Every 4-Colorable Graph with Maximum Degree 4 Has an Equitable 4-Coloring

H. A. Kierstead[1] and A. V. Kostochka[2,3]

[1] *DEPARTMENT OF MATHEMATICS AND STATISTICS*
ARIZONA STATE UNIVERSITY, TEMPE
ARIZONA 85287
E-mail: kierstead@asu.edu

[2] *DEPARTMENT OF MATHEMATICS*
UNIVERSITY OF ILLINOIS, URBANA
ILLINOIS 61801

[3] *INSTITUTE OF MATHEMATICS*
NOVOSIBIRSK, RUSSIA
E-mail: kostochk@math.uiuc.edu

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Abstract: Chen et al., conjectured that for $r \geq 3$, the only connected graphs with maximum degree at most r that are not equitably r-colorable are $K_{r,r}$ (for odd r) and K_{r+1}. If true, this would be a joint strengthening of the Hajnal–Szemerédi theorem and Brooks' theorem. Chen et al., proved that their conjecture holds for $r=3$. In this article we study properties of

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the hypothetical minimum counter-examples to this conjecture and the structure of "optimal" colorings of such graphs. Using these properties and structure, we show that the Chen–Lih–Wu Conjecture holds for $r \leq 4$.
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1. INTRODUCTION

In some applications of graph coloring, such as the mutual exclusion scheduling problem, scheduling in communication systems, construction timetables, and round-a-clock scheduling (see [1, 11, 12]), there is an additional requirement that color classes be not too large or be of approximately the same size. A model imposing such requirement is *equitable coloring*—a proper coloring such that color classes differ in size by at most one.

One of the basic results on equitable coloring is the following theorem by Hajnal and Szemerédi [3]. For a shorter proof see [4] or [8]; for an algorithm see [7].

Theorem 1. *For every positive integer r, each graph with $\Delta(G) \leq r$ has an equitable $(r+1)$-coloring.*

This theorem has interesting applications in extremal combinatorial and probabilistic problems. It is natural to ask which graphs G with $\Delta(G) = r \geq 3$ have equitable r-colorings. Certainly, such graphs are r-colorable and so do not contain the complete graph K_{r+1}. Note that if r is odd, then the complete bipartite graph $K_{r,r}$ has no equitable r-coloring. Chen et al. [2] proposed the following strengthening of Theorem 1 and Brooks' theorem.

Conjecture 2 (Chen et al. [2]). *If G is an r-colorable graph with $\Delta(G) \leq r$, then either G has an equitable r-coloring or $\omega(G) \geq r+1$ or $r=2$ and G contains an odd cycle or r is odd and G contains $K_{r,r}$.*

Using Brooks' theorem this is equivalent to:

Conjecture 3 (Chen et al. [2]). *If G is an r-colorable graph with $\Delta(G) \leq r$, then either G has an equitable r-coloring or r is odd and G contains $K_{r,r}$.*

Some partial cases of Conjecture 3 were proved in [2, 10, 13, 14, 9]. In particular, Chen et al. [2] proved that the conjecture holds for $r=3$.

Theorem 4 (Chen et al. [2]). *Let G be a connected graph with $\Delta(G) \leq 3$. Then G has no equitable 3-coloring if and only if $G = K_4$ or $G = K_{3,3}$.*

The aim of this article is twofold: (1) Prove that Conjecture 3 holds for all graphs with maximum degree at most four and (2) build a foundation for further progress on Conjecture 3 by deriving some general properties of hypothetical minimum counter-examples to it as well as properties of optimal colorings of such counter-examples.

The structure of the article is the following. In the next section we prove that minimum counter-examples to Conjecture 3 do not contain certain dense subgraphs. In Sections 3 and 4 we define *nearly equitable* and *optimal* nearly equitable colorings of minimum counter-examples to Conjecture 3 and derive some properties of such colorings. In particular, these properties yield a new proof of Theorem 4. In Section 5 we digress from the main line to discuss the following question. If a graph with maximum degree r contains $K_{r,r}$, it still may have an equitable r-coloring; an example is the graph that consists of two disjoint copies of $K_{r,r}$. Suppose that Conjecture 3 holds for all graphs with maximum degree r and at most $sr+1$ vertices and that H is an $(sr+1)$-vertex equitably r-colorable graph with maximum degree at most r. The question that we discuss is: *Under these conditions, for how many vertices $y \in V(H)$ could the graph $H-y$ have no equitable r-coloring*? The answer is 4, and we use this result to prove more properties of optimal nearly equitable colorings of minimum counter-examples to Conjecture 3. In the last section we derive some properties that yield Conjecture 3 for all graphs with maximum degree 4.

Most of our notation is standard. For example, by $\Delta(G)$, $\omega(G)$, and $\chi(G)$ we denote the maximum degree, the clique number, and the chromatic number of a graph G, respectively. All graphs are simple. If S is a set of edges whose ends are in $V(G)$, then $G+S$ denotes the graph on $V(G)$ with edge set $E(G)\cup S$. Possible deviations from standard notation include the following. For a graph G, we let $|G|:=|V(G)|$, $\|G\|:=|E(G)|$. For a vertex y and set of vertices X, $N_X(y):=N(y)\cap X$ and $d_X(y):=|N_X(y)|$. The set of edges of G linking vertices in A to vertices in B is denoted by $E_G(A,B)$ or simply by $E(A,B)$. For a set S and element x, we write $S+x$ for $S\cup\{x\}$ and $S-x$ for $S\setminus\{x\}$. For a function $f:V\to Z$, the restriction of f to $W\subseteq V$ is denoted by $f|W$. For a positive integer n, the set $\{1,\dots,n\}$ is denoted by $[n]$.

2. FORBIDDEN SUBGRAPHS

Suppose that Conjecture 3 fails, and let G be a counter-example. Then, for some integer r, $\chi(G)\le r$, and if r is odd, then $K_{r,r}\not\subseteq G$, but G has no equitable r-coloring. We may assume that $|G|$ is divisible by r, since G will still be a counter-example if we add a disjoint clique of $r-(|G| \bmod r)$ new vertices. Choose r as small as possible; then choose $G=(V,E)$ with $|G|=rs$ for some integer s with s as small as possible, and subject to this $\|G\|$ as small as possible. Then the conjecture holds for H if $\Delta(H)<r$, and also if both $\Delta(H)=r$ and $|H|\le|G|-r$. Also, G does not contain K_{r+1}. Moreover, G does not contain $K_{r,r}$: If r is odd this is by hypothesis; if r is even, it is by the minimality of G, since $K_{r,r}$, would be a component with an equitable r-coloring. Clearly the conjecture is true for $r\le 2$ and for $s=1$; so $r\ge 3$ and $s\ge 2$. In the remainder of this section we identify a collection of dense graphs that cannot be contained in our minimum counter-example G.

Proposition 5. *G is connected. Furthermore, for all nonempty proper subsets $X\subset V$, if $|X|$ is divisible by r then $|E(X,V-X)|\ge r$.*

Proof. We shall show that if either statement fails, then G has an equitable r-coloring. Consider a proper edge cut $F:=E(Y_1,Y_2)$ with $Y_2=V-Y_1$. By the minimality of G there exist equitable r-colorings f_i of $G[Y_i]$ for $i=1,2$.

First, suppose that $|Y_1|$ is divisible by r. Then all classes of f_i have the same size for $i=1,2$. Thus we can obtain an equitable r-coloring of G, if we can match the color classes of f_1 to the classes of f_2 so that there are no edges connecting a class and its mate. If $|F|<r$ this is possible because $K_{r,r}-e_1-\cdots-e_{r-1}$ has a perfect matching by Hall's theorem.

If Y_1 is not divisible by r, then each f_i has small and large classes (differing in size by 1). To obtain an equitable r-coloring (by matching classes), we must match small classes of f_1 to large classes of f_2. This is possible if $|F|=0$, i.e., G is disconnected. ■

Proposition 6. *G contains neither $K_{r+1}-e$ nor $K_{r,r}-e$.*

Proof. Assume first that $G[Q]=K_{r+1}-xy$ with $x,y\in Q$. Set $Q':=Q-y$ and $G':=G-Q'$. If x has a neighbor v in G', then it is unique. In this case set $G^+:=G'+vy$; otherwise, let $G^+:=G'$. Notice that $\Delta(G^+)\leq r$. Since $\deg_{G^+}(y)\leq 2<r$, neither K_{r+1} nor $K_{r,r}$ are contained in G^+. So by the minimality of G, G^+ has an equitable r-coloring f with $f(v)\neq f(y)$. We can extend f to an equitable r-coloring of G by coloring x with $f(y)$ and using the other $r-1$ colors on the remaining $r-1$ vertices of $Q'-x$, whose neighbors are all in Q.

Now assume $G[Q]=K_{r,r}-e$. Then $G[Q]$ has an equitable r-coloring and so $Q\subsetneq V$. Also, $|E(Q,V\setminus Q)|\leq 2<3\leq r$, which contradicts Proposition 5. ■

The following observation will be used repeatedly.

Remark 7. Let $X\subseteq V$ with $|X|=r$ and let f be an equitable r-coloring of $G':=G-X$. For $x\in X$, let $A(x):=A_f(x):=[r]\setminus\{f(v):v\in V(G')\,\text{and}\,vx\in E(G)\}$ be the set of available colors for x. If the family $\mathcal{A}:=\{A(x):x\in X\}$ has a set of distinct representatives, then G has an equitable r-coloring. Thus by Hall's theorem, if

$$|S|\leq\left|\bigcup_{x\in S}A(x)\right|\quad\text{for every } S\subseteq X \tag{1}$$

then f can be extended to an equitable r-coloring of G.

Proposition 8. *Set $k:=\lfloor r/2\rfloor$. Then G does not contain $K_r-E(K_k)$.*

Proof. Suppose not; say $Z\subseteq X\subseteq V$ satisfy $|X|=r$, $|Z|=k$ and every vertex of $Y:=X-Z$ is adjacent to every other vertex of X. Set $G':=G-X$ and $V':=V(G')$. Then $d_{V'}(y)\leq 1$ for all $y\in Y$ and $d_{V'}(z)\leq k$ for all $z\in Z$. For each $y\in Y$, if y has a neighbor in G' then it is unique; in this case denote its neighbor by y'.

First, suppose that no vertex of G' is adjacent to every vertex of Y. Choose $y_1,y_2\in Y$ so that either y_1 has no neighbor in G' or $y'_1\neq y'_2$, and set $G^+:=G'$ in the former case and $G^+:=G'+y'_1y'_2$ in the latter. Then $\Delta(G^+)\leq r$. By Proposition 6, G^+ contains neither K_{r+1} nor $K_{r,r}$, since otherwise G would contain $K_{r+1}-e$ or $K_{r,r}-e$. Thus by the minimality of G, there exists an equitable r-coloring f of G^+ (and of G') with $f(y'_1)\neq f(y'_2)$ if y'_1 exists. So $|A(z)|=r-k$ $(\geq k)$ for $z\in Z$, $|A(y)|\geq r-1$ for $y\in Y$, and $|A(y_1)\cup A(y_2)|=r$. Suppose that (1) fails for an $S\subseteq X$. Since $|A(x)|\geq k$ for every $x\in X$, there exists $y\in Y\cap S$. Since $|A(y)|\geq r-1$, $S=X$, and so $|\bigcup_{x\in S}A(x)|\geq|A(y_1)\cup A(y_2)|=r$, a contradiction.

Now suppose $y' := y'_1$ is adjacent to every vertex of Y. So the vertices of Z are interchangeable with y'. Since $X + y'$ is not an $(r+1)$-clique, $Z^+ := Z + y'$ is not a $(k+1)$-clique. We claim that we can choose $v', z_1 \in Z^+$ so that $v'z_1 \notin E$ and $N_{G'}(v') \cup N_{G'}(z_1)$ is not an r-clique: First, note that $|N_{G'}(v')|, |N_{G'}(z_1)| \leq k$. If r is odd, then $|N_{G'}(v') \cup N_{G'}(z_1)| \leq 2k < r$. Otherwise, r is even and so $r \geq 4$. Suppose the claim is false. Then by degree considerations, Z^+ is independent, and $N_{G'}(w) \cap N_{G'}(z) = \emptyset$ for all $w, z \in Z^+$. Since $r \geq 4$, $|Z^+| = k+1 \geq 3$. So each $v'' \in N_{G'}(v')$ satisfies $d(v'') \geq 3k$, a contradiction.

Set $G^+ := G' + \{v'z' : z' \in N(z_1)\}$. Then $\Delta(G^+) \leq r$ and, by our claim, $\omega(G^+) \leq r$. If r is odd, then $d_{G^+}(v') \leq 2k < r$, and so G^+ does not contain $K_{r,r}$. Thus by the minimality of G, there exists an equitable r-coloring f of G^+. Since $N_{G^+}(z_1) \subseteq N_{G^+}(v')$, $f(v') \in A(z_1)$. Using Remark 7 we can extend f to an equitable r-coloring of G: First, color z_1 with $f(v')$, then color the rest of Z, and finally color Y, all with distinct colors. ■

Proposition 9. *If $r = 2k+1$, then G does not contain $K_{2k,2k}$.*

Proof. Suppose that $G[U]$ is a copy of $K_{2k,2k}$ with bipartition $\{Y_1, Y_2\}$. Let $G' := G - U$. Each $u \in U$ has at most one neighbor in G'. Call it u' if it exists. If $k = 1$, then by Proposition 6, for $i = 1, 2$ graph $G[Y_i]$ has no edges. Otherwise, $\Delta(G[Y_i]) \leq 1 < k$. In both cases, $G[Y_i]$ has an equitable k-coloring. So we can partition U into $2k-1$ independent 2-sets $B_1, \ldots, B_{2k-1}$ and two singletons $\{u_1\}, \{u_2\}$, which must be in the same part. We claim that we can pick the partition so that $u'_1 \neq u'_2$ (maybe because u'_2 does not exist): If for some $i \in [2]$ no vertex of G' is adjacent to every vertex of Y_i, this is easy; otherwise, G contains $K_{r,r} - e$, contradicting Proposition 6. Choose notation so that $B_i = \{y_i, y^*_i\}$. Define $X := \{y_1, \ldots, y_{2k-1}, u_1, u_2\}$ and

$$G^+ := G - X + u'_1u'_2 + \{y'_iy^*_i : i \in [2k-1]\} + \{y^*_iy^*_j : 1 \leq i < j \leq 2k-1\},$$

where edges involving u' for $u \in U$ do not exist if u' does not exist. Then

$$d_{G^+}(y^*_i) \leq r - 1 \quad \text{for all } i \in [2k-1] \tag{2}$$

and $d_{G^+}(u'_i) \leq r$ for all $i \in [2]$. So $\Delta(G^+) \leq r$. Suppose that G^+ contains $K \in \{K_{r+1}, K_{r,r}\}$. Then by (2), K does not contain any y^*_i. Since $y^*_iy'_i \in E(G^+)$ for $i \in [2k-1]$, K also does not contain y'_i for any $i \in [2k-1]$. It follows that G contains $K - u'_1u'_2$, contradicting Proposition 6. Thus $\omega(G^+) \leq r$ and G^+ does not contain $K_{r,r}$. By the minimality of G, G^+ has an r-coloring f. Extend f to an equitable coloring of G by first coloring each y_i with the same color as y^*_i and then coloring u_1 and u_2 with the remaining two colors, which is possible, since $f(u'_1) \neq f(u'_2)$. This contradicts the choice of G. ■

Proposition 10. *If $r = 2k$, then G does not contain $K_{k,r-1}$ as an induced subgraph.*

Proof. We first prove the weaker statement that G does not contain $K_{k,r}$ as an induced subgraph. Suppose $G[Y \cup Y' \cup Z] = K_{k,r}$ with bipartition $\{Y \cup Y', Z\}$, where $Y = \{y_1, \ldots, y_k\}$, $Y' = \{y'_1, \ldots, y'_k\}$, and $Z = \{z_1, \ldots, z_k\}$. Let $X := Z \cup Y$ and $G' := G - X$. Obtain G^+ from G' by adding all edges between y'_i and the neighbors of y_i in G' for each $i \in [k]$. Then $\Delta(G^+) \leq r$, since each vertex in X has at most k neighbors in $G - X$. Suppose that K is an $(r+1)$-clique in G^+. Then K contains at most one vertex from the independent set $\{y'_1, \ldots, y'_k\}$. Thus K contains an r-clique of G, contradicting Proposition 8. Hence $\omega(G^+) \leq r$. Since r is even, and using the minimality of G, G^+ has an equitable r-coloring f.

It suffices to show (1). Fix $S \neq \emptyset$ and set $S' = \{y'_i : y_i \in S\}$. Note that $|A(x)| \geq k$ for all $x \in X$. Thus we may assume $|S| > k$. So there exists $z \in S \cap Z$. Then $A(z) = [r] \setminus f[Y']$. By construction, $N_{G'}(y_i) \subseteq N_{G^+}(y'_i)$, and so $f(y'_i) \in A(y_i)$ for all $i \in [k]$. Thus

$$\left|\bigcup_{x \in S} A(x)\right| \geq |[r] \setminus f[Y']| + |f[S']| \geq 2k - |f[Y' \setminus S']| \geq k + |S'| \geq |S|.$$

Now suppose that $G[Y \cup Y'' \cup Z] = K_{k,r-1}$ with bipartition $\{Y \cup Y'', Z\}$, where Y and Z are defined as above and $Y'' := \{y'_1, \ldots, y'_{k-1}\}$. Let $X := Z \cup Y$ and $G' := G - X$. Obtain G_1 from G' by adding all edges between y'_i and the neighbors of y_i in G' for each $i \in [k-1]$. Again $\Delta(G_1) \leq r$ and G_1 does not contain K_{r+1}. Since G does not contain $K_{k,r}$, $\bigcap_{z \in Z} N_{G'-Y''}(z) = \emptyset$. Each vertex of Z has at most one neighbor in $G_1 - Y''$. If $N_{G'-Y''}(Z)$ contains two distinct vertices, say z'_1, z'_2 with $z_i z'_i \in E$ for $i \in [2]$, then let $G_2 = G_1 + z'_1 z'_2$; otherwise $G_2 := G_1$. Then $\Delta(G_2) \leq r$. If G_2 contains K_{r+1}, then G_1 contains $K_{r+1} - e$, where the missing edge is $z'_1 z'_2$. In this case, G contains $K_r - e$. Since $r \geq 4$, this contradicts Proposition 6; thus $\omega(G_2) \leq r$. By the minimality of G, G_2 has an equitable r-coloring f. For every $z \in Z$, $A(z) = [r] \setminus f[Y''] - f(z')$, if z' is defined; otherwise $A(z) = [r] \setminus f[Y'']$.

It suffices to show that (1) holds. Suppose that (1) fails for some $S \subseteq X$. Let $S' = \{y'_i : y_i \in S \text{ and } i \in [k-1]\}$. Since $G[X] = K_{k,k}$, $|A(y)| \geq k$ for all $y \in S$. Thus $|S| \geq k+1$ and so S contains a vertex $z_j \in Z$. Moreover, $f(y'_i) \in A(y_i) \setminus A(z_j)$ for all $i \in [k-1]$. So

$$\left|\bigcup_{x \in S} A(x)\right| \geq |A(z_j)| + |f[S']| \geq |[r] \setminus f[Y'' \setminus S'] - f(z'_j)| \geq k + |S'|. \tag{3}$$

By the choice of S, (3) yields $|S| \geq k+1+|S'|$. It follows that $S \supseteq Z + y_k$. Recall that by the construction of G_2, $f(z'_1) \neq f(z'_2)$. Hence, similar to (3), we have

$$\left|\bigcup_{x \in S} A(x)\right| \geq |A(z_1) \cup A(z_2)| + |f[S']| \geq |[r] \setminus f[Y'' \setminus S']| \geq k+1+|S'| = |S|.$$

This contradicts the definition of S. ■

Corollary 11. *If $r = 4$, then G does not contain $K_{2,3}$.*

Proof. Suppose G contains $K_{2,3}$. By Proposition 10, $G[Q]$ does not induce $K_{2,3}$. Thus G contains $K_4 - e$, contradicting Proposition 8. ■

Proposition 12. *If $r \geq 4$, then G does not contain $K_r - E(K_{1,r-3})$.*

Proof. Suppose $X \subseteq V(G)$ and $x, y, z \in X$ are such that $X - x$ is an $(r-1)$-clique and $xy, xz \in E$. By the minimality of G, $G' := G - X$ has an equitable r-coloring f. Then $|A(v)| \geq r-2$ for every $v \in X - x$, $A(y), A(z) \geq r-1$, and $A(x) \geq 2$. We are done by Remark 7, unless $|A(v)| \leq r-1$ for all $v \in X$ and $A(y) = A(z)$. In this case, every $v \in X - x$ has a neighbor $v' \in V(G')$ and it is unique if $v \in \{y, z\}$. By Proposition 8, $\omega(G) < r$. Thus there exists $v \in X - x$ with $vy' \notin E$.

Case 1. $v = z$. Set $G^+ := G' + z'y'$. Then $\Delta(G^+) \leq r$. By Proposition 6, G^+ does not contain $K \in \{K_{r+1}, K_{r,r}\}$, since otherwise G contains $K - z'y'$.

Case 2. $y'v \notin E$ and $y' = z'$. Let $G^+ := G' + \{y'u : u \in N_{V \setminus X}(v)\}$. Then $\Delta(G^+) \leq r$, since $d_{G'}(y') \leq r-2$. By Proposition 8, G^+ does not contain K_{r+1}, since otherwise G contains $K - y' = K_r$. Suppose G^+ contains $K = K_{r,r}$. Then by degree considerations, $G[K - y']$ is an induced $K_{r,r-1}$, contradicting Proposition 9 or 10.

So in both cases G^+ induces neither K_{r+1} nor $K_{r,r}$. By the minimality of G, we can choose an equitable r-coloring f of G^+. By construction, $f(y') \in A(v) \setminus A(y)$. Hence (1) is true. ■

Proposition 13. *G does not contain a maximal independent set of size s.*

Proof. Suppose X is a maximal independent set and $|X| = s$. Let $G' = G - X$. Since X is maximal, $\Delta(G') \leq r-1$. Thus, if G' contains $K_{r-1,r-1}$ then it is induced. So by Propositions 9 and 10, G' does not contain $K_{r-1,r-1}$. By Proposition 8, $\omega(G') \leq \omega(G) \leq r-1$. Thus by the minimality of G, G' has an equitable $(r-1)$-coloring. Adding X as a color class yields an equitable r-coloring of G. ■

3. OPTIMAL COLORINGS

Recall that G is an edge-minimal counter-example to the Chen–Lih–Wu conjecture with sr vertices. A *nearly equitable* coloring is a coloring such that every color class has the same size s except for one *small* class V^- with size $s-1$ and one *large* class V^+ with size $s+1$. The following lemma (Theorem 2 in [9]) is used to show that G has a nearly equitable r-coloring.

Lemma 14 (Kostochka and Nakprasit [9]). *Let H be a graph with $\chi(H), \Delta(H) \leq r$. Let $u \in V(H)$ and f be any r-coloring of $G - u$ with color classes $V_1, \ldots, V_r$. Then there is an r-coloring of G with color classes $W_1, \ldots, W_r$ such that $|W_i| = |V_i|$ for all but one i.*

Proposition 15. *G has a nearly equitable r-coloring.*

Proof. Let $xy \in E$. By the minimality of G, $G - xy$ has an equitable r-coloring. Thus $G - x$ has an r-coloring with one class of size $s-1$ and all other classes of size s. Since G does not have an equitable r-coloring, Lemma 14 implies that G has a nearly equitable r-coloring. ■

Let f be a nearly equitable coloring of G with color classes $V^- = V_1, \ldots, V_r = V^+$. Construct an auxiliary digraph $\mathcal{H} := \mathcal{H}(G,f)$ as follows. The vertices of $\mathcal{H}$ are the color classes $V_1, \ldots, V_r$. A directed edge $V'V''$ belongs to $E(\mathcal{H})$ if some vertex $x \in V'$ has no neighbors in V''. In this case we say that x *witnesses* the edge $V'V''$. Call a color class V_i of f *accessible* if V_1 is reachable from V_i in the digraph $\mathcal{H}(G,f)$. By definition, V_1 is accessible. Let $\mathcal{A} := \mathcal{A}(f)$ denote the family of accessible classes, $\mathcal{B}$ denote the family of inaccessible classes, $A := \bigcup \mathcal{A}$, and $B := \bigcup \mathcal{B} = V - A$. If $V_r \in \mathcal{A}$, then switching witnesses along a path from V_r to V_1 yields an equitable r-coloring; so $V_r \in \mathcal{B}$. Let $a := |\mathcal{A}|$ and $b := |\mathcal{B}| = r - a$. Then $|A| = as - 1$ and $|B| = bs + 1$.

Call a nearly equitable r-coloring of G *optimal* if a is as large as possible. Fix an optimal coloring $f = (V^- = V_1, \ldots, V_r = V^+)$, where the accessible classes $V_1, \ldots, V_a$ are ordered (by breadth-first search) so that V_1 is reachable from each accessible V_i by a path in $\mathcal{H}[V_1,\dots,V_i]$. An accessible class V_i is *terminal* if V_1 can be reached from every accessible class $V_j\in\mathcal{A}-V_i$ by a path in $\mathcal{H}-V_i$. For example, V_a is terminal. Class V_1 is terminal if and only if $a=1$. A vertex $v\in V_i$ is *movable to a class* $V_j\neq V_i$ if it has no neighbors in V_j. It is *movable* if it is movable to some accessible class.

The next two propositions illustrate the utility of these definitions. The structural properties of $\mathcal{H}[A]$ allow us to find equitable a-colorings of $G[A]$, and to bound the degree of $G[B]$. In the previous articles this degree bound on $G[B]$ allowed us to apply the minimality of G to obtain an equitable b-coloring of $G[B-y]$ for any vertex $y\in B$. In the current setting this is more subtle when b is odd, since $G[B]$ may contain $K_{b,b}$. Both propositions follow immediately from the definitions.

Proposition 16. *Let u be a movable vertex in a terminal class V_t. Then $G[A-V_t+u]$ has an equitable $(a-1)$-coloring.*

Proposition 17. $\Delta(G[B])\le b$.

For $x\in V_j\in\mathcal{A}$ and $y\in B$, x is a *solo neighbor* of y, and y is a *solo neighbor* of x, if $xy\in E$ and y has no other neighbors in V_j. A vertex is a *solo vertex*, if it has a solo neighbor. For $x\in A$, let S_x be the set of solo neighbors of x.

Lemma 18. $a\ge 2$.

Proof. Assume that $a=1$, i.e., $A=V_1$. Consider the weight function $w:E(A,B)\to\mathbb{Q}$ defined by $w(xy):=1/d_A(y)$ for every $xy\in E(G)$ such that $x\in A$ and $y\in B$. By definition, $\sum_{x\in N(y)\cap A} w(xy)=1$ for every $y\in B$. Hence the total weight of all edges in $E(A,B)$ is $bs+1$, and so there exists $x\in A$ such that $\sum_{y\in N(x)} w(xy)>b=r-1$. Since $d(x)\le r$, it follows that $|S_x|\ge r-1$. If S_x is a clique, then $G[S_x+x]=K_r$, contradicting Proposition 8. Thus x has two nonadjacent solo neighbors y and y'. Move y to V_1 and move x out of V_1 to obtain V_1'. Set $G':=G[B-y]$. Then $\chi(G')\le r-1$, $\Delta(G')\le r-1$ by Proposition 17, and G' does not contain $K_{r-1,r-1}$ by Propositions 9 and 10. By the minimality of G, G' has an equitable $(r-1)$-coloring $V_2',\dots,V_r'$. Suppose $y'\in V_j'$. If $V_i'+x$ is independent for some i, then moving x to V_i' yields a nearly equitable r-coloring of G with $V_j'\in\mathcal{A}$, a contradiction to the maximality of a. Otherwise, x has exactly one neighbor in each class of G'. Move x to V_j' and y' to V_1'. This yields an equitable r-coloring of G, a contradiction to the choice of G. ■

4. IF $r\le 4$, THEN $b\ge 2$

In this section we show that if $3\le r\le 4$, then $b\ge 2$. Since $a\ge 2$, this will give a new proof of Theorem 4. About half of our argument works also for all $r\ge 3$.

We start from a new notion. A bipartite graph F with a given bipartition (X,Y) is *very special* if every vertex of Y has degree 2. A bipartite graph F with a given bipartition (X,Y) is *special* if every component of F is either very special, or a path with an even number of vertices, or a single vertex in X.

The *core* of a graph G is the maximum subgraph H with $\delta(H)\ge 2$. It exists if $|G|\le\|G\|$.

Proposition 19. *Let F be a special bipartite graph with bipartition (X,Y). Suppose $w\in X$ is a vertex in the core C of some component D of F with $d_C(w)\geq 3$. Let $F':=F-w+z$, where z is a new vertex added to X so that $d(z)\geq 3$ and all neighbors of z are in Y. If F' is special with bipartition $(X-w+z,Y)$, then $N_C(w)\subseteq N_{F'}(z)$.*

Proof. Let H be a component of $C-w$. First, suppose that H has a vertex with degree at least 3. Since F' is special, the component in F' containing H is very special, and so has no vertices of degree 1 in Y. Hence each $y\in N_H(w)$ is a neighbor of z. Now suppose $\Delta(H)\leq 2$. Since H contains a neighbor $y\in Y$ of w, H is a path. Since C is the core of D, both ends of H have degree 2 in C, and so must both be neighbors of w. In particular, they are in Y. Since no component of the special $(X-w+z,Y)$-bigraph is a path with both ends in Y, at least one of the ends of H is a neighbor of z. But then the component of F' containing H has a vertex z of degree at least 3, and hence has no vertices of degree 1 in Y. Thus both ends of H are neighbors of z. Since C is very special, no other vertex of H can be a neighbor of w. ■

Now we prove some properties of the case $b=1$. In this case $B=V_r$ and every vertex $v\in V_r$ has at least one neighbor in every other class. Thus there is at most one class in which v has two neighbors, and there are no classes in which v has more than two neighbors.

Proposition 20. *Suppose V_i is a terminal class and $b=1$. Then $G[V_i\cup V_r]$ with bipartition (V_i,V_r) is special.*

Proof. Fix $X:=V_i$, set $F:=G[X,V_r]$, and consider a component C of F. For any graph H and vertex set S, let $S_H:=S\cap V(H)$. If $C\cap V_r$ has no solo neighbors in X, then C is a single vertex in X or C is very special. So it remains to consider the case that some $y\in V_r\cap C$ has a solo neighbor $z\in X$. By definition, $d_C(y)=1$. Every vertex in V_r and every non-movable vertex in X has degree at most 2 in C.

Case 1. $C\cap X$ has no movable vertex. Then $\Delta(C)\leq 2$. So C is a path, and y is an end of C. Thus it suffices to show that $|C|$ is even. If not, then switching the vertices between X_C and Y_C yields an r-coloring with new classes $X':=(X\setminus X_C)\cup Y_C$ and $Y':=(Y\setminus Y_C)\cup X_C$ such that $|X'|=|X|+1$ and $|Y'|=|Y|-1$. Since X is accessible, it has a vertex u that is movable to $A-X$. By the case, $u\notin C$. Since X is terminal, $A-X+u$ has an equitable $(a-1)$-coloring by Proposition 16. This coloring extends to an r-coloring of G using the classes $X'-u$ and Y', a contradiction.

Case 2. $C\cap X$ has a movable vertex. Choose a movable u as close as possible to y in C and let Q be a shortest y,u-path in C. Then the path $P:=Q-u$ has an odd number of vertices. Moreover, no vertex of P is movable. Thus $d_C(v)\leq 2$ for all $v\in V(P)$, and so P is a component of $C-u$. Move u to $A\setminus V_i$, equitably color $G[(A\setminus X)+u]$, and switch the vertices of X_P and Y_P to obtain an equitable r-coloring of G. This contradicts the choice of G. ■

Call a vertex $x\in X\in\mathcal{A}$ *mobile* if the degree of x in the core of the component of $G[X\cup V_r]$ containing x is at least 3. Note that every mobile vertex is movable.

Proposition 21. *Suppose V_i is a terminal class and $b=1$. Then there exists $u_i\in V_i$ such that u_i is mobile.*

Proof. By Proposition 20, $H := G[V_i \cup V_r]$ with bipartition (V_i, V_r) is special. Since $|V_i| < |V_r|$, H has a component D with more vertices in V_r than V_i. Thus D is very special, and is not a cycle. So D has average degree >2. Its core C also has average degree >2, and so has a vertex of degree at least 3, i.e., a mobile vertex. ■

Proposition 22. *Suppose $d_{V_1}(y) \leq 2$ for all $y \in V_r$. Then there exists $u_1 \in V_1$ with $d_B(u_1) \geq 3$.*

Proof. Otherwise the components of $G[V_1 \cup V_r]$ are paths and cycles. Since $|V_r| > |V_1|$ one of them, say P, is a path, beginning and ending in V_r. Switching the vertices of P between V_1 and V_r yields an equitable r-coloring, a contradiction. ■

In the remainder of this section we consider only $r \leq 4$. We will let $u_1, \ldots, u_a$ be the distinguished vertices, whose existence is asserted by Propositions 21 and 22; if $1 < i$ and V_i is not terminal, then u_i is undefined. First, we give a new proof of Theorem 4.

Proof of Theorem 4. Let G be an edge-minimal counter-example to Conjecture 3 for $r = 3$ with $3s$ vertices. Let f be its optimal 3-coloring. By Lemma 18, $a = 2$. By definition, u_2 has no neighbors in V_1 and u_1 has no neighbors in V_2. So, switching u_1 with u_2 creates a new optimal 3-coloring of G. By Proposition 20, both graphs $G[V_2 \cup V_3]$ and $G[(V_2 - u_2 + u_1) \cup V_3]$ are special. Thus by Proposition 19, $N(u_1) = N(u_2)$, a contradiction to Proposition 9. ■

Lemma 23. *If $r = 4$, then $b \neq 1$.*

Proof. Suppose that $r = 4$ and $b = 1$. First, we show that G has an optimal 4-coloring such that both V_2 and V_3 have vertices movable to V_1 (we will call such colorings *normal*). Indeed, if our optimal coloring $f := (V_1, \ldots, V_4)$ is not normal, then $V_3V_2, V_2V_1 \in E(\mathcal{H})$. Moving a witness from V_2 to V_1 produces a normal optimal coloring of G. So assume f is normal. Then u_1, u_3, u_3 are all defined. Consider the following four cases.

Case 1. $u_2u_3 \in E$. Since $d_B(u_2), d_B(u_3) \geq 3$, u_2 and u_3 are both movable to each of $V_1, V_2 - u_2, V_3 - u_3$. So, switching u_2 and u_3 leads to another normal coloring of G. By Proposition 20, the graphs $G[V_3 \cup V_4]$ and $G[(V_3 - u_3 + u_2) \cup V_4]$ are special. Thus by Proposition 19, $N(u_3) \cap V_4 \subseteq N(u_2)$, and hence G contains $K_{2,3}$. This contradicts Corollary 11.

Case 2. Both u_2 and u_3 are movable to V_1, but $u_2u_3 \notin E$. By symmetry, we may assume that u_1 is movable to V_3. Switch u_3 with u_1. Since $u_2u_3 \notin E$, the new coloring f' is normal (u_2 and u_1 are movable to V_1'). As in Case 1, we conclude that $N(u_3) \cap V_4 \subseteq N(u_1)$, and hence G contains $K_{2,3}$.

Case 3. Neither u_2 nor u_3 is movable to V_1. Then by the definition of normal coloring, for $i = 2, 3$, there exists a vertex $x_i \neq u_i \in V_i$ that is movable to V_1. So, switching u_2 with u_3 leads to a normal coloring, and we get a contradiction exactly as in Case 1.

Case 4. Otherwise. We may assume that u_3 is movable to V_2, u_2 is movable to V_1, and $u_2u_3 \notin E$. Move u_2 to V_1. This gives a new normal coloring with small class $V_2 - u_2$ for which Case 2 holds. ■

5. GOOD VERTICES

Call a vertex $y \in B$ *good* if $G[B-y]$ has an equitable b-coloring. More generally, if H is a graph on $qs+1$ vertices and $y \in V(H)$, we say that y is *good* if $H-y$ has an equitable q-coloring; otherwise y is *bad*. Since each vertex $y \in B$ has a neighbor in every accessible class, $d_B(y) \leq b$. Moreover, $f(B)$ witnesses that $\chi(G[B]) \leq b$. By the minimality of G, y is good unless $G[B-y]$ contains $K_{b,b}$ and b is odd. However, the following theorem and corollary (a rephrasing of Theorem 4 and Corollaries 1 and 2 in [6]) shows that considerably more is true. First, we need some terminology.

A graph H is *r-equitable* if $\Delta(H) \leq r$, $\chi(H) \leq r$, and every color class of every r-coloring of H has exactly the same cardinality. An r-colorable graph H with $\Delta(H) \leq r$ is *r-quasi-equitable* if there exists a $Y \subseteq V(H)$ with $|Y| \geq 2$ such that $G-y$ is r-equitable for every $y \in Y$. In [6] 10 special graphs $F_1, \ldots, F_{10}$ are identified (see Figs. 1 and 2). A graph F is *r-basic* if $F = K_r$, or $r=5$ and $F = F_1$, or $r=4$ and $F \in \{F_2, F_3, F_4\}$, or

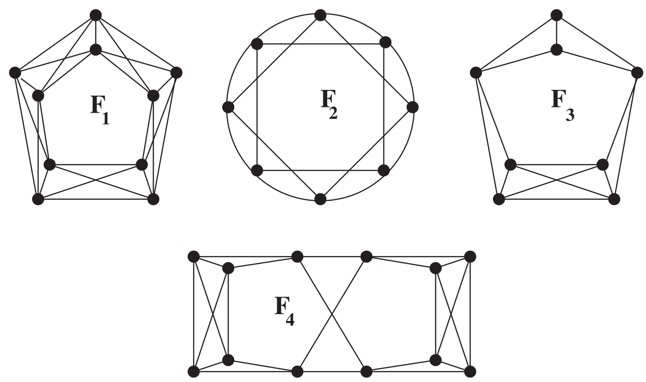

FIGURE 1. One 5-equitable and three 4-equitable basic graphs.

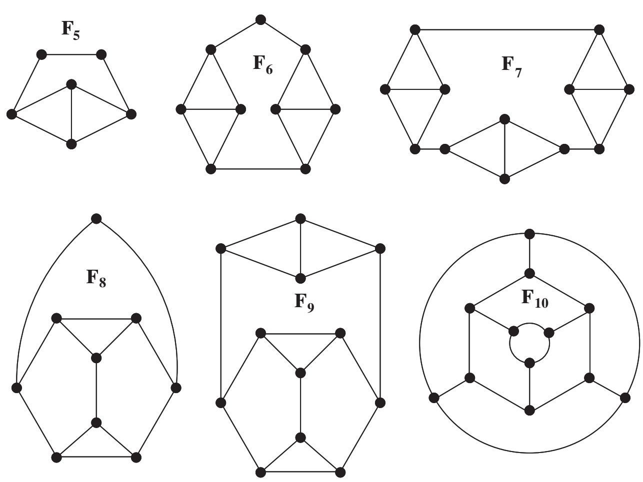

FIGURE 2. Six 3-equitable basic graphs.

$r=3$ and $F\in\{F_5,\dots,F_{10}\}$. By inspection, every r-basic graph F satisfies the following properties:

(F0) is r-equitable;
(F1) $r-1\le\delta(F)\le\Delta(F)\le r$;
(F2) the vertices with degree $r-1$ induce a complete subgraph of F;
(F3) if F is a non-clique, then at most two vertices have degree $r-1$.

Theorem 24 (Kierstead and Kostochka [6]). *Suppose H is a graph with $\Delta(H)\le r$ and $\chi(H)\le r$. Then H is r-equitable if and only if $V(H)$ has a partition $\{W_1,\dots,W_k\}$ such that $H[W_i]$ is r-basic for each $i, 1\le i\le k$. Moreover, if such a partition exists, it is unique.*

In the case that H is r-equitable, the unique partition $\{W_1,\dots,W_k\}$ is called an *r-decomposition* and each r-basic subgraph $H[W_i]$ is a *brick*. Moreover

$$\text{Every } K_3\subseteq H \text{ is contained in a brick of } H \tag{4}$$

since by (F1) all but one of the neighbors of any vertex are in the same brick as it.

Corollary 25 (Kierstead and Kostochka [6]). *Suppose that $r\ge 3$ and Conjecture 3 holds for all graphs on less than n vertices with maximum degree at most r. Let H be a graph on less than n vertices with $\Delta(H)\le r$ and $\chi(H)\le r$. Then H has no equitable r-coloring if and only if r is odd, $|H|$ is divisible by r, H contains $Q=K_{r,r}$ and $H-Q$ is r-equitable.*

We have assumed that Conjecture 3 holds for all graphs H with $\Delta(H)<r$ or both $\Delta(H)=r$ and $|H|\le|G|-r$. Suppose H is a graph on $n=qs+1$ vertices, where $q<r$, such that $\Delta(H)\le q$ and $\chi(H)\le q$. Let Y be the set of bad vertices of H and assume $|Y|\ge 2$. Then $q\ge 3$. For all $y\in Y$, $H-y$ has no equitable q-coloring, and so by Corollary 25, q is odd, $H-y$ contains $Q=K_{q,q}$, and $H-y-Q$ is q-equitable, and thus r-decomposable by Theorem 24. Set $H':=H-Q$. If $K_{q,q}\subseteq H'$, then $Q':=K_{q,q-1}\subseteq H'-y$. Using (F1), every vertex in the small part X of Q' is in the same brick B, since any two have at least three common neighbors in the large part Z. Thus all vertices in Z are also in B, since they each have at least two neighbors in $X\subseteq B$. This is a contradiction, since by inspection, no q-basic graph contains $K_{q,q-1}$. Thus H' does not contain any $K_{q,q}$. By degree considerations, any two distinct $K_{q,q}$'s in H are disjoint. Thus Q is the unique $K_{q,q}$ in H. Now consider any other $y'\in Y$. As above, using the uniqueness of Q, $H-y'-Q$ is q-equitable. Thus H' is q-quasi-equitable. Our goal is to show that $|Y|\le 4$.

Consider any q-coloring of H'. Then one color class X has size $s-1$, while all other color classes have size $s-2$. Moreover, since $H'-y$ is q-equitable for each $y\in Y$, $Y\subseteq X$. Thus

$$Y \text{ is monochromatic in every } r\text{-coloring of } H' \text{ (and so independent).} \tag{5}$$

Proposition 26. *For all distinct $y,y'\in Y$, the basic brick D_y of the decomposition of $H'-y'$ containing y, satisfies either $D_y=K_q$ or both $q=3$ and $D_y=F_5$.*

Proof. If $|E(D_y,H'-D_y)|\le q-2$, then we could add the edge yy' and still match color classes as in the proof of Proposition 5 to obtain an equitable q-coloring in which

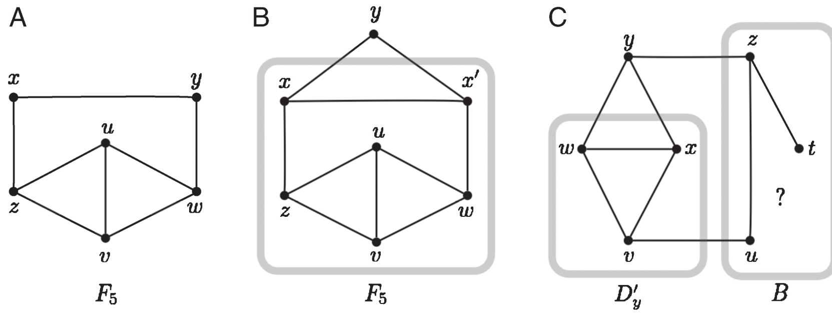

FIGURE 3.

y and y' have distinct colors, contradicting (5). So D_y has at least two vertices with degree $q-1$ in D_y. By inspection, the only possibilities are that $D_y=K_q$ or both $q=3$ and $D_y=F_5$. ■

For distinct $y,y'\in Y$, we will denote by $D_y(y')$ the brick containing y in $H'-y'$.

Proposition 27. *For each $y\in Y$ there exists a unique brick D' of $H'-y$ that contains two adjacent neighbors of y. Either $D'=K_q$ or $q=3$ and $D'=F_5$.*

Proof. First, note that two adjacent neighbors of y must be in the same brick D' of $H'-y$, since otherwise both would have degree less than $r-1$ in their respective bricks of $H'-y$, a contradiction to (F1). If $q>3$, then by Proposition 26, $D_y(y')$ is a q-clique for any $y'\in Y-y$. By (4), $D_y(y')-y$ is contained in a brick D' of $H'-y$; D' is the unique brick with two neighbors of y, since y has at most one additional neighbor. Since $q>3$, q is odd, and $\delta(D')\le q-1$, the only possibility is $D'=K_q$.

Suppose $q=3$. Then y can have two neighbors in at most one brick D' of $H'-y$; so if D' exists, it is unique. In this case $D'=K_3$ or $D'=F_5$, since the two neighbors of y have degree 2 in D'. So it suffices to show that y has two adjacent neighbors. Suppose not. Then for any $y'\in Y-y$, $D_y(y')\neq K_3$, and so $D_y(y')=F_5$ by Proposition 26. Moreover, y is one of the two vertices of $D_y(y')$ with degree 2 in $D_y(y')$; call the other x, let z be the second neighbor of x, and let w be the degree 3 neighbor of y. Let u and v be the common neighbors of z and w. See Figure 3A. Then $H[u,v,w,z]=K_4-e$. By (4), it is contained in a brick D' of $H'-y$. This is a contradiction, since no 3-basic graph has a vertex like w, which in D' has degree 2, and is contained in K_4-e. ■

For $y\in Y$, let D'_y denote the unique brick in $H'-y$ that contains two adjacent neighbors of y.

Proposition 28. *Suppose $y\in Y$ and $H'-(D'_y+y)$ contains a vertex $y'\in Y$. Then:*

(*A*) *$q=3$ and $D'_y=K_3$; let $D'_y:=H'[v,w,x]$, where $w,x\in N(y)$ (see Fig. 3C).*
(*B*) *$D_y(y')=F_5$; let $D_y(y'):=H'[u,v,w,x,y,z]$, where $z\in N(y)$ (see Fig. 3C).*
(*C*) *Let B be the brick containing z in $H'-y$. Then $B=K_3$ or $B=F_5$.*
(*D*) *$y'\in V(B-D_y(y'))$.*
(*E*) *$|Y|\le 4$.*

Proof. (A) Suppose not. If $q>3$, then $D'_y=K_q$. By (4), D'_y+y is contained in a brick B of $H'-y'$. Then $|B|>q$. So $q=5$ and $B=F_1$. Then $\omega(F_1)=4<5=\omega(D'_y)$, a contradiction.

Otherwise, $q=3$ and $D'_y=F_5$. See Figure 3B. The only vertex with degree 2 in D'_y+y is y. It follows that y is incident to a cut-edge separating y' from y. Thus it is possible to 3-color H' so that y and y' have different colors, contradicting (5).

(B) By (A) $D'_y=K_3$. So $D'_y+y=K_4-e$ is contained in $D_y(y')$ by (4). Thus by Proposition 26, $D_y(y')=F_5$.

(C) Since $y\notin B$, both other neighbors of z are in B. One is u; let t be the neighbor not in D_y. See Figure 3C. Since $uv, zy\notin E(B)$, both z and u have degree two in B. Thus either $B=K_3$ or $B=F_5$, depending on whether $tu\in E$.

(D) Since $uv, zy\notin E(B)$, we have $|E(D_y(y')+B, H'-(D_y(y')+B)|\leq 1$. Thus, using (5), $y'\in D_y(y')+B$. Since $y'\notin D_y(y')$, $y'\in V(B-D_y(y'))$.

(E) Consider $y_0\in Y-y$. Either (i) $y_0\in V(D'_y)$ or (ii) $y_0\in B$. If (ii) then $y_0=v$, since $y_0\notin N(y)$ by (5). If (iii) then, again using (5), there are at most two possibilities for y_0, since $\alpha(B-D_y)\leq 2$. ■

Theorem 29. *Let H be a subgraph of G on $qs+1$ vertices with $\Delta(H)\leq q$ and $\chi(H)\leq q$. Then H has at most four bad vertices and every $x\in V(H)$ has at most two bad neighbors.*

Proof. Let $y\in Y$. By (5), Y is monochromatic. Thus if x has three neighbors in Y, then x has no neighbors in some color class besides its own. Since x is movable, $H'-y$ is not q-equitable, a contradiction.

Now suppose $Y\geq 4$. By Proposition 27, D'_y is a complete graph or F_5. In either case, $\alpha(D'_y)\leq 2$. By (5), Y is an independent set. Thus $Y\setminus V(D'_y+y)\neq\emptyset$, and so the hypothesis of Proposition 28 holds. So $|Y|\leq 4$, by Proposition 28(E). ■

6. CASE $r=4$

In this section we assume that $b\geq 2$ and show that $a\geq 3$ for all r. In particular this shows that Conjecture 3 holds for all graphs with maximum degree at most four.

Proposition 30. *If a (possibly non-optimal) nearly equitable r-coloring of G satisfies $a\geq 2$, then its small class V^- has a vertex v with $d_B(v)\geq b+2$. In particular, v is movable.*

Proof. Suppose that $d_B(v)\leq b+1$ for every $v\in V^-$. Set $G':=G[B\cup V^-]$. Since every vertex in B has a neighbor in each class in $\mathcal{A}-V^-$, we have

$$\Delta(G')\leq b+1<a+b=r$$

since $a\geq 2$. We will obtain a contradiction by showing that G' has an equitable $(b+1)$-coloring, and hence G has an equitable r-coloring. Suppose G' does not have an equitable $(b+1)$-coloring. By Corollary 25, $b+1$ is odd, G' contains $Q:=K_{b+1,b+1}$, and every $(b+1)$-coloring of $G'-Q$ is equitable.

Let Q have bipartition (X,Y). Since by Proposition 17, $\Delta(G[B])\leq b$, we may assume that $X\cap V^-\neq\emptyset$. Then since V^- is independent, $Y\subseteq B$. Now again since $\Delta(G[B])\leq b$,

$X \subseteq V^-$. So $V^- \setminus X$ is a color class of a $(b+1)$-coloring of $G' - Q$. Since

$$(b+1)|V^- \setminus X| = (b+1)(s-1-b-1) < (b+1)(s-2) = |G' - Q|,$$

this coloring is not equitable, a contradiction. ■

Our next goal is to show that $a>2$. For a contradiction suppose $a=2$.

Proposition 31. *If $z \in V_2$ has a solo neighbor $y \in Y \in \mathcal{B}$, then z is not movable.*

Proof. Otherwise moving z to V_1 yields a new nearly equitable coloring f' with small class $V_2 - z$. Then z witnesses that $V_1 + z \in \mathcal{A}(f')$ and y witnesses that $Y \in \mathcal{A}(f')$. Thus $a \geq 3$. ■

Proposition 32. *If $y \in B$ is a good solo neighbor of $z \in V_2$, then y is adjacent to every solo neighbor of z.*

Proof. Suppose y is not adjacent to $y' \in S_z - y$. By definition, V_2 has a movable vertex u. By Proposition 31, z has a neighbor in V_1 and $u \neq z$. Move z out and y to V_2. Since y is good, $B - y$ has an equitable coloring. If possible, add z to one of the new color classes of $B - y$. This case yields a new nearly equitable coloring in which $V_2 - z + y$ and the new class of y' are both accessible. Thus $a \geq 3$. Otherwise, z has exactly one neighbor in each class of $B - y$. Moving z to the class of y', y' to $V_2 - z + y$, and u from $V_2 - z + y + y'$ to V_1 yields an equitable r-coloring. ■

Proposition 33. *If $z \in V_2$ is solo, then S_z contains at most two bad vertices.*

Proof. Let u be a movable vertex in V_2. Suppose z has three bad solo neighbors. Then, as in Section 5, $H := G[B]$ contains some $Q = K_{b,b}$, the graph $H' := H - Q$ is b-quasi-equitable, and every bad vertex is contained in the large color class $X - Q$ of H', where $X \in \mathcal{B}$. Since the solo vertex z is not movable and there exist at least three neighbors of z in X, there exists a class $W \in \mathcal{B}$ with no neighbors of z. Move z to W and move two bad vertices $y_1, y_2 \in S_z \cap X$ to V_2 and u to V_1. This yields an equitable 2-coloring of $G[A - z + y_1 + y_2]$ and a nearly equitable coloring of $H'' := H' + z - y_1 - y_2$. These colorings can be combined with a nearly equitable coloring of Q to obtain an equitable r-coloring of G, a contradiction. ■

Corollary 34. *For every $z \in V_2$, $|S_z| \leq b$.*

Proof. Suppose $|S_z| \geq b+1$. By Proposition 33, S_z contains at most two bad vertices. By Proposition 32 every good vertex in S_z is adjacent to every other vertex in S_z. Thus $G[S_z + z]$ contains $K_r - e$. Since $r = a + b \geq 4$, this contradicts Proposition 8. ■

A vertex $z \in V_2$ is *f-special* if it has at least $b+1$ neighbors in B, at least b of them are solo neighbors, and at least $b-1$ of the solo neighbors are good.

Proposition 35. *If $z \in V_2$ is f-special, then it has a unique neighbor $w \in V_1$ and w has at least two neighbors in V_2. Moreover, $S_z + z$ is an $(r-1)$-clique and w has no neighbors in S_z.*

Proof. Since z is f-special, it has at least b solo neighbors in S_z, and at least $b-1$ of them are good. By Proposition 32, each good vertex is adjacent to every vertex in S_z.

By Corollary 34, $|S_z| \leq b$. Thus $S_z + z$ is an $(r-1)$-clique. Since z is solo, it has a neighbor $w \in V_1$ by Proposition 31. Since it has $r-1$ neighbors in B, w is a unique neighbor of z in V_1. By Proposition 12, w has no neighbors in S_z, since otherwise $G[S_z + z + w]$ contains $K_r - E(K_{1,r-3})$. Suppose w has no other neighbors in A. Then switching z and w results in another nearly equitable coloring f' with $a(f') \geq 3$, since each vertex of S_z is movable to $V_2 - z + w$, a contradiction to the optimality of f. ■

Proposition 36. *If* $b \geq 2$, *then* $a \geq 3$.

Proof. Suppose that $b \geq 2$ and $a = 2$. Let $u \in V_2$ be movable. Let f' be the optimal coloring obtained from the original coloring f by moving u to V_1. Then f' has color classes $V_1' := V_2 - u, V_2' := V_1 + u, V_3' := V_3, \ldots, V_r' := V_r$. For $z \in V_2$ (respectively, $z \in V_2'$), let $\sigma(z)$ (respectively, $\sigma'(z)$) be the number of good solo neighbors of z with respect to f (respectively, f'). Let β denote the number of bad vertices in $G[B]$. For $z \in A$ let $\beta(z)$ denote the number of bad vertices in S_z. By Theorem 29, $\beta \leq 4$; moreover $\beta = 0$ if $r = b + 2 = 4$, since Conjecture 3 is true for $r = 2$. Define $\rho := 1$ if $\beta > 0$ and $\rho := 0$ otherwise. Note that $b - 2\rho > 0$, since $\beta = 0$ when $b = 2$. By Proposition 33, $\beta(z) \leq 2$ for all $z \in A$. We now do the following discharging:

(R0) Assign a total charge of $2|B| - 2\rho$ to the vertices of B so that each vertex y receives charge 2 if it is good and charge $2(1 - 1/\beta)$ if it is bad.

(R1) Each $y \in B$ distributes half of its charge evenly to its neighbors in V_2 and half of its charge evenly to its neighbors in V_2'. Note that $u \in V_2 \cap V_2'$, and so receives charge twice. Let $\mathrm{ch}_1(x)$ denote the charge of x at the end of (R1).

(R2) Each f-special vertex $z \in V_2$ forwards $\frac{1}{2}$ charge to its unique neighbor in V_1 and each f'-special vertex in V_2' forwards $\frac{1}{2}$ charge to its unique neighbor in V_1'. Let $\mathrm{ch}(x)$ denote the final charge of x.

Altogether the vertices of B send a charge of $2|B| - 2\rho$ to the vertices of A. Since u is movable in both colorings, it has no solo neighbors. Thus in the first redistribution (R1) it receives a charge of at most $\frac{1}{2}$ from each of its neighbors twice. So altogether it receives at most $r = b + 2$. Since u is movable, it has no neighbors in A, and so gains no charge in the second redistribution (R2). Thus after (R2) the set $A - u$ has total charge at least

$$2 + 2bs - 2\rho - (b+2) = 2b(s-1) + b - 2\rho > 2b(s-1) = b|A - u|.$$

So there exists a vertex $w \in A - u$ with charge $\mathrm{ch}(w) > b$. Our goal is to show that this is a contradiction.

In redistribution (R1) each vertex $z \in A - u$ receives 1 charge from each good solo neighbor, $(1 - \frac{1}{\beta})$ from each bad solo neighbor and at most $\frac{1}{2}$ from every other neighbor in B. So we have

$$\mathrm{ch}_1(z) \leq \frac{1}{2}(d_B(z) + |S_z|) - \frac{\beta(z)}{\beta} \leq \frac{1}{2}(d_B(z) + |S_z|) - \frac{\beta(z)}{4}. \tag{6}$$

If $S_z = \emptyset$, then by (6), $\mathrm{ch}_1(z) \leq \frac{1}{2} d_B(z) \leq \frac{r}{2} \leq b$. Otherwise, by Proposition 31, $d_B(z) \leq b + 1$, and by Corollary 34, $|S_z| \leq b$. Thus for any $z \in A - u$, $\mathrm{ch}_1(z) \leq b + \frac{1}{2}$. Moreover, if $\mathrm{ch}_1(z) > b$, then by (6), $\frac{1}{2}(d_B(z) + |S_z|) - \beta(z)/4 > b$. By the above, this is possible only if $d_B(z) = b + 1$, $|S_z| = b$, and $\beta(z) \leq 1$, which yields $\sigma(z) \geq b - 1$. Thus z is special.

Now consider the situation after round (R2). Each vertex z with charge $\mathrm{ch}_1(z)>b$, is special, and so loses $\frac{1}{2}$ charge to its unique (non-special) neighbor in A. Thus $\mathrm{ch}(z)\leq b$.

So, if $\mathrm{ch}(w)>b$, then w gained charge in redistribution (R2). Thus w is the unique neighbor (in A) of at least one special vertex. By symmetry, we may assume that $w\in V_1$. Using Proposition 35, $d_A(w)\geq 2$. In total w gains at most 1 from each neighbor in B and at most $\frac{1}{2}$ from each neighbor in A. Thus $d_A(w)\leq 3$ and so one of the following cases holds.

Case 1. w has exactly one f-special neighbor $z\in V_2$. By Proposition 35, w has another neighbor, say z', in V_2. In our case, w does not receive any charge from z'. Thus, to have $\mathrm{ch}(w)>b$, we need $|S_w|=b$ and $\beta(w)\leq 1$ (and so $\sigma'(w)\geq b-1$). By Proposition 32 applied to $w\in V_2'$, every good solo neighbor of w is adjacent to every other solo neighbor of w. Since $\sigma'(w)\geq b-1$, S_w+w is an $(r-1)$-clique. Choose a good $y\in S_w$. By Proposition 12, since z is adjacent to w, z is not adjacent to y. Move w out of V_1 and move both z and y into V_1. Next, using that y is good, equitably color $B-y$ with b colors. Since $z',y,z\in N(w)$, $d_{B-y}(w)\leq b-1$. So we can move w into a class of $B-y$. This yields a nearly equitable coloring of G. This coloring has at least three accessible color classes, since z and each vertex in S_z are now movable to the small class V_2-z. This contradicts the optimality of f.

Case 2. w has exactly two f-special neighbors $z_1,z_2\in V_2$. To have $\mathrm{ch}(w)>b$, w has to receive charge greater than $b-1$ from the remaining b neighbors. So we need $d_B(w)=b$ and $|S_w|\geq b-1$. Switch w with both z_1 and z_2. This yields a nearly equitable coloring with small class $V_2-z_1-z_2+w$. By Proposition 35, w has no neighbors in S_z. It follows that each vertex in S_{z_1} is movable to $V_2-z_1-z_2+w$. By Proposition 30, V_1-w has a vertex that is movable to V_2; note that it is also movable to V_2+w. So, as in Case 1, the new coloring has at least three accessible color classes.

Case 3. w has exactly three f-special neighbors $z_1,z_2,z_3\in V_2$. They give w the charge $\frac{3}{2}$. In order to receive charge greater than $b-\frac{3}{2}$ from the remaining $b-1$ neighbors, w needs $|S_w|=b-1$ and $\sigma'(w)\geq 1$. Move w out and z_1 and z_2 to V_1. Move a good $y\in S_{z_1}$ to $V_2-z_1-z_2$. Since y is good, $B-y$ has an equitable b-coloring g. Since $d_B(w)\leq r-3=b-1$, we can add w to one of the classes of g to obtain a nearly equitable r-coloring of G with small class $V_1^*:=V_2-z_1-z_2+y$. By the definition of solo vertices, $yz_2\notin E(G)$. Since $\chi(B)=b$ and S_{z_2} is a b-clique, we can choose $y'\in S_{z_2}$ so that $yy'\notin E$. So again, the new coloring has at least three accessible color classes, since both z_2 and y' are movable to V_1^*. ■

Lemma 23 and Proposition 36 yield our main result:

Theorem 37. *Conjecture* 3 *is true for* $r\leq 4$.

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