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Ore's conjecture on color-critical graphs is almost true

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ABSTRACT

A graph G is *k-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$-colorable. Let $f_k(n)$ denote the minimum number of edges in an n-vertex k-critical graph. We give a lower bound, $f_k(n) \geq F(k,n)$, that is sharp for every $n = 1 \pmod{k-1}$. The bound is also sharp for $k = 4$ and every $n \geq 6$. The result improves a bound by Gallai and subsequent bounds by Krivelevich and Kostochka and Stiebitz, and settles the corresponding conjecture by Gallai from 1963. It establishes the asymptotics of $f_k(n)$ for every fixed k. It also proves that the conjecture by Ore from 1967 that for every $k \geq 4$ and $n \geq k+2$, $f_k(n+k-1) = f_k(n) + \frac{k-1}{2}(k - \frac{2}{k-1})$ holds for each $k \geq 4$ for all but at most $k^3/12$ values of n. We give a polynomial-time algorithm for $(k-1)$-coloring of a graph G that satisfies $|E(G[W])| < F(k,|W|)$ for all $W \subseteq V(G)$, $|W| \geq k$. We also present some applications of the result.

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1. Introduction

A *proper k-coloring*, or simply *k-coloring*, of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$ such that for each $uv \in E$, $f(u) \neq f(v)$. A graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number*, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable. A graph G is *k-chromatic* if $\chi(G) = k$.

A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. Then every k-critical graph has chromatic number k and every k-chromatic graph contains a k-critical subgraph. This means that some problems for k-chromatic graphs may be reduced to problems for k-critical graphs, whose structure is more restricted. For example, every k-critical graph is 2-connected and $(k-1)$-edge-connected. Critical graphs were first defined and used by Dirac [4–6] in 1951–1952.

The only 1-critical graph is K_1, and the only 2-critical graph is K_2. The only 3-critical graphs are the odd cycles. For every $k \geq 4$ and every $n \geq k+2$, there exists a k-critical n-vertex graph. Let $f_k(n)$ be the minimum number of edges in a k-critical graph with n vertices. Since $\delta(G) \geq k-1$ for every k-critical n-vertex graph G,

$$f_k(n) \geq \frac{k-1}{2}n \tag{1}$$

for all $n \geq k$, $n \neq k+1$. Equality is achieved for $n = k$ and for $k = 3$ and n odd. Brooks' Theorem [3] implies that for $k \geq 4$ and $n \geq k+2$, the inequality in (1) is strict. In 1957, Dirac [8] asked to determine $f_k(n)$ and proved that for $k \geq 4$ and $n \geq k+2$,

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}. \tag{2}$$

The result is tight for $n = 2k-1$ and yields $f_k(2k-1) = k^2 - k - 1$. Dirac used his bound to evaluate chromatic number of graphs embedded into fixed surfaces. Later, Kostochka and Stiebitz [17] improved (2) to

$$f_k(n) \geq \frac{k-1}{2}n + k - 3 \tag{3}$$

when $n \neq 2k-1, k$. This yields $f_k(2k) = k^2 - 3$ and $f_k(3k-2) = \frac{3k(k-1)}{2} - 2$. In his fundamental papers [10,11], Gallai found exact values of $f_k(n)$ for $k+2 \leq n \leq 2k-1$:

Theorem 1. *(See Gallai [11].) If $k \geq 4$ and $k+2 \leq n \leq 2k-1$, then*

$$f_k(n) = \frac{1}{2}\big((k-1)n + (n-k)(2k-n)\big) - 1.$$

He also proved the following general bound for $k \geq 4$ and $n \geq k+2$:

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n. \tag{4}$$

For large n, this bound is much stronger than bounds (2) and (3). Gallai [10] also conjectured a lower bound on $f_k(n)$.

Conjecture 2. *(See Gallai [10].) If $k \geq 4$ and $n = 1 \pmod{k-1}$, then*

$$f_k(n) = \frac{(k^2 - k - 2)n - k(k-3)}{2(k-1)}.$$

Gallai commented that possibly this will be hard to prove. Ore [24] in 1967 observed that Hajós' construction [13] implies

$$f_k(n+k-1) \leq f_k(n) + \frac{(k-2)(k+1)}{2} = f_k(n) + \frac{k-1}{2}\left(k - \frac{2}{k-1}\right), \tag{5}$$

which yields that $\phi_k := \lim_{n\to\infty} \frac{f_k(n)}{n}$ exists and satisfies $\phi_k \leq \frac{k}{2} - \frac{1}{k-1}$.

Gallai's bound (3) gives $\phi_k \geq \frac{1}{2}(k - 1 + \frac{k-3}{k^2-3})$. Ore believed that Hajós' construction was best possible.

Conjecture 3. *(See Ore [24].) If $k \geq 4$, then*

$$f_k(n+k-1) = f_k(n) + \frac{k-1}{2}\left(k - \frac{2}{k-1}\right).$$

Much later, Krivelevich [23] improved Gallai's bound to

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-2k-1)}n$$

and demonstrated nice applications of his bound: he constructed graphs with high chromatic number and low independence number such that the chromatic numbers of all their small subgraphs are at most 3 or 4. We discuss a couple of his applications in Subsection 6.3. Then Kostochka and Stiebitz [20] proved that for $k \geq 6$ and $n \geq k+2$,

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{k^2+6k-11-6/(k-2)}n.$$

The problem of finding $f_k(n)$ is Problem 5.3 in [14] and Problem 12 in the list of 25 *pretty graph coloring problems* by Jensen and Toft [15]. It is a part of Problem P1 in [28, p. 347]. Recently, Farzad and Molloy [9] have found the minimum number of edges in 4-critical n-vertex graphs in which the set of vertices of degree 3 induces a connected subgraph.

The main result of the present paper is a bound establishing Conjecture 2:

Theorem 4. *If $k \geq 4$ and G is k-critical, then $|E(G)| \geq \lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \rceil$. In other words, if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then*

$$f_k(n) \geq F(k,n) := \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil. \tag{6}$$

This bound is exact for $k = 4$ and every $n \geq 6$. For every $k \geq 5$, the bound is exact for every $n \equiv 1 \pmod{k-1}$, $n \neq 1$. Apart from Conjecture 2 from 1963, the result confirms Conjecture 3 from 1967 for $k = 4$ and every $n \geq 6$ and also for $k \geq 5$ and all $n \equiv 1 \pmod{k-1}$, $n \neq 1$. In the second half of the paper we derive some corollaries and applications of the main result. The next two corollaries follow from Theorems 4 and 1 and from (5). Both will be proven in Section 5.

Corollary 5. *For every $k \geq 4$ and $n \geq k+2$,*

$$0 \leq f_k(n) - F(k,n) \leq \frac{k(k-1)}{8} - 1. \quad \textit{In particular, } \phi_k = \frac{k}{2} - \frac{1}{k-1}.$$

Corollary 6. *For each fixed $k \geq 4$, Conjecture 3 holds for all but at most $\frac{k^3}{12} - \frac{k^2}{8}$ values of n.*

Our proof of Theorem 4 is constructive. This allows us to give an algorithm for coloring graphs with no dense subgraphs. The idea of sparseness is expressed in terms of potentials.

Definition 7. For $R \subseteq V(G)$, define *the k-potential of R* to be

$$\rho_{k,G}(R) = (k-2)(k+1)|R| - 2(k-1)\left|E\big(G[R]\big)\right|. \tag{7}$$

When there is no chance for confusion, we will use $\rho_k(R)$. Let $P_k(G) = \min_{\emptyset \neq R \subseteq V(G)} \rho_k(R)$.

Theorem 8. *If $k \geq 4$, then every n-vertex graph G with $P_k(G) > k(k-3)$ can be $(k-1)$-colored in $O(k^{3.5}n^{6.5}\log(n))$ time.*

The restriction $P_k(G) > k(k-3)$ is sharp for every $k \geq 4$.

In Section 2 we prove several statements about list colorings that will be used in our proofs. In Section 3 we give definitions and prove several lemmas that will be used in Section 4, where we prove Theorem 4. In Section 5 we discuss the sharpness of our result. In Section 6 we present some applications. In Section 7 we prove Theorem 8. We finish the paper with some comments.

Our notation is standard. In particular, $\chi(G)$ denotes the chromatic number of graph G, $G[W]$ is the subgraph of a graph or digraph G induced by the vertex set W. For a vertex v in a graph G, $d_G(v)$ denotes the degree of vertex v in graph G, $N_G(v)$

is the set of neighbors of v and $N_G[v] = N_G(v) \cup \{v\}$. If the graph G is clear from the context, we drop the subscript.

2. Orientations and list colorings

We consider loopless digraphs. A *kernel* in a digraph D is an independent set F of vertices such that each vertex in $V(D) - F$ has an out-neighbor in F.

A digraph D is *kernel-perfect* if for every $A \subseteq V(D)$, the digraph $D[A]$ has a kernel. It is known that kernel-perfect orientations form a useful tool for list colorings. Recall that a *list* for a graph G is a mapping L of $V(G)$ into the family of finite subsets of $\mathbf{N}$. For a given list L, a graph G is *L-colorable*, if there exists a coloring $f : V(G) \to \mathbf{N}$ such that $f(v) \in L(v)$ for every $v \in V(G)$ and $f(v) \neq f(u)$ for every $uv \in E(G)$. The following fact is well known but we include its proof for completeness.

Lemma 9. *(Folklore.) If D is a kernel-perfect digraph and L is a list such that*

$$\left|L(v)\right| \geq 1 + d^+(v) \quad \textit{for every } v \in V(D), \tag{8}$$

then D is L-colorable.

Proof. We use induction on $|V(D)|$. If D has only one vertex, the statement is trivial. Suppose the statement holds for all pairs (D', L) satisfying (8) with $|V(D')| \leq n-1$. Let $|V(D)| = n$ and (D, L) satisfy (8). Let $v \in V(D)$ and α be a color present in $L(v)$. Let V_α be the set of vertices $x \in V(D)$ with $\alpha \in L(x)$. Since D is kernel-perfect, $D[V_\alpha]$ has a kernel K. Color all vertices of K with α and consider (D', L'), where $D' = D - K$ and $L'(y) = L(y) - \alpha$ for all $y \in V(D')$. Since the outdegree of every $x \in V_\alpha - K$ decreased by at least 1, (D', L') satisfies (8), and so by the induction assumption has an L'-coloring. Together with coloring of K by α, this yields an L-coloring of D, as claimed. $\square$

It is known that every orientation of a bipartite multigraph is kernel-perfect. We prove a somewhat stronger result.

Lemma 10. *Let A be an independent set in a graph G and $B = V(G) - A$. Let D be the digraph obtained from G by replacing each edge in $G[B]$ by a pair of opposite arcs and by an arbitrary orientation of the edges connecting A with B. Then D is kernel-perfect.*

Proof. Let D be a counter-example with the fewest vertices. If every $b \in B$ has an outneighbor in A, then A is a kernel. Otherwise, some $b \in B$ has no outneighbors in A. Then $N(b) = N^-(b)$. We consider $D' = D - b - N^-(b)$. By the minimality of D, D' has a kernel K. Then $K + b$ is a kernel of D. $\square$

For a graph G and disjoint vertex subsets A and B, let $G(A,B)$ denote the bipartite graph with partite sets A and B whose edges are all edges of G connecting A with B. The main result of this section is the following.

Lemma 11. *Let G be a k-critical graph. Let disjoint vertex subsets A and B be such that*

(a) at least one of A and B is independent;
(b) $d(a) = k-1$ for every $a \in A$;
(c) $d(b) = k$ for every $b \in B$.

Under these conditions,

(i) $\delta(G(A,B)) \le 2$ and
(ii) either some $a \in A$ has at most one neighbor in B or some $b \in B$ has at most three neighbors in A.

Proof. If $A \cup B = \emptyset$, then both statements are trivial. Otherwise, since G is k-critical, there exists a $(k-1)$-coloring f of $G-A-B$. Fix any such f. For every $x \in A \cup B$, let $L(x)$ be the set of colors in $\{1,\dots,k-1\}$ not used in f on neighbors of x. Let $G' = G[A \cup B]$. Then

$$\text{for every } a \in A, \quad |L(a)| \ge d_{G'}(a), \quad \text{and} \quad \text{for every } b \in B, \quad |L(b)| \ge d_{G'}(b) - 1. \tag{9}$$

Case 1: $\delta(G(A,B)) \ge 3$. Let G'' be obtained from $G(A,B)$ by splitting each $b \in B$ into $\lceil d_{G(A,B)}(b)/3 \rceil$ vertices of degree at most 3. In particular, a vertex b of degree 3 in $G(A,B)$ is not split. The graph G'' is bipartite with partite sets A and B', where B' is obtained from B by splitting. The degree of each $a \in A$ in G'' is at least 3, and the degree of each vertex $b \in B'$ is at most 3. So by Hall's Theorem, G'' has a matching M covering A. We construct a digraph D from G' as follows:

(1) replace each edge of $G[B]$ or $G[A]$ (whichever is nonempty) with two opposite arcs,
(2) orient every edge of $G(A,B)$ corresponding to an edge in M towards A,
(3) orient all other edges of $G(A,B)$ towards B.

By Lemma 10, D is kernel-perfect. Moreover, by (9), for every $a \in A$, $d^+(a) = d_{G'}(a) - 1 \le |L(a)| - 1$, and for every $b \in B$,

$$d^+(b) \le d_{G'}(b) - \left\lfloor \frac{2}{3} d_{G(A,B)}(b) \right\rfloor \le \bigl(|L(b)| + 1\bigr) - 2 = |L(b)| - 1.$$

Thus by Lemma 9, G' is L-colorable. But this means that G is $(k-1)$-colorable, a contradiction. This proves (i).

Case 2: Each $a \in A$ has at least two neighbors in B and each $b \in B$ has at least four neighbors in A. Then we obtain G'' by splitting each $b \in B$ into $\lceil d_{G(A,B)}(b)/2 \rceil$ vertices of degree at most 2. Similarly to Case 1, the graph G'' is bipartite with partite sets A and B', where B' is obtained from B. The degree of each $a \in A$ in G'' is at least 2, and the degree of each vertex $b \in B'$ is at most 2. So by Hall's Theorem, G'' has a matching M covering A. We construct the digraph D from G'' according to rules (1)–(3) in Case 1. Again, by Lemma 10, D is kernel-perfect, and by (9), for every $a \in A$, $d^+(a) = d_{G'}(a) - 1 \le |L(a)| - 1$. For every $b \in B$, since $d_{G(A,B)}(b) \ge 4$, by (9),

$$d^+(b) \le d_{G''}(b) - \left\lfloor \frac{1}{2} d_{G(A,B)}(b) \right\rfloor \le \left(\left|L(b)\right| + 1\right) - 2 = \left|L(b)\right| - 1. \qquad \square$$

Corollary 12. *Let G be a k-critical graph. Let A and B be disjoint vertex subsets such that*

(a) either A or B is independent;
(b) $d(a) = k - 1$ for every $a \in A$;
(c) $d(b) = k$ for every $b \in B$;
(d) $|A| + |B| \ge 3$.

Under these circumstances,

(i) $e(G(A,B)) \le 2(|A| + |B|) - 4$ and
(ii) $e(G(A,B)) \le |A| + 3|B| - 3$.

Proof. First we prove (i) by induction on $|A| + |B|$. If $|A| + |B| = 3$, then since $G(A,B)$ is bipartite, it has at most $2 = 2 \cdot 3 - 4$ edges. Suppose now that $|A| + |B| = m \ge 4$ and the corollary holds for $3 \le |A| + |B| \le m - 1$. By Lemma 11(i), $G(A,B)$ has a vertex v of degree at most two. By the minimality of m, $G(A,B) - v$ has at most $2(m-1) - 4$ edges. Then $e(G(A,B)) \le 2 + 2(m-1) - 4 = 2m - 4$, as claimed.

The base case $|A| + |B| = 3$ for (ii) is slightly more complicated. If $|A| = 3$, then $e(G(A,B)) = 0 = |A| + 3|B| - 3$. If $|B| \ge 1$, then $|A| + 3|B| \ge 5$ and $e(G(A,B)) \le 2 = 5 - 3 \le |A| + 3|B| - 3$. The proof of the induction step is very similar to the previous paragraph, using Lemma 11(ii). $\square$

3. Preliminary results

Fact 13. For the k-potential defined by (7), we have

1. $\rho_{k,K_k}(V(K_k)) = k(k-3)$,
2. $\rho_{k,K_1}(V(K_1)) = (k-2)(k+1)$,
3. $\rho_{k,K_2}(V(K_2)) = 2(k^2 - 2k - 1)$,
4. $\rho_{k,K_{k-1}}(V(K_{k-1})) = 2(k-2)(k-1)$.

A graph H is *smaller than* graph G, if either $|E(G)| > |E(H)|$, or $|E(G)| = |E(H)|$ and G has fewer pairs of vertices with the same closed neighborhood. The definition implies that if $|V(G)| \geq |V(H)|$, $\rho_k(V(G)) \leq \rho_k(V(H))$, and at least one of these inequalities is strict, then H is smaller than G.

Note that $(k - \frac{2}{k-1})|V(G)| > 2|E(G)| + \frac{k(k-3)}{k-1}$ is equivalent to $\rho_k(V(G)) > k(k-3)$. Let G be a minimal k-critical graph with respect to the relation "is smaller than" with $\rho_k(V(G)) > k(k-3)$. This implies that

$$\text{if } H \text{ is smaller than } G \text{ and } P_k(H) > k(k-3), \text{ then } H \text{ is } (k-1)\text{-colorable.} \tag{10}$$

Definition 14. For a graph G, a set $R \subset V(G)$ and a $(k-1)$-coloring ϕ of $G[R]$, the graph $Y(G,R,\phi)$ is constructed as follows. First, for $i = 1, \dots, k-1$, let R'_i denote the set of vertices in $V(G) - R$ adjacent to at least one vertex $v \in R$ with $\phi(v) = i$. Second, let $X = \{x_1, \dots, x_{k-1}\}$ be a set of new vertices disjoint from $V(G)$. Now, let $Y = Y(G,R,\phi)$ be the graph with vertex set $V(G) - R + X$, such that $Y[V(G) - R] = G - R$ and $N(x_i) = R'_i \cup (\{x_1, \dots, x_{k-1}\} - x_i)$ for $i = 1, \dots, k-1$.

Claim 15. *Suppose $R \subset V(G)$ and ϕ is a $k-1$ coloring of $G[R]$. Then $\chi(Y(G,R,\phi)) \geq k$.*

Proof. Let $G' = Y(G,R,\phi)$. Suppose G' has a $(k-1)$-coloring $\phi' : V(G') \to C$. By construction of G', the colors of all x_i in ϕ' are distinct. By changing the names of the colors, we may assume that $\phi'(x_i) = i$ for $1 \leq i \leq k-1$. By construction of G', for all vertices $u \in R'_i$, $\phi'(u) \neq i$. Therefore $\phi|_R \cup \phi'|_{V(G)-R}$ is a proper coloring of G, a contradiction. $\square$

Claim 16. *There is no $R \subsetneq V(G)$ with $|R| \geq 2$ and $\rho_{k,G}(R) \leq (k-2)(k+1)$.*

Proof. Let $2 \leq |R| < |V(G)|$ and $\rho_k(R) = m = \min\{\rho_k(W) : W \subsetneq V(G),\ |W| \geq 2\}$. Suppose $m \leq (k-2)(k+1)$. Then $|R| \geq k$. Since G is k-critical, $G[R]$ has a proper coloring $\phi : R \to C = \{1, \dots, k-1\}$. Let $G' = Y(G,R,\phi)$. By Claim 15, G' is not $(k-1)$-colorable. Then it contains a k-critical subgraph G''. Let $W = V(G'')$. Since $|R| \geq k > |X|$ and $\rho_k(R) < \rho_k(X)$, G'' is smaller than G. So, by the minimality of G, $\rho_{k,G'}(W) \leq k(k-3)$. Since G itself is k-critical, $W \cap X \neq \emptyset$. Since every non-empty subset of X has potential at least $(k-2)(k+1)$,

$$\rho_{k,G}(W - X + R) \leq \rho_{k,G'}(W) - (k-2)(k+1) + m \leq m - 2k + 2.$$

Since $W - X + R \supset R$, $|W - X + R| \geq 2$. Since $\rho_{k,G}(W - X + R) < \rho_{k,G}(R)$, by the choice of R, $W - X + R = V(G)$. But then $\rho_{k,G}(V(G)) \leq m - 2k + 2 \leq k(k-3)$, a contradiction. $\square$

Lemma 17. *Let $k \geq 3$ be an integer. Let $R_* = \{u_1, \dots, u_s\}$ be a vertex set and $w : R_* \to \{1, 2, \dots\}$ be an integral positive weight function on R_* such that $w(u_1) + \dots + w(u_s) \geq$*

$k-1$. *Then for each* $1 \le i \le (k-1)/2$, *there exists a graph* H *with* $V(H) = R_*$ *and* $|E(H)| = i$ *such that for every independent set* M *in* H *with* $|M| \ge 2$,

$$\sum_{u \in R_* - M} w(u) \ge i. \tag{11}$$

Proof. We may assume that $w(u_1) \ge w(u_2) \ge \cdots \ge w(u_s)$.

Case 1: $w(u_2) + \cdots + w(u_s) \le i$. We let $E(H) = \{u_1 u_j : 2 \le j \le s\}$. If M is any independent set with $|M| \ge 2$, then $u_1 \notin M$ and witnesses that (11) holds.

Case 2: $w(u_2) + \cdots + w(u_s) \ge i+1$. Choose the largest j such that $w(u_j) + \cdots + w(u_s) \ge i$. Let $\alpha = i - w(u_{j+1}) + \cdots + w(u_s)$. Since $i \le (k-1)/2$ and $w(u_1) + \cdots + w(u_s) \ge k-1$, we also have $w(u_1) + \cdots + w(u_j) \ge i + \alpha$. By the choice of j and the ordering of the vertices, $0 < \alpha \le w(u_j) \le w(u_1)$. We draw α edges connecting u_1 with u_j and $i - \alpha$ edges connecting $\{u_{j+1}, \ldots, u_s\}$ with $\{u_1, \ldots, u_j\}$ so that for each ℓ, the degree of u_ℓ in the obtained multigraph H is at most $w(u_\ell)$. Let M be any nonempty independent set in H. By the definition of H, since M is independent,

$$\sum_{u \in R_* - M} w(u) \ge \sum_{u \in R_* - M} d_H(u) \ge \frac{1}{2} \sum_{u \in R_*} d_H(u) = i,$$

as claimed. If H has multiple edges, we replace each set of multiple edges with a single edge. $\square$

Claim 18. *If* $R \subsetneq V(G)$, $|R| \ge 2$ *and* $\rho_k(R) \le 2(k-2)(k-1)$, *then* R *is a* K_{k-1}.

Proof. Let R have the minimum $\rho_k(R)$ among proper subsets of vertices with size at least 2. Suppose $m = \rho_k(R) \le 2(k-2)(k-1)$ and $G[R] \ne K_{k-1}$. Then $|R| \ge k$. Let i be the integer such that

$$1 + k(k-3) + 2i(k-1) \le \rho_k(R) \le k(k-3) + 2(i+1)(k-1). \tag{12}$$

By Claim 16, $i \ge 1$. Since for $k \ge 3$,

$$1 + k(k-3) + \frac{k-1}{2} 2(k-1) > 2(k-2)(k-1), \tag{13}$$

we have $i \le \frac{k-2}{2}$.

For $u \in R$, let $w(u) = |N(u) \cap (V(G) - R)|$. Let $R_* = \{u \in R : w(u) \ge 1\}$. Because $\kappa(G) \ge 2$, $|R_*| \ge 2$. Since G is k-critical, $\sum_{u \in R_*} w(u) = |E_G(R, V(G) - R)| \ge k-1$. So by Lemma 17, we can add to $G[R_*]$ a set E_0 of at most i edges so that for every independent subset M of R_* in $G \cup E_0$ with $|M| \ge 2$, (11) holds. Let $H = G[R] \cup E_0$. Note that $|E(G)| - |E(G[R])| \ge k-1 > i$, so H is smaller than G. By the minimality of $\rho_k(R)$ and the definition of i, for every $U \subseteq R$ with $|U| \ge 2$,

$$\rho_{k,H}(U) \geq \rho_{k,G}(U) - 2i(k-1) \geq \rho_{k,G}(R) - 2i(k-1) \geq 1 + k(k-3).$$

Thus $P_k(H) \geq 1 + k(k-3)$, and by (10) H has a proper $(k-1)$-coloring ϕ with colors in $C = \{1, \dots, k-1\}$.

As in Claim 16, we let $G' = Y(G, R, \phi)$. Since $|R| \geq k$, $|V(G')| < |V(G)|$. Since

$$\rho_{k,G'}\big(V\big(G'\big)\big) = \rho_{k,G}\big(V(G)\big) - \rho_k(R) + \rho_k(X) \geq \rho_{k,G}\big(V(G)\big),$$

$|E(G')| < |E(G)|$ and so G' is smaller than G. By Claim 15, G' is not $(k-1)$-colorable. Thus G' contains a k-critical subgraph G''. Let $W = V(G'')$. By the minimality of G, $\rho_{k,G'}(W) \leq k(k-3)$. Since G is k-critical by itself, $W \cap X \neq \emptyset$.

Since every subset of X with at least two vertices has potential at least $2(k-2)(k-1)$, if $|W \cap X| \geq 2$ then $\rho_{k,G}(W - X + R) \leq \rho_{k,G'}(W) \leq k(k-3)$, a contradiction again. So, without loss of generality, assume that $X \cap W = \{x_1\}$. But then

$$\begin{aligned}\rho_{k,G}\big(W - \{x_1\} + R\big) &\leq \big(\rho_{k,G'}(W) - (k-2)(k+1)\big) + \rho_{k,G}(R) \\ &\leq \rho_{k,G}(R) - 2k + 2.\end{aligned} \tag{14}$$

By the minimality of $\rho_{k,G}(R)$, $W - \{x_1\} + R = V(G)$. This implies that $W = V(G') - X + x_1$.

Let $R_1 = \{u \in R_* : \phi(u) = \phi(x_1)\}$. If $|R_1| = 1$, then

$$\rho_{k,G}(W - x_1 \cup R_1) = \rho_{k,H}(W) \leq k(k-3),$$

a contradiction. Thus, $|R_1| \geq 2$. Since R_1 is an independent set, by the construction of H, at least i edges connect the vertices in $R_* - R_1$ with $V(G) - R$. These edges were not counted in (14). So, in this case instead of (14), we have

$$\begin{aligned}\rho_{k,G}\big(W - \{x_1\} + R\big) &\leq \rho_{k,G'}(W) - (k-2)(k+1) - 2i(k-1) + \rho_{k,G}(R) \\ &\leq \rho_{k,G}(R) - 2k + 2 - 2i(k-1) \\ &= \rho_{k,G}(R) - 2(i+1)(k-1) \\ &\leq k(k-3),\end{aligned}$$

a contradiction. □

Claim 19. *If $d(x) = d(y) = k-1$ and x and y are in the same $(k-1)$-clique, then $N[x] = N[y]$.*

Proof. By contradiction, assume that $d(x_1) = d(x_2) = k-1$, $N(x_1) = X - x_1 + a$, $N(x_2) = X - x_2 + b$, and $a \neq b$. If $ab \in E(G)$, then define $G' = G - \{x_1, x_2\}$. Otherwise define $G' = G - \{x_1, x_2\} + ab$. Because $\rho_{k,G}(W) \geq 2(k-2)(k-1)$ for all $W \subseteq G - \{x_1, x_2\}$ with $|W| \geq 2$, and adding an edge decreases the potential of a set by $2(k-1)$,

$$P_k\big(G'\big) \geq \min\big\{(k-2)(k+1), 2(k-2)(k-1) - 2(k-1)\big\} > 1 + k(k-3).$$

So, since G' cannot contain k-critical subgraphs, it has a proper $(k-1)$-coloring ϕ' with $\phi'(a) \neq \phi'(b)$. This easily extends to a proper $(k-1)$-coloring of $V(G)$. □

Definition 20. A *cluster* is a maximal set $R \subseteq V(G)$ such that for every $x \in R$, $d(x) = k-1$ and for every pair $x, y \in R$, $N[x] = N[y]$.

Claim 21. *Let C be a cluster. Then $|C| \leq k-3$. Furthermore, if C is in a $(k-1)$-clique X, then $|C| \leq \frac{k-1}{2}$.*

Proof. A cluster with $k-2$ vertices plus its two neighbors would form a set of potential at most $k(k-3) + 2(k-1)$, which is less than $2(k-2)(k-1)$ when $k \geq 4$.

Let $\{v\} = N(C) - X$. If $|C| \geq \lceil k/2 \rceil$, then $\rho_k(X+v) \leq 2(k-2)(k-1) - 2$, a contradiction. □

Claim 22. *Let $xy \in E(G)$, $N[x] \neq N[y]$, x is in a cluster of size s, y is in a cluster of size t, and $s \geq t$. Then x is in a $(k-1)$-clique. Furthermore, $t = 1$.*

Proof. Assume that x is not in a $(k-1)$-clique. Let $G' = G - y + x'$, where $N[x'] = N[x]$. We have $|E(G')| = |E(G)|$. If two vertices z and z' distinct from y had the same closed neighborhood in G, then they would also have the same closed neighborhood in G'. Thus, since the cluster containing x is at least as large as the one containing y, G' is smaller than G in our ordering. If G' has a $(k-1)$-coloring $\phi' : V(G') \to C = \{1, 2, \dots, k-1\}$, then we extend it to a proper $(k-1)$-coloring ϕ of G as follows: define $\phi|_{V(G)-x-y} = \phi'|_{V(G')-x-x'}$, then choose $\phi(y) \in C - (\phi'(N(y)-x))$, and $\phi(x) \in \{\phi'(x), \phi'(x')\} - \{\phi(y)\}$.

So, $\chi(G') \geq k$ and G' contains a k-critical subgraph G''. Let $W = V(G'')$. Since G'' is smaller than G, $\rho_{k,G'}(W) \leq k(k-3)$. Since G'' is not a subgraph of G, $x' \in W$. Then $\rho_{k,G}(W - x') \leq k(k-3) - (k-2)(k+1) + 2(k-1)(k-1) = 2(k-2)(k-1)$. This contradicts Claim 18 because $y \notin W - x'$ and so $W - x' \neq V(G)$. □

4. Proof of Theorem 4

4.1. Case $k = 4$

Claim 23. *Each edge of G is in at most 1 triangle. Moreover, each cluster has only one vertex.*

Proof. The vertex set of a subgraph with 4 vertices and 5 edges has potential 10, which contradicts Claim 18. A cluster of size two would create an edge shared by two triangles. □

Claim 24. *Each vertex with degree 3 has at most 1 neighbor with degree 3.*

Proof. This follows directly from Claims 23 and 22. □

We will now use discharging to show that $|E(G)| \geq \frac{5}{3}|V(G)|$, which will finish the proof to the case $k = 4$. Each vertex begins with charge equal to its degree. If $d(v) \geq 4$, then v gives charge $\frac{1}{6}$ to each neighbor with degree 3. Note that v will be left with charge at least $\frac{5}{6}d(v) \geq \frac{10}{3}$. By Claim 24, each vertex of degree 3 will end with charge at least $3 + \frac{2}{6} = \frac{10}{3}$. □

4.2. Case $k = 5$

Claim 25. *Each cluster has only one vertex.*

Proof. Assume $N[x] = N[y]$ and $d(x) = d(y) = 4$. Because G does not contain a K_5, there exist $a, b \in N[x]$ such that $ab \notin E(G)$. We obtain G' from G by deleting x and y and gluing a with b into a single vertex $a * b$. If G' is 4-colorable, then so is G. This is because a 4-coloring of G' will have at most 2 colors on $N[x] - \{x, y\}$ and therefore could be extended greedily to x and y.

So G' contains a k-critical subgraph G''. Let $W = V(G'')$. Since G'' is smaller than G, $\rho_{5,G'}(W) \leq 10$. Since G'' is not a subgraph of G, $a * b \in W$. But then $\rho_{5,G}(W - a * b + a + b + x + y) \leq 10 + 54 - 40 = 24$. Because $ab \notin E(G)$, $W - a * b + a + b + x + y$ is not a K_4. By Claim 18, $W - a * b + a + b + x + y = V(G)$. But then we did not account for two of the edges incident to $\{x, y\}$, so $\rho'_G(W - a * b + a + b + x + y) \leq 24 - 2 \cdot 8 = 8$, a contradiction. □

Claim 26. *Each K_4-subgraph of G contains at most one vertex with degree 4. If $d(x) = d(y) = 4$ and $xy \in E(G)$, then each of x and y is in a K_4.*

Proof. The first statement follows from Claims 19 and 25. The second statement follows from Claims 22 and 25. □

Definition 27. We define $H \subseteq V(G)$ to be the set of vertices of degree 5 not in a K_4, and $L \subseteq V(G)$ to be the set of vertices of degree 4 not in a K_4. Set $\ell = |L|$, $h = |H|$ and $e_0 = |E(L, H)|$.

Claim 28. $e_0 \leq 3h + \ell$.

Proof. This is trivial if $h + \ell \leq 2$ and follows from Corollary 12(ii) and Claim 26 for $h + \ell \geq 3$. □

We will do discharging in two stages. Let every vertex $v \in V(G)$ have initial charge $d(v)$. The first half of discharging has one rule:

Rule R1: Each vertex in $V(G) - H$ with degree at least 5 gives charge $1/6$ to each neighbor.

Claim 29. *After the first round of discharging, each vertex in $V(G) - H - L$ has charge at least* 4.5*.*

Proof. Let $v \in V(G) - H - L$. If $d(v) = 4$, then v receives $1/6$ from at least 3 neighbors and gives no charge. If $d(v) = 5$, then v gives $1/6$ to 5 neighbors, but receives $1/6$ from at least 2 neighbors. If $d(v) \geq 6$, then v is left with charge at least $5d(v)/6 \geq 5 \geq 4.5$. □

For the second round of discharging, all charge in $H \cup L$ is taken up and distributed evenly among the vertices in $H \cup L$.

Claim 30. *After the first round of discharging, the sum of the charges on the vertices in $H \cup L$ is at least $4.5|H \cup L|$.*

Proof. By Rule R1, vertices in L receive from outside of $H \cup L$ the charge at least $\frac{1}{6}(4\ell - |E(H,L)|)$. By Claim 28, $|E(H,L)| \leq 3h + \ell$. So, the total charge on $H \cup L$ is at least

$$5h + 4\ell + \frac{1}{6}\bigl(4\ell - (3h + \ell)\bigr) = 4.5(h + \ell),$$

as claimed. □

Combining Claims 29 and 30, the average degree of the vertices in G is at least 4.5, a contradiction.

4.3. Case $k \geq 6$

Claim 31. *Let T be a cluster in G and $t = |T| \geq 2$.*

(a) If $N(T) \cup T$ does not contain K_{k-1}, then $d_G(v) \geq k - 1 + t$ for every $v \in N(T) - T$;
(b) If $N(T) \cup T$ contains a K_{k-1} with vertex set X, then $d_G(v) \geq k - 1 + t$ for every $v \in X - T$.

Proof. Let $v \in N(T) - T$ such that $k \leq d(v) \leq k - 2 + t$ and if $N(T) \cup T$ contains a K_{k-1} with vertex set X, then $v \in X$. Since $\rho_{k,G}(N(T) \cup T) > (k+1)(k-2)$, T is contained in at most one $(k-1)$-clique, and so

$$N(T) \cup T - v \text{ does not contain } K_{k-1}. \tag{15}$$

By the choice of v, $|N(v) - T| \leq k - 2$. Let $u \in T$ and $G' = G - v + u'$, where $N[u'] = N[u]$. Suppose G' has a $(k-1)$-coloring $\phi' : V(G') \to C = \{1, \ldots, k-1\}$. Then there is a $(k-1)$-coloring ϕ of G as follows: set $\phi|_{V(G)-T-v} = \phi'|_{V(G')-T-u'}$, $\phi(v) \in C - \phi'(N(v) - T)$, and then color T using colors from $\phi'(T \cup u') - \phi(v)$. This is a contradiction, so there is no $(k-1)$-coloring of G'. Thus G' contains a k-critical subgraph G''. Let $W = V(G'')$.

Because $d_G(v) \geq k$ and $d_{G'}(u') = k-1$, $|E(G')| < |E(G)|$. So, G'' is smaller than G and hence $\rho_{k,G'}(W) \leq k(k-3)$. Since G'' is not a subgraph of G, $u' \in W$. By symmetry, it follows that $T \subset W$. But then

$$\rho_{k,G}(W - u') \leq k(k-3) - (k-2)(k+1) + 2(k-1)(k-1) = 2(k-2)(k-1).$$

This implies that $G[W - u']$ is a K_{k-1}, a contradiction to (15). □

Claim 32. *If X is a $(k-1)$-clique with a unique vertex of degree $k-1$, then X contains at least $(k-1)/2$ vertices with degree at least $k+1$.*

Proof. Let v be the unique vertex of X that has degree $k-1$, and let $\{u\} = N(v) - X$. By way of contradiction, assume that X contains at least $k/2-1$ vertices with degree k. Note that $|N(u) \cap X| < k/2$, so there exists a $w \in X$ such that $uw \notin E(G)$ and $d(w) \leq k$. Let $N(w) - X = \{a, b\}$. Let G' be obtained from $G - v$ by adding edges ua and ub.

If G' is not $(k-1)$-colorable, then it contains a k-critical subgraph G''. Let $W = V(G'')$. Since $|E(G')| < |E(G)|$, G'' is smaller than G and so, $\rho_{k,G'}(W) \leq k(k-3)$. If $W = V(G')$, then $\rho_{k,G}(V(G)) \leq k(k-3) + (k-2)(k+1)(1) - 2(k-1)(k-3) < k(k-3)$ when $k \geq 6$. If $W \neq V(G')$ then $\rho_{k,G}(W) \leq k(k-3) + 2(k-1)(2) < 2(k-2)(k-1)$, a contradiction.

Thus G' has a $(k-1)$-coloring f. If $f(u)$ is not used on $X-w-v$, then we recolor w with $f(u)$. So, v will have two neighbors of color $f(u)$, and we can extend the $(k-1)$-coloring to v. □

Claim 33. *If $k = 6$ and a cluster C is contained in a 5-clique X, then $|C| = 1$.*

Proof. By Claim 21, assume that $C = \{v_1, v_2\}$. Let $N(v_1) - X = \{y\}$ and $\{u, u', u''\} = X - C$. Obtain G' from $G - C$ by gluing u to y.

Suppose that G' has a 5-coloring. We will extend this coloring to a coloring on G by greedily assigning colors to C. This can be done because only 3 different colors appear on the vertices $\{u, u', u'', y\}$. So we may assume that $\chi(G') \geq 6$. Then G' contains a k-critical subgraph G''. Let $W = V(G'')$. Because $|E(G')| < |E(G)|$, $\rho_{6,G'}(W) \leq 18$. Since G'' is not a subgraph of G, $u * y \in W$. Let $t = |\{u', u''\} \cap W|$.

Case 1: $t = 0$. Then $\rho_{6,G}(W - u*y + y + X) \leq 18 + 28(5) - 10(12) = 38$. By Claim 18, $W - u*y + y + X = V(G)$. But then we did not account for edges in $E(\{u', u''\}, V(G) - X)$. Thus $\rho_{6,G}(V(G)) \leq 38 - 2 \cdot 10 = 18$.

Case 2: $t = 1$. Then $\rho_{6,G}(W - u * y + y + u + C) \leq 18 + 28(3) - 10(7) = 32$. This is a contradiction to Claim 18 because $V(G) \neq (W - u * y + y + u + C)$.

Case 3: $t = 2$. Then $\rho_{6,G}(W - u * y + y + u + C) \leq 18 + 28(3) - 10(9) = 12$, which is a contradiction. □

Definition 34. We partition $V(G)$ into four classes: L_0, L_1, H_0, and H_1. Let H_0 be the set of vertices with degree k, H_1 be the set of vertices with degree at least $k+1$, and $H = H_0 \cup H_1$. Let

$$L = \{u \in V(G) : d(u) = k-1\},$$
$$L_0 = \{u \in L : N(u) \subseteq H\},$$

and

$$L_1 = L - L_0.$$

Set $\ell = |L_0|$, $h = |H_0|$ and $e_0 = |E(L_0, H_0)|$.

Claim 35. *$e_0 \le 2(\ell + h)$.*

Proof. This is trivial if $h + \ell \le 2$ and follows from Corollary 12(i) for $h + \ell \ge 3$. □

Let every vertex $v \in V(G)$ have initial charge $d(v)$. We first do a half-discharging with two rules:

Rule R1: Each vertex in H_1 keeps charge $k - 2/(k-1)$ to itself and distributes the rest equally among its neighbors of degree $k-1$.

Rule R2: If a K_{k-1}-subgraph C contains s $(k-1)$-vertices adjacent to a $(k-1)$-vertex x outside of C and not in a K_{k-1}, then each of these s vertices gives charge $\frac{k-3}{s(k-1)}$ to x.

Claim 36. *Each vertex in H_1 donates a charge of at least $\frac{1}{k-1}$ to each neighbor of degree $k-1$.*

Proof. If $v \in H_1$, then v donates at least $\frac{d(v)-k+2/(k-1)}{d(v)}$ to each neighbor. Note that this function increases as $d(v)$ increases, so the charge is minimized when $d(v) = k+1$. But then each vertex gets charge at least $(1 + 2/(k-1))/(k+1) = 1/(k-1)$. □

Claim 37. *Each vertex in L_1 has charge at least $k - 2/(k-1)$.*

Proof. Let $v \in L_1$ be in a cluster C of size t.

Case 1: v is in a $(k-1)$-clique X and $t \ge 2$. By Claim 33, this case only applies when $k \ge 7$.

By Claim 31 each vertex in $X - C$ has degree at least $k - 1 + t \ge k+1$, and therefore $X - C \subseteq H_1$. Furthermore, each vertex in $X - C$ has at least $k-2-t$ neighbors with degree at least k. Therefore each vertex $u \in (X - C)$ donates charge at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+2+t}$ to each neighbor of degree $k-1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u) = k-1+t$. It follows that u gives charge at least $\frac{t-1+2/(k-1)}{2t+1}$ to v.

So, v has charge at least $k-1+(k-1-t)(\frac{t-1+2/(k-1)}{2t+1})-\frac{k-3}{t(k-1)}$, which we claim is at least $k-2/(k-1)$. Let

$$g_1(t)=(k-1-t)\big((t-1)(k-1)+2\big)-(2t+1)(k-3)\left(1+\frac{1}{t}\right).$$

We claim that $g_1(t)\geq 0$, which is equivalent to v having charge at least $k-2/(k-1)$. Let

$$\widetilde{g}_1(t)=(k-1-t)\big((t-1)(k-1)+2\big)-(2t+1)(k-3)(3/2).$$

Note that $\widetilde{g}_1(t)\leq g_1(t)$ when $t\geq 2$, so we need to show that $\widetilde{g}_1(t)\geq 0$ on the appropriate domain. The function $\widetilde{g}_1(t)$ is quadratic with a negative coefficient at t^2, so it suffices to check its values at the boundaries. They are

$$\widetilde{g}_1(2)=(k-3)(k-6.5)$$

and

$$\begin{aligned}4\widetilde{g}_1\left(\frac{k-1}{2}\right)&=(k-1)\big((k-3)(k-1)+4\big)-6k(k-3)\\&=k^3-11k^2+29k-7\\&=(k-7)\big(k^2-4k+1\big).\end{aligned}$$

Each of these values is non-negative when $k\geq 7$.

Case 2: $t\geq 2$ and v is not in a $(k-1)$-clique. By Claim 31, each neighbor of v outside of C has degree at least $k-1+t\geq k+1$ and is in H_1. Therefore v has charge at least $k-1+(k-t)(\frac{t-1+2/(k-1)}{k-1+t})$. We define

$$\begin{aligned}g_2(t)&=(k-t)\left(t-1+\frac{2}{k-1}\right)-\frac{k-3}{k-1}(k-1+t)\\&=t(k-t)-2\left(1-\frac{2}{k-1}\right)(k-1)\\&=t(k-t)-2(k-3).\end{aligned}$$

Note that $g_2(t)\geq 0$ is equivalent to v having charge at least $k-2/(k-1)$. The function $g_2(t)$ is quadratic with a negative coefficient at t^2, so it suffices to check its values at the boundaries. They are

$$g_2(2)=2(k-2)-2(k-3)=2$$

and

$$g_2(k-3) = (k-3)(3) - 2(k-3) = k-3.$$

Each of these values is positive.

Case 3: $t=1$. If v is not in a $(k-1)$-clique X, then by Claim 22 the vertex adjacent to v with degree $k-1$ is in a $(k-1)$-clique and cluster of size at least 2. In this case v will receive charge $(k-3)/(k-1)$ in total from that cluster. Therefore we may assume that v is in a $(k-1)$-clique X.

By Claim 32, there exists a $Y \subset X$ such that $|Y| \geq \frac{k-1}{2}$ and every vertex in Y has degree at least $k+1$. Furthermore, each vertex in Y has at least $k-3$ neighbors with degree at least k. Therefore each vertex $u \in Y$ donates a charge of at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+3}$ to each neighbor of degree $k-1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u) = k+1$. It follows that u gives a charge at least $\frac{1+2/(k-1)}{4}$ to v, and v has charge at least

$$k-1+\frac{k-1}{2}\left(\frac{1+2/(k-1)}{4}\right) = k+\frac{k-7}{8},$$

which is at least $k-2/(k-1)$ when $k \geq 6$. $\square$

We then observe that after the half-discharging,

a) the charge of each vertex in $H_1 \cup L_1$ is at least $k-2/(k-1)$;
b) the charges of vertices in H_0 did not decrease;
c) along every edge from H_1 to L_0 a charge of at least $1/(k-1)$ is sent.

Thus by Claim 35, the total charge F of the vertices in $H_0 \cup L_0$ is at least

$$\begin{aligned} kh+(k-1)\ell+\frac{1}{k-1}\big(\ell(k-1)-e(G_0)\big) &\geq k(h+\ell)-\frac{1}{k-1}2(h+\ell) \\ &= (h+\ell)\left(k-\frac{2}{k-1}\right), \end{aligned}$$

and so by a), the total charge of all the vertices of G is at least $n(k-\frac{2}{k-1})$, a contradiction. $\square$

5. Sharpness

The next statement shows some cases when the bound (6) of Theorem 4 is exact.

Theorem 38. *If one of the following holds:*

1. $n \equiv 1 \pmod{k-1}$ *and* $n \geq k$,
2. $k=4$, $n \neq 5$, *and* $n \geq 4$, *or*
3. $k=5$, $n \equiv 2 \pmod 4$, *and* $n \geq 10$,

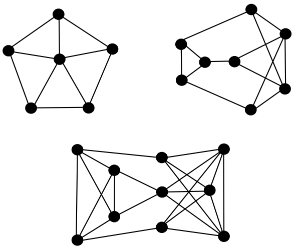

Fig. 1. Minimal k-critical graphs.

then

$$f_k(n) = F(k,n) = \left\lceil \frac{1}{2}\left(\left(k - \frac{2}{k-1}\right)n - \frac{k(k-3)}{k-1}\right)\right\rceil.$$

Proof. By (5), we only need to show that (6) is tight when

1. $n = k$,
2. $k = 4$, $n = 6$,
3. $k = 4$, $n = 8$, and
4. $k = 5$, $n = 10$.

The first case follows from K_k. The other three cases follow from Fig. 1. □

By Theorem 1, (6) is not sharp when $k \geq 5$ and $k + 2 \leq n \leq 2k - 2$. We suspect that (6) is sharp only in the cases covered by Theorem 38.

Now we prove Corollary 5. First, we restate it:

Corollary 5. *For $k \geq 4$, $0 \leq f_k(n) - F(k,n) \leq (1 + o(1))\frac{k^2}{8}$. In particular, $\phi_k = \frac{k}{2} - \frac{1}{k-1}$.*

Proof. By Theorem 38, the corollary holds for $k = 4$. Let $k \geq 5$. By (5) and Theorem 4, for every $n \geq k$, $n \neq k + 1$,

$$f_k\bigl(n + (k-1)\bigr) - F\bigl(k, n + (k-1)\bigr) \leq f_k(n) - F(k,n).$$

Thus, it is enough to check the inequality for $k + 2 \leq n \leq 2k$. There exists a k-critical $2k$-vertex graph with $k^2 - 3$ edges. So,

$$f_k(2k) - F(k,2k) \leq k^2 - 3 - \frac{(k+1)(k-2)2k - k(k-3)}{2(k-1)} \leq \frac{k(k-3)}{2(k-1)} < \frac{k-2}{2},$$

and by the integrality of f_k and F, $f_k(2k) - F(k,2k) \leq \frac{k-3}{2}$.

By Theorems 4 and 1, for $k + 2 \leq n \leq 2k - 1$,

$$f_k(n) - F(k,n) \leq \left(\frac{1}{2}\bigl((k-1)n + (n-k)(2k-n)\bigr) - 1\right) - \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}$$

$$= -1 + \frac{1}{2}\left[(n-k)\left(2k - \frac{k-3}{k-1} - n\right)\right]. \tag{16}$$

For every fixed k, the maximum of the last expression (quadratic in n) is attained at $n = \frac{1}{2}(k + 2k - \frac{k-3}{k-1})$. If $k \geq 5$, then the closest half-integer to this point is $\frac{3k-1}{2}$. Thus,

$$f_k(n) - F(k,n) \leq f_k\left(\frac{3k-1}{2}\right) - F\left(k, \frac{3k-1}{2}\right) \leq -1 + \frac{1}{2}\left[\frac{k-1}{2}\left(\frac{k+1}{2} - \frac{k-3}{k-1}\right)\right]$$
$$< -1 + \frac{k-1}{4}\frac{k}{2} = -1 + \frac{k(k-1)}{8}. \qquad \square$$

In particular, by the integrality of f_k and F, $f_5(n) - F(5,n) \leq 1$ for all $n \geq 7$.

Now we prove Corollary 6. First, we restate it:

Corollary 6. *If $k \geq 4$, then for all but $\frac{k^3}{12} - \frac{k^2}{8}$ values of $n \geq k+2$,*

$$f_k(n+k-1) = f_k(n) + (k-1)\left(k - \frac{2}{k-1}\right)/2.$$

Proof. By Theorem 38, the corollary holds for $k = 4$. Let $k \geq 5$. By (5) and Theorem 4, for every $n \geq k$, $n \neq k+1$,

$$f_k\big(n + (k-1)\big) - F\big(k, n + (k-1)\big) \leq f_k(n) - F(k,n).$$

So the number of times when $f_k(n+k-1) < f_k(n) + (k-1)(k - \frac{2}{k-1})/2$ is bounded by

$$\sum_{i=k+2}^{2k} f_k(n) - F(k,n).$$

Expanding (16), the above bound is at most

$$\frac{1}{2}\sum_{i=k+2}^{2k-2}\left(-i^2 + 3ik + \frac{k-3}{k-1}(k-i) - 2k^2 - 2\right) + 0 + \frac{k-2}{2}$$
$$\leq \frac{-1}{12}\left(14k^3 - 45k^2 + 13k - 12\right) + \frac{9k^3 - 27k^2}{4} - \left(\frac{k^2 - 3k}{4}\cdot\frac{k-3}{k-1}\right)$$
$$- k^3 + 3k^2 - k + 3 + \frac{k-2}{2}$$
$$\leq \frac{k^3}{12} - \frac{k^2}{8} - \frac{11k}{6} + 7 \leq \frac{k^3}{12} - \frac{k^2}{8}. \qquad \square$$

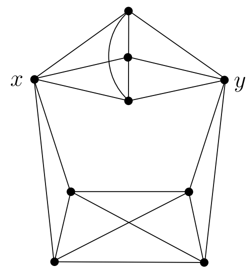

Fig. 2. The graph O_5.

6. Some applications

6.1. Ore-degrees

The *Ore-degree*, $\Theta(G)$, of a graph G is the maximum of $d(x)+d(y)$ over all edges xy of G. Let $\mathcal{G}_t = \{G : \Theta(G) \le t\}$. It is easy to prove (see, e.g. [16]) that $\chi(G) \le 1 + \lfloor t/2 \rfloor$ for every $G \in \mathcal{G}_t$. Clearly $\Theta(K_{d+1}) = 2d$ and $\chi(K_{d+1}) = d+1$. The graph O_5 in Fig. 2 is the only 9-vertex 5-critical graph with Θ at most 9. We have $\Theta(O_5) = 9$ and $\chi(O_5) = 5$. A natural question is to describe the graphs in $\mathcal{G}_{2d+1}$ with chromatic number $d+1$. Each $(d+1)$-chromatic graph G contains a $(d+1)$-critical subgraph G'. Since $\delta(G') \ge d$ and $\Theta(G') \le \Theta(G) \le 2d+1$,

$$\Delta\big(G'\big) \le d+1, \quad \text{and} \quad \text{vertices of degree } d+1 \text{ form an independent set.} \tag{17}$$

Thus a description of the graphs in $\mathcal{G}_{2d+1}$ with chromatic number $d+1$ is equivalent to a description of $(d+1)$-critical graphs satisfying (17). Kierstead and Kostochka [16] solved the problem for $d \ge 6$ and Rabern [25] extended the result to $d = 5$:

Theorem 39. *(See [16,25].) For $d \ge 5$, the only $(d+1)$-critical graph satisfying (17) is K_{d+1}.*

The case $d = 4$ was settled by Kostochka, Rabern, and Stiebitz [22]:

Theorem 40. *(See [22].) The only 5-critical graphs satisfying (17) are K_5 and O_5.*

Theorem 4 and Corollary 12 yield simpler proofs of Theorems 39 and 40. The key observation is the following.

Lemma 41. *Let $d \ge 4$ and G' be a $(d+1)$-critical graph satisfying (17). If G' has n vertices of which $h > 0$ vertices have degree $d+1$, then*

$$(i) \quad h \ge \left\lceil \frac{(d-2)n - (d+1)(d-2)}{d} \right\rceil \quad \text{and} \quad (ii) \quad h \le \left\lfloor \frac{n-3}{d-1} \right\rfloor. \tag{18}$$

Proof. By definition, $2e(G') = dn + h$. So, by Theorem 4 with $k = d + 1$,

$$dn + h \geq \left(d + 1 - \frac{2}{d}\right)n - \frac{(d+1)(d-2)}{d},$$

which yields (18)(i).

Let B be the set of vertices of degree $d + 1$ in G' and $A = V(G') - B$. By (17), $e(G'(A, B)) = h(d + 1)$. So, by Corollary 12(ii) with $k = d + 1$,

$$h(d + 1) \leq 3h + (n - h) - 3 = 2h + n - 3,$$

which yields (18)(ii). □

Another ingredient is the following old observation by Dirac.

Lemma 42. *(See Dirac [7].) Let $k \geq 3$. There are no k-critical graphs with $k+1$ vertices, and the only k-critical graph (call it D_k) with $k + 2$ vertices is obtained from the 5-cycle by adding $k - 3$ universal vertices.*

Suppose G' with n vertices of which h vertices have degree $d+1$ is a counter-example to Theorem 39 or 40. Since the graph D_{d+1} from Lemma 42 has a vertex of degree $d+2$, $n \geq d + 4$. So since $d \geq 4$, by (18)(i),

$$h \geq \left\lceil \frac{(d-2)(d+4) - (d+1)(d-2)}{d} \right\rceil = \left\lceil \frac{3(d-2)}{d} \right\rceil \geq 2.$$

On the other hand, if $n \leq 2d$, then by (18)(ii), $h \leq \lfloor \frac{2d-3}{d-1} \rfloor = 1$. Thus $n \geq 2d + 1$.

Combining (18)(i) and (18)(ii), we get $[(d - 2)n - (d + 1)(d - 2)]/d \leq (n - 3)/(d - 1)$. Solving with respect to n, we obtain

$$n \leq \left\lfloor \frac{(d+1)(d-1)(d-2) - 3d}{d^2 - 4d + 2} \right\rfloor. \tag{19}$$

For $d \geq 5$, the RHS of (19) is less than $2d+1$, a contradiction to $n \geq 2d+1$. This proves Theorem 39.

Suppose $d = 4$. Then (19) yields $n \leq 9$. So, in this case, $n = 9$. By (18), we get $h = 2$. Let $B = \{b_1, b_2\}$ be the set of vertices of degree 5 in G'. By a theorem of Stiebitz [26], $G' - B$ has at least two components. Since $|B| = 2$ and $\delta(G') = 4$, each such component has at least 3 vertices. Since $|V(G') - B| = 7$, we may assume that $G' - B$ has exactly two components, C_1 and C_2, and that $|V(C_1)| = 3$. Again because $\delta(G') = 4$, $C_1 = K_3$ and all vertices of C_1 are adjacent to both vertices in B. So, if we color both b_1 and b_2 with the same color, this can extended to a 4-coloring of $G' - V(C_2)$. Thus to have G' 5-chromatic, we need $\chi(C_2) \geq 4$ which yields $C_2 = K_4$. Since $\delta(G') = 4$, $e(V(C_2), B) = 4$. So, since each of b_1 and b_2 has degree 5 and 3 neighbors in C_1, each of them has exactly two neighbors in C_2. This proves Theorem 40. □

6.2. Local vs. global graph properties

Krivelevich [23] presented several nice applications of his lower bounds on $f_k(n)$ and related graph parameters to questions of existence of complicated graphs whose small subgraphs are simple. We indicate here how to improve two of his bounds using Theorem 4.

Let $f(\sqrt{n}, 3, n)$ denote the maximum chromatic number over n-vertex graphs in which every $\sqrt{n}$-vertex subgraph has chromatic number at most 3. Krivelevich proved that for every fixed $\epsilon > 0$ and sufficiently large n,

$$f(\sqrt{n}, 3, n) \geq n^{6/31-\epsilon}. \tag{20}$$

He used his result that every 4-critical t-vertex graph with odd girth at least 7 has at least $31t/19$ edges. If instead of this result, we use our bound on $f_4(n)$, then repeating almost word by word Krivelevich's proof of his Theorem 4 (choosing $p = n^{-0.8-\epsilon'}$), we get that for every fixed ϵ and sufficiently large n,

$$f(\sqrt{n}, 3, n) \geq n^{1/5-\epsilon}. \tag{21}$$

Another result of Krivelevich is:

Theorem 43. *(See [23].) There exists $C > 0$ such that for every $s \geq 5$ there exists a graph G_s with at least $C(\frac{s}{\ln s})^{\frac{33}{14}}$ vertices and independence number less than s such that the independence number of each 20-vertex subgraph is at least 5.*

He used the fact that for every $m \leq 20$ and every m-vertex 5-critical graph H,

$$\frac{|E(H)| - 1}{m - 2} \geq \frac{\lceil 17m/8 \rceil - 1}{m - 2} \geq \frac{33}{14}.$$

From Theorem 4 we instead get

$$\frac{|E(H)| - 1}{m - 2} \geq \frac{\lceil \frac{9m-5}{4} \rceil - 1}{m - 2} \geq \frac{43}{18}.$$

Then repeating the argument in [23] we can replace $\frac{33}{14}$ in the statement of Theorem 43 with $\frac{43}{18}$.

6.3. Coloring planar graphs

One of the basic results on 3-coloring of planar graphs is Grötzsch's Theorem [12]: every triangle-free planar graph is 3-colorable. The original proof of this theorem is somewhat sophisticated. There were subsequent simpler proofs (see, e.g. [1] or [27] and the references therein), but Theorem 4 yields a half-page proof. A disadvantage of this proof is that the proof of Theorem 4 itself is not too simple. In [21], we give a shorter proof of the fact $f_4(n) = F(4, n)$ and a short proof of Grötzsch's Theorem. In [2], we use Theorem 4 to give short proofs of some other known and new results on 3-colorability of planar graphs.

7. Algorithm

Recall that $\rho_{k,G}(W) = (k+1)(k-2)|W| - 2(k-1)|E(G[W])|$ and that $P_k(G)$ is the minimum of $\rho_{k,G}(W)$ over all nonempty $W \subseteq V(G)$. We will also use the related parameter $\widetilde{P}_k(G)$ which is the minimum of $\rho_{k,G}(W)$ over all $W \subset V(G)$ with $2 \leq |W| \leq |V(G)| - 1$.

7.1. Procedure R1

The input of the procedure $R1_k(G)$ is a graph G. The output is one of the following five:

(S1) a nonempty set $R \subseteq V(G)$ with $\rho_{k,G}(R) \leq k(k-3)$, or
(S2) conclusion that $k(k-3) < \widetilde{P}_k(G) < (k+1)(k-2)$ and a nonempty set $R \subsetneq V(G)$ with $\rho_{k,G}(R) = \widetilde{P}_k(G)$, or
(S3) conclusion that $\widetilde{P}_k(G) < 2(k-1)(k-2)$, and a set $R \subset V(G)$ with $2 \leq |R| \leq n-1$ and $\rho_{k,G}(R) = \widetilde{P}_k(G)$, or
(S4) conclusion that $\widetilde{P}_k(G) = 2(k-1)(k-2)$, and a set $R \subset V(G)$ with $k \leq |R| \leq n-1$ and $\rho_{k,G}(R) = 2(k-1)(k-2)$, or
(S5) conclusion that $\widetilde{P}_k(G) \geq 2(k-1)(k-2)$ and that every set $R \subseteq V(G)$ with $\rho_{k,G}(R) = 2(k-1)(k-2)$ has size $k-1$ and induces K_{k-1}.

First we calculate $\rho_k(V(G))$, and if it is at most $k(k-3)$, then we are done. Suppose

$$(k+1)(k-2)\left|V(G)\right| - 2(k-1)\left|E(G)\right| \geq 1 + k(k-3). \tag{22}$$

Consider the auxiliary network $H = H(G)$ with vertex set $V \cup E \cup \{s,t\}$ and the set of arcs $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{sv : v \in V\}$, $A_2 = \{et : e \in E\}$, and $A_3 = \{ve : v \in V,\ e \in E,\ v \in e\}$. The capacity c of each $sv \in A_1$ is $(k+1)(k-2)$, of each $et \in A_2$ is $2(k-1)$, and of each $ve \in A_3$ is ∞.

Since the capacity of the cut $(\{s\}, V(H) - s)$ is finite, H has a maximum flow f. Let $M(f)$ denote the value of f, and let (S,T) be the minimum cut in it. By definition, $s \in S$ and $t \in T$. Let $S_V = S \cap V$, $S_E = S \cap E$, $T_V = T \cap V$, and $T_E = T \cap E$.

Since $c(ve) = \infty$ for every $v \in e$,

$$\text{no edge of } H \text{ goes from } S_V \text{ to } T_E. \tag{23}$$

It follows that if $e = vu$ in G and $e \in T_E$, then $v, u \in T_V$. On the other hand, if $e = vu$ in G, $v, u \in T_V$ and $e \in S_E$, then moving e from S_E to T_E would decrease the capacity of the cut by $2(k-1)$, a contradiction. So, we get

Claim 44. $T_E = E(G[T_V])$.

By the claim,

$$M(f) = \min_{W \subseteq V} \{(k+1)(k-2)|W| + 2(k-1)\big(|E| - \big|E\big(G[W]\big)\big|\big)\}$$
$$= 2(k-1)|E| + \min\{P_k(G), 0\}. \tag{24}$$

So, if $M(f) < 2(k-1)|E|$, then $P_k(G) < 0$ and any minimum cut gives us a set with small potential. Otherwise, consider for every $e_0 \in E$ and every vertex v_0 not incident to e_0, the network H_{e_0,v_0} that has the same vertices and edges and differs from H in the following:

(i) the capacity of the edge e_0t is not $2(k-1)$ but $2(k-1) + 2(k-1)(k-2) = 2(k-1)^2$;
(ii) for every $v \in V(G) - v_0$, the capacity of the edge sv is $(k+1)(k-2) - \frac{1}{2n}$;
(iii) the capacity of the edge sv_0 is $(k+1)(k-2) - \frac{1}{2n} + 2(k-1)(k-2) + 1$.

Then for every $e_0 \in E$ and $v_0 \in V(G)$, the capacity of the cut $(V(H_{e_0,v_0}) - t, t)$ is $2(k-1)|E| + 2(k-1)(k-2)$. Since this is finite, H_{e_0,v_0} has a maximum flow f_{e_0,v_0}. As above, let $M(f_{e_0,v_0})$ denote the value of f_{e_0,v_0}, and let (S,T) be the minimum cut in H_{e_0,v_0}. By definition, $s \in S$ and $t \in T$. Let $S_V = S \cap V$, $S_E = S \cap E$, $T_V = T \cap V$, and $T_E = T \cap E$. By the same argument as above, (23) and Claim 44 hold. Let $M_k(G)$ denote the minimum value over $M(f_{e_0,v_0})$.

By (22), for every $e_0 \in E$ and $v_0 \in V(G)$, the capacity of the cut $(s, V(H_{e_0,v_0}) - s)$ is at least

$$\left((k+1)(k-2) - \frac{1}{2n}\right)n + 2(k-1)(k-2) + 1 \geq 2(k-1)|E| + 2(k-1)(k-2) + \frac{1}{2}.$$

If the potential of some nonempty $W \neq V$ is less than $(k+1)(k-2)$, then $G[W]$ contains some edge e_0 and there is $v_0 \in V - W$. So, in the network H_{e_0,v_0}, the capacity of the cut $(\{s\} \cup (V - W) \cup (E - E(G[W])), W \cup E(G[W]) \cup \{t\})$ is

$$\left((k+1)(k-2) - \frac{1}{2n}\right)|W| + 2(k-1)\big(|E| - \big|E\big(G[W]\big)\big|\big)$$
$$= 2(k-1)|E| + \rho_{k,G}(W) - \frac{|W|}{2n}.$$

On the other hand, for every nonempty $W \neq V$, every edge e_0 and every $v_0 \in V$, the capacity of the cut $(\{s\} \cup (V - W) \cup (E - E(G[W])), W \cup E(G[W]) \cup \{t\})$ is at least

$$\left((k+1)(k-2)-\frac{1}{2n}\right)|W|+2(k-1)\left(|E|-\left|E\left(G[W]\right)\right|\right)>2(k-1)|E|+\rho_{k,G}(W)-\frac{1}{2}.$$

Thus if $M_k(G)\le k(k-3)+2(k-1)|E|$, then (S1) holds and if $k(k-3)+2(k-1)|E|<M_k(G)<(k+1)(k-2)+2(k-1)|E|$, then (S2) holds. Note that if a nonempty W is independent, then $E(G[W])=\emptyset$, and the capacity of the cut $(\{s\}\cup(V-W)\cup(E-E(G[W])), W\cup E(G[W])\cup\{t\})$ is at least

$$2(k-1)|E|+2(k-1)(k-2)+(k+1)(k-2).$$

Thus, if

$$(k+1)(k-2)+2(k-1)|E|\le M_k(G)<2(k-1)(k-2)-1+2(k-1)|E|,$$

then (S3) holds.

Similarly, if

$$2(k-1)(k-2)-1+2(k-1)|E|\le M_k(G)<2(k-1)(k-2)+2(k-1)|E|-\frac{k-1}{2n},$$

then there exists $W\subset V$ with $k\le|W|\le n-1$ with potential $2(k-1)(k-2)$. Then (S4) holds. Finally, if $M_k(G)\ge 2(k-1)(k-2)+2(k-1)|E|-\frac{k-1}{2n}$, then (S5) holds.

Since the complexity of the max-flow problem is at most $Cn^2\sqrt{|E|}$ and $|E|\le kn$, the procedure takes time at most $Ck^{1.5}n^{4.5}$.

7.2. *Outline of the algorithm*

We consider the outline for $k\ge 7$. For $k\le 6$, everything is quite similar and easier.

Let the input be an n-vertex e-edge graph G. The algorithm will be recursive. The output will be either a coloring of G with $k-1$ colors or return a nonempty $R\subseteq V(G)$ with $\rho_{k,G}(R)\le k(k-3)$. The algorithm runs through 7 steps, which are listed below. If a step is triggered, then a recursive call is made on a smaller graph G'. Some steps will then require a second recursive call on another graph G''.

The algorithm does not make the recursive call if $|E(G')|\le k^2/2$. In this case, G' is either $(k-2)$-degenerate or K_k minus a matching, and so is easily $(k-1)$-colorable in time $O(k|V(G')|^2)$. This also holds for G''.

After all calls have been made, the algorithm will return a coloring or a subgraph with low potential, skipping the other steps.

(1) We check whether G is disconnected or has a cut-vertex or has a vertex of degree at most $k-2$. In the case of any “yes,” we consider smaller graphs (and at the end will reconstruct the coloring).
(2) We run $R1_k(G)$ and consider possible outcomes. If the outcome is (S1), we are done.

(3) Suppose the outcome is (S2). The algorithm makes a recursive call on $G' = G[R]$, which returns a $(k-1)$-coloring ϕ. Let G'' be the graph $Y(G, R, \phi)$ described in Definition 14. The proof of Claim 16 yields that $P_k(G'') \geq k(k-3)$, and thus the recursive call will return with a coloring. Let ϕ' be the coloring returned. It is straightforward to combine the colorings ϕ and ϕ' into a $(k-1)$-coloring of G.

(4) Suppose the outcome is (S3) or (S4). We choose i using (12) and add i edges to $G[R]$ as in the proof of Claim 18. Denote the new graph G'. The algorithm makes a recursive call on $G' = G[R]$, which returns a $(k-1)$-coloring ϕ. Let G'' be the graph $Y(G, R, \phi)$ described in Definition 14. The proof of Claim 18 yields that $P_k(G'') \geq k(k-3)$, and thus the recursive call will return with a coloring. Let ϕ' be the coloring returned. It is straightforward to combine the colorings ϕ and ϕ' into a $(k-1)$-coloring of G.

(5) So, the only remaining possibility is (S5). For every $(k-1)$-vertex $v \in V(G)$, check whether there is a $(k-1)$-clique $K(v)$ containing v (since (S5) holds, such a clique is unique, if it exists). We certainly can do this in $O(kn^2)$ time. Let a_v denote the neighbor of v not in $K(v)$ and T_v denote the set of $(k-1)$-vertices in $K(v)$. Then for every pair $(v, K(v))$ such that $d(v) = k-1$ and $K(v)$ exists, do the following:

(5.1) If there is $w \in T_v - v$ with $a_w \neq a_v$, then consider the graph $G' = G - v - w + a_v a_w$. By Claim 19, $P_k(G') > k(k-3)$. So, the algorithm will return with a $(k-1)$-coloring of G', which we then extend to G.

(5.2) Suppose that $|T_v| \geq 2$ and $K(v) - T_v$ contains a vertex x of degree at most $k - 2 + |T_v|$. Let $G' = G - x + v'$, where the closed neighborhood of v' is the same as of v. By Claim 31, $P_k(G') > k(k-3)$, so the algorithm returns a $(k-1)$-coloring of G', which is then extended to G as in the proof of Claim 31.

(5.3) Suppose that $T_v = \{v\}$ and $K(v)$ contains at least $k/2 - 1$ vertices of degree k. Since (S5) holds, there is $x \in K(v) - v$ of degree at most k not adjacent to a_v. Let x_1 and x_2 be the neighbors of x outside of K_v. Let G' be obtained from $G - v$ by adding edges $a_v x_1$ and $a_v x_2$. By the proof of Claim 32, $P_k(G') > k(k-3)$, so the algorithm finds a $(k-1)$-coloring of G', which is then extended to G as in the proof of Claim 32.

(6) Let C_v denote the *cluster of* v, i.e. the set of vertices that have the same closed neighborhood as v. We certainly can find C_v for every $(k-1)$-vertex $v \in V(G)$ in $O(kn^2)$ time. Then for every pair (v, C_v) such that $d(v) = k-1$, do the following:

(6.1) Suppose that $|C_v| \geq 2$ and $N(v) - C_v$ contains a vertex x of degree at most $k - 2 + |T_v|$. Then do the same as in (5.2).

(6.2) Suppose that $N(v) - C_v$ contains a $(k-1)$-vertex w and that $|C_w| \leq |C_v|$. If v is not in a $(k-1)$-clique, then consider $G' = G - w + v'$, where the v' is a new vertex whose closed neighborhood is the same as that of v. By the proof of Claim 22, $P_k(G') > k(k-3)$, and so we find a $(k-1)$-coloring of G' and then extend it to G as in the proof of Claim 22.

(7) Let L_0, H_0, and e_0 be as defined in Definition 34. If $e_0 \geq 2(|L_0| + |H_0|)$, then iteratively remove vertices in L_0 with at most two neighbors in H_0 and vertices

in H_0 with at most two neighbors in L_0. Let H be the graph that remains, and $G' = G - V(H)$. Clearly $P_k(G') > k(k-3)$, so the recursive call returns a coloring of G'. Give each vertex $v \in V(H)$ a list of colors $L(v) = \{c_1, \dots, c_{k-1}\}$, then remove from that list the colors on $N(v) \cap V(G')$. Orient the edges of H as in Case 1 of the proof of Lemma 11. Then extend the coloring of G' to a coloring of G by list coloring H using the system described in the proof to Lemma 9.

7.3. Analysis of correctness and running time

The proof of Theorem 4 consists in proving that at least one of the situations in steps (1) through (7) described above must happen. Moreover, the main theorem proves that $G', G'' \prec G$ by a partial order with finite descending chains, and therefore the algorithm will terminate. We claim that the algorithm makes at most $O(k^2n^2\log(n))$ recursive calls, and each call only takes $O(k^{1.5}n^{4.5})$ time, so the algorithm runs in $O(k^{3.5}n^{6.5}\log(n))$ time.

If a call of the recursive algorithm terminates on step (2), we will refer to this as 'Type 1', a call terminating on step (1), (3), (4), (5.1), (5.3), (6.1), or (7) is 'Type 2', and a call terminating on step (5.2) or (6.2) is 'Type 3'. If a call is made on a Type 1, then the whole algorithm stops.

If a Type 3 happens, then the algorithm makes one recursive call with a graph with the same number of edges and strictly more pairs of vertices with the same closed neighborhood. The proof of Claim 21 shows that the number of pairs of vertices with the same closed neighborhood is bounded by kn. Then at least one out of every kn consecutive recursive calls is Type 1 or 2.

Consider an instance of a Type 2 call with input graph H. If H' is the graph in the first recursive call and H'' is the graph in the second call (if necessary), then $|E(H')|, |E(H'')| < |E(H)|$ and $|E(H)| \geq |E(H')| + |E(H'')| - k^2/2$. Let $g_k(e,i)$ denote the number of Type 2 recursive calls made on graphs with i edges. Note that if $i \leq k^2/2$ then $g_k(e,i) = 0$ and $g_k(e,e) = 1$. By tracing calls up through their parent calls, it follows that

$$e \geq i + \bigl(g_k(e,i) - 1\bigr)\bigl(i - k^2/2\bigr)$$

when $i > k^2/2$. Therefore

$$g_k(e,i) < \frac{e}{(i - k^2/2)}.$$

The total number of calls that our algorithm makes is at most

$$kn \sum_{i=k^2/2+1}^{e} g_k(e,i) < kne\log(e).$$

Because $e \leq nk$, we have that the total number of calls is $O(k^2n^2\log(n))$.

A call may run algorithm R1 once, which will take $O(k^{1.5}n^{4.5})$ time. Constructing the appropriate graphs for recursion in steps (3), (4), (5), and (6) will take $O(kn^2)$ time. Combining colorings in steps (1), (3), (4), (5), and (6) will take $O(n)$ time. Coloring a degenerate graph will take $O(kn^2)$ time, which happens at most twice. The only thing left to consider is step (7). Iteratively removing vertices will take $O(n^2)$ time. Splitting the vertices and orienting the edges using network flows will take $O(n^{2.5}k^{0.5})$ time. Finding a kernel will take $O(n^2)$ time, which happens at most n times. Therefore each instance of the algorithm takes $O(k^{1.5}n^{4.5})$ time.

8. Concluding remarks

Many open questions remain:

1. It would be good to find exact values of $f_k(n)$ for all k and n.

2. Similar questions for list coloring look much harder. Some results are in [19]. Very recently, Kierstead and Rabern obtained new impressive bounds.

3. One can ask how few edges can there be in an n-vertex k-critical graph not containing a given subgraph, for example, with bounded clique number. Krivelevich [23] has interesting results on the topic. Very recently, Postle obtained interesting results in this direction.

4. Brooks-type results (characterizing the graphs for which (6) is sharp) would be interesting.

5. A similar problem for hypergraphs was considered in [18,20], but the bounds there are good only for large k.

6. In our coloring algorithm, testing for subgraphs with low potential for every vertex–edge pair seems needlessly expensive, and it is likely that there are algorithms with much better performance than ours.

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