Equitable List Coloring of Graphs with Bounded Degree

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Abstract: A graph G is equitably k-choosable if for every k-list assignment L there exists an L-coloring of G such that every color class has at most $\lceil |G|/k \rceil$ vertices. We prove results toward the conjecture that every graph with maximum degree at most r is equitably $(r+1)$-choosable. In particular, we confirm the conjecture for $r \leq 7$ and show that every graph with maximum degree at most r and at least r^3 vertices is equitably

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$(r+2)$-choosable. Our proofs yield polynomial algorithms for corresponding equitable list colorings. J. Graph Theory 74: 309–334, 2013

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1. INTRODUCTION

In several applications of graph coloring such as the mutual exclusion scheduling problem, scheduling in communication systems, construction timetables, and round-a-clock scheduling (see [1, 11, 12]), there is an additional requirement that color classes be not so large or be of approximately the same size. A model imposing such a requirement is *equitable coloring*—a proper coloring such that color classes differ in size by at most one. Perhaps these applications make even more sense in the model of *equitable list coloring.*

First we formalize these notions. Let G be a graph. An *equitable k-coloring* of G is a proper coloring such that any two color classes differ in size by at most one. In particular, each color class in an equitable k-coloring of G has size at most $\lceil |G|/k \rceil$. Hajnal and Szemerédi [4] answered a question of Erdős [2] by proving that if the maximum degree, $\Delta(G)$, of a graph G is strictly less than k, then G has an equitable k-coloring. In this article, we investigate list-coloring versions of this theorem.

A *list-assignment* for G is a function L that assigns a set (list) $L(v)$ of colors to each $v \in V(G)$. For a list-assignment L, an *L-coloring* f of G is a proper coloring of G such that $f(v) \in L(v)$ for each $v \in V(G)$. An L-coloring f is defined to be *t-equitable* if the size of every color class is at most $\lceil |G|/t \rceil$. A *k-list-assignment* for G is a list-assignment L such that $|L(v)| = k$ for each $v \in V(G)$. The graph G is *equitably k-choosable* if it has a k-equitable L-coloring for every k-list assignment L. Notice that in the list setting, it is unrealistic to require that any two color classes differ in size by at most one, since the list of some vertex might only contain colors that are in no other lists, while all other lists could be identical. In [8] the following theorem is proved:

Theorem 1 (Kostochka, Pelsmajer, and West [8]). *If G is a graph and $k \geq \max\{\Delta(G), |G|/2\}$, then G is equitably k-choosable unless G contains K_{k+1} or is $K_{k,k}$ (with k odd in the latter case). In particular, if $k \geq \max\{\Delta(G)+1, |G|/2\}$ then G is equitably k-choosable.*

Since we will use the last sentence of Theorem 1, we give its short proof: Fix a k-list assignment L for G. By the Hajnal–Szemerédi Theorem, G has an equitable k-coloring p with classes of size at most 2. Let H be the complete k-partite graph whose parts are the color classes of p. Erdős, Rubin, and Taylor [3] proved that every complete k-partite graph whose partite sets have size at most 2 is k-list-colorable. So, H has an L-coloring f, and f must be a k-equitable coloring of G, since the classes of f are refinements of the classes of p.

The purpose of this article is to attack the following conjecture:

Conjecture 2 (Kostochka, Pelsmajer, and West [8]). *Every graph G with $\Delta(G) \leq r$ is equitably $(r+1)$-choosable.*

Some progress has already been made. Pelsmajer [10] and independently Wang and Lih [13] proved the following special cases:

Theorem 3. *Let G be a graph with $\Delta(G) \le r$.*

(a) *([10] and [13]) If $r \le 3$ then G is equitably $(r+1)$-choosable.*
(b) *([10]) G is equitably $(2+\binom{r}{2})$-choosable and ([13]) equitably $(r-1)^2$-choosable.*

In this article, we strengthen these results by proving:

Theorem 4. *Let G be a graph with $\Delta(G) \le r$.*

(a) *If $r \le 7$ and $k \ge r+1$ then G is equitably k-choosable.*
(b) *G is equitably k-choosable, if*

$$k \ge r + \begin{cases} 1 + \dfrac{r-1}{7} & \text{if } r \le 30 \\ \dfrac{r}{6} & \text{if } r \ge 31 \end{cases}$$

(c) *If $|G| \ge r^3$ and $k \ge r+2$, then G is equitably k-choosable.*
(d) *If $\omega(G) \le r$ and $|G| \ge 3(r+1)r^8$, then G is equitably $(r+1)$-choosable.*

If we could remove the restriction on the size of $|G|$ in (d) then Conjecture 2 would follow (see Proposition 41). The proof of Theorem 4 is based on our short proof (see [6]) of the Hajnal–Szemerédi Theorem. However, the list setting introduces new complications—not just that some colors may not be available for a vertex, but also that there may be classes of various sizes. These difficulties require generalizations of our previous techniques.

We shall prove each part of Theorem 4 using the same general set-up. This set-up will be developed in the next three sections. In Section 5, we prove Statements (a), (b), and (c), and in Section 6 we prove (d).

The original proof of the Hajnal–Szemerédi Theorem did not provide a polynomial time algorithm for the coloring. Recently, Mydlarz and Szemerédi [9] and independently the authors [5] provided such algorithms. Ideas of both teams were combined in [7]:

Theorem 5 ([7])**.** *There exists an $O(rn^2)$ time algorithm that finds an $(r+1)$-equitable coloring for each n-vertex graph with maximum degree at most r.*

The proofs of Theorem 3 in [10] yield a polynomial time algorithm that for $r \le 3$, a graph G with $\Delta(G) \le r$ and an $(r+1)$-list L for G produces an equitable L-coloring. In Section 7 (Theorem 40), we provide polynomial time algorithms for each of the colorings whose existence is asserted by Theorem 4(a), (b), and (c). In Section 8, we use Theorem 4(d) to prove the following theorem.

Theorem 6. *There is a deterministic algorithm that decides for each positive integer r which of the following possibilities is true:*

(a) *Conjecture 2 holds for r, i.e., every graph G with $\Delta(G) \le r$ is $(r+1)$-choosable.*
(b) *Only finitely many graphs G with $\Delta(G) \le r$ are not $(r+1)$-choosable.*
(c) *Infinitely many graphs G with $\Delta(G) \le r$ are not $(r+1)$-choosable.*

We also use the techniques for the proof of Theorem 6 to provide (Theorem 44), for each fixed integer r, a polynomial time algorithm that for any graph G with $\Delta(G) \le r$ and any $(r+1)$-list assignment L either L-colors G or decides that this is impossible.

A. Notation

Our notation is mostly standard, but some possible exceptions include the following. Let $G = (V, E)$ be a graph with $x, y \in V$ and $A, B \subseteq V$. Set $|G| := |V|$ and $\|G\| := |E|$. The subgraph of G induced by A is denoted by $G[A]$. Let $N(x) := N_G(x)$ denote the neighborhood of x, and set $N_A(x) := N(x) \cap A$, when G is clear from the context. Similarly, $d(x) := |N(x)|$ and $d_A(x) := |N_A(x)|$. For digraphs, $N^-(x)$ denotes the in-neighborhood of x and $d^-(x) := |N^-(x)|$ denotes the in-degree of x. Also $E(A, B) := \{xy \in E : (x, y) \in A \times B\}$. If x has a neighbor in A, we say that x is adjacent to A and write $x \sim A$; otherwise x is nonadjacent to A, and we write $x \nsim A$. An x, y-path is a path that begins with x and ends with y. An A, B-path is an x, y-path with $x \in A$ and $y \in B$ that has no internal vertices in $A \cup B$.

A coloring of G is a function $f : V \to \mathcal{C}$, where $\mathcal{C}$ is a set of colors. We identify f with the family of *labeled* classes $\{V_\gamma : \gamma \in \mathcal{C}\}$, where each $V_\gamma := V_\gamma(f) := \{v \in V : f(v) = \gamma\}$ is a (color) class of f. For a list assignment L, we may somewhat abuse notation by treating V_γ and γ interchangeably, for example, by writing $W \in L(v)$ when we mean $W = V_\gamma$ and $\gamma \in L(v)$.

We write $A - x$ for $A \setminus \{x\}$ and $A + x$ for $A \cup \{x\}$. As usual for graphs, $G - x := G[V - x]$. For a positive integer n, the set $\{1, \ldots, n\}$ is denoted by $[n]$.

2. SET-UP

We argue by contradiction. Let $r \geq 3$ and $k \geq r + 1$ be nonnegative integers, $g = k - r - 1$, and suppose there exists a graph $G := (V, E)$ with $\Delta(G) \leq r$ that is not equitably k-choosable. (Conjecture 2 asserts that this is impossible even when $g = 0$; the parameter g is the gap between the conjecture and what we are trying to prove). Choose such a G that is edge-minimal, and let L be a k-list assignment that witnesses that G is not equitably k-choosable. Let $\mathcal{C} := \bigcup_{x \in V} L(x)$ be the set of colors that appear in the lists of L. Let s be the integer such that

$$k(s - 1) < |G| \leq ks. \tag{1}$$

Thus every color class should have size at most s (but since G is a minimal counter example, one will not). By Theorem 1, we may assume $s \geq 3$.

By the minimality of G, for each $xy \in E(G)$, the graph $G - xy$ has an equitable L-coloring with x and y in the same color class. Since $|L(y)| > d(y)$, when we add back the edge xy, we can move y to some other color class $W \in L(y)$ to obtain a new coloring with exactly one color class V^+ having size $s + 1$. Such a coloring f is said to be *nearly equitable*. An h-class is a class with exactly h members. Similarly, an h^--class is a *nonempty* class with *less than* h members; an h^*-class is a class with *at least* h members.

Construct the auxiliary digraph $\mathcal{H} := \mathcal{H}(f)$ as follows. The vertices of $\mathcal{H}$ are the classes of f. A directed edge $V'V''$ belongs to $E(\mathcal{H})$ if there exists a vertex $x \in V'$ such that $V'' \in L(x)$ and $x \nsim V''$. In this case x is called a *witness* for the edge $V'V''$, and notice that if x is a witness for $V'V''$ then we can obtain a new proper L-coloring by moving x from V' to V''.

Let $\mathcal{A}_0 := \mathcal{A}_0(f)$ denote the family of s^--classes. Call a class U *accessible*, if there exists a $U, \mathcal{A}_0$-path $\mathcal{P}$ in the digraph $\mathcal{H}$. We say that the path $\mathcal{P}$ *witnesses* that U is accessible. For every class $U \in \mathcal{A}_0$, the one-vertex path U witnesses that U is accessible.

Let $\mathcal{A} := \mathcal{A}(f)$ denote the family of accessible classes, $A_0 := \bigcup \mathcal{A}_0$ and $A := \bigcup \mathcal{A}$. Denote the number of accessible s-classes by $c := |\mathcal{A} \setminus \mathcal{A}_0|$.

Let $\mathcal{F}$ be a spanning subgraph of $\mathcal{H}[\mathcal{A}]$ with no directed cycles such that every vertex has out-degree at most one in $\mathcal{F}$. We call $\mathcal{F}$ a *directed forest*, and the vertices with out-degree zero are called *roots*. Furthermore, choose $\mathcal{F}$ so that its roots are exactly the classes of $\mathcal{A}_0$. Then for every class $U \in \mathcal{A}$, there exists a $U, \mathcal{A}_0$-path in $\mathcal{F}$. For $Z \in \mathcal{A} \setminus \mathcal{A}_0$ let $p(Z)$ be the out-neighbor of Z in $\mathcal{F}$ and p_Z be a fixed witness of the edge $Zp(Z)$. We say that $p(Z)$ is the *parent* of Z and Z is a *child* of $p(Z)$. Let $N^-_{\mathcal{F}}(W) = \{Z \in \mathcal{A} : p(Z) = W\}$ and $d^-_{\mathcal{F}}(W) = |N^-_{\mathcal{F}}(W)|$.

The following lemma shows the first of three ways that we can make progress toward an equitable coloring. However, since we are arguing by contradiction, it is phrased negatively.

Lemma 7. *The large class V^+ is not accessible.*

Proof. Suppose for a contradiction that $\mathcal{P} = V^+U_1 \dots U_t$ is a path in $\mathcal{H}$ that witnesses that V^+ is accessible. Then moving the witness p_{V^+} to U_1 and each witness p_{U_i} to U_{i+1} results in an equitable L-coloring, contradicting the choice of G. ■

Set $B := V(G) - A$ and let $\mathcal{B}$ be the set classes contained in B. Note that all classes in $\mathcal{B} - V^+$ are s-classes, and by Lemma 7 the $(s+1)$-class V^+ is also in $\mathcal{B}$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the family of classes U such that there exists a V^+, U-path in $\mathcal{H}$. The one-vertex path V^+ witnesses that V^+ is in $\mathcal{B}'$. Let $B' = \bigcup \mathcal{B}'$. Set $b := |\mathcal{B}|$ and $b' := |\mathcal{B}'|$. Then $|B| = sb + 1$, $|B'| = sb' + 1$, and $|A| \leq (k-b)s - 1$. It is convenient to set $\widetilde{a} := k - b$. Then $|A| \leq \widetilde{a}s - 1$ and $|L(v) \cap \mathcal{A}| \geq \widetilde{a}$ for all vertices v. However, $|\mathcal{A}|$ might be bigger than $\widetilde{a}$. It follows that:

$$\forall v \in B'\ \forall W \in L(v) \setminus \mathcal{B}',\quad v \sim W,\quad d_{V \setminus B'}(v) \geq k - b' \quad \text{and} \quad d_{B'}(v) \leq b' - 1 - g. \quad (2)$$

$$\forall v \in B\ \ \forall W \in L(v) \cap \mathcal{A},\quad v \sim W, d_A(v) \geq k - b = \widetilde{a} \quad \text{and} \quad d_B(v) \leq b - 1 - g. \quad (3)$$

The following proposition shows the utility of B'. In the proofs of nonlist versions [4, 6], one could assume inductively that $G[B] - y$ had an equitable b-coloring for any $y \in B$. In the list setting, we only get this for $y \in B'$. For a set of classes $\mathcal{D} \subseteq \mathcal{C}$, let $L_{\mathcal{D}}$ be the restriction of L to $\mathcal{D}$, i.e., $L_{\mathcal{D}}(v) = L(v) \cap \mathcal{D}$.

Proposition 8. *For every $y \in B'$, graph $G[B'] - y$ has a b'-equitable $L_{\mathcal{B}'}$-coloring and graph $G[B] - y$ has a b-equitable $L_{\mathcal{B}}$-coloring.*

Proof. Obtain an equitable $L_{\mathcal{B}'}$-coloring of $G[B'] - y$ by switching witnesses along a $V^+, V_{f(y)}$-path in $\mathcal{H}[\mathcal{B}]$. ■

For a class $W \in \mathcal{A}$, let $\mathcal{M}(W)$ be the set of classes $X \in \mathcal{A} - W$ such that there is no $X, \mathcal{A}_0$-path in $\mathcal{H} - W$. Call W *terminal* if $\mathcal{M}(W) = \emptyset$. Let $\mathcal{A}'$ be the set of terminal classes and $A' := \bigcup \mathcal{A}'$. Classes in $\mathcal{A} \setminus \mathcal{A}'$ are called *nonterminal*. Let $t := |\mathcal{A} \setminus \mathcal{A}'|$ be the number of nonterminal classes, and $q := |\mathcal{A}_0 \setminus \mathcal{A}'|$ be the number of nonterminal s^--classes.

An edge zy is *solo* if $Z := V_{f(z)} \in L(y) \cap \mathcal{A}$, $y \in B$, and $N_Z(y) = \{z\}$, i.e., z is the only neighbor of y in Z. Ends of solo edges are called *solo* vertices, and are called *solo neighbors* of each other. Let S_z denote the set of solo neighbors of $z \in A$ and S^y denote the set of solo neighbors of $y \in B$. For $y \in B$, set

$$\mathcal{S}^y := \{X \in L(y) \cap \mathcal{A} : d_X(y) = 1\} \quad \text{and} \quad \mathcal{T}^y = \{X \in \mathcal{A} : d_X(y) \geq 1\} \setminus \mathcal{S}^y.$$

So $\mathcal{S}^y = \{V_{f(z)} : z \in S^y\}$. Every class in $L(y) \cap \mathcal{A}$ contains a neighbor of y; each class $X \in \mathcal{T}^y$ contains an *extra* neighbor of y, in the sense that either $X \notin L(y)$ or $d_X(y) \geq 2$.

Let $y \in B'$. Since y has neighbors in at least $k - b'$ color classes in $L(y) \setminus \mathcal{B}'$, we have

$$|\mathcal{T}^y| \leq r - (k - b') - d_{B'}(y) = b' - g - 1 - d_{B'}(y) \tag{4}$$

and hence

$$|\mathcal{S}^y| = |L(y) \cap \mathcal{A} \setminus \mathcal{T}^y| \geq \widetilde{a} - |\mathcal{T}^y| \geq \widetilde{a} - b' + g + 1 + d_{B'}(y). \tag{5}$$

The following lemma and its corollary show the second way that we can make progress. Again they are phrased negatively.

Lemma 9. *Suppose zy is a solo edge with $z \in Z \in \mathcal{A}$ and $y \in B'$. (a) For all $U \in \mathcal{A} \cap L(z) - Z$, if $z \nsim U$ then $U \in \mathcal{M}(Z)$. In this case, (b) if w witnesses an edge WZ on a $U, \mathcal{A}_0$-path in $\mathcal{H}$ then $yw \in E(G)$. In particular, (c) there exists $W \in \mathcal{M}(Z) \cap N^-_{\mathcal{F}}(Z)$ with $yp_W \in E$.*

Proof. By Proposition 8, $B - y$ has a b-equitable $L_{\mathcal{B}}$-coloring. Suppose $z \nsim U$. If $U \notin \mathcal{M}(Z)$ then there exists a $U, \mathcal{A}_0$-path $\mathcal{P}$ in $\mathcal{H} - Z$. Moving z to U and y to $Z - z$ yields an a-equitable $L_{\mathcal{C} \setminus \mathcal{B}}$-coloring of $A + y$, and a $(U + z), \mathcal{A}_0$-path $\mathcal{P}$, contradicting Lemma 7. So z is only movable to classes in $\mathcal{M}(Z)$.

Let $\mathcal{Q}$ be a $U, \mathcal{A}_0$-path in $\mathcal{H}$. Then there is an edge $WZ \in E(\mathcal{Q})$, and possibly another edge $ZX \in E(\mathcal{Q})$. In the latter case, we have just seen that z does not witness ZX. If w is a witness for WZ and $yw \notin E(G)$ then moving z to U and y to $Z - z$ yields the same contradiction as before. ■

For $y \in B'$, set

$$\widetilde{\mathcal{T}}^y := \{W \in \mathcal{A} : yp_W \in E(G) \wedge W \in \mathcal{M}(p(W))\} \quad \text{and} \quad \widetilde{\mathcal{S}}^y := \mathcal{S}^y \setminus \{p(W) : W \in \widetilde{\mathcal{T}}^y\}.$$

Notice that $\widetilde{\mathcal{T}}^y \subseteq \mathcal{T}^y$: If $W \in L(y)$ and p_W is the only neighbor of y in W, then we can move y into W, move the witnesses along the $W, \mathcal{A}_0$-path in $\mathcal{F}$, and apply Proposition 8 to $B' - y$. This would yield an equitable L-coloring of G, a contradiction. Moreover, $\mathcal{S}^y \cap \mathcal{A}' \subseteq \widetilde{\mathcal{S}}^y$: If $Z \in \mathcal{A}'$ then $\mathcal{M}(Z) = \emptyset$, and so $Z \neq p(W)$ for any $W \in \widetilde{\mathcal{T}}^y$.

The vertices in $\widetilde{S}^y := S^y \cap (\bigcup \widetilde{\mathcal{S}}^y)$ are called *good* solo vertices. If $z \in S^y$ is in a terminal class then z is a good solo vertex. Good solo vertices in A play a role similar to solo vertices in A' in our previous proof [6], as seen by the following corollary.

Corollary 10. *Let $y \in B'$ and $z \in Z \in \widetilde{\mathcal{S}}^y$. If $z \in \widetilde{S}^y$, then z is adjacent to every class in $L(z) \cap \mathcal{A} - Z$, and so $d_B(z) \leq b - g$.*

Proof. Suppose $z \nsim U \in \mathcal{A} \cap L(z) - Z$. By Lemma 9(a,c), $U \in \mathcal{M}(Z)$ and $yp_W \in E(G)$ for some $W \in N^-_{\mathcal{F}}(Z)$. By the definition of $\widetilde{\mathcal{T}}^y$, $W \in \widetilde{\mathcal{T}}^y$. Thus $p(W) = Z \notin \widetilde{\mathcal{S}}^y$, a contradiction. ■

Corollary 11. *Every class $W \in \mathcal{A}$ has at most $s - 1$ vertices with solo neighbors in B'.*

Proof. If $|W| < s$ then this is trivial. Otherwise $|W| = s$ and p_W is defined. Since p_W is movable to $p(W) \notin \mathcal{M}(W)$, Lemma 9(a) implies p_W has no solo neighbor in B'. ■

Among all nearly equitable L-colorings of G, choose a coloring f such that

(C0) $|\mathcal{A}_0|$ is as large as possible;
(C1) subject to (C0), $|A|$ is as large as possible.

The following proposition collects easy facts about the above definitions.

Proposition 12. *For all $v \in X \in \mathcal{A} \setminus \mathcal{A}_0$ and $Z \in \mathcal{A} \setminus \mathcal{A}'$:*

(a) *If $W \in L(v)$ is an $(s-1)^-$-class then $v \sim W$.*
(b) *Each class $W \in \mathcal{M}(Z)$ is an s-class.*
(c) *Z is an $(s-1)^*$-class.*
(d) $\mathcal{M}(Z) \cap \mathcal{A}' \neq \emptyset$.

Proof. Suppose (a) fails. Then after moving v from X to W, both $X - v$ and $W + v$ are s^--classes. Thus $|\mathcal{A}_0|$ increases by one, contradicting (C0).
Since $Z \notin \mathcal{A}'$, we have $\mathcal{M}(Z) \neq \emptyset$.

(b) Since there is no $W, \mathcal{A}_0$-path in $\mathcal{H} - Z$, $W \notin \mathcal{A}_0$; so W is an s-class.
(c) Let $W \in \mathcal{M}(Z)$ and $\mathcal{P}$ be a $W, \mathcal{A}_0$-path in $\mathcal{H}$. Then only the last class of $\mathcal{P}$ is an s^--class and by (a) it is an $(s-1)$-class. Since Z is a vertex of $\mathcal{P}$, it is an $(s-1)^*$-class, proving (c).
(d) Choose $W \in \mathcal{M}(Z)$ such that $\mathcal{M}(W)$ is minimal. Then $\mathcal{M}(W) = \emptyset$, since $X \in \mathcal{M}(W)$ implies $\mathcal{M}(X) \subset \mathcal{M}(W)$, contradicting the minimality of W. So $W \in \mathcal{A}'$, proving (d). ∎

The following lemma and its corollary (phrased negatively) show our third way of making progress.

Lemma 13. *Suppose zy and zy' are solo edges with $z \in Z \in \mathcal{A}$, $y \in B$, $y' \in B'$, and $yy' \notin E(G)$. Then there exists $W \in \mathcal{M}(Z) \cap N^-_{\mathcal{F}}(Z)$ such that y' is adjacent to every witness of WZ.*

Proof. Suppose that (*) there is no such W. Let f' be the equitable L-coloring of $G - z$ obtained by moving y' to Z and applying Proposition 8 to $B' - y'$.

We claim $\mathcal{A}(f) \subseteq \mathcal{A}(f')$ (after identifying Z with $Z - z + y'$): Consider $U \in \mathcal{A}(f)$. Let $\mathcal{P}$ be a U, V^--path in $\mathcal{H}(f)$; if possible choose $\mathcal{P}$ so that $Z \notin V(\mathcal{P})$; otherwise $U \in \mathcal{M}(f)(Z)$, and we choose $\mathcal{P} \subseteq \mathcal{F}(f)$. If $Z \notin V(\mathcal{P})$ then $\mathcal{P}$ is also a U, V^--path in $\mathcal{H}(f')$, and so $U \in \mathcal{A}(f')$. So suppose $ZY \subseteq V(\mathcal{P})$. If z were the witness for ZY, then we would obtain an equitable L-coloring of G from f' by moving z to Y and switching witnesses along $Y\mathcal{P}V^-$, a contradiction. Finally suppose $WZ \subseteq \mathcal{P}$. Then $W \in N^-_{\mathcal{F}(f)}(Z)$. By (*) WZ has a witness, even in f'. So again $\mathcal{P} \subseteq \mathcal{H}(f')$, and $U \in \mathcal{A}(f')$.

Since $k > r$, z has no neighbors in some class $U \in L(z)$ of f'. If $U \in \mathcal{A}(f')$, then we move z into U and move the witnesses along the $U, \mathcal{A}_0$-path in $\mathcal{H}(f')$ to obtain an equitable L-coloring of G, a contradiction. Thus $U \in \mathcal{B}(f')$. Moving z to U extends f' to a nearly equitable L-coloring f'' of G. Since yz was a solo edge and $yy' \notin E$, y is nonadjacent to $Z - z + y'$. Thus y witnesses that $V_{f''(y)} \in \mathcal{A}(f'') \setminus \mathcal{A}(f)$, contradicting (C1). ∎

In light of Lemma 13, we call a solo edge zy' with $y' \in B'$ *useful* if there exists $y \in S_z \cap B$ such that $yy' \notin E$. Then by Lemma 13, there exists $Z' \in N^-_{\mathcal{F}}(Z)$ such that $y'p_{Z'} \in E$.

Corollary 14. *If $y' \in B'$ and $z \in \widetilde{S}^{y'}$ then y' is adjacent to every $y \in S_z$. In particular, $|S_z \cap B'| \leq b' - g$.*

Proof. By the definition of $z \in \widetilde{S}^{y'}$, if $W \in \mathcal{M}(Z) \cap N^-_{\mathcal{F}}(Z)$ then y' is not adjacent to the witness $p_W(Z)$ of WZ. So the contrapositive of Lemma 13 implies y' is adjacent to every $y \in S_z$.

Let $y' \in S_z \cap B'$. Using (2), yields

$$|S_z \cap B'| \leq d_{B'}(y') + 1 \leq b' - g.$$

■

Proposition 15. *There exist at least two s^--classes.*

Proof. We may assume $|\mathcal{C}| > r + 1$, since otherwise all lists are identical and we are done by the Hajnal–Szemerédi Theorem. Since $|V^+| = s + 1$ and $|G| \leq ks$, there is at least one s^--class W. If W is the only s^--class, then by Proposition 12(c), $|W| = s - 1$ and there is an empty class $X \in \mathcal{C}$; suppose $X \in L(x)$ and $x \in U$. Move x to X. Then X is an s^--class of the new L-coloring. Moreover, since $s \geq 3$, each s^--class of f is still an s^--class. So the new coloring has a larger $|\mathcal{A}_0|$. This contradicts (C0). ■

Proposition 16. *$b \leq r - 1$ and so $\widetilde{a} \geq 2 + g$. Moreover, $c \leq \widetilde{a} - 2$.*

Proof. Suppose $b \geq r$. By (3) every vertex $v \in B$ satisfies $d_A(v) \geq \widetilde{a}$. Thus $e(A, B) \geq \widetilde{a}|B| > \widetilde{a}rs$. Since $|A| < \widetilde{a}s$, there exists $x \in A$ with $d(x) > r$, a contradiction. This also shows that if $c = 0$ then $c \leq \widetilde{a} - 2$. If $c \geq 1$ then there exists an $\mathcal{A} \setminus \mathcal{A}_0$, $\mathcal{A}_0$-path, and it ends in an $(s-1)$-class by Proposition 12(a). By Proposition 15, there is another nonempty class. Thus $|A_0| \geq s$. It follows that $c \leq \frac{|A|-s}{s} < \widetilde{a} - 1$. ■

A. Review of Notation

Unfortunately our arguments require a large amount of notation. The following table provides a quick reference for the more common items.

Graphs	Sets of color classes	Vertex sets	Numbers
$G = (V, E)$	$\mathcal{A}_0 = \{V : \lvert V\rvert < s\}$	$A_0 = \bigcup \mathcal{A}_0$	$c = \lvert\mathcal{A} \setminus \mathcal{A}_0\rvert$
$\Delta(G) \leq r$	$\mathcal{A} = \{V : V \text{ reaches } \mathcal{A}_0\}$	$A = \bigcup \mathcal{A}$	$\widetilde{a} = k - b$
$n = \lvert G\rvert = \lvert V\rvert$	$\mathcal{A}' = \{V \in \mathcal{A} : V \text{ is terminal}\}$	$A' = \bigcup \mathcal{A}'$	$t = \lvert\mathcal{A} \setminus \mathcal{A}'\rvert$
$k = r + 1 + g$	$\mathcal{M}(V) = \{W \in \mathcal{A} : V \text{ cuts } W\}$	$\lvert V^+\rvert = s + 1$	$q = \lvert\mathcal{A}_0 \setminus \mathcal{A}'\rvert$
$k(s-1) < n \leq ks$	$\mathcal{B} = \overline{\mathcal{A}} = \mathcal{C} \setminus \mathcal{A}$	$B = \bigcup \mathcal{B}$	$b = \lvert\mathcal{B}\rvert$
$\mathcal{H}$ auxiliary digraph	$\mathcal{B}' = \{V \in \mathcal{B} : V^+ \text{ reaches } V\}$	$B' = \bigcup \mathcal{B}'$	$b' = \lvert\mathcal{B}'\rvert$
$\mathcal{F}$ spanning forest			

3. LOWER BOUNDS ON b'

In this section, we prove propositions that are useful for the case $r > b + b'$, or equivalently, $\widetilde{a} \geq b' + g + 2$.

Proposition 17. *Let J be a maximal independent set in $G[B']$. Then the number of solo edges in $E(A, J)$ is at least*

$$\sum_{y\in J} |\mathcal{S}^y| \geq |J|(\widetilde{a} - b' + g) + sb' + 1. \tag{6}$$

Proof. Since J is a maximal independent subset of B',

$$\sum_{y\in J}(d_{B'}(y) + 1) \geq |B'| = sb' + 1. \tag{7}$$

By (5) and (7)

$$\sum_{y\in J} |\mathcal{S}^y| \geq |J|(\widetilde{a} - b' + g) + \sum_{y\in J}(d_{B'}(y) + 1) \geq |J|(\widetilde{a} - b' + g) + sb' + 1. \qquad \blacksquare$$

Proposition 18. (a) $b' > \widetilde{a} - t + g$ *and* (b) $2b' \geq \widetilde{a} + 2g + 1$. *Furthermore,* (c) *if $gs + 1 \geq r(r-2)$ then $b' \geq \widetilde{a} + g$.*

Proof. Let J be a maximal independent set in $G[B']$ with $V^+ \subseteq J$.
Suppose (a) fails; then $\widetilde{a} - t + g \geq b'$. Using this, (6) and $|J| \geq s$, we have

$$\begin{aligned}\sum_{y\in J} |\mathcal{S}^y \cap \mathcal{A}'| &\geq \sum_{y\in J}(|\mathcal{S}^y| - |\mathcal{S}^y \setminus \mathcal{A}'|) \geq \sum_{y\in J} |\mathcal{S}^y| - |J|t \\ &\geq |J|(\widetilde{a} + g - b') + sb' + 1 - |J|t \geq s(\widetilde{a} + g - t) + 1. \end{aligned} \tag{8}$$

By Proposition 12(c), $|A - A'| = ts - q$, and so we have $|A'| \leq s(\widetilde{a} - t) + q - 1$. By Proposition 12(d,b), the number of s-classes in $\mathcal{A}'$ is at least q, and by Corollary 11 each contains a vertex with no solo neighbor in B'. So A' has at most $s(\widetilde{a} - t) - 1$ solo vertices. By (8), some vertex $z \in A'$ has two solo neighbors in $J \subseteq B'$. This contradicts Corollary 14, since J is independent.

Next suppose (b) fails; then $2b' \leq \widetilde{a} + 2g$. Using (4), (5), (7) and $|\widetilde{\mathcal{T}}^y| \leq |\mathcal{T}^y|$, we have

$$\sum_{y\in J} |\widetilde{\mathcal{S}}^y| \geq \sum_{y\in J}(|\mathcal{S}^y| - |\mathcal{T}^y|) \geq |J|(\widetilde{a} - 2b' + 2g) + 2b's + 2.$$

Since $2b' \leq \widetilde{a} + 2g$ and $|J| \geq s$,

$$\sum_{y\in J} |\widetilde{\mathcal{S}}^y| = \sum_{y\in J} |\widetilde{\mathcal{S}}^y| \geq s(\widetilde{a} - 2b' + 2g) + 2b's + 2 > \widetilde{a}s > |A|.$$

Thus there exist $y, y' \in J$ with $\widetilde{\mathcal{S}}^y \cap \widetilde{\mathcal{S}}^{y'} \neq \emptyset$. Since J is independent, this contradicts Corollary 14. We conclude that $2b' \geq \widetilde{a} + 2g + 1$.

Finally, suppose $gs + 1 \geq r(r-2)$, and (c) fails; then $b' < \widetilde{a} + g$. This time we estimate $\sum_{y\in J} |\widetilde{\mathcal{T}}^y|$ and $\sum_{y\in J} |\widetilde{\mathcal{S}}^y|$ differently:

$$\sum_{y\in J}|\widetilde{\mathcal{T}}^y| \le \sum_{Z\in\mathcal{A}-\mathcal{A}_0} d_{B'}(p_Z) \le (\widetilde{a}-2)r \le (r-2)r.$$

$$\sum_{y\in J}|\widetilde{\mathcal{S}}^y| \ge \sum_{y\in J}(|\mathcal{S}^y| - |\widetilde{\mathcal{T}}^y|) \ge |J|(\widetilde{a}-b'+g)+sb'+1-\sum_{y\in J}|\widetilde{\mathcal{T}}^y|.$$

$$\sum_{y\in J}|\widetilde{\mathcal{S}}^y| = \sum_{y\in J}|\widetilde{\mathcal{S}}^y| \ge \widetilde{a}s - b's + sb' + 1 + gs - r(r-2)$$
$$\ge \widetilde{a}s > |A|.$$

So again we contradict Corollary 14. ■

4. DISCHARGING

In this section, we study the case $r \le b+b'$. This is equivalent to $\widetilde{a} \le b'+1+g$. Before proceeding, we refine the choice of the spanning forest $\mathcal{F}$. For $X \in \mathcal{C}$, let $\mathcal{F}[X]$ be the subtree of $\mathcal{F}$ with root X, and $\mathcal{F}(X) := F[X]-X$. Notice that $\mathcal{M}(X) \subseteq \mathcal{F}(X)$, but the containment may be proper. Let $U_1,\dots,U_q$ be the sequence of all s^--classes in $\mathcal{A}\setminus\mathcal{A}'$ ordered so that $m(U_1)\ge\cdots\ge m(U_q)$, where $m(U) := |\mathcal{F}[U]|$. By Proposition 12(c), every U_i is an $(s-1)$-class. By Proposition 16, $2+c\le\widetilde{a}$. Thus,

$$\text{if } 2\le j\le q \quad\text{then}\quad 2m(U_j)\le m(U_1)+m(U_j)\le 2+c\le\widetilde{a}. \tag{9}$$

Among all eligible choices for $\mathcal{F}$, choose one that:
(D1) minimizes the sum of the in-degrees of $U_2,\dots,U_q$.

Remark 1. The complexity of finding $\mathcal{F}$ could be high, but the proof, and in particular the algorithm in Section 7, will use only the weaker property that the sum of the in-degrees of $\mathrm{U}_2,\dots,\mathrm{U}_\mathrm{q}$ in each eligible $\mathcal{F}'$ that can be obtained from $\mathcal{F}$ by deleting one edge and adding one edge is at least this sum in $\mathcal{F}$.

For a set $\mathcal{D}\subseteq\mathcal{A}$, the $(2,1)$*-discharging from* (B',B) *to* $\mathcal{D}$ is defined as follows: for every $y\in B$ and $W\in(L(y)\cap\mathcal{D})$, the vertex y distributes a charge of 1 evenly among its neighbors in W, and in addition, if $y\in B'$ then y distributes another charge of 1 evenly among its neighbors in W. For $z\in Z\in\mathcal{D}$, let $ch(z)$ be the total charge distributed to z and $ch(Z)=\sum_{z\in Z}ch(z)$. Then

$$ch(z)\le d_B(z)+|S_z\cap B'|.$$

Proposition 19. *Suppose* $x\in X\in\mathcal{A}'$. *Consider the* $(2,1)$*-discharging from* (B',B) *to* $\{X\}$. *Then the following hold:*

(a) *if* $S_x\cap B'\ne\emptyset$ *then* $ch(x)\le b+b'-2g$;
(b) *if* $X\in\mathcal{M}(U_j)$ *and* $S_x\cap B'=\emptyset$ *then* $ch(x)\le r-\widetilde{a}+m(U_j)$; *and*
(c) *if* $X\in\mathcal{M}(U_j)$ *and* $2\le j\le q$ *then* $ch(X)<s(b+b'-2g)-1$.

Proof.

(a) Since $x\in S^y$, by Corollary 10, $d_B(x)\le b-g$. By Corollary 14, $S_x\cap B'$ is a clique. By (2), $d_{B'}(y)\le b'-1-g$ for all $y\in B'$. Thus $|S_x\cap B'|\le b'-g$. By (4),

$$ch(x)\le d_B(x)+|S_x\cap B'|\le b-g+(b'-g)\le b+b'-2g.$$

(b) Since $X \in \mathcal{M}(U_j)$, x has a neighbor in every class of $L(x) \cap \mathcal{A} \setminus \mathcal{F}[U_j]$. Thus

$$ch(x) \leq d_B(x) + |S_z \cap B'| \leq d_B(x) \leq r - \widetilde{a} + m(U_j).$$

(c) By (9), $m(U_j) \leq \frac{\widetilde{a}}{2}$. By Proposition 18(b), $\frac{\widetilde{a}}{2} < b' - g$. Thus

$$r - \widetilde{a} + m(U_j) \leq b - g - 1 + \frac{\widetilde{a}}{2} < b - g - 1 + b' - g.$$

By Lemma 9(a), p_X has no solo neighbor in B'. Thus (a) and (b) imply

$$ch(X) < s(b + b' - 2g) - 1.$$

■

Proposition 20. *If $r \leq b + b'$, then $\mathcal{A} \setminus \mathcal{A}'$ contains at least two $(s-1)$-classes.*

Proof. Recall $t = |\mathcal{A} \setminus \mathcal{A}'|$. By Proposition 12, $\mathcal{A}' \neq \emptyset$ and every $X \in \mathcal{A} \setminus \mathcal{A}'$ is an $(s-1)^*$-class. Suppose that $\mathcal{A} \setminus \mathcal{A}'$ contains at most one $(s-1)$-class U_1. Then

$$|B \cup (A \setminus A')| \geq (bs + 1) + (ts - 1) = s(b + t),$$

and so

$$0 < |A'| = |G| - |B \cup (A \setminus A')| \leq (\widetilde{a} - t)s.$$

Consider the (2, 1)-discharging from (B', B) to $\mathcal{A}'$. For each $y \in B$, $|L(y) \cap \mathcal{A}'| \geq \widetilde{a} - t$. Thus the total charge distributed from B is at least

$$(\widetilde{a} - t)\left(s(b' + b) + 2\right) > (\widetilde{a} - t)s(b + b') \geq |A'|(b + b').$$

Thus $ch(x) > b + b' \geq r$ for some vertex $x \in A'$. Using (4), x has a solo neighbor in B', contradicting Proposition 19(a). ■

Lemma 21. *Suppose $r \leq b + b'$. Let $1 < j \leq q$ and $u \in U_j$. Consider the (2, 1)-discharging from (B', B) to $\{U_j\}$. Then*

(a) *if $S_u \cap B' = \emptyset$ then $ch(u) \leq r$;*
(b) *if $ch(u) > b + b' - 2g$ then for all $y \in S_u \cap B'$ there exists $W \in N^-_{\mathcal{F}}(U_j) \cap \mathcal{M}(U_j)$ such that $yp_W \in E$ and $d_B(p_W) \leq b - g + m(W)$;*
(c) *$ch(U_j) \leq (s-1)(b + b') + d^-_{\mathcal{F}}(U_j)(b - g) - 1$.*

Proof.

(a) By (4), $ch(u) \leq d_B(u) + |S_u \cap B'| \leq r$.
(b) Suppose $y \in S_u \cap B'$. We first show that there exists $W \in \mathcal{M}(U_j) \cap N^-_{\mathcal{F}}(U_j)$ with $yp_W \in E$. If there exists $Z \in L(u) \cap \mathcal{A} - U_j$ such that $u \nsim Z$, then by Lemma 9(c), there exists $W \in \mathcal{M}(U_j) \cap N^-_{\mathcal{F}}(U_j)$ with $yp_W \in E$. Otherwise $u \sim Z$ for every $Z \in L(u) \cap \mathcal{A} - U_j$. Then $d_B(u) \leq r - (\widetilde{a} - 1) = b - g$. So

$$b + b' - 2g < ch(u) \leq d_B(u) + |S_u \cap B'| \leq b - g + |S_u \cap B'|.$$

It follows that $|S_u \cap B'| > b' - g$. By (2), $\Delta(G[B']) \leq b' - g - 1$. So for each $y \in S_u \cap B'$ there exists another $y' \in S_u \cap B'$ with $yy' \notin E(G)$. Thus every $y \in S_u \cap B'$ is useful, and so by Lemma 13, there exists $W \in \mathcal{M}(U_j) \cap N^-_{\mathcal{F}}(U_j)$ such that $yp_W \in E$.

Since $W \in \mathcal{M}(U_j)$, p_W is not movable to any class of $\mathcal{A} \setminus \mathcal{F}[U_j]$. By (D1), p_W is not movable to any class $X \in \mathcal{F}(U_j) \setminus \mathcal{F}[W]$, since otherwise $\mathcal{F} - WU_j + WX$ would be a better choice than $\mathcal{F}$. Thus $d_A(p_W) \geq \widetilde{a} - m(W) - 1$, and so $d_B(p_W) \leq b + m(W) - g$.

(c) Let $U' := \{u \in U_j : ch(u) > b + b'\}$. First note that $d^-_{\mathcal{F}}(U_j)(b - g) - 1 \geq 0$: by Proposition 12(d), $d^-_{\mathcal{F}}(U_j) \geq 1$, and by Propositions 18(b) and 16, $b - g \geq (\widetilde{a} + 1)/2 \geq 1$. If $ch(U_j) \leq (s - 1)(b + b')$ then we are done. So suppose $ch(U_j) > (s - 1)(b + b')$; thus $U' \neq \emptyset$. By the definition of solo edges, if $u \neq u'$ then $S_u \cap S_{u'} = \emptyset$. By (4),

$$ch(U_j) \leq (s - 1 - |U'|)(b + b') + \sum_{u \in U'} (d_B(u) + |S_u \cap B'|). \tag{10}$$

Suppose $u \in U'$. By (a), $S_u \cap B' \neq \emptyset$. Thus by Lemma 9, $d_A(u) \geq \widetilde{a} - m(U_j)$. By (9), $m(U_j) \leq \widetilde{a}/2$. So

$$\forall u \in U',\ \ d_B(u) \leq r - \widetilde{a} + m(U_j) \leq r \leq b + b'. \tag{11}$$

By (b), each $y \in S_u \cap B'$ is adjacent to a witness p_W with $W \in N^-_{\mathcal{F}}(U_j)$, and each p_W is adjacent to at most $b - g + m(W)$ such vertices in $y \in S_u \cap B'$. So

$$\begin{aligned} \sum_{u \in U'} |S_u \cap B'| &\leq \sum_{W \in N^-_{\mathcal{F}}(U_j)} (b - g + m(W)) \\ &\leq d^-_{\mathcal{F}}(U_j)(b - g) + \sum_{W \in N^-_{\mathcal{F}}(U_j)} m(W) \\ &\leq d^-_{\mathcal{F}}(U_j)(b - g) + m(U_j) - 1. \end{aligned} \tag{12}$$

Combining (10), (11), (12), $U' \neq \emptyset$ and (9), we have

$$\begin{aligned} ch(U_j) &\leq (s - 1 - |U'|)(b + b') + \sum_{u \in U'} (r - \widetilde{a} + m(U_j)) + d^-_{\mathcal{F}}(U_j)(b - g) + m(U_j) - 1 \\ &\leq (s - 1)(b + b') + |U'|(-b - b' + r - \widetilde{a} + m(U_j)) \\ &\quad + d^-_{\mathcal{F}}(U_j)(b - g) + m(U_j) - 1 \\ &\leq (s - 1)(b + b') - \widetilde{a} + m(U_j) + d^-_{\mathcal{F}}(U_j)(b - g) + m(U_j) - 1 \\ &\leq (s - 1)(b + b') + d^-_{\mathcal{F}}(U_j)(b - g) - 1. \end{aligned}$$

■

Proposition 22. *If $r \leq b + b'$ then $b > (2s + 1)g$.*

Proof. Let $\mathcal{U} := \{U_2, \ldots, U_q\}$, $\mathcal{Q} = \mathcal{A}' \cup \mathcal{U}$, $Q = \bigcup \mathcal{Q}$, $\mathcal{P} = \mathcal{A} \setminus \mathcal{Q}$, $P = \bigcup \mathcal{P}$, and $p = |\mathcal{P}|$. By Proposition 12, every $Z \in \mathcal{P}$ is an s-class, except U_1. So

$$|B \cup P| = (1 + bs) + (ps - 1) = (b + p)s \tag{13}$$

$$0 < |Q| \leq (\widetilde{a} - p)s. \tag{14}$$

By Proposition 12, for $2 \leq j \leq q$ and $W \in N^-_{\mathcal{F}}(U_j)$ there exists $X_W \in (\{W\} \cup \mathcal{M}(W)) \cap \mathcal{A}'$. Set $\mathcal{A}'_j = \{X_W : W \in N^-_{\mathcal{F}}(U_j)\}$, $\mathcal{A}^* := \mathcal{A}' \setminus \bigcup_{j=2}^q \mathcal{A}'_j$, $A'_j := \bigcup \mathcal{A}'_j$, and $A^* := \bigcup \mathcal{A}^*$. Then $Q = A^* \cup \bigcup_{j=2}^q (U_j \cup A'_j)$.

Consider the $(2, 1)$-discharging from (B', B) to Q. Each vertex $x \in B$ distributes charge at least 1, and at least 2 if $v \in B'$, to each class in $L(v) \setminus \mathcal{B}$. There are at least $\widetilde{a}$ of these classes. So the total charge distributed to Q is

$$ch(Q) \geq (\widetilde{a} - p)\big(2 + s(b' + b)\big) > |Q|(b + b'). \tag{15}$$

By Proposition 19 and (4), $ch(x) \leq r \leq b + b'$ for each $x \in A^*$. So $ch(A^*) \leq (b + b')|A^*|$. It follows that there exists j with $2 \leq j \leq q$ such that,

$$ch(U_j \cup A'_j) > (b + b')(|U_j| + |A'_j|) = (b + b')(s - 1 + d^-_{\mathcal{F}}(U_j)s). \tag{16}$$

By Proposition 19 and Lemma 21,

$$\begin{aligned} ch(U_j \cup A'_j) &< (s - 1)(b + b') + d^-_{\mathcal{F}}(U)(b - g) + d^-_{\mathcal{F}}(U_j)s(b + b' - 2g) \\ &< (b + b')(s - 1 + d^-_{\mathcal{F}}(U_j)s) + d^-_{\mathcal{F}}(U_j)(b - g - 2gs). \end{aligned} \tag{17}$$

Combining (16) and (17), we have $b > (2s + 1)g$. ■

5. PROOF OF THEOREM 4(a)–(c)

In this section, we prove the first three of the four statements of Theorem 4. The fourth proof is long and will take the next section.

A. Proof of Theorem 4(a)

Recall that c is the number of s-classes in $\mathcal{A}$, t is the number of nonterminal classes and q is the number of nonterminal $(s - 1)$-classes. By Proposition 16, $c \leq \widetilde{a} - 2$. Also, there are $t - q$ nonterminal s-classes. For all distinct, nonterminal $(s - 1)$-classes U_i and U_j, both $\mathcal{F}(U_i)$ contains a terminal s-class, and $\mathcal{F}(U_i) \cap \mathcal{F}(U_j) = \emptyset$. By Proposition 18(a), $\widetilde{a} - t < b'$. It follows that

$$\widetilde{a} - b' + 1 \leq t \leq c \leq \widetilde{a} - 2. \tag{18}$$

Suppose $r \leq 7$ and $g = 0$. We consider two cases.

Case 1: $\widetilde{a} \geq b' + 2$. Then by Proposition 18(b),

$$b' + 2 \leq \widetilde{a} \leq 2b' - 1 \leq b' + b - 1 \leq b' + r - \widetilde{a} \leq b' + 7 - \widetilde{a}.$$

By the first two inequalities, $b' \geq 3$, and thus $\widetilde{a} \geq 5$. So equality holds throughout. In particular, $\widetilde{a} = 5$ and $b' = 3 = b$. Thus by (18), $t = 3 = c$.

Let $J \subseteq B'$ be a maximal independent set. The number σ of solo vertices in A satisfies $\sigma \leq \widetilde{a}s - 1 - c$, since witnesses are not solo (Lemma 9(a)). The number τ of solo vertices in A with at least two solo neighbors in J satisfies $\tau \geq \frac{1}{r-1}(\sum_{y \in J} |S^y| - \sigma)$. By Proposition 17, $\sum |S^y| \geq |J|(\widetilde{a} - b') + sb' + 1$. Since J is independent, the number μ of useful solo edges incident to J is

$$\begin{aligned}\mu &\geq \sum_{y\in J} |S^y| - \sigma + \tau \\ &\geq \left(1+\frac{1}{r-1}\right)\left(\sum_{y\in J}|S^y| - \sigma\right) \\ &\geq \left(1+\frac{1}{r-1}\right)((s+|J|-s)(\widetilde{a}-b')+b's+1)-(\widetilde{a}s-1-c)) \\ &\geq \left(1+\frac{1}{r-1}\right)((|J|-s)(\widetilde{a}-b')+c+2). \qquad (19)\end{aligned}$$

For each useful solo edge zy with $z \in Z \in \mathcal{A}$, there exists $W \in \mathcal{M}(Z)$ with $p(W) = Z$ and $p_W y \in E$. Thus $W \in \widetilde{\mathcal{T}}^y \subseteq \mathcal{T}^y$. If $z'y$ with $z' \in Z'$ is another useful solo edge incident to y then $Z \neq Z'$. Thus, the number of useful solo edges incident to y is at most $|\mathcal{T}^y|$. Thus, using Equations (4) and (7), we have

$$\mu \leq \sum_{y\in J} |\mathcal{T}^y| \leq b'|J| - \sum_{y\in J}(1+d_{B'}(y)) = b'(|J|-s)-1. \qquad (20)$$

Combining (19) and (20), and substituting $\widetilde{a} = 5$, $b = 3 = c$, yields

$$b'(|J|-s)-1 \geq \mu \geq \left(1+\frac{1}{r-1}\right)((|J|-s)(\widetilde{a}-b')+c+2) \qquad (21)$$

$$|J|-s = (|J|-s)(2b'-\widetilde{a}) > c+3 = 6. \qquad (22)$$

Substituting (22) into (19) yields $\mu \geq \lceil \frac{7}{6}(2\times 7+5)\rceil = 23$ useful edges. Each of them is incident to one of c witnesses, and so some witness w satisfies $d(w) \geq \lceil \frac{23}{c} \rceil = 8$, a contradiction.

Case 2: $\widetilde{a} \leq b'+1$. Then $r \leq b'+b$. Using Proposition (16),

$$4 \leq 2\tilde{a} \leq b'+1+\tilde{a} = b'+1+(r+1-b) \leq 9,$$

and so $\tilde{a} \leq \lfloor 9/2 \rfloor = 4$. By Proposition 20, there exist at least two $(s-1)$-classes $U_1, U_2 \in \mathcal{A}\setminus\mathcal{A}'$. Thus $\mathcal{A}'$ has two s-classes $W_j \in \mathcal{M}(U_j)\cap\mathcal{A}'$ for $j = 1, 2$. So $\widetilde{a} \geq 4$; thus $\widetilde{a} = 4$. This accounts for all, but possibly one vertex u, of the vertices of A. If u exists, its class is $\{u\}$, and $\{u\}$ is terminal, by Proposition 12.

Consider the $(2, 1)$-discharging from (B', B) to A'. Set $\beta := b + b' \geq r$. By Proposition (19), each vertex in A' gets charge at most β. Moreover, the two movable witnesses p_{W_j}, $j = 1, 2$ have no solo neighbors in B' by Lemma 13, and so have charge at most $ch(p_{W_j}) \leq r - \widetilde{a} + m(U_j) = r-2$. So the total charge received by A' is at most $\beta(2s-1)+2r-4$. The total charge distributed to A' is at least $(\widetilde{a}-2)(|B|+|B'|) = 2(\beta s+2)$. Thus,

$$0 \leq (\beta(2s-1)+2r-4)-2(\beta s+2) = 2r-\beta-8 \leq r-8 < 0,$$

a contradiction.

If $g \geq 1$ then Theorem 4(b) applies, since $1 \geq \max\{\frac{r}{6}-1, \frac{r-1}{7}\}$ when $r \leq 8$.

B. Proof of Theorem 4(b)

Assume g is an integer satisfying $g \geq \max\{\frac{r}{6}-1, \frac{r-1}{7}\}$.

Proposition 23. *If* $g \geq \frac{r}{6} - 1$ *then* $r \leq b + b'$.

Proof. If $b' \geq 3g + 3$ then $b + b' \geq 6(g + 1) \geq r$. So suppose $b' \leq 3g + 2$. By Proposition 18(b),

$$r = \widetilde{a} + b - g - 1 \leq 2\left(b' - g - \frac{1}{2}\right) + b - g - 1 \leq b + b' + (b' - 3g - 2) \leq b + b'.$$

■

Recall that $s \geq 3$. By Proposition 23, $r \leq b + b'$. Thus by Propositions 16 and 22, we have the contradiction

$$r - 1 \geq b > (2s + 1)g \geq 7g \geq r - 1.$$

So G is k-choosable, for

$$k \geq r + \begin{cases} 1 + \dfrac{r-1}{7} & \text{if } \; r \leq 30 \\ \dfrac{r}{6} & \text{if } \; r \geq 31 \end{cases}.$$

C. Proof of Theorem 4(c)

Suppose $|G| \geq r^3$ and $g \geq 1$. Then

$$gs = g\left\lceil \frac{|G|}{r + 1 + g} \right\rceil \geq r^2 - 2r = r(r - 2).$$

Thus by Proposition 18(c), $b + b' \geq r$. So by Proposition 22, $b > g(2s + 1) > r$, a contradiction. So G is k-choosable, for $k = r + 1 + g \geq r + 2$.

6. PROOF OF THEOREM 4(d)

Assume s is sufficiently large. A set is said to contain *almost no elements,*if it has less than m elements, where m is a constant that does not depend on s. A set whose cardinality is unbounded as s increases is said to contain *many* elements. Recall that G is a counter example to Conjecture 2 when $g = 0$. It suffices to show the following technical statement.

Theorem 24. *Let* $g = 0$. *If* s *is sufficiently large then* $\omega(G) = r + 1$.

We shall prove a sequence of results that ends with the conclusion of the theorem. Call a vertex $y \in B'$ *dense* if it is in a b'-clique $Q \subseteq B'$; otherwise it is *sparse*. Call a vertex v *rich* if $|L(v) \cap \mathcal{A}| > \widetilde{a}$; otherwise it is *poor*.

Proposition 25. *If* $v \in V$ *is poor then* $L(v) \cap \mathcal{B} = \mathcal{B}$.

Proposition 26. *If* $y \in B'$ *is dense then it is not movable.*

Proof. By definition y is not movable to any class in $\mathcal{A} \cup \mathcal{B} \setminus \mathcal{B}'$, and since it is in a b'-clique contained in B', it is not movable to any class of $\mathcal{B}'$. ■

Proposition 27. *If a vertex* v *is not movable then it has exactly one neighbor in each class of* $L(v) - f(v)$, *and thus no neighbors in any other classes.*

Corollary 28. *The neighbors in A of a dense vertex $y \in B'$ are solo neighbors of y, and are not movable to any class in $\mathcal{A}$. In particular, they do not witness any edge of $\mathcal{H}[\mathcal{A}]$.*

Proof. By Propositions 26 and 27, the neighbors in A of y are solo neighbors of y. Suppose $z \in N_A(y)$ and $z \in Z \in \mathcal{A}$. If z is movable to a class $X \in \mathcal{A}$ then by Lemma 9 $X \in \mathcal{M}(Z)$ and y is adjacent to a witness w of an edge WZ. Then $Z \notin \mathcal{M}(W)$ and this contradicts Lemma 9 applied to W. ■

Corollary 29. *Each dense vertex $y \in B'$ is a member of a unique b'-clique contained in B', and has no other neighbors in B'.*

For each dense vertex $y \in B'$, let $Q^y \subseteq B'$ be the unique b'-clique to which it belongs.

Corollary 30. *No vertex $x \in A$ has two dense neighbors y and y' in different b'-cliques of B'.*

Proof. Suppose not. By Corollary 29, $yy' \notin E$. By Corollary 28, xy and xy' are solo. Moreover, y is not adjacent to any witness of any edge in $\mathcal{H}[\mathcal{A}]$. This contradicts Lemma 13. ■

Let $B'_g \subseteq B'$ be the set of dense, poor vertices of B'.

Proposition 31. *Almost no vertices of B' are sparse or rich; indeed $|B' \setminus B'_g| < r^4$.*

Proof. Let s' be the the number of b'-cliques in B'. Using Corollary 29, B' has $s'b'$ dense vertices and $(s - s')b' + 1$ sparse vertices.

Case 1: $\widetilde{a} > b'$. By Propositions 18(b) and 16, we have $b' \geq \lceil \frac{\widetilde{a}+1}{2} \rceil \geq 2$. For each b'-clique Q choose a vertex $v \in Q$, preferring a rich vertex. Let J_1 be the set of chosen vertices and $J_0 \subseteq J_1$ be the set of rich chosen vertices. By Corollary 29, J_1 is independent. Then B' has at most $b'|J_0| + (s - s')b' + 1$ vertices that are either sparse or rich.

Let $H \subseteq G$ be the subgraph induced by the sparse vertices. By (2), $\Delta(H) \leq b' - 1$. Since $\omega(H) \leq b' - 1$, Brooks' Theorem implies that $\chi(H) \leq b' - 1$. Thus, H has a maximum independent set J_2 with

$$|J_2| \geq \frac{(s - s')b'}{b' - 1} = s - s' + \frac{s - s'}{b' - 1}.$$

Set $J := J_1 \cup J_2$. Then J is a maximum independent subset of $G[B']$ and $|J| \geq s + \frac{s-s'}{b'-1}$.

Set $\Delta(y) := |L(y) \cap \mathcal{A}| - \widetilde{a}$. Then, as in (5), for each $y \in B'$,

$$|\mathcal{S}^y| = |L(y) \cap \mathcal{A} \setminus \mathcal{T}^y| = \widetilde{a} + \Delta(y) - |\mathcal{T}^y| \geq \widetilde{a} + \Delta(y) - b' + 1 + d_{B'}(y).$$

Since by Corollary 14 $\widetilde{S}^y \cap \widetilde{S}^{y'} = \emptyset$ for distinct $y, y' \in J$, $\widetilde{a}s - 1 \geq |A| \geq \sum_{y\in J} |\widetilde{S}^y|$. As at the end of Section 3,

$$\begin{aligned}\widetilde{a}s - 1 &\geq \sum_{y\in J} |\widetilde{\mathcal{S}}^y| \geq \sum_{y\in J}(|\mathcal{S}^y| - |\widetilde{\mathcal{T}}^y|)\\ &\geq \sum_{y\in J}(\widetilde{a} + \Delta(y) - b' + 1 + d_{B'}(y) - |\widetilde{\mathcal{T}}^y|)\\ &\geq |J|(\widetilde{a} - b') + |J_0| + sb' + 1 - \sum_{y\in J} |\widetilde{\mathcal{T}}^y|\\ &\geq \widetilde{a}s - b's + |J| - s + |J_0| + sb' + 1 - r(r-2)\\ &\geq \widetilde{a}s + |J| - s + |J_0| + 1 - r^2.\end{aligned}$$

Since $|J| \geq s + \frac{s-s'}{b'-1}$, we conclude that

$$r^2 \geq \frac{s - s'}{b' - 1} + |J_0|.$$

It follows that the number of vertices that are sparse or rich is at most $b'^2 r^2 \leq r^4$. So $|B' \setminus B'_g| < r^4$.

Case 2: $\widetilde{a} \leq b'$. Let j be the number of rich vertices in B'. As in the proof of Proposition 22, and using the same notation, consider the $(2, 1)$-discharging from (B', B) to Q. As in the justification of (15), each vertex $v \in B$ sends a charge of 1 or 2 to each class in $L(v) \setminus \mathcal{B}$, and there are at least $\widetilde{a}$ of these classes. But if v is rich then $v \in B'$ and $|L(v) \setminus \mathcal{B}| \geq \widetilde{a} + 1$. In this case, (15) undercounts the number of classes to which v distributes charge 2. So the total charge distributed to Q is bounded by

$$ch(Q) \geq (\widetilde{a} - p)(2 + s(b' + b)) + 2j > |Q|(b + b') + 2j.$$

Set $U := \bigcup \mathcal{U}$. By Lemma 21(c),

$$\begin{aligned}ch(U) &\leq \sum_{j=2}^{q}((s-1)(b+b') + b \cdot d^-_{\mathcal{F}}(U_j))\\ &\leq |U|(b+b') + b\sum_{j=2}^{q} d^-_{\mathcal{F}}(U_j) \leq |U|(b+b') + br.\end{aligned}$$

By Proposition 19 and (4), each $z \in A'$ receives charge $ch(z) \leq b + b'$; moreover, if $S_z \cap B'$ is not a b'-clique then $ch(z) \leq b + b' - 1$ by (4). If z and z' are in the same class of $\mathcal{A}$ then $S_z \cap S_{z'} = \emptyset$. Thus

$$ch(A') \leq \sum_{Z\in\mathcal{A}'}(s(b + b' - 1) + s') = |A'|(b + b' - s + s').$$

Since the classes in $\mathcal{A}'$ are terminal and those in $\mathcal{U}$ are nonterminal,

$$(b + b')|Q| + 2j < ch(Q) \leq (b + b')|Q| + br - s + s',$$

which means

$$2j + s - s' < br.$$

Thus $|B' \setminus B'_g| \leq j + (s - s')b' + 1 \leq r^3$. ∎

Call a vertex $z \in A$ *defective* if it has a neighbor in $B' \setminus B'_g$ or fewer than b' neighbors in B'; otherwise z is *nondefective*. In the latter case, set $Q_z := N_{B'}(z)$. By Corollary 30, Q_z is a b'-clique. So $Q_z + z$ is a clique. Let $A_g \subseteq A$ be the set of nondefective vertices.

Proposition 32. *If z is a nondefective vertex whose neighbors in B are not movable, then $N_B(z)$ is a clique.*

Proof. Let $y \in N_{B'}(z)$, $y' \in N_{B \setminus B'}(z)$ and $y'' \in N_B(z)$. Since z is nondefective, $|N_{B'}(z)| = b$ and each vertex of $N_{B'}(z)$ is dense. So Corollary 30 implies $N_{B'}(z)$ is the b'-clique $Q_z = Q^y$. Since the neighbors of z in B are not movable, Proposition 27 implies they are all solo neighbors of z. By Lemma 13 and Corollary 28, both $N_{B'}(z) + y'$ and $N_{B'}(z) + y''$ are cliques. Moreover, $E(Q^y, B' \setminus Q^y) = \emptyset$. It follows that y and y' are neither witnesses nor adjacent to witness of any edge in $\mathcal{H}[B']$. Thus switching y with y' yields a new nearly equitable L-coloring f^* with $B'(f) = B'(f^*)$ after identifying the class W of y in f with the class $W - y + y'$ in f^*. Applying the above argument to f^* shows $y'y'' \in E$. Since the arbitrary vertex $y'' \in N_B(z)$ is adjacent to the arbitarty vertices $y \in N_{B'}$ and $y' \in N_B \setminus B''$, it follows that $N_B(z)$ is a clique. ■

Proposition 33. *Almost no $z \in A$ is defective; indeed $|A \setminus A_g| \leq 2r^5$.*

Proof. Let α be the number of vertices in A that have a neighbor in $B' \setminus B'_g$. By Proposition 31, $\alpha < r|B' \setminus B'_g| < r^5$. Let β be the number of vertices of A with fewer than b' neighbors in B'. Each vertex in B'_g is poor, and so has exactly $\widetilde{a}$ neighbors in A. Thus

$$\widetilde{a}(b's - r^4) \leq |E(A, B'_g)| < \widetilde{a}sb' - \beta,$$

which yields $\beta \leq \widetilde{a}r^4 < r^5$. It follows that $|A \setminus A_g| \leq \alpha + \beta \leq 2r^5$. ■

Now assume that apart from (C0) and (C1), f satisfies

(C2) among nearly equitable L-colorings satisfying (C0) and (C1), f has the maximal b'.

Proposition 34. *No $z \in A_g$ is movable.*

Proof. Suppose that $z \in A_g$ is movable. By Corollary 28, since z is nondefective, it is not movable to a class in $\mathcal{A}$. So z is movable to a class $Y \in \mathcal{B}$. Again, since z is nondefective, $Y \in \mathcal{B} \setminus \mathcal{B}'$. Let v_z be the unique vertex in $Q_z \cap V^+$. We can obtain a new nearly equitable L-coloring f^* by moving z to Y and v_z to $Z := V_{f(z)}$. By Propositions 26 and 27, $Z \in L(v_z)$, and so this is allowed. Moreover, $\mathcal{A}(f^*) \supseteq \mathcal{A}(f)$ (where Z is replaced by $Z - z + v_z$): By Corollary 28, z does not witness any edge of $\mathcal{H}[\mathcal{A}]$; so removing z does not destroy any edge of $\mathcal{H}[\mathcal{A}]$. Also by Corollary 28, v_z is not adjacent to a witness of any edge of $\mathcal{H}[\mathcal{A}]$, and so does not destroy edge of $\mathcal{H}[\mathcal{A}]$.

By the maximality of A, $\mathcal{A}(f^*) = \mathcal{A}(f)$. Recall that $z \in V^+(f^*)$ and is movable to $V^+(f) - v_z$, since v_z was the only neighbor of z in $V^+(f)$. Furthermore, since every class $Y' \in \mathcal{B}'(f)$ was reachable from $V^+(f)$ and v_z was in a b'-clique, Y' is still reachable from $V^+(f) - v_z$. But then $\mathcal{B}'(f^*) \supseteq \mathcal{B}'(f) + (Y + z)$. This contradicts the maximality of b'. ■

Proposition 35. *Almost no $z \in A_g$ is rich; indeed at most r^5 such z are rich.*

Proof. Let R be the set of rich vertices in A_g and $\mathcal{Q} := \{Q_z : z \in R\}$. Suppose $|R| > r^5$. Since $|\{z : Q = Q_z\}| \leq r$ for each $Q \in \mathcal{Q}$, we have $|\mathcal{Q}| > r^4$. Consider any $z \in R$ and $y \in Q_z \cap V^+$. By Propositions 34 and 26, neither z nor y is movable, and so by Proposition 27, $V_{f(y)} \in L(z)$ and $V_{f(z)} \in L(y)$. Moreover, y is the only neighbor of z in $V_{f(y)}$ and z is the only neighbor of y in $V_{f(z)}$. So switching z with y yields a new nearly equitable coloring f'. Since y is solo, by Lemma 9(a), no neighbor of y (including z) witnesses an edge of $\mathcal{F}$. So $|\mathcal{A}(f)| = |\mathcal{A}(f')|$ and $|A(f)| = |A(f')|$, and so (C0) and (C1) are maintained. Since y was not movable in f and $y \in V^+(f)$, we have $b'(f) = b'(f')$, and so (C2) is maintained. Repeating this construction $|\mathcal{Q}| > r^4$ times contradicts Proposition 31. ■

Now refine the choice of f so that
(C3) subject to (C0), (C1), and (C2), B' has as many sparse vertices as possible.

Proposition 36. *Let $R := B' - B'_g$ be the set of vertices in B' that are rich or sparse. Let $y \in (B - B') \cap (N(B') - N(R))$. Then y is not movable to B'.*

Proof. Suppose that such a y belongs to a class $Y \in \mathcal{B} - \mathcal{B}'$ and is movable to a class $Y' \in \mathcal{B}'$. By hypothesis, y has a dense, poor neighbor $v \in B'$. Let $v' \in Q^v \cap Y'$. Then $v'y \notin E$, since y is movable to Y'. Since v is dense, y is its unique neighbor in Y (Propositions 27, 26). Thus moving v to Y, y to Y', and v' to $V_{f(v)}$, yields a new proper coloring f^*. By Propositions 27 and 26, f^* is an L-coloring. Since the sizes of color classes did not change, and classes in $\mathcal{A}$ did not change at all, f^* is a nearly equitable coloring satisfying (C0) and (C1). Moreover, since all witnesses for edges of $\mathcal{H}[\mathcal{B}'](f)$ were in R (Corollary 29), and the changed vertices were in $Q^v + y$, and so had no neighbors in R, we have $\mathcal{B}'(f^*) \supseteq \mathcal{B}'(f)$. So, f^* satisfies (C2). But the $b' \geq 2$ vertices of $Q^v(f) + y - v$ are not dense in $B'(f^*)$. Moreover, since y has no neighbors in R, no vertices of $B'(f)$ became dense. This contradicts (C3). ■

Observation 37. *Let Q be a b'-clique in $\mathcal{B}'$. Then permuting vertices of Q within $\mathcal{B}'$, we again obtain a nearly equitable L-coloring of G satisfying (C0)–(C3).*

Proof. Let f^* be the new coloring. Since $G[Q]$ is a component of $G[B']$, f^* is proper. By Propositions 27, 26, f^* is an L-coloring. Since $f|_A$ does not change, (C0) and (C1) hold for f^*. Since all witnesses for edges of $\mathcal{H}[\mathcal{B}'](f)$ were in $B'(f) \setminus B'_g(f)$, and had no neighbors in Q, $\mathcal{B}'$ did not change, and so (C2) and (C3) continue to hold. ■

Proposition 38. *Let $R := B' - B'_g$ be the set of vertices in B' that are rich or sparse. At most r^6 vertices $x \in (B - B') \cap (N(B') - N(R))$ are movable.*

Proof. Let X be the set of movable vertices $x \in (B - B') \cap (N(B') - N(R))$, and suppose $|X| > r^6$. There exists a class $W \in \mathcal{B} \setminus \mathcal{B}'$ such that $X' := X \cap W$ satisfies $|X'| > |X|/r > r^5$. For each $x \in X'$, let $Y_x \in \mathcal{B}$ be a class to which x can be moved. By Proposition 36, $Y_x \notin \mathcal{B}'$. There exists a class $Y \in \mathcal{B} \setminus \mathcal{B}'$ such that $X'' := \{x \in X' : Y_x = Y\}$ satisfies $|X''| > |X'|/r > r^4$. For $x \in X''$ choose $v_x \in B'_g$ so that $xv_x \in E$, which is possible by the definition of X. Since v_x is poor and dense, it has a unique neighbor in every class of $\mathcal{B} - V_{f(v_x)}$ and $\mathcal{B} \subseteq L(v_x)$ by Propositions 25, 26, 27. In particular, this holds for W and Y. So $V' := \{v_x : x \in X''\}$ satisfies $|X'| = |V'|$. For $x \in X''$ let u_x be the unique neighbor of v_x in Y, and set $U := \{u_x : x \in X''\}$. Then $|U| = |V'| > r^4$.

Consider any $x \in X''$. By Observation 37, we may assume $v_x \in V^+$. Let f_x be the coloring obtained from f by moving x to Y and v_x to W. Then f_x is a nearly equitable L-coloring with large class $V^+(f_x) = Y + x$ that satisfies (C0) and (C1). Applying Proposition 31 to $f(x)$ yields $|B'(f_x) \setminus B'_g(f_x)| < r^4 < |U|$. Thus there exists $z \in U$ such that $\mathcal{B} \subseteq L(z)$. If $z = x$ then z is movable to $V^+(f)$, and thus $\mathcal{B}'(f) + V^+(f_x) \subseteq \mathcal{B}'(f_x)$, contradicting (C2). Otherwise, replace x by z in the above argument (which does not change $L(z)$). ■

Proposition 39. *If $s \geq 3r^8$ then $\omega(G) = r + 1$.*

Proof. Suppose that $v \in B$ is movable. Then v is not dense. So $v \in R = B' - B'_g$ or $v \in B \setminus B'$. In the latter case, by Proposition 38, $v \in N(R)$ or is one of at most r^6 exceptional vertices. By Proposition 31, $|R| = |B' - B'_g| \leq r^4$, and so $|N(R)| \leq r^5$. Thus at most $r^4 + r^5 + r^6 \leq 2r^6$ vertices in B are movable or rich. So at most $2r^7$ vertices of A have movable neighbors in B. At most $2r^5$ vertices of A are defective. By Proposition 35, at most r^5 nondefective vertices of A are rich. Thus less than $3r^8$ vertices of B' have neighbors in A that are rich, defective or adjacent to movable vertices of B.

Since $s \geq 3r^8$, we can choose $y \in B'$ whose neighbors in A are all poor, nondefective, and not adjacent to movable vertices of B. We will show that $K := N(y) + y$ is an $(r + 1)$-clique. First note that y is dense and not movable. Since y is not movable, $|N(y) + y| = r + 1$. Since y is dense, Q^y is defined, and by Propositions 16, 18, $b' \geq 2$. Choose $y' \in Q^y - y$. Consider any $z \in N_A(y)$. Since y is dense, Corollary 28 implies $z \in S^y$ and z is not movable to any class in $\mathcal{A}$. By Proposition 32, $N_B(z)$ is a clique, and so y is the only neighbor of z in $V_{f(y)}$. Thus, we can switch z and y to obtain a new nearly equitable coloring f'. Since z is poor, and y is not movable, Propositions 25 and 27 imply that f' is an L-coloring. Since z is not movable to any class in $A(f)$, it does not witness an edge of $\mathcal{F}[\mathcal{A}](f)$. By Lemma 9(a), y is not adjacent to the witness of any edge in $\mathcal{F}[\mathcal{A}](f)$. Thus $\mathcal{H}[\mathcal{A}](f) \subseteq \mathcal{H}[\mathcal{A}](f')$, and so f' satisfies (C0) and (C1). Moreover, no $z' \in N_{A(f)}(y)$ is movable to a class of $A(f')$.

By Proposition 32, $N_{B(f')}(y) + y = N_{B(f)}(y) + z + y$ is a clique. Since this argument applies to every $z \in N_A(y)$, it suffices to show that $zz' \in E$ for all distinct $z, z' \in N_{A(f)}(y)$. Consider $z' \in N_{A(f)}(y)$. We have just shown that $y' \in N(z)$. So the choice of y implies that y' is not movable in f. Nor is y' movable in f', since f' is obtained by switching two neighbors of y'. As above, obtain a new nearly equitable coloring f^* from f' that satisfies (C0) and (C1) by switching y' and z'. Since $z, z' \in N_{B(f^*)}(y')$, Proposition 32 implies $z, z' \in E$. ■

This completes the proof of Theorem 4(d).

7. ALGORITHM

In fact, the proof of Theorem 4 is algorithmic. In this section, we show how to adapt this proof to derive the following.

Theorem 40. *There exists a polynomial time (in terms of n) algorithm that for any k-list assignment L of any n-vertex graph G with $\Delta(G) \leq r$, finds an equitable L-coloring of G in each of the following cases:*

(a) *$r \leq 7$ and $k = r + 1$;*

(b)

$$k \geq r + \begin{cases} 1 + \dfrac{r-1}{7} & \textit{if} \quad r \leq 30 \\ \dfrac{r}{6} & \textit{if} \quad r \geq 31 \end{cases};$$

(c) $|G| \geq r^3$ *and* $k \geq r + 2$.

A. General Outline

The input is an n-vertex graph $G = (V, E)$ with maximum degree at most r and a k-list L for G. If r, k, and n satisfy the conditions in Theorem 4(a), (b), or (c), then the output is an equitable L-coloring.

Let $V := \{v_1, \ldots, v_n\}$. We start by greedily constructing an equitable L-coloring f_0 for the edgeless spanning subgraph $G_0 \subseteq G$. The main part of the algorithm is the following Main Procedure: it takes a graph with a list assignment L satisfying the conditions of one of (a), (b), or (c) of Theorem 4 together with a nearly equitable L-coloring f and produces an equitable L-coloring f^*. The procedure will be described in the next subsection.

For $i = 1, \ldots, n-1$, let G_i denote the spanning subgraph of G whose edges are exactly the edges of G incident with at least one of $v_1, \ldots, v_i$. Note that $G_{n-1} = G$. Suppose that we have an equitable L-coloring f_{i-1} of G_{i-1}. Consider it as an improper L-coloring of G_i. By definition, every monochromatic edge contains v_i. If there are no such edges, we already have f_i. Suppose that there are conflicts. Since $|L(v_i)| \geq r + 1$, there is a color class $Z \in L(v_i)$ not containing neighbors of v_i. Moving v_i into Z, we obtain a nearly equitable proper L-coloring f of G_i. Now applying Main Procedure, we produce an equitable L-coloring f_i of G_i. So, after the initialization, it is enough to perform Main Procedure at most $n - 1$ times.

B. Main Procedure

As mentioned above, the input is a graph G_i together with the list assignment L and a nearly equitable proper L-coloring f of G_i. The output is an equitable L-coloring f^* of G_i. Let $\mathcal{C} := \bigcup_{v \in V} L(v)$. By definition, $|\mathcal{C}| \leq kn \leq n^2$. The procedure starts by constructing the auxiliary digraph $\mathcal{H} := \mathcal{H}(f)$ with the vertex set $\mathcal{C}$ as described in Section 2. If $\mathcal{H}$ has a (directed) path from the large class V^+ to a class in $\mathcal{A}_0$, then we recolor the witnesses along this path, and the procedure ends. We need to work harder when there is no such path. Then we define $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$ and related parameters as in Section 2. Then we find the related directed forest $\mathcal{F}$. By Remark 1 to Property (D1), we can find such a forest in polynomial time.

Then we do **Check 1:** We check whether for every solo edge zy with $z \in Z \in \mathcal{A}$ and $y \in B'$, the statement of Lemma 9 holds. If at least once it does not, then we return an equitable L-coloring f_i of G_i. There are at most rn solo edges and each check takes time at most n^2.

If Check 1 is successful, we call our Main Subroutine. This subroutine receives our f and returns a proper L-coloring f^* such that either

(i) f^* is an equitable L-coloring of G_i (and then the Main Procedure ends), or

(ii) f^* is nearly equitable and $|\mathcal{A}_0(f^*)| > |\mathcal{A}_0(f)|$, or
(iii) f^* is nearly equitable, $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| > |A(f)|$.

Since $|\mathcal{C}| \leq n^2$, Main Subroutine will be called at most n^3 times. So, it is enough to prove that Main Subroutine works efficiently.

C. Main Subroutine

First, we do **Check 2:** We check whether for every $v \in X \in \mathcal{A} \setminus \mathcal{A}_0$, the statement of Proposition 12(a) holds and whether there exist at least two s^--classes. If at least once the answer is "No", then we return f^* with $|\mathcal{A}_0(f^*)| > |\mathcal{A}_0(f)|$. By Theorem 5, Check 2 needs fewer that n^3 elementary operations.

If Check 2 is successful, then we perform **Check 3:** We check whether for every two solo edges zy and zy' with $z \in Z \in \mathcal{A}$, $y \in B$, $y' \in B'$ and $yy' \notin E(G)$, the statement of Lemma 13 holds. If at least once it does not, then we return f^* with $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| > |A(f)|$.

We perform Check 3 for fewer than r^2n pairs of solo edges, and for each pair, it takes at most Cn^3 elementary operations.

The rest of the proofs in Sections 2– 5 is devoted to showing that it is impossible that all Checks 1, 2, and 3 are successful. So, each time Main Subroutine will need time at most Cn^6 to complete. Hence Main Procedure will need time at most Cn^9, and the whole algorithm will take time at most Cn^{10}. This is a rough estimate and can be improved.

8. ALGORITHM FOR A FIXED r AND $k = r + 1$

Finally, we prove Theorem 6 and an algorithmic version of Theorem 4(d). Throughout this section, G is a graph with $\Delta(G) \leq r$, and L is an $(r+1)$-list assignment for G, where r is either the input to an algorithm or a fixed parameter, depending on the context, and $|G| = (r+1)s$.

Call a vertex of G *strongly dense* if it is in an $(r+1)$-clique of G; otherwise it is *strongly sparse*. Define the *core* $G' := (V', E')$ of G to be the subgraph of G induced by the strongly sparse vertices and the *remainder* $G'' := (V'', E'')$ to be the subgraph of G induced by the strongly dense vertices. Set $s' := |G'|/(r+1)$ and $s'' := |G''|/(r+1)$. Then G'' is the disjoint union of s'' disjoint $(r+1)$-cliques, $G = G' + G''$ and $s = s' + s''$. By Theorem 4(d), if $|G'| \geq c_1 := 3(r+1)r^8$ then G' is equitably $(r+1)$-choosable. Since no color can be used more than once on an $(r+1)$-clique, we have the following:

Proposition 41. *Suppose H is a graph such that $G' \subseteq H \subseteq G$ and every vertex of $G - H$ is strongly dense in $G - H$. Then every equitable L-coloring of H has an extension to an L-coloring of G, and every such extension is an equitable L-coloring of G.*

A *blocking set* Γ for (G, L) is a set of colors such that for any L-coloring of G there exists a class $\alpha \in \Gamma$ such that $|V_\alpha| > s$. Then G has an equitable L-coloring if and only if there is no blocking set for (G, L).

Proposition 42. *If there exists an $(r+1)$-list assignment L' for G' with a blocking set Γ such that $|\Gamma| \leq r + 1$, then G is not equitably $(r+1)$-choosable.*

Proof. We may assume $|\Gamma| = r + 1$. Extend L' to an $(r + 1)$-list assignment L by setting $L(v) := \Gamma$ for each strongly dense vertex v. Then for every L-coloring f there exists a color $\gamma \in \Gamma$ such that $|V_\gamma| = |V_\gamma \cap V'| + |V_\gamma \cap V''| > s' + s'' = s$. ■

Set $c_2 := c_1(r + 1)^3$.

Proposition 43. *If no $(r + 1)$-list assignment for G' has a blocking set of size at most $r + 1$, and $|G| \geq c_2 = c_1(r + 1)^3$, then G is equitably $(r + 1)$-choosable.*

Proof. Consider any $(r + 1)$-list assignment L, and let L' be the restriction of L to G'. If $|G'| \geq c_1$ then G' has an $(r + 1)$-equitable L'-coloring f', and by Proposition 41 f' can be extended greedily to an equitable L-coloring f of G.

So suppose $|G'| < c_1$. Consider any L-coloring f'' of G'', and let Γ be a set consisting of the $r + 1$ largest color classes of f'' (breaking ties arbitrarily). By hypothesis, Γ is not a blocking set for (G', L'). So there exists an L'-coloring f' of G' such that no class in Γ is oversized. Set $f := f' \cup f''$. We claim that f is an $(r + 1)$-equitable L-coloring of G.

Consider any color α. If $\alpha \in \Gamma$ then $|V_\alpha| \leq s' + s'' = s$. If $\alpha \notin \Gamma$ then $V_\alpha \cap V''$ is not one of the $r + 1$ largest classes of f'', and so $|V_\alpha \cap V''| \leq (r + 1)s''/(r + 2)$. Also, $|V_\alpha \cap V'| \leq |V'| = (r + 1)s'$. Since $s = |G|/(r + 1) \geq c_2/(r + 1) = c_1(r + 1)^2$,

$$\begin{aligned} |V_\alpha| &\leq \frac{(r+1)(s-s')}{r+2} + (r+1)s' \leq \frac{(r+1)s + (r+1)^2 s'}{r+2} \\ &\leq \frac{(r+1)s + c_1(r+1)^2}{r+2} \leq s. \end{aligned}$$

■

Proof of Theorem 6. Consider a fixed graph G as above. If $|G'| \geq c_1 = 3(r + 1)r^8$ then by Theorem 4(d), G' is equitably $(r + 1)$-choosable. Otherwise, since for every $(r + 1)$-list assignment L of G', the total number of colors in all vertex lists is at most $(r + 1)c_1$, we can check in constant time (for fixed r) whether G' has a list assignment with a blocking set of size at most $r + 1$. If it does, then by Proposition 42, G is not equitably $(r + 1)$-choosable. Moreover, as explained in Proposition 42, in this case there are infinitely many graphs G containing G' that are not equitably $(r + 1)$-choosable. If G' has no such list assignment, and $|G| \geq c_2$, then G is equitably $(r + 1)$-choosable by Proposition 43. This leaves open the case that $|G| < c_2$ (and $\omega(G) = r + 1$), but again this can be checked in constant time. So to resolve the conjecture it suffices to:

1. Check all graphs G' with $|G'| \leq c_1$ and $\omega(G) \leq r$ to see if any has an $(r + 1)$-list assignment with a nonempty blocking set of size at most $r + 1$. If there is such a graph the conjecture is false for infinitely many graphs. Otherwise continue.
2. Check all graphs G with $|G| \leq c_2$ to see if any is not equitably $(r + 1)$-choosable. In this case the conjecture is false only for finitely many graphs. Otherwise it is true.

This proves Theorem 6. ■

The next result is based on the proofs of Theorem 4(d) and Theorem 6.

Theorem 44. *Let r be a fixed parameter. Then there exists a polynomial time (in terms of n) algorithm that for any $(r + 1)$-list assignment L of any n-vertex graph G with $\Delta(G) \leq r$, either finds an equitable L-coloring of G or produces a certificate that shows that G has no such coloring.*

A. Algorithm

The input is an n-vertex graph $G = (V, E)$ with maximum degree at most r, and an $(r + 1)$-list assignment L for G. The output is either an equitable L-coloring or a certificate that demonstrates that G does not have such a coloring. Recall that G' is the core graph of G and that $c_1 = 3r^8$. Set $c_3 := c_1 + (r + 1)^2 c_1^2$.

> **Case 1:** $n \le c_3 = c_1 + (r + 1)^2 c_1^2$. Try all $(r + 1)^n \le (r + 1)^{c_3}$ L-colorings of G, and if at least one of them is proper and equitable, then return it; otherwise, all these colorings together witness that G is not equitably L-colorable.
> **Case 2:** $n > c_3$ and $|G'| \le c_1$. Let $\mathcal{C}' := \bigcup_{v\in V'} L(v)$. Then $|\mathcal{C}'| \le c_1(r + 1)$. For each color $\gamma \in \mathcal{C}'$, let $\mathcal{K}(\gamma)$ be the set of $(r + 1)$-cliques Q such that $\gamma \notin \bigcup_{v\in Q} L(v)$ or $|\bigcup_{v\in Q} L(v)| > r + 1$. Observe that if Q is an $(r + 1)$-clique, then $Q \in \mathcal{K}(\gamma)$ if and only if Q can be L-colored so that no vertex is colored with γ. Choose a maximum subset $\mathcal{K}_0(\gamma) \subseteq \mathcal{K}(\gamma)$ subject to $|\mathcal{K}_0(\gamma)| \le c_1$. Let $\mathcal{C}^- = \{\gamma \in \mathcal{C}' : |\mathcal{K}_0(\gamma)| < c_1\}$, $V^* := V' \cup \bigcup\{Q : Q \in \mathcal{K}_0(\gamma), \gamma \in \mathcal{C}'\}$ and $G^* := G[V^*]$. Then $|G^*| \le c_1 + c_1(r + 1)|\mathcal{C}'| = c_3$. It suffices to show that G has an equitable L-coloring if and only if G^* does: Then we can test by exhaustive search in constant time whether G^* has an equitable L-coloring. If we find one, we can greedily extend it to an L-coloring of G in linear time, and by Proposition 41 this extension is equitable; otherwise the record of the exhaustive search demonstrates that G has no equitable L-coloring.

Since $G' \subseteq G^* \subseteq G$ and every vertex of $G - G^*$ is strongly dense in $G - G^*$, Proposition 41 implies that if G^* has an equitable L-coloring, so does G. So suppose G has an equitable L-coloring f. It suffices to extend $f' := f|V'$ to an L-coloring f^* of G^* so that

$$|V_\gamma(f^*)| \le s^* := |V^*|/(r + 1) \text{ for all classes } V_\gamma(f^*) \text{ with } \gamma \in \mathcal{C}'. \tag{23}$$

Recall that $s' = |G'|/(r + 1)$. Let $\mathcal{O} := \{\gamma \in \mathcal{C}' : |V_\gamma(f')| > s'\}$ be the set of oversized classes of f', and for $\gamma \in \mathcal{O}$, set $s_\gamma := |V_\gamma(f')| - s'$. Then $s_\gamma > 0$.

For $\gamma \in \mathcal{C}'$, consider three cases depending on whether $\gamma \in \mathcal{C}' \setminus \mathcal{O}$, $\gamma \in \mathcal{O} \cap \mathcal{C}^-$ or $\gamma \in \mathcal{O} \setminus \mathcal{C}^-$. If $\gamma \in \mathcal{C}' \setminus \mathcal{O}$ then $|V_\gamma(f^*)| \le s^*$ for any extension of f' to V^*. So suppose $\gamma \in \mathcal{O}$.

Now consider all $\gamma \in \mathcal{O} \cap \mathcal{C}^-$. For such γ, $\mathcal{K}_0(\gamma) = \mathcal{K}(\gamma)$. Since f is equitable, there is a s_γ-subset $\mathcal{K}_1(\gamma) \subseteq \mathcal{K}(\gamma) = \mathcal{K}_0(\gamma)$ with $f(v) \ne \gamma$ for each $v \in \bigcup \mathcal{K}_1(\gamma)$. Let $\mathcal{W} := \bigcup_{\gamma\in\mathcal{O}\cap\mathcal{C}^-} \mathcal{K}_1(\gamma)$. Then $W := \bigcup \mathcal{W} \subseteq V^*$, and so $|V_\gamma(f^*)| \le s^*$ for every L-coloring f^* extending $f|(V' \cup W)$.

Finally, consider all $\gamma \in \mathcal{O} \setminus \mathcal{C}^-$. For such γ, $|\mathcal{K}_0(\gamma)| = c_1$. Choose disjoint s_γ-sets $\mathcal{K}_2(\gamma) \subseteq \mathcal{K}_0(\gamma) \setminus \mathcal{W}$. This is possible since $|\mathcal{W}| \le \sum_{\gamma\in\mathcal{O}\cap\mathcal{C}^-} s_\gamma$ and $\sum_{\gamma\in\mathcal{O}} s_\gamma \le c_1 = |\mathcal{K}_0(\gamma)|$. Then any L-coloring f^* of G^* that extends $f|(V' \cup W)$ so that $f^*(v) \ne \gamma$ for all $v \in \bigcup \mathcal{K}_2(\gamma)$ satisfies (23).

> **Case 3:** $n > c_3$ and $|G'| > c_1$. Then we know that G has an equitable L-coloring and in fact G' has an equitable L-coloring. We will construct an equitable L-coloring of G' which together with any L-coloring of $G - G'$ will yield the required equitable L-coloring of G. To do this, we repeat for G' the general outline in Section 7. Moreover, we then repeat for G' subsection *Main Procedure* of Section 7, but now our *Main Subroutine* requires more checks.

This subroutine receives a nearly equitable proper L-coloring f of G'_i and returns a proper L-coloring f^* such that either

(i) f^* is an equitable L-coloring of G'_i (and then the Main Procedure ends), or
(ii) f^* is nearly equitable and $|\mathcal{A}_0(f^*)| > |\mathcal{A}_0(f)|$, or
(iii) f^* is nearly equitable, $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| > |A(f)|$, or
(iv) f^* is nearly equitable, $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$, $|A(f^*)| = |A(f)|$, and $b'(f^*) > b'(f)$, or
(v) f^* is nearly equitable, $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$, $|A(f^*)| = |A(f)|$, and $b'(f^*) = b'(f)$ and $B'(f^*)$ has more sparse vertices than $B'(f)$.

B. Main Subroutine

First, we do for G'_i Check 2 as in Section 7, and if it fails at least once, then we return f^* with $|\mathcal{A}_0(f^*)| > |\mathcal{A}_0(f)|$. By Theorem 5, for a fixed r Check 2 takes Cn^2 elementary operations. Then we do Check 3 as in Section 7, and if it fails at least once, then we return f^* with $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| > |A(f)|$. Since we have a constant number (depending on r) of color classes of size at least s', Check 3 for a given pair of edges incident to a vertex $z \in A$ takes at most Cn_1^2 elementary operations, and we need to check at most r^2n_1 of such pairs.

Now we find the set R of vertices in B' that are rich or sparse and the set A_g of the vertices in A that are nonneighbors of R and have at least b' neighbors in B'. It will take at most Cn_1^3 elementary operations. Then we perform **Check 4:** We check whether any $z \in A_g$ is nonmovable. Again, if the check fails at least once (i.e., if some $z \in A_g$ is movable), then we stop and return f^* with $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$, $|A(f^*)| = |A(f)|$ and $b'(f^*) > b'(f)$. Check 4 needs at most Cn_1^2 elementary operations, and we do it at most $|A_g|$ times.

Next we do **Check 5:** We check whether at most r^6 vertices $y \in (B - B') \cap (N(B') - N(R))$ are movable. If the check fails, then we return f^* with $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| = |A(f)|$ that has either $b'(f^*) > b'(f)$ or $b'(f^*) = b'(f)$ and more sparse vertices in $B'(f^*)$ than in $B'(f)$. Since r is a constant, Check 5 (including the construction of f^* if needed) takes at most Cn_1^2 elementary operations.

In the next **Check 6**, we check whether for every nondefective vertex z whose neighbors in B are not movable, the set $N_B(z)$ is a clique. Again, if the check fails, then we return a nearly equitable coloring f^* of G'_i with $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| > |A(f)|$. Finally, we do **Check 7:** We check whether for every $y \in B'$ whose neighbors in A are all poor, nondefective, and not adjacent to movable vertices of B, $zz' \in E$ for all distinct $z, z' \in N_A(y)$. If the check fails for some such y, then we return a nearly equitable coloring f^* of G'_i with $|\mathcal{A}_0(f^*)| = |\mathcal{A}_0(f)|$ and $|A(f^*)| > |A(f)|$. Again, since r is a constant, Checks 6 and 7 (including the construction of f^* if needed) take at most Cn_1^2 elementary operations.

The proof of Theorem 4(d) yields that if all Checks 1–7 are successful, then G' contains a K_{r+1}, a contradiction to its choice. So, at least one of the checks fails and Main Subroutine returns a proper L-coloring f^* such that one of (i)–(v) holds. By the above estimates, one such run of Main Subroutine (including the construction of f^* if needed) takes at most $C'n_1^3$ elementary operations. The subroutine can improve criterion (v) without changing (ii)–(iv) at most n_1 times, can improve (iv) without changing (ii) and (iii) at most r times, can improve (iii) without changing (ii) at most n_1 times, and can improve

(ii) without achieving (i) at most n_1 times. Thus (since r is a constant) Main Procedure needs at most $C''n_1^6$ elementary operations. We have n_1 graphs G_i', and so run Main Procedure at most n_1 times. Therefore, for the whole G, the complexity is at most $C''n^7$.

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