Decomposition of Sparse Graphs into Forests and a Graph with Bounded Degree

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Abstract: For a loopless multigraph G, the *fractional arboricity* $\mathrm{Arb}(G)$ is the maximum of $\frac{|E(H)|}{|V(H)|-1}$ over all subgraphs H with at least two vertices. Generalizing the Nash-Williams Arboricity Theorem, the Nine Dragon Tree Conjecture asserts that if $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests with one having maximum degree at most d. The conjecture was previously proved for $(k, d) \in \{(1,1), (1,2)\}$; we prove it for $d = k+1$ and when $k = 1$ and $d \le 6$. For $(k, d) = (1, 2)$, we can further restrict one forest to have at most two edges in each component.

For general (k, d), we prove weaker conclusions. If $d > k$, then $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$ implies that G decomposes into k forests plus a multigraph (not necessarily a forest) with maximum degree at most d. If $d \le k$, then $\mathrm{Arb}(G) \le k + \frac{d}{2k+2}$ implies that G decomposes into $k+1$ forests, one having maximum degree at most d. Our results generalize earlier results about decomposition of sparse planar graphs. J. Graph Theory 74: 369–391, 2013

Keywords: *graph decomposition; forest; fractional arboricity; maximum average degree; discharging*

1. INTRODUCTION

Throughout this article, we use the model of "graph" in which multiedges but no loops are allowed. A *decomposition* of a graph G consists of edge-disjoint subgraphs with union G. The *arboricity* of G, written $\Upsilon(G)$, is the minimum number of forests needed to decompose it. The famous Nash-Williams Arboricity Theorem [13] states that a necessary and sufficient condition for $\Upsilon(G) \le k$ is that no subgraph H has more than $k(|V(H)|-1)$ edges.

Payan [14] defined $\mathrm{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$ as the *fractional arboricity* of G. The Nash-Williams Theorem says $\Upsilon(G) = \lceil \mathrm{Arb}(G) \rceil$. If G has fractional arboricity $k + \epsilon$ (with $k \in \mathbb{N}$ and $0 < \epsilon \le 1$), then $k+1$ forests are needed to decompose it. When ϵ is small, one may hope to place some restrictions on the last forest, since k forests are "almost" enough to decompose G. The *Nine Dragon Tree (NDT) Conjecture*[1] asserts that one can bound the maximum degree of the last forest in terms of ϵ. Call a graph *d-bounded* if its maximum degree is at most d.

Conjecture 1.1 (NDT Conjecture [11]). *If $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests, one of which is d-bounded.*

Montassier, Ossona de Mendez, Raspaud, and Zhu [11] posed the conjecture, showed that the condition cannot be relaxed, and proved the conjecture for $(k, d) \in \{(1,1), (1,2)\}$. In this article, we prove it in the following cases:

- $d = k+1$;
- $k = 1$ and $d \le 6$.

[1]The Nine Dragon Tree is a banyan tree atop a mountain in Kaohsiung, Taiwan; it is far from acyclic.

For $(k, d) = (1, 2)$, we prove the stronger result that if $\text{Arb}(G) \leq 1 + \frac{1}{2}$, then G decomposes into two forests, one of which has at most two edges in each component.

For general (k, d), we prove weaker versions of the conjecture.

- If $d > k$ and $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into k forests and a d-bounded graph (instead of a d-bounded forest). In fact, we prove that a weaker condition implied by $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ also suffices, and that result is sharp.
- If $d \leq k$ and $\text{Arb}(G) \leq k + \frac{d}{2k+2}$, then G decomposes into k forests and a d-bounded forest.

Our results generalize earlier results about decomposition of planar graphs. For convenience, let a (k, d)-*decomposition* of a graph G be a decomposition of G into k forests and one d-bounded graph, and let a $(k, d)^*$-*decomposition* be a decomposition of G into k forests and one d-bounded forest. Graphs having such decompositions are (k, d)-*decomposable* or $(k, d)^*$-*decomposable*, respectively.

Motivated by an application to "game coloring number," He et al. [8] initiated the study of $(1, d)$-decomposition of planar graphs with large girth, proving that those with girth at least 5 are $(1, 4)$-decomposable. For $d = 2$, planar graphs were shown to be $(1, 2)$-decomposable when they have girth at least 7 (in [8]) and at least 6 (in [10]), with non-$(1, 2)$-decomposable examples of girth 5 in [10, 11]. Planar graphs were shown to be $(1, 1)$-decomposable when they have girth at least 11 (in [8]), 10 (in [2]), 9 (in [6]), or 8 (in [11, 15]). Planar graphs with girth 7 that are not $(1, 1)$-decomposable appear in [10, 11].

Gonçalves [7] proved that every planar graph is $(2, 4)^*$-decomposable, as conjectured by Balogh et al. [1]. He also proved that those with girth at least 6 are $(1, 4)^*$-decomposable and those with girth at least 7 are $(1, 2)^*$-decomposable. With fractional arboricity of planar graphs arbitrarily close to 3, the NDT Conjecture cannot guarantee them being $(2, d)^*$-decomposable for any constant d. However, girth at least 6 or 7 yields $\text{Arb}(G) < 6/4$ or $\text{Arb}(G) < 7/5$, respectively, in which case it guarantees $(1, 4)^*$- or $(1, 2)^*$-decompositions. Hence our proof of the NDT Conjecture for (k, d) with $k = 1$ and $d \leq 6$ implies Gonçalves' results for $(1, d)^*$-decomposition but not for $(2, 4)^*$-decomposition.

The *maximum average degree* of a graph G, denoted $\text{Mad}(G)$, is $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$; it is the maximum over subgraphs H of the average vertex degree in H. Many conclusions on planar graphs with large girth need only the corresponding bound on $\text{Mad}(G)$. A planar graph on n vertices with finite girth g has at most $\frac{g}{g-2}(n-2)$ edges, by Euler's Formula; thus it satisfies $\text{Mad}(G) < \frac{2g}{g-2}$. Subgraphs of graphs with girth g have girth at least g. In particular, forests can be viewed as having infinite girth, and their average degree is less than 2. Thus $\text{Mad}(G) < \frac{2g}{g-2}$ when G is planar with girth at least g.

As stated above, the sharp requirements on girth of a planar graph G for being $(1, d)$-decomposable are 8, 6, and 5 when d is 1, 2, or 4, respectively. Such a graph has $\text{Mad}(G)$ less than 8/3, 3, or 10/3, respectively. By our results, these bounds imply that G is $(1, d)$-decomposable. Thus our results for bounded $\text{Mad}(G)$ generalize the earlier sharp results on decomposition of planar graphs with large girth.

In [12], Montassier et al. posed the question of finding the loosest upper bound on $\text{Mad}(G)$ that guarantees decomposition into a forest and a d-bounded graph. They proved that $\text{Mad}(G) < 4 - \frac{8d+12}{d^2+6d+6}$ is sufficient and that $\text{Mad}(G) = 4 - \frac{4}{d+2}$ is not sufficient (as seen by subdividing every edge of a $(2d + 2)$-regular graph). The case $k = 1$ of our Theorem 1.2 completely solves this problem, implying that $\mathrm{Mad}(G) < 4 - \frac{4}{d+2}$ is sufficient.

Upper bounds on $\mathrm{Arb}(G)$ or $\mathrm{Mad}(G)$ are "sparseness" conditions. These parameters are similar but not identical: $\mathrm{Mad}(G) < 2\mathrm{Arb}(G)$ always holds, but $\mathrm{Mad}(G) < 2\rho$ is a bit weaker than $\mathrm{Arb}(G) \le \rho$. Our main result uses another sparseness condition between them.

To compute $\mathrm{Arb}(G)$ or $\mathrm{Mad}(G)$, it suffices to perform the maximization only over induced subgraphs. Letting $G[A]$ denote the subgraph of G induced by a vertex set A, we write $\|A\|$ for the number of edges in $G[A]$ (and $|A|$ for the number of vertices). We restate the conditions $\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$ and $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$ as integer inequalities and in the same format define an intermediate condition of being (k, d)-*sparse* (the restatement of the conditions uses the equality $k(k + d + 1) + d = (k + 1)(k + d)$):

Condition	Equivalent constraint (when imposed for all $A \subseteq V(G)$)
$\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$	$(k+1)(k+d)\lvert A\rvert - (k+d+1)\|A\| - (k+1)(k+d) \ge 0$
(k, d)−sparse	$(k+1)(k+d)\lvert A\rvert - (k+d+1)\|A\| - k^2 \ge 0$
$\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d)\lvert A\rvert - (k+d+1)\|A\| - 1 \ge 0$

With this definition, we can state our general result for $d > k$.

Theorem 1.2. *For $d > k$, every (k, d)-sparse graph is (k, d)-decomposable. Furthermore, the condition is sharp.*

Since $(k + 1)(k + d) > k^2 \ge 1$, the condition on $\mathrm{Arb}(G)$ implies (k, d)-sparseness, which in turn implies the condition on $\mathrm{Mad}(G)$. By showing that (k, d)-sparseness suffices, Theorem 1.2 thus implies that $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$ suffices for G to be (k, d)-decomposable, but $\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$ might not. However, since $k^2 = 1$ when $k = 1$, the condition of $(1, d)$-sparseness is equivalent to $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2} = 4 - \frac{4}{d+2}$.

In Section 2, we prove Theorem 1.2 by giving an inductive proof of a more general statement in which varying degree bounds are imposed on the vertices in the d-bounded graph. Further motivation for the additive constant in the definition of (k, d)-sparseness comes from the sharpness example in Section 2.

Theorem 1.2 omits the case $d \le k$. In Section 3, we prove a result implying that a stronger condition on $\mathrm{Arb}(G)$ than that in the NDT Conjecture guarantees a $(k, d)^*$-decomposition when $d \le k + 1$. The condition is $\mathrm{Arb}(G) \le k + \frac{d}{2k+2}$. When $d = k + 1$, this bound equals $k + \frac{d}{k+d+1}$, so this theorem implies the case $d = k + 1$ of the NDT Conjecture.

Around the same time that we obtained our result, Király and Lau [9] also considered (k, d)-decomposability. In our terminology, they showed that G is (k, d)-decomposable when $\mathrm{Arb}(G) \le k + \frac{d-1}{k+d}$. Our result is stronger when $d \ge k$, but for $d < k$ neither implies the other; their arboricity bound is looser than ours, applying to more graphs, but our conclusion is stronger, guaranteeing that the last graph is a d-bounded forest rather than just a d-bounded graph. Their methods are different, applying linear programming and a result on matroids.

In Sections 4–6, we prove the NDT Conjecture for $(k, d) = (1, d)$ with $d \le 6$, in a form that requires only (k, d)-sparseness as long as small graphs violating $\mathrm{Arb}(G) \le k + 1$ are forbidden. Meanwhile, the *Strong NDT Conjecture* asserts that $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$ guarantees a $(k, d)^*$-decomposition in which every component of the d-bounded forest has at most d edges. We prove this for $(k, d) = (1, 2)$ in Section 7 (the result of [11] implies it for $(k, d) = (1, 1)$). The results of Sections 4–7 use reducible configurations and discharging.

2. (k, d)-DECOMPOSITION FOR $d > k$

We begin with a general example showing that Theorem 1.2 is sharp. This example also motivates the additive constant in the condition for (k, d)-sharpness. We restate the definition of (k, d)-sparseness to introduce convenient notation for the proofs.

Definition 2.1. *A graph G is (k, d)-sparse if $\beta(A) \geq 0$ for every nonempty vertex subset A, where*

$$\beta(A) = (k+1)(k+d)|A| - (k+1+d)\|A\| - k^2. \tag{1}$$

For each choice of $k, d \in \mathbb{N}$, we construct a sequence of bipartite graphs that are not (k, d)-decomposable and yet just barely fail to be (k, d)-sparse. Figure 1 illustrates the construction for $(k, d) = (2, 1)$. Vertices in X have degree 3, so the graph has 42 edges. With 19 vertices, only 36 edges can be absorbed by two forests. A 1-bounded subgraph (a matching) has at most five edges, so at most 41 edges can be absorbed by a $(2, 1)$-decomposition.

Proposition 2.2. *Fix $k, d, t \in \mathbb{N}$ with $t > k$. Let $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$, where $s = t(k+d) - k + 1$. For $1 \leq i \leq s$, make x_i adjacent to $y_i, \ldots, y_{i+k}$, with indices in Y taken modulo t. The resulting bipartite graph G is not (k, d)-decomposable, even though $\beta(A) \geq 0$ for every nonempty proper vertex subset A, but $\beta(V(G)) = -1$.*

Proof. Every vertex in X has degree $k+1$, so $|E(G)| = (k+1)(k+d)t - k^2 + 1$. Since $|Y| = t$, a d-bounded subgraph of G has at most dt edges. Deleting a d-bounded subgraph thus leaves at least $k(k+d)t + kt - k^2 + 1$ edges. Since $|V(G)| = t(k+d+1) - k + 1$, any k forests in G cover at most $k[t(k+d+1) - k]$ edges. Hence G is not (k, d)-decomposable.

Now consider $\beta(A)$ for $A \subseteq V(G)$. Choose A to minimize β among nonempty subsets of $V(G)$, and suppose $\beta(A) < 0$. If $|A| = 1$, then $\beta(A) = kd + k + d > 0$, so $|A| > 1$. If some vertex $v \in A$ has at most k neighbors in A, then $\beta(A - v) \leq \beta(A) - d$, contradicting the choice of A. Therefore, all $k+1$ neighbors of each vertex in $A \cap X$ are also in A. Let $s' = |A \cap X|$ and $t' = |A \cap Y|$. Now

$$\begin{aligned}\beta(A) &= (k+1)(k+d)(s'+t') - (k+d+1)(k+1)s' - k^2 \\ &= (k+1)(k+d)t' - s'(k+1) - k^2 = (k+1)[(k+d)t' - s' - k + 1] - 1.\end{aligned}$$

For such A, the condition $\beta(A) < 0$ becomes $s' > (k+d)t' - k$. If $Y \subseteq A$, then $t' = t$, and we have $\beta(A) < 0$ if and only if $s' = s$ and $A = V(G)$.

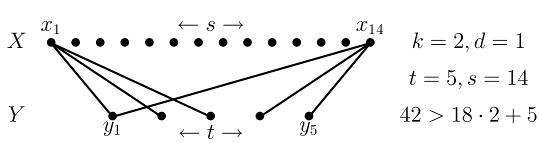

FIGURE 1. A graph with no (k, d)-decomposition.

If $t' < t$, then each vertex of $Y - A$ forbids all its neighbors from A. For fixed t', the largest s' and hence smallest $\beta(A)$ occurs when $Y \cap A = \{y_1, \ldots, y_{t'}\}$ (this makes the subsets forbidden from X overlap as much as possible, allowing X to be larger). Writing $i = qt + r$ with $q \geq 0$ and $1 \leq r \leq t$, we then have $x_i \in A$ only when $1 \leq r \leq t' - k$. With $s = t(k + d) - k + 1$, this yields $s' \leq (k + d)(t' - k) < (k + d)t' - k$. We conclude that $\beta(A) \geq 0$ except when $A = V(G)$. ■

Using k^2 as the additive term in the definition of β has enabled us to construct a non-(k, d)-decomposable graph with the smallest possible failure of (k, d)-sparseness.

Theorem 1.2 allows degree d at each vertex in the special subgraph. We prove the theorem in a seemingly more general form to facilitate the inductive proof and avoid the discharging method, considering varying "capacities" at vertices. Lemma 4.7 will show that the more general form is equivalent to Theorem 1.2.

Definition 2.3. *Fix positive integers d and k. A* capacity function *on a graph G is a function $f : V(G) \to \{0, \ldots, d\}$. A (k, f)*-decomposition *of G decomposes it into k forests and a graph D such that each vertex v has degree at most $f(v)$ in D. For each vertex set A in G, let*

$$\beta_f(A) = (k + 1)\sum_{v \in A}(k + f(v)) - (k + 1 + d)\|A\| - k^2. \tag{2}$$

A capacity function f on G is feasible *if $\beta_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$.*

When $f(v) = d$ for all $v \in V(G)$, the value of $\beta_f(A)$ in (2) reduces to the value of $\beta(A)$ in (1), and the condition for feasibility of f becomes the condition for (k, d)-sparseness of G.

Capacity functions can reserve an edge uv for use in D: we delete uv and reduce the capacity of its endpoints. In order to reduce the problem to a case where we can show feasibility of the reduced function f' on $G - uv$, we will apply the induction hypothesis to various subgraphs of G obtained by contraction or deletion of subgraphs.

Definition 2.4. *For $B \subseteq V(G)$, let G_B denote the graph obtained by contracting B into a new vertex z. Let $f|_B$ denote the restriction of f to B.*

Note that the degree of z in G_B is the number of edges joining B to $V(G) - B$ in G. Edges of G with both endpoints in B disappear.

Lemma 2.5. *If f is a feasible capacity function on G, and B is a proper subset of $V(G)$ such that $|B| \geq 2$ and $\beta_f(B) \leq k$, then f^* is a feasible capacity function on G_B, where $f^*(z) = 0$ and f^* agrees with f on $V(G) - B$.*

Proof. Consider $A \subseteq V(G_B)$. If $z \notin A$, then $\beta_{f^*}(A) = \beta_f(A) \geq 0$. When $z \in A$, we compare $\beta_{f^*}(A)$ with $\beta_f(A')$, where $A' = (A - \{z\}) \cup B$ (see Figure 2). Every edge in $G[A']$ appears in $G_B[A]$ or $G[B]$; hence the edges contribute the same amount to both sides of the equation below. Comparing the terms for constants and the terms for vertices

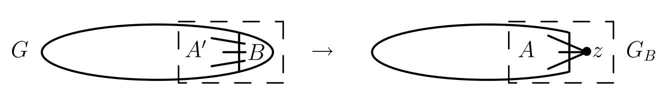

FIGURE 2. Feasibility of contractions.

(using $f^*(z) = 0$) yields

$$\beta_f(A') = \beta_{f^*}(A) - (k+1)k + \beta_f(B) + k^2.$$

If $\beta_f(B) \le k$, then $\beta_{f^*}(A) \ge \beta_f(A') \ge 0$. ■

Definition 2.6. *When writing (F, D) for a decomposition of a graph, the subgraph F will denote a union of k forests.*

Lemma 2.7. *Let f be a capacity function on a graph G, and let B be a proper subset of $V(G)$. If $G[B]$ is $(k, f|_B)$-decomposable and G_B is (k, f^*)-decomposable, where f^* is defined from f as in Lemma 2.5, then G is (k, f)-decomposable.*

Proof. Let (F, D) be a $(k, f|_B)$-decomposition of $G[B]$, and let (F', D') be a (k, f^*)-decomposition of G_B. Each edge of G is in $G[B]$ or G_B, becoming incident to z in G_B if it joins B to $V(G) - B$ in G. View $(F \cup F', D \cup D')$ as a decomposition of G by viewing edges incident to z in F' as corresponding edges in G.

The resulting decomposition is a (k, f)-decomposition of G. Since $f^*(z) = 0$, vertex z has degree 0 in D', and all edges joining B to $V(G) - B$ lie in F'. Hence the restrictions from f are satisfied by $D \cup D'$. For each forest F_i among the k forests in F, its union with the corresponding forest F_i' in F' is still a forest, since otherwise contracting the portion in F_i of a resulting cycle would yield a cycle through z in F_i' when viewed as a forest in G'. ■

When β_f is small on some nontrivial subset, we will use Lemmas 2.5 and 2.7 to produce the (k, f)-decomposition. Otherwise, when u and v are adjacent and have positive capacity, we will apply the induction hypothesis to $G - uv$ with their capacity reduced. That leaves the case where the vertices with positive capacity are independent; in this case a (k, f)-decomposition consists of k forests, and Nash-Williams' Arboricity Theorem applies.

Theorem 2.8. *If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.*

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most k edges. Hence we may assume $|E(G)| > k$.

If $\beta_f(B) \le k$ for some proper subset B of $V(G)$ with $|B| \ge 2$, then the capacity function f^* on G_B that agrees with f except for $f^*(z) = 0$ is feasible, by Lemma 2.5. Since $G[B]$ is an induced subgraph of G, the restriction of f to B is feasible on $G[B]$. Since G_B and $G[B]$ are smaller than G, by the induction hypothesis G_B is $(k, f|_B)$-decomposable and G_B is (k, f^*)-decomposable. By Lemma 2.7, G is (k, f)-decomposable.

Hence we may assume that $\beta_f(B) \ge k + 1$ when $B \subset V(G)$ and $|B| \ge 2$. Let $S = \{v \in V(G) : f(v) > 0\}$. If S has adjacent vertices u and v, then let f' be the capacity function on $G - uv$ that agrees with f except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. If f' is feasible, then since $G - uv$ is smaller than G, it has a (k, f')-decomposition, and we add uv to the degree-bounded subgraph to obtain a (k, f)-decomposition of G.

To show that f' is feasible, consider $A \subseteq V(G') = V(G)$. If $u, v \notin A$, then $\beta_{f'}(A) = \beta_f(A)$. If $u, v \in A$, then the reduction in f and loss of one edge yield $\beta_{f'}(A) = \beta_f(A) - 2(k+1) + (k+d+1) \ge \beta_f(A)$, where the last inequality uses $d > k$. If exactly one of

$\{u, v\}$ is in A, then A is a proper subset of $V(G)$. If $|A| \geq 2$, then $\beta_{f'}(A) = \beta_f(A) - (k + 1) \geq 0$. If $|A| = 1$, then $\beta_{f'}(A) \geq k$, since G' has no loops.

Hence we may assume that S is independent. In this case, we decompose G into k forests, yielding a (k, f)-decomposition of G in which the f-bounded subgraph has no edges. If no such decomposition exists, then by Nash-Williams' Theorem $V(G)$ has a minimal subset A such that $\|A\| \geq k(|A| - 1) + 1$ (note that $|A| \geq 2$). By the minimality of A, every vertex of A has at least $k + 1$ neighbors in A. Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k + 1)|A'|$. Taking $k + 1$ times the first lower bound on $\|A\|$ plus d times the second yields

$$(k + 1 + d)\|A\| \geq (k + 1)k(|A| - 1) + (k + 1) + d(k + 1)|A'|.$$

Now we compute

$$\begin{aligned}
\beta_f(A) &= (k + 1)k|A| + (k + 1)\sum_{v \in A'} f(v) - (k + d + 1)\|A\| - k^2 \\
&\leq (k + 1)k|A| + (k + 1)d|A'| - (k + 1)k(|A| - 1) - (k + 1) - d(k + 1)|A'| - k^2 \\
&= (k + 1)k - (k + 1) - k^2 = -1.
\end{aligned}$$

This contradicts the feasibility of f, and hence the desired decomposition of G exists. ■

3. $(k, d)^*$-DECOMPOSITION FOR $d \leq k + 1$

The capacity function f in Section 2 controls vertex degrees to facilitate inductive construction of a (k, d)-decomposition, but it cannot prevent creation of cycles when we return a deleted edge. We introduce another property to do this.

Definition 3.1. *A strong $(k, f)^*$-decomposition is a $(k, f)^*$-decomposition in which each component of the degree-bounded forest contains at most one vertex v such that $f(v) < d$.*

We will now apply the induction hypothesis to $G - uv$ with reduced capacity function f' only when at least one endpoint of uv has capacity d. In $G - uv$, these endpoints will be the only vertices with capacity less than d in their components in D. We will thus be able to add uv to D without introducing cycles. Since u or v has capacity d in f, the resulting decomposition of G is also strong. This approach will reduce the problem to the case where no edge joins a vertex with capacity d to a vertex with positive capacity; in this case, we will decompose G into k forests as in the final step of Theorem 2.8.

We must also strengthen the sparseness condition. If G consists of two vertices and an edge of multiplicity $k + 2$, and $f(u) = f(v) = d$, then $\beta(A) \geq 0$ for all A, but G does not decompose into $k + 1$ forests. Another auxiliary function will exclude such examples.

Definition 3.2. *Given a capacity function f on $V(G)$ using capacities at most d, let $S = \{v \in V(G) : f(v) = d\}$. For $A \subseteq V(G)$, let $f(A) = \sum_{v \in A} f(v)$ and $\hat{f}(A) = \min\{f(x) : x \in A\}$. Define α_f and β_f^* on subsets of G as follows:*

$$\begin{aligned}
\alpha_f(A) &= k|A| - k - \|A\| + |A \cap S|, \\
\beta_f^*(A) &= (2k + 2 - d)k|A| + (k + 1)(f(A) - 2\|A\|) - (k - 1)(2k + 2 - d).
\end{aligned}$$

Say that f is strongly feasible when $\beta^*_f(A) > 0$ and $\alpha_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$, with strict inequality $\alpha_f(A) > 0$ whenever $A \subseteq S$.

The condition on α_f is necessary for a strong $(k, f)^*$-decomposition. Requiring $\alpha_f(A) \geq 0$ allows A to induce k forests plus $|A \cap S|$ edges in the last forest. However, the last forest cannot have $|A|$ edges within A, so when $A \subseteq S$ we must require $\alpha_f(A) > 0$.

With these definitions, we can state the main result of this section.

Theorem 3.3. *If $d \leq k+1$ and f is a strongly feasible capacity function on a graph G, then G has a strong $(k, f)^*$-decomposition.*

To compare with fractional arboricity, the condition $\mathrm{Arb}(G) \leq k + \frac{d}{2k+2}$ is equivalent to

$$(2k+2-d)k|A| + (k+1)(d|A| - 2\|A\|) - k(2k+2) - d \geq 0 \quad \text{for } A \subseteq V(G).$$

When $f(v) = d$ for all v, this condition becomes $\beta^*_f(A) \geq 2k + kd + 2$ for $A \subseteq V(G)$, which is more restrictive than strong feasibility.

Corollary 3.4. $\mathrm{Arb}(G) \leq k + \frac{d}{2k+2}$ *guarantees that G is $(k, d)^*$-decomposable. In particular, the NDT Conjecture holds when $d = k+1$.*

Proof. We have observed that $\mathrm{Arb}(G) \leq k + \frac{d}{2k+2}$ implies $\beta^*_f(A) > 0$ for all A when $f(v) = d$ for all v. With this capacity function, $|A \cap S| = |A|$ for all $A \subseteq V(G)$, and the condition $\alpha_f(A) \geq 1$ (since $A \subseteq S$) becomes $\|A\| \leq (k+1)(|A| - 1)$, which is true for all A when $\mathrm{Arb}(G) < k+1$. Hence Theorem 3.3 applies.

When $d = k+1$, we have $d + k + 1 = 2k+2$, so $\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$ is sufficient. ■

We next prove a useful bound on β^*_f in terms of α_f for the case $d \leq k+1$.

Lemma 3.5. *Given $d \leq k+1$, a capacity function f on a graph G, and a set $A \subseteq V(G)$ with $|A| \geq 2$,*

$$\beta^*_f(A) \leq (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S| + 1].$$

In particular, if $\alpha_f(A) \leq 0$ and $\beta^*_f(A) > 0$ with $A \not\subseteq S$, then $f(x) \geq |A \cap S|$ for all $x \in A$.

Proof. Substituting $\|A\| = k|A| - k - \alpha_f(A) + |A \cap S|$ into the formula for $\beta^*_f(A)$ yields

$$\beta^*_f(A) = -dk|A| + (k+1)[2\alpha_f(A) + f(A) - 2|A \cap S|] + (2k+2) + d(k-1).$$

Summing capacities over $x \in A$ yields $f(A) \leq (d-1)|A| + |A \cap S| + \hat{f}(A) - (d-1)$ (the inequality is strict when $A \subseteq S$). Substituting this into the formula above yields

$$\begin{aligned}
\beta^*_f(A) &\leq -dk|A| + (k+1)(d-1)|A| + (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] \\
&\quad + 3k + 3 - 2d \\
&= (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + (d-k-1)|A| + 3k + 3 - 2d \\
&\leq (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + k + 1,
\end{aligned}$$

where the last inequality uses $|A| \geq 2$ and $d \leq k+1$. ■

We need analogues of Lemmas 2.5 and 2.7, with G_B as defined in Section 2.

Lemma 3.6. *For $d \le k+1$, let f be a strongly feasible capacity function on G, and let B be a proper subset of $V(G)$ with $|B| \ge 2$. Define f^* and $\bar{f}$ on G_B by $f^*(z) = \hat{f}(B) - |B \cap S|$ and $\bar{f}(z) = 0$, letting both functions agree with f on $V(G) - B$. If $\alpha_f(B) = 0$, then f^* is strongly feasible. If $\beta_f^*(B) \le k+1$, then $\bar{f}$ is strongly feasible.*

Proof. First, consider the case $\alpha_f(B) = 0$. As observed in Lemma 3.5, $\hat{f}(B) \ge |B \cap S|$ when $\alpha_f(B) = 0$. Hence $f^*(z) \ge 0$, so f^* is a capacity function. Since f is strongly feasible and $\alpha_f(B) = 0$, we have $\beta_f^*(B) > 0$ and $B \not\subseteq S$. Since $\hat{f}(B) = d$ only if $B \subseteq S$, we must have $f^*(z) < d$, so the set S is the same for f^* and f.

If $z \notin A \subseteq V(G_B)$, then $\beta_{f^*}^*(A) = \beta_f^*(A)$ and $\alpha_{f^*}(A) = \alpha_f(A)$. When $z \in A$, we compute $\alpha_{f^*}(A)$ and $\beta_{f^*}^*(A)$ from $\alpha_f(A')$ and $\beta_f^*(A')$, where $A' = (A - \{z\}) \cup B$. As in Lemma 2.5, $|A'| = |A| - 1 + |B|$ and $\|A'\| = \|A\| + \|B\|$, where $\|A\|$ counts edges in G_B. Hence

$$\alpha_f(A') = \alpha_{f^*}(A) + \alpha_f(B);$$
$$\beta_f^*(A') = \beta_{f^*}^*(A) + \beta_f^*(B) - (k+1)f^*(z) - (2k+2-d).$$

Since $\alpha_f(B) = 0$, we obtain $\alpha_{f^*}(A) = \alpha_f(A') \ge 0$, as desired since $f^*(z) < d$. By Lemma 3.5, $\alpha_f(B) = 0$ implies $\beta_f^*(B) \le (k+1)[\hat{f}(B) - |B \cap S| + 1] = (k+1)f^*(z) + (k+1)$. Now $\beta_{f^*}^*(A) \ge \beta_f^*(A') + k + 1 - d \ge \beta_f^*(A') > 0$.

For $\bar{f}$, again it suffices to check A with $z \in A \subseteq V(G_B)$ and let $A' = (A - \{z\}) \cup B$. Now

$$\beta_f^*(A') = \beta_{\bar{f}}^*(A) + \beta_f^*(B) - (2k+2-d) \le \beta_{\bar{f}}^*(A),$$

where we have used $\beta_f^*(B) \le k+1$, $\bar{f}(z) = 0$, and $k+1-d \ge 0$. We also need $\alpha_{\bar{f}}(A) \ge 0$. With $\beta_{\bar{f}}^*(A) \ge \beta_f^*(A') > 0$ and $\bar{f}(z) = 0$, this follows from Lemma 3.5. ■

Lemma 3.7. *Let f be a capacity function on G, and let B be a proper subset of $V(G)$ with $|B| \ge 2$. If $G[B]$ is strongly $(k, f|_B)^*$-decomposable and G_B is strongly $(k, f^*)^*$-decomposable, with f^* defined from f as in Lemma 3.6, then G is strongly $(k, f)^*$-decomposable.*

Proof. Let (F, D) be a strong $(k, f|_B)^*$-decomposition of $G[B]$, and let (F', D') be a strong $(k, f^*)^*$-decomposition of G_B. Each edge of G is in $G[B]$ or G_B, becoming incident to z in G_B if it joins B to $V(G) - B$ in G. Viewing F' and D' as subgraphs of G, we show that $(F \cup F', D \cup D')$ is a strong $(k, f)^*$-decomposition of G.

As in Lemma 2.7, the union of any forest F_i in F with the corresponding forest F_i' in F' is still a forest, since otherwise contracting the portion in F_i of a resulting cycle would yield a cycle through z in F_i' when viewed as a forest in G'. This argument applies also to $D \cup D'$.

Recall that $S = \{v \in V(G) : f(v) = d\}$. If $\hat{f}(B) = d$, then $B \subseteq S$; we conclude that $f^*(z) < d$. Since (F', D') is a strong $(k, f^*)^*$-decomposition, $f^*(z) < d$ implies that vertices other than z in its component in D' lie in S. Therefore, each component of $D \cup D'$ in G has at most one vertex outside S.

Since $D \subseteq G[B]$ and each component of D has at most one vertex outside S, each vertex v of B has at most $|B \cap S|$ neighbors in D. By the definition of $f^*(z)$, vertex v

gains at most $\hat{f}(B) - |B \cap S|$ neighbors in D'; together it has at most $f(v)$ neighbors in $D \cup D'$. ■

***Proof of Theorem* 3.3.** *If $d \leq k+1$ and f is a strongly feasible capacity function on a graph G, then G has a strong $(k, f)^*$-decomposition.*

Proof. We use induction on $|V(G)| + |E(G)|$; statement is trivial when $|E(G)| \leq k$. Hence we may assume $|E(G)| > k$. Recall that $S = \{v \in V(G) : f(v) = d\}$. Let $R = \{v \in V(G) : f(v) = 0\}$, and let $T = V(G) - S - R$. We prove:

$$\text{If } G \text{ has no strong}(k, f)^*\text{-decomposition,}$$
$$\text{then } G \text{ has no edge } uv \text{ with } u \in S \text{ and } v \in S \cup T. \qquad (*)$$

Suppose that G has such an edge uv. We choose uv with $v \in T$ if possible; otherwise, $v \in S$. Let $G' = G - uv$, and let f' be the capacity function on G' that agrees with f except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. Note that $u \notin \{x : f'(x) = d\}$. If we can show that f' is strongly feasible, then since $G - uv$ is smaller than G, it has a strong $(k, f')^*$-decomposition (F, D). Since $f'(u) < d$ and $f(u) = d$, adding the edge uv to D yields a strong $(k, f)^*$-decomposition of G.

To prove $(*)$, it thus suffices to show that f' is strongly feasible. We consider $\alpha_{f'}(A)$ and $\beta^*_{f'}(A)$. If $|A| = 1$, then $\alpha_{f'}(A) = |A \cap S|$ (positive if $A \subseteq S$). Also, $\beta^*_{f'}(A) = (2k + 2 - d) + (k+1)f(A) \geq 2k + 2 - d > 0$, since $d \leq k+1$.

Next consider $A = V(G)$. Since $u, v \in A$, we have $\beta^*_{f'}(A) = \beta^*_f(A) > 0$. Also, $\alpha_{f'}(A) < \alpha_f(A)$ requires $u, v \in S$. Not all vertices satisfy $f'(x) = d$, since $f'(u) < d$. Therefore, having $\alpha_f(A) \geq 1$ and $\alpha_{f'}(A) \geq 0$ suffices, so we may assume $\alpha_f(A) = 0$. With $A = V(G)$ and $u, v \in S$, the choice of uv in defining f' implies that no edges join S and T. Since $\alpha_f(A) = 0$ implies $A \not\subseteq S$, we have $R \cup T \neq \varnothing$. If $R \neq \varnothing$, then $\hat{f}(A) = 0$, contradicting Lemma 3.5. Hence $R = \varnothing$. Since no edges join S and T, now G is disconnected, and we can combine strong decompositions of the components obtained from the induction hypothesis.

Finally, suppose $2 \leq |A| < |V(G)|$. If $\alpha_f(A) = 0$, then the capacity function f^* on G_A that agrees with f except for $f^*(z) = \hat{f}(A) - |A \cap S|$ is strongly feasible, by Lemma 3.6. Also, the restriction of f to A is strongly feasible on $G[A]$. Since G_A and $G[A]$ are smaller than G, by the induction hypothesis $G[A]$ is strongly $(k, f|_A)^*$-decomposable and G_A is strongly $(k, f^*)^*$-decomposable. By Lemma 3.7, G is strongly $(k, f)^*$-decomposable.

Hence we may assume that $\alpha_f(A) > 0$. If $\alpha_{f'}(A) < \alpha_f(A)$, then u or v is in $A \cap S$, and the difference is at most 1. Hence $\alpha_{f'}(A) \geq 0$, which is good enough since $f'(u), f'(v) < d$. If $\beta^*_{f'}(A) > 0$, then A causes no problem.

Otherwise, $\beta^*_f(A) \leq k+1$, since reduction of β^* requires $|A \cap \{u, v\}| = 1$, and the reduction is then by $k+1$. Now Lemma 3.6 implies that $\bar{f}$ is strongly feasible on G_A, where $\bar{f}(z) = 0$ and otherwise $\bar{f}$ agrees with f. By the induction hypothesis, G_A has a strong $(k, \bar{f})^*$-decomposition (F, D), and $G[A]$ has a strong $(k, f|_A)^*$-decomposition (F', D'). As in Lemma 3.7, $(F \cup F', D \cup D')$ is a strong $(k, f)^*$-decomposition of G; since z is isolated in D, the components of D' do not extend.

Hence we may assume that S is independent and that no edge joins S and T. As in Theorem 2.8, we claim that G decomposes into k forests, completing the desired decomposition. Otherwise, we find a set A such that $\beta^*_f(A) \leq 0$, contradicting strong feasibility.

Note that $\beta_f^*(A) = (2k+2-d)g(A) + h(A)$, where $g(A) = k(|A|-1) - \|A\| + 1$ and $h(A) = (k+1)f(A) - d\|A\|$. It suffices to find A such that $g(A) \le 0$ and $h(A) \le 0$.

If $\Upsilon(G) > k$, then $V(G)$ has a minimal subset A such that $\|A\| \ge k(|A|-1)+1$; that is, $g(A) \le 0$. Minimality implies that every vertex of A has at least $k+1$ neighbors in A.

If $A \cap T = \varnothing$, then $\|A\| \ge (k+1)|A \cap S| = (k+1)f(A)/d$, which simplifies to $h(A) \le 0$. If $A \subseteq T$, then $|A \cap S| = 0$, so $\alpha_f(A) = g(A) - 1 < 0$, contradicting strong feasibility of f.

Hence we may assume that $A \cap T$ is a nonempty proper subset of A. The minimality of A implies that $\|A - T\| \le k(|A - T| - 1)$, and hence more than $k|A \cap T|$ edges of $G[A]$ are incident to T. From the independence of S and the absence of edges joining S and T, we now have $\|A\| > (k+1)|A \cap S| + k|A \cap T|$. Since $f(v) = d$ for $v \in S$ and $f(v) \le d-1$ for $v \in T$, this yields $\|A\| \ge (k+1)\frac{f(A\cap S)}{d} + k\frac{f(A\cap T)}{d-1}$. Multiplying by d, we obtain

$$d\|A\| \ge (k+1)f(A \cap S) + kf(A \cap T)\frac{d}{d-1} \ge (k+1)f(A),$$

using $d/(d-1) \ge (k+1)/k$ and $f(R) = 0$. Thus $h(A) \le 0$, which as we noted suffices to complete the proof. ■

4. APPROACH TO $(k, d)^*$-DECOMPOSITION

When $d > k+1$, an inequality in the proof of Lemma 3.6 fails, and we need a different approach to $(k, d)^*$-decomposition. For $k = 1$, the needed sparseness condition reduces to a condition on $\mathrm{Mad}(G)$. We will obtain *reducible configurations* (forbidden from minimal counterexamples) and then use the *discharging method* (reallocating vertex degrees) to show that the average degree in a graph avoiding those configurations is higher than assumed.

In this discussion, we modify β by removing the constant term, and we drop the notation for the capacity function because each vertex will have capacity d.

Definition 4.1. *For a set A of vertices in a graph G, the sparseness $\beta_G(A)$ is defined by*

$$\beta_G(A) = (k+1)(k+d)|A| - (k+d+1)\|A\|.$$

In addition, let $m_{k,d} = 2k + \frac{2d}{k+d+1}$.

Note that G is (k, d)-sparse if and only if $\beta_G(A) \ge k^2$ for all A. We have noted that $(\mathrm{Arb}(G) \le m_{k,d}/2) \Rightarrow ((k, d)\text{-sparse}) \Rightarrow (\mathrm{Mad}(G) < m_{k,d})$; here we present an example that separates these conditions. Recall that we allow multiedges in graphs.

Example 4.2. Form H from the star $K_{1,q}$ by replacing each edge with $k+1$ parallel edges (see Figure 3). Note that H decomposes into $k+1$ forests, but only with each forest being $K_{1,q}$ and having maximum degree q. Since $\mathrm{Arb}(H) = k+1$, the NDT Conjecture does not apply (and $\mathrm{Arb}(H) \le m_{k,q}/2$ is not necessary for $(k, q)^*$-decomposition).

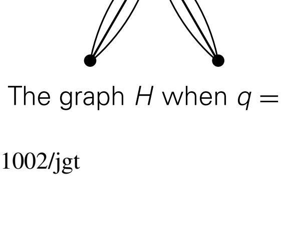

FIGURE 3. The graph H when $q = 4$ and $k = 2$.

Since $\beta_H(V(H)) = (k+1)(k+d)|V(H)| - (k+d+1)\|V(H)\| = (k+1)(k+d-q)$, the graph H is (k, d)-sparse if and only if $q \leq d$, which is precisely when it has a $(k, d)^*$-decomposition.

However, $\text{Mad}(H) = 2(k+1)\frac{q}{q+1}$, so $\text{Mad}(H) < m_{k,d}$ when $q < k+d$. For $d < q < k+d$, this shows that $\text{Mad}(H) < m_{k,d}$ does not imply $(k, d)^*$-decomposition. Note that $d < q < k+2$ requires $k \geq 2$; "$(1, d)$-sparse" and "$\text{Mad}(G) < m_{1,d}$" are equivalent.

Remark 4.3. It is possible for a (k, d)-sparse graph to have no decomposition into $k+1$ forests. Violating $\Upsilon(G) \leq k+1$ requires for some r a subgraph having r vertices and more than $(k+1)(r-1)$ edges. If G is also (k, d)-sparse, then

$$(k+1)(k+d)r - (k+d+1)[(k+1)(r-1)+1] \geq k^2,$$

which simplifies to $r \leq \frac{k(d+1)}{k+1}$. Thus a (k, d)-sparse graph with $\Upsilon(G) > k+1$ must have a small dense subgraph.

Remark 4.3 suggests a strengthening of the NDT Conjecture using sparseness. First, we introduce a term for "dense subgraph."

Definition 4.4. *For $k \in \mathbb{N}$ and a graph G, a set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.*

Graphs with overfull sets are not $(k, d)^*$-decomposable. We have noted that $\text{Arb}(G) \leq m_{k,d}/2$ both implies (k, d)-sparseness and prohibits overfull sets. By Remark 4.3, (k, d)-sparseness prohibits overfull sets with more than $\frac{k(d+1)}{k+1}$ vertices. Hence the conjecture below strengthens the NDT Conjecture.

Conjecture 4.5. *Fix $k, d \in \mathbb{N}$. If G is (k, d)-sparse and has no overfull set with at most $\frac{k(d+1)}{k+1}$ vertices, then G is $(k, d)^*$-decomposable.*

We will prove Conjecture 4.5 when $k = 1$ and $d \leq 6$. When $k = 1$ also $k^2 = 1$, so (k, d)-sparseness becomes "$\beta_G(A) > 0$ for all A" and is equivalent to $\text{Mad}(G) < 2 + \frac{2d}{d+2} = m_{1,d}$. We can then use the discharging method; we will also use properties of submodular functions.

The basic framework of the proof holds for general k, so we maintain the general language in this section before specializing to $k = 1$. We do this to suggest generalization to larger k and because the proofs of these lemmas are as short for general k as they are for $k = 1$.

Instead of capacity functions, we use a different device to control vertex degrees. We will show that it is essentially equivalent to reducing capacity on vertices.

Definition 4.6. *With k and d fixed, a* ghost *is a vertex of degree $k+1$ having only one neighbor (via $k+1$ edges).* Adding a ghost neighbor *at a vertex v means adding to the graph a vertex of degree $k+1$ whose only neighbor is v.*

Note that H in Example 4.2 is constructed by adding q ghost neighbors to a single vertex.

Lemma 4.7. *With k and d fixed, let v be a vertex in a graph G with a capacity function f. Form G' and f' from G and f by adding $d - f(v)$ ghost neighbors at v and letting f' agree with f except for equaling d at v and the new vertices. The following statements hold:*

(a) G is (k, f)-decomposable if and only if G' is (k, f')-decomposable.
(b) G is $(k, f)^*$-decomposable if and only if G' is $(k, f')^*$-decomposable.
(c) If f is feasible on G, then f' is feasible on G' (recall Definition 2.3).

Proof. (a,b) In such a decomposition (F, D) of G', each ghost vertex has at most k incident edges in F (and can have k), requiring an incident edge in D. Hence D can have only $f(v)$ edges of G at v. Thus the decompositions of G and G' correspond, whether D is required to be a forest or not.

(c) Consider β_f and $\beta_{f'}$. Given a set $A' \subseteq V(G')$, let $A = A' \cap V(G)$ and $a = |A' - V(G)|$. If $v \notin A$, then $\beta_{f'}(A') = \beta_f(A) + (k+1)(k+d)a$. If $v \in A$, then adding a ghost neighbor of v adds 1 to the size of the set and $k+1$ to the number of edges induced. Thus

$$\begin{aligned}\beta_{f'}(A') &= \beta_f(A) + (k+1)[d - f(v) + a(k+d)] - (k+1+d)(k+1)a \\ &= \beta_f(A) + (k+1)(d - f(v) - a).\end{aligned}$$

Since $0 \le a \le d - f(v)$, we conclude that $\beta_f(A) \ge 0$ implies $\beta_{f'}(A') \ge 0$. ■

Although the generality of the capacity function facilitates the inductive proof of Theorem 2.8, and the desired statement about (k, d)-decomposition in Theorem 1.2 is the special case of Theorem 2.8 when all capacities equal d, Lemma 4.7 shows that in fact the special case with capacity d for all v implies the general statement, making Theorem 1.2 and Theorem 2.8 equivalent.

Corollary 4.8. *Theorem 1.2 implies Theorem 2.8.*

Proof. Assume that (k, d)-sparseness implies (k, d)-decomposability. Let G be a graph with a feasible capacity function f. Form G' by giving $d - f(v)$ ghost neighbors to each vertex v. By repeated application of Lemma 4.7, the capacity function with capacity d at each vertex of G' is feasible. That is, G' is (k, d)-sparse. By Theorem 1.2, G' has a (k, d)-decomposition. Deleting the ghost vertices yields a (k, f)-decomposition of G. ■

In essence, we have shown that ghosts have the same effect as reduced capacity on the existence of decompositions. Lemma 4.7 allows us to translate the key lemmas of Section 2 into the language of adding ghosts instead of reducing capacity.

Definition 4.9. *For $B \subseteq V(G)$, let $\hat{G}_B$ denote the graph obtained by contracting B into a new vertex z and adding d ghost neighbors at z. That is, form the graph G_B as in Definition 2.4 and then add the ghost neighbors (see Figure 4).*

Lemma 4.10. *If G is (k, d)-sparse and $\beta_G(B) \le k(k+1)$, then $\hat{G}_B$ is (k, d)-sparse.*

Proof. Let f be the capacity function on G that assigns capacity d to each vertex. The statement that G is (k, d)-sparse is the statement that $\beta_f(A) \ge 0$ for $A \subseteq G$. The

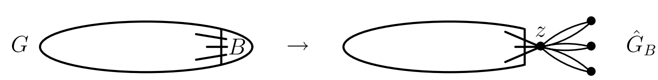

FIGURE 4. (k, d)-Sparseness of contractions with added ghosts.

hypothesis $\beta_G(B) \le k(k+1)$ is the statement $\beta_f(B) \le k$. Now Lemma 2.5 followed by Lemma 4.7 implies that $\hat{G}_B$ is (k, d)-sparse. ■

Lemma 4.11. *For $B \subseteq V(G)$, if $G[B]$ and $\hat{G}_B$ are $(k, d)^*$-decomposable, then G is $(k, d)^*$-decomposable.*

Proof. Let f and $\hat{f}$ be capacity functions on G and $\hat{G}_B$ assigning capacity d everywhere. The hypothesis states that $G[B]$ and $\hat{G}_B$ are $(k, f|_B)^*$-decomposable and $(k, \hat{f})^*$-decomposable, respectively. By Lemma 4.7, G_B is $(k, f^*)^*$-decomposable, where f^* assigns capacity 0 to z and capacity d elsewhere. By Lemma 2.7, G is $(k, f)^*$-decomposable (the combining argument there is also valid when the d-bounded subgraphs are forests). That is, G is $(k, d)^*$-decomposable. ■

The hypotheses of Conjecture 4.5 remain satisfied under discarding of edges or vertices. Next, we study the behavior of ghosts in minimal counterexamples.

Definition 4.12. *Among the non-$(k, d)^*$-decomposable graphs satisfying the hypotheses of Conjecture 4.5, a* minimal counterexample *is one that has the fewest non-ghost vertices and among such counterexamples has the fewest ghosts.*

Lemma 4.13. *A minimal counterexample G is $(k+1)$-edge-connected (and hence has minimum degree at least $k+1$).*

Proof. If G has an edge cut Q with size at most k, then $(k, d)^*$-decompositions of the components of $G - Q$ combine to form an $(k, d)^*$-decomposition of G by allowing each forest to acquire at most one edge of the cut. ■

Corollary 4.14. *In a minimal counterexample G, a vertex with degree at most $2k+1$ cannot be a neighbor of a ghost.*

Proof. If such a vertex v is also a ghost, then G has two vertices and is $(k, d)^*$-decomposable. Otherwise, the edges incident to v but not to the neighboring ghost form an edge cut of size at most k, contradicting Lemma 4.13. ■

Definition 4.15. *A set $A \subseteq G$ is* nontrivial *if A contains at least two non-ghosts but not all non-ghosts in G.*

When referring to vertex degrees, we avoid confusion with the overall parameter d by always using the relevant graph as a subscript in the expression $d_G(v)$.

Lemma 4.16. *Let A be a vertex set in a minimal counterexample G. If A is nontrivial, then $\beta_G(A) > k(k+1)$. If A is trivial with exactly one non-ghost vertex v and $\beta_G(A) \le k(k+1)$, then $d_G(v) \ge (k+1)(d+1)$.*

Proof. Suppose that $\beta_G(A) \le k(k+1)$. By Lemma 4.10, $\hat{G}_B$ is (k, d)-sparse. If A is nontrivial, then $\hat{G}_A$ has fewer non-ghosts than G. The minimality of G then implies that both $\hat{G}_A$ and $G[A]$ are $(k, d)^*$-decomposable. By Lemma 4.11, also G would be $(k, d)^*$-decomposable.

Hence we may assume that A is trivial with non-ghost vertex v, so A consists of v and h ghost neighbors of v, for some h. Now $\beta_G(A) = (k+1)(k+d-h)$, so

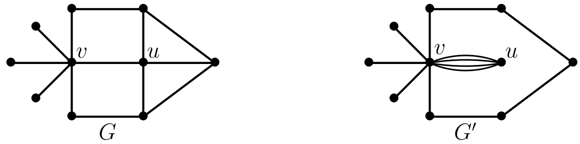

FIGURE 5. Transformation for Lemma 4.17 when $k = 3$.

$\beta_G(A) \le k(k+1)$ requires $h \ge d$. If $h > d$, then already $d_G(v) \ge (k+1)(d+1)$. If $h = d$ and $A = V(G)$, then G is explicitly $(k,d)^*$-decomposable. In the remaining case, G has vertices outside A, and the only vertex of A with outside neighbors is v. Since G is $(k+1)$-edge-connected (by Lemma 4.13), we again have $d_G(v) \ge (k+1)(d+1)$. ■

Lemma 4.17. *If v is a vertex in a minimal counterexample G and $d_G(v) < (k+1)(k+d)$, then v has no non-ghost neighbor with degree $(k+1)$.*

Proof. Let u be a non-ghost neighbor of v having degree $k+1$, and let W be the set of k neighbors of u other than v. Since $d_G(u) = k+1$, no vertex in $W \cup \{v\}$ is a ghost. Form G' from G by deleting the edges incident to u and then adding $k+1$ edges joining u to v; this makes u a ghost neighbor of v in G' (see Figure 5). Note that G' and G have the same numbers of edges and vertices, but G' has fewer non-ghost vertices than G, since u and its neighbors are non-ghosts in G and at least u becomes a ghost in G'.

If G' is (k,d)-sparse, then the minimality of G implies that G has a $(k,d)^*$-decomposition (F,D). Since u is a ghost, we may assume that each forest in F has one edge incident to u, as does D. Modify (F,D) as follows: keep copies of uv including the copy in D to make the multiplicity of uv as it is in G, but replace the other copies of uv in G' with edges to W, assigned to the same forests. Still each forest has one edge incident to u, so we obtain a $(k,d)^*$-decomposition of G.

It thus suffices to show that G' is (k,d)-sparse. We need only consider A such that $u, v \in A$ and $W \not\subseteq A$; otherwise, $\beta_{G'}(A) \ge \beta_G(A) \ge k^2$, since G is (k,d)-sparse. With $u \in A$, we have $\beta_{G'}(A) = \beta_G(A-u) - (k+1)$, since adding a ghost neighbor costs $k+1$. We worry only if $\beta_G(A-u) \le k(k+1)$. Since $W \not\subseteq A$, the set A does not contain all non-ghosts in G. If v is the only non-ghost in $A-u$, then $d_G(v) \ge (k+1)(k+d)$, by Lemma 4.16. Since our hypothesis is $d_G(v) < (k+1)(k+d)$, we conclude that $A-u$ is nontrivial, and now Lemma 4.16 yields $\beta_G(A-u) > k(k+1)$. ■

Lemma 4.18. *If a minimal counterexample G has a vertex v with q ghost neighbors, where $q \ge 1$, then $d_G(v) > kq + k + d$.*

Proof. Form G' from G by deleting the ghost neighbors of v. Since G' is an induced subgraph of G, it is (k,d)-sparse. Forming G' does not increase the number of non-ghost vertices, but it decreases the numbers of vertices and edges, so G' has an $(k,d)^*$-decomposition (F', D').

By Lemma 4.13, $d_{G'}(v) \ge k+1$. We may assume that $d_{D'}(v) \le d_{G'}(v) - k$, since edges of D' at v can be moved arbitarily to F' until F' has at least k edges at v. Now restore each ghost vertex by adding one incident edge to each forest in F' and the remaining incident edge to D', yielding (F, D).

FIGURE 6. Transformation for Lemma 4.19 when $k = 3$.

Since F is a union of k forests and D is a forest, but G has no $(k, d)^*$-decomposition, we must have $d_D(v) > d$. Since $d_D(v) = d_{D'}(v) + q \le d_{G'}(v) - k + q = d_G(v) - kq - k$, we conclude $d_G(v) > kq + k + d$. ■

If v has q ghost neighbors, then $d_G(v) \ge (k+1)q$. Hence the lower bound in Lemma 4.18 strengthens the trivial lower bound when $q \le k + d$.

Lemma 4.19. *If G is a minimal counterexample, then two vertices in G are joined by $k + 1$ edges only when one of them is a ghost.*

Proof. Since G has no overfull set, edge-multiplicity is at most $k + 1$. If two ghosts are adjacent, then G has two vertices and is $(k, d)^*$-decomposable.

Suppose that non-ghosts u and v are joined by $k + 1$ edges. Obtain G' from G by contracting them into a single vertex z and adding a ghost neighbor w of z (see Figure 6).

We claim that G' is (k, d)-sparse and has no overfull set. If $A \subseteq V(G') - \{z\}$, then $\beta_{G'}(A) \ge \beta_G(A - \{w\}) \ge k^2$. If $z \in A \subseteq V(G')$, then $\beta_{G'}(A) \ge \beta_{G'}(A \cup \{w\}) = \beta_G(A') \ge k^2$, where $A' = (A - \{z, w\}) \cup \{u, v\}$. Hence G' is (k, d)-sparse.

Since G has no overfull set, an overfull set in G' must contain z, and a smallest such set A does not contain w. Let $A' = (A - \{z\}) \cup \{u, v\}$. Now A' has one more vertex than A and induces $k + 1$ more edges in G than A induces in G'. Hence A' is overfull if and only if A is overfull. We conclude that G' has no overfull set.

Since G' has the same numbers of vertices and edges as G, but G' has fewer non-ghosts than G, minimality of G now implies that G' has a $(k, d)^*$-decomposition (F, D). At w there is one edge in each forest in F and one edge in D. Replacing these with the edges joining u and v (one in each forest) yields a $(k, d)^*$-decomposition of G, since the new degree of u or v in D is at most $d_D(z)$, and an edge joining u and v completes a cycle in its forest only if contracting that edge yields a cycle in the corresponding forest in (F, D). ■

5. DISCHARGING ARGUMENT AND SUBMODULARITY

In this section, we outline a discharging argument aimed at showing that a graph having the properties known for a minimal non-$(k, d)^*$-decomposable graph also has average degree at least $m_{k,d}$. Section 6 will complete the argument for the case $k = 1$ and $d \le 6$. Perhaps the approach can be extended to work at least for $k = 1$ and all d.

For convenience, we say that a *j-vertex* is a vertex of degree j, and a *j-neighbor* of a vertex is a neighbor that is a j-vertex. Give each vertex initial charge equal to its degree in G (by Lemma 4.13, each vertex has degree at least $k + 1$). We aim to redistribute charge so that the final charge $\mu(v)$ for each vertex v is at least $m_{k,d}$. This motivates our first discharging rule.

Rule 1: A vertex of degree $k+1$ takes charge $\frac{m_{k,d}}{k+1}-1$ along each incident edge from the other endpoint of that edge. This amount equals $\frac{k+d-1}{k+d+1}$.

In particular, a ghost takes total charge $m_{k,d}-(k+1)$ from its neighbor. Rule 1 increases the charge of each $(k+1)$-vertex to $m_{k,d}$, since Lemma 4.17 implies that $(k+1)$-vertices are not adjacent unless $|V(G)|=2$.

If all neighbors of v have degree $k+1$, then $\mu(v)=d_G(v)\frac{2}{k+d+1}$, since each edge takes $\frac{k+d-1}{k+d+1}$. In this case, $\mu(v)\geq m_{k,d}$ if and only if $d_G(v)\geq(k+1)(k+d)$.

It remains to consider v such that $k+1<d_G(v)<(k+1)(k+d)$. Vertices with degree at most $2k+1$ need additional charge, and vertices with degree less than $(k+1)(k+d)$ must not lose too much. Fortunately, low-degree vertices must have high-degree neighbors.

For example, if $d_G(v)<(k+1)(k+d)$, then v cannot have only $(k+1)$-neighbors. By Lemma 4.17, v has no non-ghost $(k+1)$-neighbor. If v has only ghost neighbors, then G consists of one vertex plus ghost neighbors, but such a graph is $(k,d)^*$-decomposable or is the non-(k,d)-sparse graph H of Example 4.2. Hence v has neighbor(s) with higher degrees.

When $(k,d)=(1,1)$, only 2-vertices need charge. By Lemma 4.17, their neighbors have high enough degree that Rule 1 completes the discharging argument. Since a forest with maximum degree 1 is a matching, this proves the result of [11] that the Strong NDT Conjecture holds when $(k,d)=(1,1)$.

When $k=1$ and $d>1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition for completing the discharging argument.

Theorem 5.1. *For $d>k=1$, let G be a minimal counterexample in the sense of Section 4. If each 3-vertex in G has a neighbor with degree at least $d+2$, then* $\mathrm{Mad}(G)\geq m_{1,d}=2+\frac{d}{d+2}$.

Proof. We specialize Rule 1 for $k=1$ and add a rule to satisfy 3-vertices.

Rule 1: Each 2-vertex receives $\frac{d}{d+2}$ along each incident edge.

Rule 2: If $d_G(v)=3$, and v has neighbor u with $d_G(u)\geq d+2$, then v receives $\frac{d-2}{d+2}$ from u.

We show that the final charge of each vertex is at least $m_{1,d}$. Rules 1 and 2 ensure that $\mu(v)\geq m_{1,d}$ when $d_G(v)\in\{2,3\}$ (since $3+\frac{d-2}{d+2}=2+\frac{2d}{d+2}$). Since $\frac{d-2}{d+2}<\frac{d}{d+2}$, the general argument for vertices with degree at least $2d+2$ also remains valid.

If $4\leq d_G(v)\leq 2d+1$, then v has no non-ghost 2-neighbor, by Lemma 4.17. If v has q ghost 2-neighbors, with $q\geq 1$, then $d_G(v)\geq q+d+2$, by Lemma 4.18. Hence $\mu(v)=d_G(v)>m_{1,d}$ if $4\leq d_G(v)\leq d+1$, since Rule 2 takes no charge from v.

If $d+2\leq d_G(v)\leq 2d+1$, then v may give charge to q ghost neighbors (along the two edges to each) and to $d_G(v)-2q$ neighbors of degree 3. Using Lemma 4.18,

$$\begin{aligned}\mu(v)&\geq d_G(v)-\frac{d}{d+2}2q-[d_G(v)-2q]\frac{d-2}{d+2}=\frac{4(d_G(v)-q)}{d+2}\\&\geq\frac{4(d+2)}{d+2}=4>m_{1,d}.\end{aligned}$$

Thus $\mu(v)\geq m_{1,d}$ for all v, and there is no (k,d)-sparse minimal counterexample. ∎

This reduces Conjecture 4.5 for the case $k = 1$ to proving that in a minimal counterexample G, each 3-vertex has a neighbor with degree at least $d + 2$. Our proofs of this fact depend on d. In each case, we will use submodularity properties of the function β_G.

Definition 5.2. *A function β on the subsets of a set is* submodular *if $\beta(X \cap Y) + \beta(X \cup Y) \leq \beta(X) + \beta(Y)$ for all subsets X and Y.*

Lemma 5.3. *For a graph G, the sparseness function β_G on subsets of $V(G)$ is submodular.*

Proof. To compare $\beta_G(X \cap Y) + \beta_G(X \cup Y)$ with $\beta_G(X) + \beta_G(Y)$, note first that $|X \cup Y| + |X \cap Y| = |X| + |Y|$. Hence it suffices to show that $\|X \cup Y\| + \|X \cap Y\| \geq \|X\| + \|Y\|$. All edges contribute equally to both sides except edges joining $X - Y$ and $Y - X$, which contribute 1 to the left side but 0 to the right. ■

6. NEIGHBORS OF 3-VERTICES WHEN $k = 1$

Now restrict to $k = 1$, where (k, d)-sparseness reduces to the statement that $\beta_G(A) = (2d + 2)|A| - (d + 2)\|A\| \geq 1$ for $A \subseteq V(G)$. For a minimal counterexample G in this setting, Lemma 4.16 yields $\beta_G(A) \geq 3$ when A is nontrivial (meaning that A contains at least two non-ghosts but not all non-ghosts). Also, if d is even, then always $\beta_G(A)$ is even, which yields $\beta_G(A) \geq 4$ when A is nontrivial. By Theorem 5.1, to prove the NDT Conjecture when $k = 1$ it suffices to prove that every 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. The two lemmas of this section accomplish this for $d \leq 6$. The first concerns an auxiliary function on vertex subsets of induced subgraphs.

Definition 6.1. *When G' is an induced subgraph of G, define the* potential function $\rho_{G'}$ *by $\rho_{G'}(X) = \min\{\beta_G(W) : X \subseteq W \subseteq V(G')\}$, where $X \subseteq V(G')$.*

Lemma 6.2. *Fix d with $2 \leq d \leq 6$, and let G be a minimal counterexample. If v is a 3-vertex in G and has no neighbor with degree at least $d + 2$, then v has two neighbors u and u' such that $\rho_{G'}(\{u, u'\}) \geq d + 3$, where $G' = G - v$.*

Proof. Together, Corollary 4.14 and Lemma 4.19 imply that every 3-vertex has three distinct neighbors. Let U be the neighborhood of v, with $U = \{u_1, u_2, u_3\}$. Let $Z_i = U - \{u_i\}$. Suppose that $\rho_{G'}(U_i) \leq d + 2$ for all i.

For $k \in \{1, 2, 3\}$, let X_k be a subset of $V(G')$ such that $\rho_{G'}(Z_k) = \beta_G(X_k)$. Letting i and j be distinct elements of $\{1, 2, 3\}$,

$$2d + 4 \geq \beta_G(X_i) + \beta_G(X_j) \geq \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j).$$

For $X' \subseteq V(G')$, let $X = X' \cup \{v\}$. If $U \subseteq X' \subseteq V(G')$, then $\beta_G(X') = \beta_G(X) + d + 4$. If $X' \neq V(G')$, then $X \neq V(G)$, and X is nontrivial if it has at least two non-ghosts. By Lemma 4.16, this would yield $\beta_G(X') \geq d + 7 + \epsilon$, where $\epsilon = 1$ if d is even and $\epsilon = 0$ if d is odd. However, if $X' = V(G')$, then we only have $\beta_G(X') \geq d + 5 + \epsilon$.

Since each edge vu_i has multiplicity 1, no vertex in U is a ghost, and neither is v. Since the vertex of $X_i \cap X_j$ has degree less than $d + 2$ in G, Lemma 4.16 implies $\beta_G(X_i \cap X_j) \geq 3 + \epsilon$. Since $U \subseteq X_i \cup X_j$, we also conclude $\beta_G(X_i) + \beta_G(X_j) \geq d + 8 + 2\epsilon$ for all d, and the lower bound increases by 2 if $X_i \cup X_j \neq V(G')$.

Thus $\rho_{G'}(X_i) + \rho_{G'}(X_j) \geq d + 8 + 2\epsilon$. If $d \leq 4$, then $d + 8 + 2\epsilon > 2d + 4$, and the desired conclusion follows. Hence we may assume $d \in \{5, 6\}$; furthermore, $X_i \cup X_j = V(G')$ for all i, j, since otherwise the lower bound on $\beta_G(X_i) + \beta_G(X_j)$ again exceeds $2d + 4$.

In more detail, the computation of Lemma 5.3 is

$$\beta_G(X_i) + \beta_G(X_j) = \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j) + (k + d + 1)m,$$

where m is the number of edges joining $X_i - X_j$ and $X_j - X_i$. If $m \geq 1$, then we obtain $\beta_G(X_i) + \beta_G(X_j) \geq 2d + 10 > 2d + 4$, which yields the desired conclusion. Hence $m = 0$ in each case. That is, each $X_i \cap X_j$ is a separating set in G'. (If G' is disconnected, then some edge incident to v is a cut-edge, which contradicts Lemma 4.13.) Furthermore,

$$\beta_G(X_i \cap X_j) = \beta_G(X_i) + \beta_G(X_j) - \beta_G(X_i \cup X_j) \leq 2d + 4 - (d + 5 + \epsilon) = d - 1 - \epsilon.$$

Now let $Z = X_1 \cap X_2 \cap X_3$. Since $X_i \cup X_j = V(G')$, any vertex of $V(G') - Z$ misses exactly one of the three sets, so $\{Z, \overline{X}_1, \overline{X}_2, \overline{X}_3\}$ is a partition of $V(G')$. Since $\beta_G(X_i) \leq d + 2$ and $\beta_G(V(G')) \geq d + 5$, each $\overline{X}_i$ is nonempty, so $Z \neq V(G')$. If Z contains only one non-ghost, then (k, d)-sparseness requires it to have at most d ghost neighbors, and $\beta_G(Z) \geq 2$. Otherwise, since $v \notin Z$, we conclude that Z is nontrivial, and hence $\beta_G(Z) \geq 3$.

Now, since $\overline{X}_i \subseteq X_j \cap X_k$, submodularity yields

$$2d + 1 - \epsilon \geq \beta_G(X_i) + \beta_G(X_j \cap X_k) \geq \beta_G(V(G')) + \beta_G(Z) \geq d + 7.$$

We conclude that $d \geq 6 + \epsilon$, so the desired conclusion holds when $d \leq 6$. ■

Lemma 6.3. *If $3 \leq d \leq 6$ and G is a minimal counterexample, then every 3-vertex has a neighbor with degree at least $d + 2$.*

Proof. Let u_1, u_2, u_3 be the neighbors of a 3-vertex v, and let $U = \{u_1, u_2, u_3\}$. Suppose that $d_G(u) \leq d + 1$ for $u \in U$. Since each edge vu_i has multiplicity 1, no vertex in U is a ghost vertex, and any edge induced by U has multiplicity 1 (Lemma 4.19).

Let $G' = G - v$. By Lemma 6.2, we may assume by symmetry that $\rho_{G'}(\{u_1, u_2\}) \geq d + 3$. Form H from G' by adding an extra edge joining u_1 and u_2 (see Figure 7). For $A \subseteq V(H) = V(G')$, we have $\beta_H(A) = \beta_G(A)$ unless $u_1, u_2 \in A$, but in the remaining case $\rho_{G'}(\{u_1, u_2\}) \geq d + 3$ yields $\beta_H(A) \geq 1$.

Hence H is (k, d)-sparse, and it has fewer non-ghosts than G. We can obtain a $(1, d)^*$-decomposition of H if H has no overfull sets of size at most $(d + 1)/2$, which is at most 3. There are no triple-edges in H, since G has no double-edges within U. An overfull triple in H must include u_1 and u_2, since G has no overfull triple. The third vertex w must

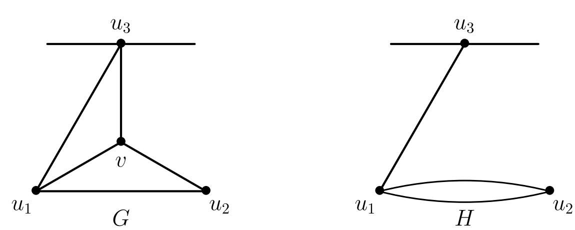

FIGURE 7. Transformation for Lemma 6.3.

be adjacent to u_1 or u_2 by two edges in G. Since those vertices are also adjacent to v, we have contradicted $d_G(u_1) = d_G(u_2) = 3$. Hence H has no such overfull set, and we conclude by the minimality of G that H has an $(1, d)^*$-decomposition.

Let (F, D) be an $(1, d)^*$-decomposition of H. Obtain a decomposition of G by (1) replacing the added edge u_1u_2 with vu_1 and vu_2 in whichever of F and D contains it, and (2) placing vu_3 in the other subgraph. Let (F', D') be the resulting decomposition. Note that $d_{D'}(u_i) = d_D(u_i)$ for $i \in \{1, 2\}$, and cycles through v would correspond to cycles in the decomposition of H. The only worry is $d_{D'}(u_3)$, since this exceeds $d_D(u_3)$ if the added edge in H belonged to F. If $d_{D'}(u_3) = d + 1$, then we have obtained a neighbor of v with degree at least $d + 2$ unless $d_G(u_3) = d + 1$, but now we can move any one edge incident to u_3 from D' to F' to complete a $(1, d)^*$-decomposition of G. ■

7. THE STRONG NDT CONJECTURE FOR $(k, d) = (1, 2)$

Here, we prove our strongest conclusion for our most restrictive hypothesis. Many of the steps are quite similar to our previous arguments, so we combine them in a single proof.

Theorem 7.1. *The Strong NDT Conjecture holds when* $(k, d) = (1, 2)$*. That is, if G is* $(1, 2)$*-sparse, then G has a* strict decomposition (F, D)*, meaning a* $(1, 2)^*$*-decomposition in which every component of D has at most two edges.*

Proof. Since $m_{1,2} = 3$, $(1, 2)$-sparseness is equivalent to $\mathrm{Mad}(G) < 3$. Let G be a counterexample with the fewest non-ghosts. By the argument of Lemma 4.13, G is 2-edge-connected.

If G has adjacent 2-vertices u and v, then at least one is not a ghost. Letting $G' = G - \{u, v\}$, the minimality of G yields a strict decomposition (F, D) of G'. Adding the edge uv to D and the other edges incident to u and v to F yields a strict decomposition of G.

If G has a vertex with three ghost neighbors, then G is not $(1, 2)$-sparse, so every vertex has at most two ghost neighbors. If G has only one non-ghost, then G explicitly has a strict decomposition. Hence we may assume that G has at least two non-ghosts.

Since d is even, always β_G is even, so $(1, 2)$-sparseness can be stated as $\beta_G(A) \geq 2$ for $A \subseteq V(G)$ (here $\beta_G(A) = 6|A| - 4\|A\|$). A set A is *tight* if $\beta_G(A) = 2$. A set consisting of a vertex with two ghost neighbors is a trivial tight set.

By Lemma 4.10, if A is a tight set, then G_A is $(1, 2)$-sparse. The same argument as in Lemma 4.11 shows that if G is a minimal counterexample, $A \subseteq V(G)$, and G_A has a strict decomposition, then G has a strict decomposition. Hence we may assume, as in the earlier proofs, that $\beta_G(A) \geq 4$ for every nontrivial set A.

Suppose that G has a non-ghost 2-vertex v. Each neighbor of v has degree at least 3. If a neighbor u of v has at most one ghost neighbor, then form G' from $G - v$ by giving u one additional ghost neighbor w (see Figure 8). Now G and G' have the same numbers of vertices and edges, but G' has fewer non-ghost vertices.

FIGURE 8. Transformation for Theorem 7.1.

We claim also that G' is $(1, 2)$-sparse. If $u \notin A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \notin A$, and then $\beta_{G'}(A) = \beta_G(A) \geq 2$. If $u \in A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \in A$, and then $\beta_{G'}(A) \geq \beta_G(A - \{w\} \cup \{v\}) - 2 \geq 2$, since $A - \{w\} \cup \{v\}$ is nontrivial.

We conclude that G' has a strict decomposition (F, D), by the minimality of G. Each of F and D must have one edge incident to w. We obtain a strict decomposition of G by deleting w, adding vu to D, and adding the other edge at v to F.

We may therefore assume that every neighbor of a non-ghost 2-vertex has at least two ghost neighbors. Since G is 2-edge-connected, a q-vertex cannot have $(q - 1)/2$ ghost neighbors. In particular, a vertex with at least two ghost neighbors must have degree at least 6, so every neighbor of a non-ghost 2-vertex has degree at least 6.

Once again we have derived many properties of a minimal counterexample. We complete the proof by using discharging to show that if G has these properties, then $\text{Mad}(G) \geq 3$. This contradicts $(1, 2)$-sparseness, which is equivalent to $\text{Mad}(G) < 3$; hence there is no minimal counterexample.

The initial charge of each vertex is its degree; we manipulate charge so that the final charge $\mu(v)$ of each vertex v is at least 3. The only discharging rule is that a 2-vertex takes charge $1/2$ along each incident edge from the other endpoint of that edge. Hence the final charge of a 2-vertex is 3.

Since each neighbor of a non-ghost 2-vertex has degree at least 6, vertices of degree 3, 4, or 5 give charge only to ghosts. If $d_G(v) = 3$, then v has no ghost neighbors, and $\mu(v) = 3$. If $d_G(v) \in \{4, 5\}$, then v has at most one ghost neighbor, and $\mu(v) \geq d_G(v) - 1 \geq 3$. If $d_G(v) \geq 6$, then v gives at most $1/2$ along each edge, so $\mu(v) \geq d_G(v) - d_G(v)/2 \geq 3$. ■

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