# A Hypergraph Version of a Graph Packing Theorem by Bollobás and Eldridge

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**Abstract:** Two $n$-vertex hypergraphs $G$ and $H$ *pack*, if there is a bijection $f : V(G) \to V(H)$ such that for every edge $e \in E(G)$, the set $\{f(v) : v \in e\}$ is not an edge in $H$. Extending a theorem by Bollobás and Eldridge on graph packing to hypergraphs, we show that if $n \geq 10$ and $n$-vertex hypergraphs

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$G$ and $H$ with $|E(G)| + |E(H)| \leq 2n - 3$ with no edges of size 0, 1, $n - 1$ and $n$ do not pack, then either

(i) one of $G$ and $H$ contains a spanning graph-star, and each vertex of the other is contained in a graph edge, or
(ii) one of $G$ and $H$ has $n - 1$ edges of size $n - 2$ not containing a given vertex, and for every vertex $x$ of the other hypergraph some edge of size $n - 2$ does not contain $x$.



Keywords: *graph packing; hypergraph*

## 1. INTRODUCTION

By a *hypergraph* we mean a pair $(V, E)$ where $V$ is a finite set (elements of $V$ are called *vertices*) and $E$ is a family of subsets of $V$ (members of $E$ are called *edges*). An empty edge is also allowed. An important instance of combinatorial packing problems is that of *(hyper)graph packing*. Two $n$-vertex hypergraphs $G$ and $H$ *pack*, if there is a bijection $f : V(G) \to V(H)$ such that for every edge $e \in E(G)$, the set $\{f(v) : v \in e\}$ is not an edge in $H$. For graphs, this means that $G$ is a subgraph of the complement $\overline{H}$ of $H$, or, equivalently, $H$ is a subgraph of the complement $\overline{G}$ of $G$.

Some milestone results on extremal graph packing problems were obtained in the seventies. At the same time, fundamental papers by Bollobás and Eldridge [1] and Sauer and Spencer [7] have appeared. The papers gave sufficient conditions for packing of graphs under different conditions. Some of these results were also obtained by Catlin in his Ph.D. Thesis [3] and in [2]. Surveys on the topic are [10] and [9].

In particular, Sauer and Spencer [7] proved the following.

**Theorem 1** ([7]). *Let $G$ and $H$ be $n$-vertex graphs with $|E(G)| + |E(H)| < 1.5n - 1$. Then $G$ and $H$ pack.*

The result is sharp, since if $H$ is the star $K_{1,n-1}$ and $G$ is the graph with $\lceil n/2 \rceil$ edges and minimum degree 1, then $|E(G)| + |E(H)| = \lceil 1.5n \rceil - 1$ but $G$ and $H$ do not pack. An important feature of this example is that $H$ has a universal vertex. By a *universal vertex* in a hypergraph $F$ we mean a vertex $v$ such that for every other vertex $w \in V(F)$, the graph edge $vw$ belongs to $E(F)$.

Bollobás and Eldridge [1] obtained the following refinement of Theorem 1.

**Theorem 2** ([1]). *Let $G$ and $H$ be $n$-vertex graphs with $|E(G)| + |E(H)| \leq 2n - 3$. If neither of $G$ and $H$ has a universal vertex, and the pair $\{G, H\}$ is none of the seven pairs in Figure 1, then $G$ and $H$ pack.*

Corollary 1 in [1] yields that Theorem 2 can be restated as follows.

**Theorem 3** ([1]). *Let $G$ and $H$ be $n$-vertex graphs with $|E(G)| + |E(H)| \leq 2n - 3$. Then $G$ and $H$ do not pack if and only if either $\{G, H\}$ is one of the seven pairs in Figure 1, or one of $G$ and $H$ has a universal vertex and the other has no isolated vertices.*

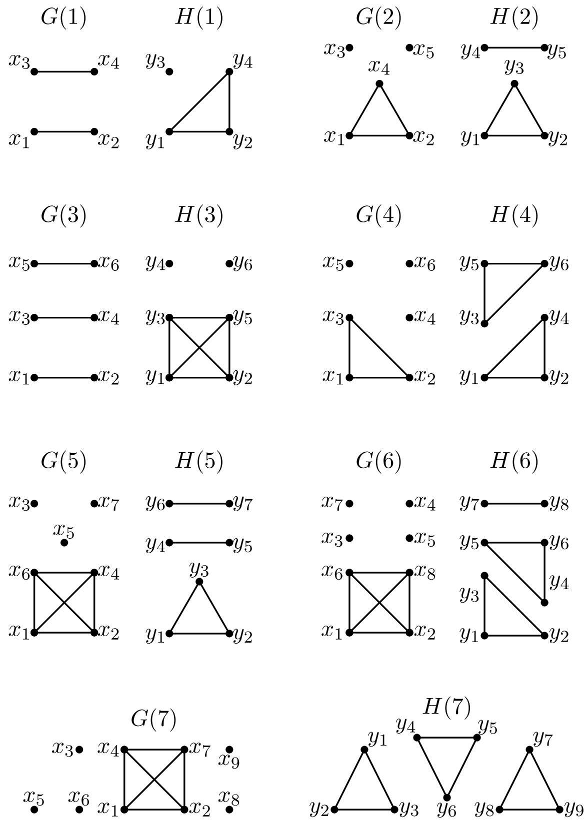

FIGURE 1. Bad pairs in Theorem 2.

To see that Theorem 3 yields Theorem 1, observe that for each pair $(G, H)$ in Figure 1, $|E(G)| + |E(H)| = 2n - 3 \geq 1.5n - 1$ and that if $H$ has a universal vertex and $G$ has no isolated vertices, then $|E(G)| + |E(H)| \geq (n-1) + \lceil n/2 \rceil$.

If $G$ and $H$ are $n$-vertex nonuniform hypergraphs, then packing may become more complicated. By *i-edge* we will mean an edge of size $i$. Sometimes, edges of size 2 will be called *graph edges*, and edges of size at least 3 will be called *hyperedges*.

Edges of size $0, 1, n-1$ or $n$ make harder for hypergraphs to pack. For example, if $V(G)$ is an edge in $G$ and $V(H)$ is an edge in $H$, then $G$ and $H$ do not pack. Similarly, if $\emptyset$ is an edge in both $G$ and $H$, then $G$ and $H$ do not pack. Also if the total number of 1-edges or the total number of $n-1$ edges in $G$ and $H$ is at least $n+1$, then $G$ and $H$ again do not pack. These examples indicate that edges of size $i$ and $n-i$ behave similarly. Indeed, a bijection $f : V(G) \to V(H)$ maps edge $e \in E(G)$ onto edge $g \in E(H)$ if and only if it maps set $V(G) - e$ onto $V(H) - g$. This motivates our notion of the *orthogonal hypergraph*: For a hypergraph $F$, the *orthogonal hypergraph* $F^{\perp}$ has the same set of vertices as $F$ and $E(F^{\perp}) := \{V(F) - e : e \in E(F)\}$. By definition, two $n$-vertex hypergraphs $G$ and $H$ pack if and only if $G^{\perp}$ and $H^{\perp}$ pack.

Pilśniak and Woźniak [5] proved that if an $n$-vertex hypergraph $G$ has at most $n/2$ edges and neither $\emptyset$ nor $V(G)$ is an edge in $G$, then $G$ packs with itself. They also asked whether every such $G$ packs with each $n$-vertex hypergraph $H$ satisfying the same conditions. Recently, Naroski [4] proved the following stronger result.

**Theorem 4.** *Let $G$ and $H$ be $n$-vertex hypergraphs with no 0-edges and $n$-edges. If $|E(G)| + |E(H)| \leq n$, then $G$ and $H$ pack.*

By the above examples, the bound of $n$ in Theorem 4 is sharp. We will prove a corresponding bound for $n$-vertex hypergraphs with no 0-, 1-, $(n-1)$-, and $n$-edges. This result also generalizes Theorem 3 and extends it to hypergraphs.

We define a *bad pair* of hypergraphs to be either one of the pairs $(G(i), H(i))$ in Figure 1, or one of the pairs $(G(i)^{\perp}, H(i)^{\perp})$.

Our main result is the following.

**Theorem 5.** *Let $G$ and $H$ be $n$-vertex hypergraphs with $|E(G)| + |E(H)| \leq 2n - 3$ containing no 0-, 1-, $(n-1)$-, and $n$-edges. Let $|E(G)| \leq |E(H)|$. Then $G$ and $H$ do not pack if and only if either*

*(i) $(G, H)$ or $(H, G)$ is a bad pair, or*
*(ii) $H$ has a universal vertex and every vertex of $G$ is incident to a graph edge, or*
*(iii) $H^{\perp}$ has a universal vertex and every vertex of $G^{\perp}$ is incident to a graph edge.*

Since each of the graphs in Figure 1 has at most nine vertices, for $n \geq 10$, the theorem says that ... *$G$ and $H$ do not pack if and only if either $H$ has a universal vertex and every vertex of $G$ is incident to a graph edge or $H^{\perp}$ has a universal vertex and every vertex of $G^{\perp}$ is incident to a graph edge.* Note that the theorem is sharp even for graphs: for infinitely many $n$ there are $n$-vertex graphs $G_n$ and $H_n$ such that $|E(G)| + |E(H)| = 2n - 2$, neither of $G_n$ and $H_n$ has a universal vertex, and $G_n$ and $H_n$ do not pack (see, e.g., [1, 8]).

In the same way Theorem 3 yields Theorem 1, Theorem 5 yields the following extension of Theorem 1 to hypergraphs.

**Corollary 1.** *Let $G$ and $H$ be $n$-vertex hypergraphs with $|E(G)| + |E(H)| < n - 1 + \lceil n/2 \rceil$ containing no 0-, 1-, $(n-1)$-, and $n$-edges. Then $G$ and $H$ pack.*

Very recently we have learned that Pilśniak and Woźniak [6] independently obtained a weaker version of Theorem 5. They proved that *every $n$-vertex hypergraph with $n - 2$ edges not containing 0-, 1-, $(n-1)$-, and $n$-edges packs with itself.*

To prove Theorem 5, we consider a counterexample $(G, H)$ with the fewest vertices. In the next section, we set up the proof and derive simple properties of $(G, H)$. In Section 3, we prove two more advanced properties of $(G, H)$. In the last section, we deliver the proof of Theorem 5.

## 2. PRELIMINARIES

Consider a counterexample $(G, H)$ to Theorem 5 with the least number of vertices $n$. This means that $|E(G)| + |E(H)| \leq 2n - 3$, $|E(G)| \leq |E(H)|$, neither $(G, H)$ nor $(H, G)$ is a bad pair, $G$ and $H$ do not pack, and if $H$ (respectively, $H^{\perp}$) has a universal vertex, then $G$ (respectively, $G^{\perp}$) has a vertex not incident with graph edges. If at least one of $G$, $H$, $G^{\perp}$, and $H^{\perp}$ is an ordinary graph, then the statement holds by Theorem 3. So we will assume that

$$\text{each of } G, H, G^{\perp}, \text{ and } H^{\perp} \text{ has at least one hyperedge.} \tag{1}$$

Naroski [4] used the following hypergraph operation: For an $n$-vertex hypergraph $F$, the hypergraph $\widetilde{F}$ is obtained from $F$ by replacing each edge $e \in E(F)$ of size at least $(n+1)/2$ with $V(F) - e$ and deleting multiple edges if they occur. This operation has the following useful property.

**Lemma 1** ([4]). *Let $F_1$ and $F_2$ be $n$-vertex hypergraphs with no edge with size less than $k$ and no edge with size greater than $n-k$. Then*

*(a) $|E(\widetilde{F_1})| \leq |E(F_1)|$ and $|E(\widetilde{F_2})| \leq |E(F_2)|$,*
*(b) both $\widetilde{F_1}$ and $\widetilde{F_2}$ have no edges of size less than $k$ and no edges of size greater than $\lfloor \frac{n}{2} \rfloor$, and*
*(c) if $\widetilde{F_1}$ and $\widetilde{F_2}$ pack, then $F_1$ and $F_2$ pack.*

**Lemma 2.** *If $\widetilde{H}$ has a universal vertex and every vertex of $\widetilde{G}$ is incident to a graph edge, then $G$ and $H$ pack.*

***Proof.*** Let $S$ be the set of 2-edges of $\widetilde{G}$ and $\widetilde{H}$ that are 2-edges in $G$ and $H$. Let $S'$ be the set of 2-edges of $\widetilde{G}$ and $\widetilde{H}$ whose complementary $(n-2)$-edges exist in $G$ and $H$. Suppose that $\widetilde{H}$ contains a universal vertex $v$. Then $\widetilde{G}$ contains at most $n-2$ edges and hence some vertex of $\widetilde{G}$ is contained in at most one 2-edge. We consider two cases.

*Case 1:* All 2-edges in $\widetilde{H}$ that contain $v$ are contained in $S$ (respectively, $S'$). By the symmetry between $H$ and $H^{\perp}$, we may assume that they all are in $S$. Then under the conditions of the theorem, some vertex $w \in V(\widetilde{G})$ is not contained in any edge in $S$. We let $H'$ be the hypergraph obtained from $H$ by deleting $v$, and all 2-edges containing $v$, and replacing each hyperedge $e \in E(H)$ that contains $v$ by $e - v$. We let $G'$ be the hypergraph obtained from $G$ by deleting $w$ and replacing each edge $e \in E(G)$ containing $w$ by the edge $e - w$. Then since $|E(G')| + |E(H')| \leq 2n - 3 - (n-1) = n-2$, Theorem 4 yields that $G'$ and $H'$ pack. We extend this packing to a packing of $G$ and $H$ by mapping $v$ to $w$.

*Case 2:* Vertex $v$ is contained in a 2-edge of $\widetilde{H}$ that is not in $S$ and in a 2-edge of $\widetilde{H}$ that is not in $S'$. Let $w_1$ be a vertex of $\widetilde{G}$, which is contained in exactly one 2-edge (if no such vertex exists, then some vertex $w$ of $\widetilde{G}$ is not incident to 2-edges at all, and we proceed as in Case 1 (deleting all 2-edges of $\widetilde{H}$ incident with $v$)). Let $w_1w_2$ be the 2-edge in $\widetilde{G}$ containing $w_1$. By symmetry, we may assume that $w_1w_2 \in S$. Let $vv'$ be an edge of $\widetilde{H}$, which is not in $S$. We let $H''$ be the hypergraph obtained from $H^{\perp}$ by first deleting $v$, $v'$, and all 2-edges containing $v$ and then removing $v$ and $v'$ from each edge $e$ that contains any of them. We let $G''$ be the hypergraph obtained from $G^{\perp}$ by first deleting $w_1$, $w_2$, and the edge $w_1w_2$ and then truncating all edges containing either of $w_1$ and $w_2$. Then since $|E(G')| + |E(H')| \leq 2n - 3 - (n-1) - 1 = n-3$, Theorem 4 yields that $G''$ and $H''$ pack. We extend this packing to a packing of $G$ and $H$ by mapping $v$ to $w_1$ and $v'$ to $w_2$. ■

In view of Lemmas 1 and 2, we will assume that $G$ and $H$ have no edges of size greater than $\frac{n}{2}$. We will study properties of the pair $(G, H)$ and finally come to a contradiction.

Throughout the proof, for $i \in \{2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, $G_i$ (respectively, $H_i$) denotes the subgraph of $G$ (respectively, of $H$) formed by all of its edges of size $i$, and $d_i(v, G)$ (respectively, $d_i(v, H)$) denotes the degree of vertex $v$ in $G_i$ (respectively, in $H_i$). In particular, $G_2$ and $H_2$ are formed by graph edges in $G$ and $H$, respectively. Then we let $l_i := |E(G_i)|$ and $m_i := |E(H_i)|$. Also, for brevity, let $m := \sum_{i=2}^{n} m_i$, $l := \sum_{i=2}^{n} l_i$, $\overline{m} = m - m_2$ and

$\bar{l} = l - l_2$. In other words, $\bar{l}$ is the number of *hyperedges* in $G$, and $\overline{m}$ is the number of hyperedges in $H$. Recall that by the choice of $G$,

$$l \leq n - 2. \tag{2}$$

For $n$-vertex hypergraphs $F_1$ and $F_2$, let $x(F_1, F_2)$ denote the number of bijections from $V(F_1)$ onto $V(F_2)$ that are not packings. Since we have chosen $G$ and $H$ that do not pack,

$$x(G, H) = n!. \tag{3}$$

A nice observation of Naroski is

**Lemma 3** ([4]).

$$x(G, H) \leq 2(n-2)!\, m_2 l_2 + 3!(n-3)!\, \overline{m}\bar{l}. \tag{4}$$

***Proof.*** For edges $e \in G$ and $f \in H$, let $X_{ef}$ be the set of bijections in $X$ that map the edge $e$ onto the edge $f$. Then

$$\begin{aligned} x(G,H) &= \left| \bigcup_{e \in E(G), f \in E(H)} X_{ef} \right| \leq \sum_{e,f} |X_{ef}| = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{e,f:|e|=|f|=i} |X_{ef}| \\ &= \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{e,f:|e|=|f|=i} i!(n-i)! = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} m_i l_i i!(n-i)! \leq 2(n-2)! m_2 l_2 \\ &+ 3!(n-3)! \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor} m_i l_i \leq 2(n-2)! m_2 l_2 + 3!(n-3)! \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor} m_i \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor} l_i \\ &= 2(n-2)! m_2 l_2 + 3!(n-3)!\, \overline{m}\bar{l}. \end{aligned}$$

■

**Lemma 4.** *The number $n$ of vertices in $G$ is at least* 8.

***Proof.*** If $n \leq 5$, then $\lfloor \frac{n}{2} \rfloor \leq 2$, and $G$ and $H$ are graphs, a contradiction to (1). Suppose now that $n = 7$. By (4), $x(G, H) \leq 2 \cdot 5! m_2 l_2 + (3!)(4!)\overline{m}\bar{l}$. By (1), $\overline{m} \geq 1$ and $\bar{l} \geq 1$. And the maximum of the expression $2 \cdot 5! m_2 l_2 + (3!)(4!)\overline{m}\bar{l}$ under the conditions that $m_2 + l_2 + \overline{m} + \bar{l} \leq 11$, $\overline{m} \geq 1$ and $\bar{l} \geq 1$ is attained at $l_2 = 4$, $m_2 = 5$, $\overline{m} = \bar{l} = 1$ and is equal to

$$2 \cdot 5! \cdot 4 \cdot 5 + (3!)(4!) = 4800 + 144 < 5040 = 7!,$$

a contradiction to (3).

Finally, suppose that $n = 6$. Similarly to the case for $n = 7$, $x(G, H) \leq 2 \cdot 4! m_2 l_2 + (3!)^2 \overline{m}\bar{l}$, $\overline{m} \geq 1$ and $\bar{l} \geq 1$. Since $2 \cdot 4! \geq (3!)^2$, for nonnegative integers $m_2$, $l_2$ and positive integers $\overline{m}$, $\bar{l}$, the maximum of the expression $2 \cdot 4! m_2 l_2 + (3!)^2 \overline{m}\bar{l}$ under the condition that $m_2 + l_2 + \overline{m} + \bar{l} \leq 9$ is exactly $6!$ and is attained only if $m_2 = l_2 = 0$, $\bar{l} = 4$, and $\overline{m} = 5$. So, $G$ and $H$ are 3-uniform hypergraphs with 4 and 5 edges, respectively.

Now we show that even in this extremal case $x(G, H) < 6!$. In the proof of Lemma 3, for every pair of edges $e \in G$ and $f \in H$, we considered the cardinality of the set of bijections $X_{ef}$ from $V(G)$ onto $V(H)$ that map the edge $e$ onto the edge $f$ and estimated $\Sigma := \sum_{e\in E(G)} \sum_{f\in E(H)} |X_{ef}|$. We will show that some bijection $F : V(G) \to V(H)$ maps at least two edges of $G$ onto two edges of $H$, thus this bijection is counted at least twice in $\Sigma$. For this, it is enough to (and we will) find edges $e_1, e_2 \in E(G)$ and $f_1, f_2 \in E(H)$ such that $|e_1 \cap e_2| = |f_1 \cap f_2|$, since in this case we can map $e_1$ onto $f_1$ and $e_2$ onto $f_2$.

If $G$ has two disjoint edges $e$ and $e'$, then any third edge of $G$ shares one vertex with one of $e$ and $e'$ and two vertices with the other. So, we may assume that any two edges in $G$ intersect. Similarly, we may assume that any two edges in $H$ intersect.

Now we show that

$H$ has a pair of edges with intersection size 1 and a pair of edges with intersection size 2. (5)

If the intersection of each two distinct edges in $H$ contains exactly one vertex, then each vertex belongs to at most two edges, which yields $|E(H)| \le 2 \cdot 6/3 = 4$, a contradiction to $\overline{m} = 5$. Finally, suppose that $|f_1 \cap f_2| = 2$ for all distinct $f_1, f_2 \in E(H)$. If two vertices in $H$, say $v_1$ and $v_2$, are in the intersection of at least three edges, then every other edge also must contain both $v_1$ and $v_2$. Since $n = 6$ and $\overline{m} = 5$, this is impossible. Hence, we may assume that each pair of vertices is the intersection of at most two edges. Given the edges $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$, every other edge must contain $v_3$, $v_4$, and one of $v_1$ or $v_2$. Hence, each edge of $H$ is contained in $\{v_1, v_2, v_3, v_4\}$. Thus, $H$ has at most 4 edges, a contradiction. This proves (5). Hence, the lemma holds. ■

**Lemma 5.** *$m_2 l_2 > \frac{(n-2)^2}{2}$, where $l_2$ (respectively, $m_2$) is the number of graph edges in $G$ (respectively, in $H$).*

***Proof.*** Suppose that $m_2 l_2 = C \le \frac{(n-2)^2}{2}$. It suffices to show that $x(G, H) < n!$. So, by Lemmas 3 and 4, it is enough to show that for $n \ge 8$ and any nonnegative integers $m_2, l_2$ and positive integers $\overline{m}, \overline{l}$ such that $m_2 + l_2 + \overline{m} + \overline{l} \le 2n - 3$, the expression $Y := 2(n-2)!\, m_2 l_2 + 3!(n-3)!\, \overline{m}\overline{l}$ is less than $n!$. Since $C \le \frac{(n-2)^2}{2}$, $m_2 + l_2 \ge 2\sqrt{C}$. Therefore, $\overline{m} + \overline{l} \le 2n - 3 - 2\sqrt{C}$ and so $\overline{m}\overline{l} \le (n - 1.5 - \sqrt{C})^2$. It follows that

$$\begin{aligned} Y &\le 2!\,(n-3)!((n-2)C + 3(n-1.5-\sqrt{C})^2) \\ &= 2!(n-3)!((n+1)C + 3(n-1.5)^2 - 6(n-1.5)\sqrt{C}). \end{aligned}$$

The second derivative w.r.t. $C$ of the last expression is positive, and so it is enough to check $C = 0$ and $C = \frac{(n-2)^2}{2}$. If $C = 0$, then $Y \le 2!\,(n-3)!3(n-1.5)^2$, which is less than $n!$ for $n \ge 8$. Similarly, if $C = \frac{(n-2)^2}{2}$ and $n \ge 8$, then

$$\begin{aligned} \frac{Y}{n!} &< \frac{2(n-2)!\frac{(n-2)^2}{2} + 3!(n-3)!\left(n - \frac{n-2}{\sqrt{2}}\right)^2}{n!} = \frac{(n-2)^3 + 6\left(n - \frac{n-2}{\sqrt{2}}\right)^2}{n(n-1)(n-2)} \\ &= \frac{n^3 - 6n^2 + 12n - 8 + 6n^2 - 6n(n-2)\sqrt{2} + 3(n-2)^2}{n(n-1)(n-2)} \\ &= \frac{n^3 - 6n(n-2)\sqrt{2} + 3n^2 + 4}{n(n-1)(n-2)} < 1, \end{aligned}$$

a contradiction to 3. ■

**Corollary 2.** *The number $m_2$ of graph edges in $G$ is greater than $n/2$.*

***Proof.*** Suppose that $m_2 \leq n/2$. By Lemma 5, $l_2 m_2 > \frac{(n-2)^2}{2}$. Therefore

$$l_2 > \frac{(n-2)^2}{2} \cdot \frac{2}{n} > n-4.$$

Also, by (2) and (1), $l_2 \leq n-3$. So, $l_2 = n-3$, and thus $l = n-2$ and $m \leq n-1$. Hence by Lemma 3, for $n \geq 8$

$$\begin{aligned} x(G,H) &\leq 2(n-2)!\, m_2(n-3) + 3!(n-3)!\, (m-m_2)\cdot 1 \\ &\leq 2\cdot(n-3)!((n-2)(n-3)m_2 + 3(n-1-m_2)) \end{aligned}$$

$$\leq 2\cdot(n-3)!((n-2)(n-3)\frac{n}{2} + 3(0.5n-1)) = (n-2)!((n-3)n+3) < n!,$$

a contradiction to 3. ∎

## 3. TWO MORE LEMMAS

We need some definitions.

**Definition.** *For a hypergraph $F$ without 1-edges and $A \subset V(F)$, the hypergraph $F - A$ has vertex set $V(F) - A$ and $E(F-A) := \{e - A \,:\, e \in E(F)$ and $|e - A| \geq 2\}$, where multiple edges are replaced with a single edge.*

An edge $e$ of $G$ *belongs to* a component $C$ of $G_2$ if strictly more than $|e|/2$ vertices of $e$ are in $V(C)$. By definition, each $e$ belongs to at most one component of $G_2$. A component $C$ of $G_2$ is *clean* if no hyperedge belongs to $C$. A *clean tree-component* of $G$ is a clean component of $G_2$, which is a tree. In particular, each single-vertex component of $G_2$ is a clean tree-component. By definition, for each component $C$ of $G_2$, at least $|V(C)| - 1$ graph edges belong to $C$. Moreover,

if exactly $|V(C)| - 1$ edges belong to $C$, then $C$ is a clean tree-component. (6)

Since $l_2 \leq n-3$, $G_2$ has at least three tree-components. Since $l \leq n-2$, by (6), at least two components of $G_2$ are clean tree-components. Since each nonclean component has at least two vertices,

the smallest clean tree-component of $G_2$ has at most $\max\{\frac{n}{3}, \frac{n-2}{2}\} = \frac{n-2}{2}$ vertices. (7)

**Lemma 6.** *Among the smallest clean tree-components of $G_2$, there exists a component $T$ such that $G - T$ does not have a universal vertex.*

***Proof.*** Let $T$ be the vertex set of a smallest clean tree-component of $G_2$ and let $|V(T)| = t$.

*Case 1:* $|E(G)| \leq n-3$. Since $G - T$ is an $n-t$ vertex hypergraph containing only $n-t-2$ edges, $G-T$ cannot have a universal vertex.

*Case 2:* $|E(G)| = n-2$. Assume that $G-T$ contains a universal vertex, say $w$. Since $G-T$ has at most $n-t-1$ edges, each edge in $G-T$ is a graph edge connecting $w$ with some other vertex. In particular, every hyperedge in $G$ has all but two of its vertices in $T$. Hence for each hyperedge $e$ in $G$, the edge $e - T$ connects an isolated vertex of $G_2$ to $w$. Since $G_2$ contains at least three components, we get that $G_2$ contains at least one isolated vertex. Then since any isolated vertex is a clean tree-component, $t = 1$.

Assume that $G_2$ contains $k$ isolated vertices $v_1, v_2, \ldots, v_k$. Each of these vertices then forms a smallest clean tree-component. If $G - v_i$ does not contain a universal vertex for some $i \le k$, we are done. Hence, we may assume that $G - v_i$ contains a universal vertex $w_i$ for each $i \le k$. It follows that every edge of $G$ has size at most 3 and contains $w_i$ for every $i$. In particular, $G_2$ has at most one nonsingleton component. Since $l_2 \le l - 1 \le n - 3$, $G_2$ has at least three components. Hence, $k \ge 2$. Furthermore, each of the $v_i's$ is contained in each 3-edge, hence $k \le 3$. If $k = 3$, then we have exactly one 3-edge $v_1v_2v_3$ in $G$. But then one the vertices of this edge is $w_i$ for some $i$ and hence is incident with $n - 3$ graph edges. Since $n \ge 8$, vertex of degree $n - 2$ is not isolated. So, $k = 2$.

Since $G$ contains a 3-edge, we have an edge $v_1v_2w$ where $w$ is necessarily the universal vertex in $G - v_1$ and in $G - v_2$. Thus, $v_1v_2w$ is the only 3-edge in $G$, and so $wu$ is an edge of $G_2$ for every $u \in V(G) - v_1 - v_2 - w$.

*Case 2.1:* $H_2$ contains an isolated vertex $y$. Since $m = n - 1$ and $n \ge 8$, there exist vertices $y_1$ and $y_2$ such that $\{y, y_1, y_2\}$ is not a 3-edge in $H$. Then, we may map $w$ to $y$, $v_1$ to $y_1$, and $v_2$ to $y_2$, and the rest of $V(G)$ arbitrarily to the rest of $V(H)$ to get a packing of $G$ and $H$, a contradiction to their choice.

*Case 2.2:* $H_2$ has no isolated vertices. Since $|E(H_2)| \le n - 2$, $H_2$ necessarily contains a vertex $y$ of degree 1. Suppose $yy_1 \in E(H_2)$. Since $H$ contains at most $n - 1 - n/2$ 3-edges, there exists some $y_2 \in V(H)$ which is not in a 3-edge with $y$ and $y_1$. Then we may pack $G$ and $H$ as in Case 2.1. ■

**Lemma 7.** *Let $t \le (n-2)/2$. Let $T$ be a $t$-vertex clean tree in $G_2$ and let $S \subset V(H)$ with $|S| = t$ be such that $S$ intersects at least $t + 1$ graph edges. If* $G[T]$ *and* $H[S]$ *pack, then either $G' := G - T$ or $H' := H - S$ has a universal vertex.*

***Proof.*** Assume that the lemma does not hold. Since the (graph) edges of $T$ and the graph edges in $H$ incident with $S$ do not correspond to any edge in $G'$ and $H'$, we have

$$|E(G')| + |E(H')| \le |E(G)| + |E(H)| - (t-1) - (t+1) \le 2(n-t) - 3. \quad (8)$$

We claim that if $G'$ and $H'$ pack, then so do $G$ and $H$. Indeed suppose that $\sigma'$ is a packing of $G'$ onto $H'$ and $\sigma''$ is a packing of $G[T]$ onto $H[S]$. We will check that $\sigma' \cup \sigma''$ is a packing of $G$ onto $H$. Suppose the contrary: that an edge $A$ of $G$ is mapped onto edge $B$ of $H$. If $A \subset T$, this is impossible, since $\sigma''$ is a packing of $G[T]$ onto $H[S]$. So, suppose $A' := A \cap V(G') \ne \emptyset$ and $B' := B \cap V(H') \ne \emptyset$. Since $T$ is a clean component of $G_2$, $|A'| \ge 2$. So, $|B'|$ is also at least 2. Then, by the definition of $G - T$ and $H - S$, $A'$ is an edge of $G'$ and $B'$ is an edge of $H'$. Hence $\sigma'$ does not send $A'$ to $B'$, a contradiction to the choice of $A$ and $B$. Thus, since $G$ and $H$ do not pack, neither do $G'$ and $H'$. So by (8) and the minimality of $n$, either $(G', H')$ is a bad pair or the lemma holds. Hence we may assume that $(G', H')$ is a bad pair.

Let $k = n - t$. Note that for each bad pair $(G(i), H(i))$ in Figure 1, the total number of edges in $G(i)$ and $H(i)$ is $2|V(G(i))| - 3 = 2|V(H(i))| - 3$. Hence, $|E(H)| - |E(H - S)| = t + 1$ and $S$ covers exactly $t + 1$ graph edges. Then

$$|E(G(i))| + |E(H(i))| = 2k - 3 \quad \text{and} \quad |V(G)| = |V(H)| \le 2k - 2. \quad (9)$$

By the definition of bad pairs, either all edges in $G'$ and $H'$ are graph edges or all of them are $(k-2)$-edges. In the latter case, $H$ has only $t+1 \le n/2$ graph edges, a contradiction to Corollary 2. Thus, we may assume that $\{G', H'\} = \{G(i), H(i)\}$ in Figure 1 for some $i \in \{1, \dots, 8\}$.

*Case 1:* $\bar{l} + \overline{m} \ge 2k-3$. Then $l_2 + m_2 \le (2n-3) - (2k-3) = 2n-2k$, and hence $l_2 m_2 \le (n-k)^2$. Since $4 \le k \le 9$ and $k \ge (n+2)/2$, we get

$$l_2 m_2 \le (n-k)^2 \le \left(\frac{n-2}{2}\right)^2 < \frac{(n-2)^2}{2},$$

a contradiction to Lemma 5.

Since we proved that $\bar{l} + \overline{m} < 2k-3$ at least one edge of $G'$ or $H'$ is a graph edge in $G$ or $H$. Furthermore, since $T$ was a clean component, all the hyperedges of $G$ become graph edges of $G'$. Let $e_G$ be some such edge of $G'$. If none of the edges of $H'$ was obtained from a hyperedge of $H$, then it is enough to pack $G' - e_G$ with $H'$, which is possible by Theorem 3. So, there are $e \in E(G')$ and $f \in E(H')$ such that one of them is a graph edge and the other is a hyperedge in $(G, H)$.

*Case 2:* $(G', H')$ is one of the unordered pairs $\{G(1), H(1)\}$, $\{G(3), H(3)\}$, $\{G(4), H(4)\}$, $\{G(7), H(7)\}$. By symmetry, we may assume that $e = x_1x_2$ and $f = y_1y_2$. In all cases, we define mapping $\phi(x_j) = y_j$ for $j = 1, \dots, k$. This mapping together with the packing of $G[T]$ with $H[S]$ yields a packing of $G$ with $H$, a contradiction.

*Case 3:* $(G', H')$ is one of the unordered pairs $\{G(2), H(2)\}$, $\{G(5), H(5)\}$, $\{G(6), H(6)\}$. By symmetry, we may assume that $e = x_1x_2$ and either $f = y_1y_2$ or $f = y_{k-1}y_k$. If $f = y_1y_2$, then we let $\phi(x_j) = y_j$ for $j = 1, \dots, k$, and if $f = y_{k-1}y_k$, then we let $\phi(x_j) = y_{k+1-j}$ for $j = 1, \dots, k$. ■

**Remark.** Practically, the same proof will verify the lemma with the roles of $G$ and $H$ switched, that is, with $T$ being be a $t$-vertex clean tree in $H_2$ and $S$ being a subset of $V(G)$ with $|S| = t$ such that $S$ intersects at least $t+1$ graph edges in $G$. The only difference is that if all edges of $G'$ and $H'$ are $(k-2)$-edges, then $H$ has only $t-1 \le n/2$ graph edges (those that are the graph edges of $T$), and we get the same contradiction to Corollary 2.

## 4. PROOF OF THEOREM 5

By Lemma 6, there is a smallest clean tree-component $T$ of $G_2$ such that

$$G - T \text{ does not contain a universal vertex.} \tag{10}$$

We let $t = |V(T)|$.

**Case 1:** $t = 1$. Let $V(T) = \{u\}$. By Corollary 2, $\Delta(H_2) \ge 2$. Let $w \in V(H)$ with $d_2(w, H) = \Delta(H_2)$. Let $G' = G - u$ and let $H' = H - w$. By Lemma 7 and (10), $H'$ contains a universal vertex, say $w'$.

Let $y = \Delta(H_2)$. Since $H$ contains at least $n-2$ edges forming the star in $H'$ plus $y$ graph edges incident to $w$, we get that $l + (n-2) + y \le l + m \le 2n-3$. Since $l_2 \le l-1$, we get $l_2 + y \le n-2$. By Lemma 5, $m_2 > \frac{(n-2)^2}{2l_2}$. Also, $w'$ is contained in at least $n-2-y$ 3-edges, hence

$$(l_2 + 1) + \frac{(n-2)^2}{2l_2} + (n-2-y) < l + m \le 2n-3,$$

which gives that $l_2 - y + \frac{(n-2)^2}{2l_2} < n-2$. Adding these expressions gives

$$(l_2 + y) + \big(l_2 - y + \frac{(n-2)^2}{2l_2}\big) < 2(n-2)$$

or $l_2 + \frac{(n-2)^2}{4l_2} < n-2$. This can be rewritten as $(2l_2 - (n-2))^2 < 0$ which is false. This contradiction finishes Case 1, so below we assume that $t > 1$.

**Case 2:** $t = 2$. Let $V(T) = \{v_1, v_2\}$. If $H$ contains a vertex $w$ with $d_2(w, H) > n/2$, let $w'$ be a nonneighbor of $w$ in $H_2$. Then $G' = G - v_1 - v_2$, and $H' = H - w - w'$ are $(n-2)$-vertex graphs with $|E(G')| + |E(H')| < \frac{3(n-2)-2}{2}$, so $G'$ and $H'$ pack by the minimality of $n$ (we simply apply Corollary 1). Mapping $v_1$ to $w$ and $v_2$ to $w'$ will complete the packing of $G$ with $H$. So, $\Delta(H_2) \leq n/2$.

*Case 2.1:* $\Delta(H_2) \geq 3$. Given nonadjacent vertices $w_1$ and $w_2$ in $H_2$ with $d_2(w_1, H) = \Delta(H_2)$, we let $G' = G - v_1 - v_2$ and $H' = H - w_1 - w_2$. By Lemma 7 and (10), $H'$ contains a universal vertex.

Let $y = \Delta(H_2) \leq n/2$. Then $l + (n-3) + y \leq l + m \leq 2n - 3$. Since $H'$ contains a universal vertex, $m - m_2 \geq n - 3 - y$, so $l + m_2 + (n-3-y) \leq l + m \leq 2n-3$. Adding these gives $2(2n-3) \geq 2l + m_2 + 2(n-3)$, or

$$2n \geq 2l + m_2. \tag{11}$$

By Lemma 5, $l_2 > \frac{(n-2)^2}{2m_2}$. So if $l - l_2 \geq 2$ or $m - m_2 \geq n - 1 - y$, then $2n > 4 + m_2 + \frac{(n-2)^2}{m_2}$. And since $m_2 + \frac{n-2)^2}{m_2} \geq 2(n-2)$, we get $2n > 2n$, a contradiction. Hence we may assume that $l - l_2 = 1$ and that $m - m_2 \leq n - 2 - y$. Furthermore, if $l_2 m_2 \leq \frac{(n-1)^2}{2}$, Lemma 3 gives

$$\begin{aligned} x(G,H) &\leq 2(n-2)!\,\frac{(n-1)^2}{2} + 3!(n-3)!\,1(n-2-y) \\ &\leq 2(n-2)!\,\frac{(n-1)^2}{2} + 3!(n-3)!\,1(n-5) \\ &= (n-1)!\left[(n-1) + \frac{6(n-5)}{(n-1)(n-2)}\right] \\ &< n! \text{ (since } n \geq 8), \end{aligned}$$

a contradiction to (3). Thus $l_2 m_2 > \frac{(n-1)^2}{2}$ which gives $l = 1 + l_2 > 1 + \frac{(n-1)^2}{2m_2}$. Applying this to (11), we obtain $2n > 2 + m_2 + \frac{(n-1)^2}{m_2} \geq 2 + 2(n-1) = 2n$, a contradiction.

*Case 2.2:* $\Delta(H_2) \leq 2$. By Corollary 2, $\Delta(H_2) \geq 2$. Thus $\Delta(H_2) = 2$. Let $w_1$ be a vertex with $d_2(w_1, H) = 2$. If there exists some $w_2$ in $H$ with $w_1 w_2 \notin E(H)$ and $d_2(w_2, H) \geq 1$, then we proceed as in Case 2.1. Hence we may assume that every vertex in $H_2$ that is not adjacent to $w_1$ is an isolated vertex. We then have that $m_2 \leq 3$, and $m_2 l_2 \leq 3(n-3)$. Lemma 5 then gives that $3(n-3) > (n-2)^2/2$ or $(n-5)^2 < 3$, a contradiction to $n \geq 8$.

**Case 3:** $t \geq 3$ and $H_2$ has an isolated vertex $w$. Let $y$ be a leaf of $T$ and let $x$ be the neighbor of $y$ in $G_2$. Let $G' = G - x$ and let $H' = H - w$. Since $t \geq 3$, $d_2(x, G) \geq 2$ and hence $|E(G')| \leq n - 4$. Therefore, $|E(G')| + |E(H')| \leq 2(n-1) - 3$, and $G'$ does not have a universal vertex. Thus by the remark to Lemma 7, $H'$ has a universal vertex, say

$w'$. Let $G'' = G' - y$ and let $H'' = H' - w'$. Since $w'$ was universal in $H'$,

$$|E(G'')| + |E(H'')| = |E(G')| + |E(H')| - (n-2) \leq 2(n-1) - 3 - (n-2)$$
$$= n - 3 < \frac{3(n-2)-2}{2}.$$

So by the minimality of $n$ and Corollary 1, $G''$ and $H''$ pack. We may then extend the packing of $G''$ and $H''$ to a packing of $G$ and $H$ by mapping $x$ to $w$ and $y$ to $w'$. This finishes Case 3.

If $n_1$ vertices of $G$ are in clean tree-components, then $l \geq \frac{n_1(t-1)}{t} + (n - n_1)$. Moreover, if $n = n_1$, then (since $G$ has a hyperedge) $l \geq 1 + \frac{n_1(t-1)}{t} \geq 2 + \frac{(n-2)(t-1)}{t}$. Since $n - n_1 \neq 1$, we conclude that $l \geq n - \lfloor \frac{n-2}{t} \rfloor$. So

$$m \leq 2n - 3 - l \leq n - 3 + \left\lfloor \frac{n-2}{t} \right\rfloor. \tag{12}$$

We consider two cases depending on the maximum degree of $H_2$.

**Case 4:** $t \geq 3$ and $\Delta(H_2) \geq \lfloor \frac{n-2}{t} \rfloor$. Let $w_1$ be a vertex of maximum degree in $H_2$. Let $v_1$ be a leaf in $T$ and choose $v_2, v_3, \ldots, v_t$ in $T$ so that for each $i$ with $2 \leq i \leq t$, the set $\{v_1, v_2, \ldots, v_i\}$ induce a tree in $G_2$ with $v_i$ as a leaf with neighbor $v_{(i-1)'}$. We map $v_1$ to $w_1$ and proceed by induction to pack $V(T)$ into $V(H)$ so that for every $i = 1, \ldots, t$, the image, $W_i$, of $\{v_1, v_2, \ldots, v_i\}$ is incident to at least $\lfloor \frac{n-2}{t} \rfloor + i - 1$ graph edges. Assume that $v_1, v_2, \ldots, v_i$ have been mapped in this way to $w_1, w_2, \ldots, w_i$, so that $W_i = \{w_1, w_2, \ldots, w_i\}$. In particular, $W_i$ is incident to at least $\lfloor \frac{n-2}{t} \rfloor + i - 1$ graph edges in $H$.

*Case 4.1:* $W_i$ is incident to at least $\lfloor \frac{n-2}{t} \rfloor + i$ graph edges. It suffices to map $v_{i+1}$ to a vertex $w_{i+1}$ in $V(H)$ such that for each $j \leq i$, $w_j \neq w_{i+1}$ and $w_j w_{i+1}$ is not an edge. Since $v_{i+1}$ is adjacent only to $v_{i'}$ in $\{v_1, v_2, \ldots, v_i\}$, if $i + d_2(w_{i'}, H - W_i) < n$, then we can choose as $w_{i+1}$ any vertex in $V(H) - W_i$ not adjacent to $w_{i'}$ in $H_2$. Hence we may assume that $d_2(w_{i'}, H - W_i) \geq n - i$. Since $G_2$ contains no isolated vertices, by the choice of $G$ and $H$, $\Delta(H_2) \leq n - 2$, so $i \neq 1$. Since $v_1$ is a leaf in $T$ and $i \geq 2$, $i' \neq 1$. So, by the choice of $w_1$,

$$m_2 \geq d_2(w_{i'}, H - W_i) + d_2(w_1, H - w_{i'}) \geq 2d_2(w_{i'}, H - W_i) \geq 2(n - i).$$

Also, $i \leq t - 1$. Hence $m \geq 1 + m_2 \geq 1 + 2(n - i) \geq 2n - 2t + 3$. So, by 12, $2n - 2t + 3 \leq n - 3 + \frac{n-2}{t}$. This gives $0 \leq 2t^2 - (n+6)t + (n-2)$, but for $2 \leq t \leq \frac{n-2}{2}$, this expression is at most $-6$.

*Case 4.2:* $W_i$ is incident to exactly $\lfloor \frac{n-2}{t} \rfloor + i - 1$ graph edges. If there exists some $w_{i+1} \in V(H) - W_i$ not adjacent to $W_i$ in $H_2$, then we can map $v_{i+1}$ onto this $w_{i+1}$. Hence, we may assume that $i + \lfloor \frac{n-2}{t} \rfloor + i - 1 \geq n$. This yields $0 \leq 2t^2 - (n+3)t + (n-2)$, but for $2 \leq t \leq \frac{n-2}{2}$, this expression is at most $-3$.

So, we can pack $T$ into $H$ in such a way that at least $\lfloor \frac{n-2}{t} \rfloor + t - 1$ graph edges of $H$ are covered. Let $G' = G - v_1 - v_2 - \cdots - v_t$ and $H' = H - w_1 - w_2 - \cdots - w_t$. Since by (7), $\lfloor \frac{n-2}{t} \rfloor \geq 2$, Lemma 7 and (10) yield that $H'$ has a universal vertex. But

$$|E(H')| \leq n - 3 + \left\lfloor \frac{n-2}{t} \right\rfloor - \left\lfloor \frac{n-2}{t} \right\rfloor - t + 1 = n - t - 2,$$

a contradiction.

**Case 5:** $t \geq 3$ and $\Delta(H_2) \leq \lfloor \frac{n-2}{t} \rfloor - 1$. By Corollary 2, $\Delta(H_2) \geq 2$. Hence $2 \leq \lfloor \frac{n-2}{t} \rfloor - 1$, which yields $t \leq (n-2)/3$. Define $v_1, v_2, \ldots, v_t$ as in Case 4. We map $v_1$ to a vertex $w_1$ of maximum degree in $H_2$. Since $\Delta(H_2) \geq 2$, we may proceed as in Case 4, to get a packing of $T$ into $H$, which covers at least $\Delta(H_2) + t - 1 \geq t + 1$ graph edges in $H$. Again by Lemma 7 and (10), $H'$ has a universal vertex, say $z$. Then $z$ is contained in at least $n - t - 1 - \Delta(H_2)$ hyperedges in $H$. Hence, $m - m_2 \geq n - t - \lfloor \frac{n-2}{t} \rfloor \geq n - t - \frac{n-2}{t}$. We also have that $m - m_2 \leq 2n - 3 - (l_2 + m_2) - (l - l_2)$. These inequalities together give

$$(l_2 + m_2) + (l - l_2) \leq n - 3 + t + \frac{n-2}{t}. \tag{13}$$

By Lemma 5, $l_2 + m_2 > \sqrt{2}(n-2)$.

We consider two cases.

*Case 5.1:* $l - l_2 \geq 2$. Then by (13) and Lemma 5 we have $\sqrt{2}(n-2) + 2 < n - 3 + t + \frac{n-2}{t}$. As $n - 3 + t + \frac{n-2}{t}$ achieves its maximum for extremal values of $t$, we need only to check the inequality for $t = 3$ and $t = \frac{n-2}{3}$. For $t = 3$ we get $\sqrt{2}(n-2) < (4/3)(n-2)$ and for $t = \frac{n-2}{3}$ we get $\sqrt{2} < 4/3$; both inequalities are false.

*Case 5.2:* $l - l_2 = 1$. By (13), we have $l_2 + m_2 \leq n - 2 + t + \frac{n-2}{t}$. For fixed $n$, the expression $n - 2 + t + \frac{n-2}{t}$ achieves its maximum at extremal values of $t$. So, we check $t = 3$ and $t = \frac{n-2}{3}$. In either case,

$$l_2 + m_2 \leq \frac{4(n-2)}{3} + 1. \tag{14}$$

Since $l - l_2 = 1$ and $l + m \leq 2n - 3$, by Lemma 3, the number $x(G, H)$ of "bad" bijections from $V(G)$ onto $V(H)$ satisfies

$$\begin{aligned} x(G,H) &\leq m_2 l_2 2(n-2)! + 3!(n-3)!(m - m_2) \\ &\leq m_2 l_2 2(n-2)! + 3!(n-3)!(2n - 3 - l_2 - 1 - m_2). \end{aligned}$$

So, denoting $y := (l_2 + m_2)/2$, we have

$$x(G,H) \leq h(y) := y^2 2 \cdot (n-2)! + 3!(n-3)!(2n - 4 - 2y).$$

Since $y \geq m_2/2 > n/4 \geq 2$, we have $h'(y) = 4 \cdot (n-2)! y - 3!(n-3)! 2 = 4 \cdot (n-3)!((n-2)y - 3) > 0$. Thus by (14),

$$\begin{aligned} \frac{x(G,H)}{n!} &\leq \frac{h(2(n-2)/3 + 1/2)}{n!} = \frac{|X|}{n!} \\ &\leq \frac{1}{n!}\left[2(n-2)!\left(\frac{2}{3}(n-2) + \frac{1}{2}\right)^2 + 3!(n-3)!\frac{2n-7}{3}\right] \\ &= \frac{16n^3 - 72n^2 + 177n - 302}{18n(n-1)(n-2)}. \end{aligned}$$

As this is less than 1 for $n \geq 8$, $x(G, H) < n!$, a contradiction to (3).

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