based on w and $x_1,\ldots,x_{d_1}$ be the peripheral (k,j)-hosts adjacent to w. Let y be the remaining neighbor of w.

Suppose first that $h\leqslant k-1$. Recall that by Lemma 3, $h+d_1\geqslant k+1$. Thus by Claim 3, the graph G' obtained from G by deleting $w,x_1,\ldots,x_{d_1}$ together with all $(j+1)$-flags based on them and then adding one $(j+1)$-flag F based on y satisfies (3). By construction, G' is smaller than G. So by the minimality of G, G' has a (j,k)-coloring c'. Since y has a neighbor of color k in F, when we color in G vertex w with k, there will be no conflict at y. Now we can color each x_i with j and all vertices in all $(j+1)$-flags based on x_i with k. Finally, for each $s=1,\ldots,h$, we color a vertex of degree $j+1$ in F_s-w with k and all other vertices in F_s-w with j. Since $h\leqslant k-1$, this will be a (j,k)-coloring of G.

Suppose now that $d_1\leqslant j-1$. By Claim 3, the graph G'' obtained from G by deleting $x_1,\ldots,x_{d_1}$ together with all $(j+1)$-flags based on them and then adding d_1 $(j+1)$-flags $F'_1,\ldots,F'_{d_1}$ based on w satisfies (3). Since the number of vertices of $\widetilde{G}$ decreases, G'' is smaller than G. So by the minimality of G, G'' has a (j,k)-coloring c''. Since w is in $h+d_1\geqslant k+1$ $(j+1)$-flags, we have $c''(w)=j$. Now we delete flags $F'_1-w,\ldots,F'_{d_1}-w$ and extend c'' to the whole G as follows: color each x_i with j

and all vertices in all $(j+1)$-flags based on x_i with k. To be on the safe side, recolor each vertex (apart from w) in each flag based on w with k. Again, we get a (j,k)-coloring of G.

Finally, suppose that $h=k$ and $d_1=j$. Let G^* be obtained from G by deleting $w, x_1, \dots, x_{d_1}$ together with all $(j+1)$-flags based on them. Since G^* is an induced subgraph of G, it is smaller than G and satisfies (3). So, it has a (j,k)-coloring c^*. If $c^*(y)=j$, then we color the rest as in the proof of (a), and if $c^*(y)=k$, then we color the rest as in the proof of (b). $\square$

3.3. *Discharging procedure*

By (3), we have

$$\sum_{v\in V(G)}\left(d(v)-2\left(2-\frac{k+2}{(j+2)(k+1)}\right)\right) < \frac{2}{k+1}. \tag{8}$$

The *initial charge* of each vertex v of G is $\mu(v)=d(v)-2(2-\frac{k+2}{(j+2)(k+1)})$, and the *final charge* $\mu^*(v)$ is determined by applying the following rules:

(R1) Every $w\in V(\widetilde{G})$ gives to the vertices of each $(j+1)$-flag F based on it the exact amount α such that together with their own initial charges the total charge of vertices in $F-w$ would become 0.

(R2) If $j\geqslant 1$, then for every peripheral (k,j)-host x, its neighbor y in $\widetilde{G}$ gives to x the exact amount β to make the resulting charge of x equal to 0. If $j=0$, then nothing happens.

First observe that by Lemma 4, Rule (R2) does not create confusion.

Second, let us understand what are the values of α and β. By definition, for each $(j+1)$-flag F based on w, we have

$$\sum_{v\in F-w}\mu(v)=(j+2)+2(j+1)-(j+2)2\left(2-\frac{k+2}{(j+2)(k+1)}\right)=-j-2+\frac{2}{k+1}.$$

So, $\alpha=j+2-\frac{2}{k+1}$. We will view this as if from the $j+2$ edges connecting w with $F-w$, w leaves for itself $\frac{2}{k+1}$ of degree, and gives α to the vertices of $F-w$, and they share their charges so that their modified charges are zeros.

After a peripheral (k,j)-host x leaves for itself $\frac{2}{k+1}$ from each of the $k+1$ $(j+1)$-flags based on x, it also has degree 1 from the incident edge in $\widetilde{G}$. Thus after applying (R1), the charge of x is

$$2+1-2\left(2-\frac{k+2}{(j+2)(k+1)}\right)=-1+\frac{2(k+2)}{(j+2)(k+1)}.$$

So if $j\geqslant 1$, then $\beta=1-\frac{2(k+2)}{(j+2)(k+1)}$. We view it as if the neighbor of x from the edge connecting it to x leaves for itself $\frac{2(k+2)}{(j+2)(k+1)}$ and gives β to x to make its charge zero.

Now we evaluate the final charges of vertices. By above, the charges of all ghost vertices are zeros. For $j\geqslant 1$, the charge of each peripheral (k,j)-host is zero, and for $j=0$, it is $\frac{1}{k+1}$.

Let w be a vertex in $\widetilde{G}$ with $\widetilde{d}(w)\geqslant 2$. Let $h:=h(w)$, $d_1:=d_1(w)$, and $d_2=d_2(w)$. Let $F_1,\dots,F_h$ be the $(j+1)$-flags based on w and $x_1,\dots,x_{d_1}$ be the peripheral (k,j)-hosts adjacent to w. Let $y_1,\dots,y_{d_2}$ be the remaining neighbors of w.

Case 0: $d_2=0$. Then $\widetilde{G}=K_{1,d_1}$ with the center w. If $d_1\leqslant j$, then we can color $w,x_1,\dots,x_{d_1}$ with j, and the remaining vertices of G with k. If $h\leqslant k$, then we can color $x_1,\dots,x_{d_1}$ with j, the remaining vertices in $(j+1)$-flags based on them with k, color w and one vertex in each $(j+1)$-flag based on w with k, and the remaining vertices in the $(j+1)$-flags based on w with j. In both cases, we obtain a (j,k)-coloring of G. So, we may assume that $h\geqslant k+1$ and $d_1\geqslant j+1$. Then G contains the graph $G_1(j,k)$ for which (3) fails, a contradiction.

Case 1: $d_2 = 1$. Since $\alpha > \beta$, by Lemma 5, we know that w leaves for itself at least $(k+1)\frac{2}{k+1} + j\frac{2(k+2)}{(j+2)(k+1)}$ from the edges connecting it with $F_1, \dots, F_h$ and $x_1, \dots, x_{d_1}$. It also has 1 from the edge connecting it with y_1. So since $k \geqslant 2j+2$,

$$\mu^*(w) \geqslant (k+1)\frac{2}{k+1} + j\frac{2(k+2)}{(j+2)(k+1)} + 1 - 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$$
$$= -1 + 2\frac{(j+1)(k+2)}{(j+2)(k+1)} = \frac{j(k+3)+2}{(j+2)(k+1)} \geqslant \frac{j(2j+5)+2}{(j+2)(k+1)}.$$

The last expression equals $\frac{1}{k+1}$ when $j = 0$, and exceeds $\frac{2}{k+1}$ when $j \geqslant 1$.

Case 2: $d_2 \geqslant 2$. Since $h + d_1 + d_2 \geqslant k+2$ and $\alpha > \beta$, we have (since $k \geqslant 2j+2$)

$$\mu^*(w) \geqslant 2 + k\frac{2}{k+1} - 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right) = -\frac{2}{k+1} + \frac{2(k+2)}{(j+2)(k+1)}$$
$$= \frac{2(k+2-j-2)}{(j+2)(k+1)} \geqslant \frac{2((2j+2)-j)}{(j+2)(k+1)} = \frac{2}{k+1}.$$

Thus, in particular $\mu^*(w) \geqslant 0$ for every $w \in V(G)$. By (8), no vertex gets final charge at least $\frac{2}{k+1}$ and at most one gets final charge at least $\frac{1}{k+1}$. Hence for $j \geqslant 1$ none of Cases 0, 1, and 2 may occur. This contradicts Lemma 4.

Suppose now that $j = 0$. By Corollary 2, $\widetilde{G}$ has at least two (non-isolated) vertices, and by the analysis above, each of them gets charge at least $\frac{1}{k+1}$. This contradicts (8).

The theorem is proved.

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