$K_{s,t}$ Minors in $(s+t)$-Chromatic Graphs, II

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Received January 16, 2012; Revised March 12, 2013

Published online 30 May 2013 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21744

Abstract: Let $K^*_{s,t}$ denote the graph obtained from the complete graph K_{s+t} by deleting the edges of some K_t-subgraph. The author proved earlier that for each fixed s and $t > t_0(s) := \max\{4^{15s^2+s}, (240s\log_2 s)^{8s\log_2 s+1}\}$, every graph with chromatic number $s+t$ has a $K^*_{s,t}$ minor. This confirmed a partial case of the corresponding conjecture by Woodall and Seymour. In this paper, we show that the statement holds already for much smaller t, namely, for $t > C(s\log s)^3$. © 2013 Wiley Periodicals, Inc. J. Graph Theory 75: 377–386, 2014

Keywords: *graph coloring; graph minors; complete bipartite graphs*

Mathematical Subject Classification (MSC) 2000: *05C15, 05C83*

Grant sponsor: NSF; grant number: DMS-0965587; grant sponsor: Russian Foundation for Basic Research; grant numbers: 12-01-00448 and 12-01-00631.

1. Introduction

A graph is *k-chromatic* if its chromatic number is exactly k. A *minor* of a graph G is a graph H that can be obtained from G by a sequence of vertex and edge deletions and edge contractions. If H is a minor of G, we will also say that *G has an H minor*. In search of ways to attack Hadwiger's Conjecture, Woodall [12] and independently Seymour [10] suggested to prove the following weaker statement.

Conjecture 1. *Every $(s+t)$-chromatic graph has a $K_{s,t}$ minor.*

For convenience, we will always assume that $s \leq t$. The conjecture is evident for $s = 1$. The validity of the conjecture for $s = 2$ and all t (and even of the list-coloring version of it) was proved by Woodall [12]. It also follows from an observation of Seymour (see Lemma 17 below) and the following result by Chudnovsky, Reed, and Seymour [1].

Theorem 1. *Let G be a graph with $n \geq 3$ vertices such that $e(G) > \frac{1}{2}(t+1)(n-1)$. Then G has a $K_{2,t}$ minor.*

Note that this result for $t > 10^{29}$ was earlier proved by Myers [9]. In [6], it was proved that for $t \geq 6500$, every $(3+t)$-chromatic graph has a $K^*_{3,t}$ minor, where $K^*_{s,t}$ denotes the graph obtained from $K_{s,t}$ by adding all edges between the vertices of the partite set of size s. In other words, $K^*_{s,t} = K_{s+t} - E(K_t)$.

The author [4] showed that for every fixed s and huge (in comparison with s) t, Conjecture 1 holds in a slightly stronger form:

Theorem 2. *Let s and t be positive integers such that*

$$t > t_0(s) := \max\left\{4^{15s^2+s}, (240s\log_2 s)^{8s\log_2 s+1}\right\}. \quad (1)$$

Then every $(s+t)$-chromatic graph has a $K^*_{s,t}$ minor.

In this paper, we show that the Woodall–Seymour Conjecture holds already when t is polynomial in s. Our main result is:

Theorem 3. *Let s and t be positive integers such that*

$$t > t_1(s) := 5(200s\log_2(200s))^3. \quad (2)$$

Then every $(s+t)$-chromatic graph has a $K^*_{s,t}$ minor.

The proof is based on ideas from [4, 5, 7]. It repeats a lot of the proof in [4] with the following main changes:

(1) In order to handle dense graphs, instead of the result in [5], which implied the lower bound $(240s\log_2 s)^{8s\log_2 s+1}$ on t in (1), we use Theorem 6 below from a recent paper [7].
(2) Instead of using Ramsey Theorem to estimate the connectivity of a minimum counterexample, which implied the lower bound 4^{15s^2+s} on t in (1), we use a theorem of Wollan [11] on rooted minors (see Theorem 8) and an old bound in [3] (see Theorem 9 below).
(3) The main lemma (Lemma 13) uses some new ideas.

In the next two sections, we introduce notation and cite or prove auxiliary statements. In Section 4, we prove the key lemma on minors in dense subgraphs of moderate order. We deliver the main proof in Section 5.

2. Preliminaries

For a graph G, $V(G)$ is the set of its vertices, $E(G)$ is the set of its edges, and $e(G) = |E(G)|$. By $G[X]$ we denote the subgraph of G induced by the vertex set X. For $v \in V(G)$, we let $N_G(v)$ denote the set of neighbors of v in G, $d_G(v) = |N_G(v)|$, and $N_G[v] = N_G(v) \cup \{v\}$.

For a graph G, a set $T \subseteq V(G)$ is *totally dominating*, if every vertex of G has a neighbor in G. The following simple fact is a variation of Lemma 12 in [6].

Lemma 4. *Let G be an n-vertex graph with minimum degree $k \geq 1$. Then:*

(a) *G contains a totally dominating set T with $|T| \leq \lfloor \log_{n/(n-k)} n \rfloor + 1$; and*
(b) *G contains a totally dominating set T' with $|T'| \leq 2 \log_{n/(n-k)} n$ such that for each component H of G, the subgraph $G[T' \cap V(H)]$ is connected. In particular, if G is connected, then $G[T']$ is connected.*

Proof. Let $A \subseteq V(G)$. The total number of neighbors of vertices in A counted with multiplicities is at least $k|A|$. Hence

$$\text{there exists } v_A \in V(G) \text{ that is adjacent to at least } k|A|/n \text{ vertices in } A. \tag{3}$$

Consider the sequence $A_0, A_1, \dots,$ where $A_0 = V(G)$ and for $i \geq 1$, $A_i = A_{i-1} - N(v_{A_{i-1}})$. By (3), for every $i \geq 1$, $|A_i| \leq \frac{n-k}{n}|A_{i-1}|$. It follows that for $i_0 = \lfloor \log_{n/(n-k)} n \rfloor + 1$,

$$|A_{i_0}| \leq n \left(\frac{n-k}{n}\right)^{i_0} < n \left(\frac{n-k}{n}\right)^{\log_{n/(n-k)} n} = 1,$$

and so $A_{i_0} = \emptyset$. Hence $T = \{v_{A_0}, v_{A_1}, \dots, v_{A_{i_0-1}}\}$ is totally dominating. This proves (a).

Let $C_1, \dots, C_m$ be the vertex sets of the components of $G[T]$. Since T is totally dominating, each C_j has at least two vertices. It follows that $m \leq i_0/2$. We will add at most $2(m-1)$ vertices to T to obtain a set satisfying (b). Let $H_1, \dots, H_r$ be the components of G. Choose in each H_h a vertex set C_h of a component of $G[T]$ contained in $V(H_h)$. Let $T' = T$ and $C_0 = \bigcup_{h=1}^{r} C_h$. We do the following iterations: If C_0 dominates $V(G)$, then Stop. Otherwise, choose any vertex w at distance exactly two from C_0. Let w' be the intermediate vertex on a shortest path from C_0 to w. By the choice of T, w has a neighbor $z \in T - C_0$. By definition, z belongs to some C_j. Add to T' vertices w and w', and let the new C_0 be obtained by adding to the old one vertices w and w' and all C_j that contain these vertices or are adjacent to them. This increases $|T'|$ by two and decreases the number of components in $G[T']$ by at least one.

After at most $m - r$ iterations, we obtain a totally dominating set T'. By construction, $|T'| \leq |T| + 2(m-r) \leq i_0 + 2(i_0/2 - 1) = 2i_0 - 2 \leq 2\log_{n/(n-k)} n$. ∎

Applying Lemma 4 s times, we obtain the following corollary.

Lemma 5. *Let s, k, and u be positive integers. Suppose $u > k > 2$. Let H be a graph of order u with $\delta(H) \geq k + 2(s-1)\log_{u/(u-k)} u$. Then $V(H)$ contains s pairwise-disjoint subsets $A_1, \dots, A_s$ such that, for every $i = 1, \dots, s$,*

(i) *A_i dominates $H - \bigcup_{j=1}^{i-1} A_j$,*
(ii) *$|A_i| \leq 2\log_{u/(u-k)} u$,*

(iii) *for every component H_ℓ of $H - \bigcup_{j=1}^{i-1} A_j$, the set $A_{i,\ell} = A_i \cap V(H_\ell)$ is contained in one connected component of $H[A_i]$.*

An important tool will be the following result from [7].

Theorem 6 ([7]). *Let s and t be positive integers with*

$$t > 10^3 s \log_2 t.$$

Let G be a graph such that $e(G) \geq \frac{t+8s\log_2 s}{2}(n(G) - s + 1)$. Then G has a $K^*_{s,t}$ minor.

As it was mentioned in the introduction, it is known that Conjecture 1 holds for $s \leq 2$ and all t. For $s = 1$, graph $K^*_{s,t}$ equals $K_{s,t}$. To extend the base of induction a bit more, we use the following result for $s = 2, 3$ also mentioned above.

Theorem 7 ([6]). *Let $t \geq 6500$. Then every $(3+t)$-chromatic graph has a $K^*_{3,t}$ minor and every $(2+t)$-chromatic graph has a $K^*_{2,t}$ minor.*

3. Rooted minors

It follows from the definition of minors that a graph G contains a graph H as a minor if and only if there exist pairwise disjoint sets $\{S_v \subseteq V(G) \ : \ v \in V(H)\}$ such that for every $v \in V(H)$, $G[S_v]$ is a connected subgraph of G and for every edge uv in H, there exists an edge in G with one end in S_u and the other end in S_v. The sets S_v will be called the *branch sets* of a given H minor in G.

Let G and H be graphs and $X \subseteq V(G)$ with $|X| = |V(H)|$. Let $\phi : V(H) \to X$ be a bijection. Then we say that *the pair (G, X) contains a ϕ-rooted H minor* if there exist the branch sets $\{S_v \subseteq V(G) \ : \ v \in V(H)\}$ of an H minor in G such that $\phi(v) \in S_v$ for every $v \in V(H)$.

Wollan [11] proved the following theorem.

Theorem 8 ([11]). *Let H be a fixed graph and $c \in R$, $c = c(H) \geq 1$ be a constant such that every graph on n vertices with at least cn edges contains H as a minor. If G is any graph such that G is $|V(H)|$-connected and has at least $(9c + 26, 833|V(H)|)|V(G)|$ edges, then for all sets $X \subseteq V(G)$ with $|X| = |V(H)|$ and for all bijective maps $\phi : V(H) \to X$, (G, X) contains a ϕ-rooted H minor.*

The following result gives a bound on $c(K_r)$ for $r \geq 4$.

Theorem 9 ([3]). *For every integer $n \geq k \geq 2$, each n-vertex graph with at least kn edges has a K_r minor, where $r \geq 0.064k/\sqrt{\ln k} + 1$.*

Indeed, if $k \geq 60$ and $r \geq 0.064k/\sqrt{\ln k}$, then $\sqrt{\ln k} \geq 2$ and

$$\ln r \geq \ln k + \ln 0.064 - \frac{1}{2}\ln\ln k \geq (\sqrt{\ln k} - 2.5)^2.$$

It follows that $c(K_r) \leq \frac{r(\sqrt{\ln r}+2.5)}{0.064}$. This and Theorem 8 yield:

Corollary 10. *Let $r \geq 16$. If G is any r-connected graph with average degree at least $300(\sqrt{\ln r} + 190)r$, then for all sets $X \subseteq V(G)$ with $|X| = r$, G has a K_r minor such that all vertices in X are in distinct branch sets.*

4. Dense subgraphs of moderate order

Let $\mathcal{U} = \{U_1, U_2, \dots, U_q\}$ be a family of pairwise-disjoint sets of vertices in a graph G. Then a path P is a *strict $\mathcal{U}$-path* if the ends of P are in distinct members of $\mathcal{U}$ and all internal vertices of P are disjoint from $\bigcup_{i=1}^{q} U_i$. Furthermore, a family $(P_1, \dots, P_{q-1})$ of paths is *$\mathcal{U}$-connecting* if all paths in the family are strict $\mathcal{U}$-paths and the graph whose vertices are $U_1, U_2, \dots, U_q$ and two vertices are adjacent if they are connected by a P_j is connected.

The following statement is Lemma 8 in [4]:

Lemma 11 ([4]). *Let s and q be positive integers. Let G be an $s(q-1)$-connected graph and let $\mathcal{U} = \{U_1, U_2, \dots, U_q\}$ be a family of pairwise-disjoint sets of vertices in G such that $|U_i| \geq s(q-1)$ for $i = 1, \dots, q$. Then G contains s vertex-disjoint $\mathcal{U}$-connecting families of paths.*

Observation 12. *If $t \geq t_1(s)$, then*

$$t > (200s)^3 \log_2^2 s \ln t \tag{4}$$

and

$$t > 200^3 s^2 \log_2 s \ln^{3/2} t. \tag{5}$$

Proof. If

$$t \leq (200s)^3 \log_2^2 s \ln t, \tag{6}$$

then

$$\ln t \leq 3\ln(200s) + \ln(\log_2^2 s) + \ln\ln t. \tag{7}$$

Since $t \geq t_1 > 10^8$, $\ln t > 6 \ln\ln t$. For $s \geq 4$, $200s \geq \log_2^2 s$. Thus, (7) yields $\frac{5}{6}\ln t \leq 4\ln(200s)$, and so $\ln t < 5\ln(200s)$. But then (6) contradicts (2).

Similarly, if

$$t \leq 200^3 s^2 \log_2 s \ln^{3/2} t, \tag{8}$$

then $\ln t \leq 3\ln(200s) - \ln s + \ln\log_2 s + \frac{3}{2}\ln\ln t$. Again since $\ln t > 6\ln\ln t$, we conclude that $\frac{2}{3}\ln t \leq 3\ln(200s)$. Plugging $\ln t \leq 4.5\ln(200s) < 3.2\log_2(200s)$ into (8), we get

$$\begin{aligned} t &\leq 200^3 s^2 \log_2 s (3.2\log_2(200s))^{3/2} \\ &< (200s)^3 \frac{1}{s} \log_2^{5/2}(200s) 3.2^{3/2} < (200s\log_2(200s))^3 \cdot 2, \end{aligned}$$

a contradiction to (2). ■

Lemma 13. *Let s and t be positive integers satisfying (4). Let G be a $150s^2 \log s \ln t$-connected graph. Suppose that G contains a vertex subset U with*

$$t + 200s^2 \log s \ln t \leq |U| \leq 2.5t$$

such that $\delta(G[U]) \geq t/(1 + 16s\log_2 s)$. Then G has a $K^*_{s,t}$ minor.

Proof. Let $H = G[U]$. We will check that H satisfies the conditions of Lemma 5 for $k = \lceil \frac{t}{20s \log_2 s} \rceil$ and $u = |U|$. Indeed, by (4) for $s \geq 4$ and the above k,

$$\frac{t}{1 + 16s \log_2 s} - k \geq \frac{t(4s \log_2 s - 1)}{20s \log_2 s(1 + 16s \log_2 s)} - 1 > \frac{t}{100s \log_2 s} \geq 80000s^2 \log_2 s \ln t. \tag{9}$$

On the other hand,

$$\frac{u}{u-k} \geq \frac{2.5t}{2.5t - t/20s \log_2 s} = 1 + \frac{1/20s \log_2 s}{2.5 - 1/20s \log_2 s} = 1 + \frac{1}{50s \log_2 s - 1},$$

and hence

$$2s \log_{\frac{u}{u-k}} u \leq \frac{2s \ln u}{\ln\left(1 + \frac{1}{50s \log_2 s}\right)} \leq \frac{2s \ln \frac{5t}{2}}{\frac{1}{50s \log_2 s}} < 100s^2 \log_2 s(1 + \ln t). \tag{10}$$

Thus by Lemma 5, $G[U]$ contains s pairwise-disjoint subsets $A_1, \ldots, A_s$ such that for every $i = 1, \ldots, s$,

(i) A_i dominates $H - \bigcup_{j=1}^{i-1} A_j$,
(ii) $|A_i| \leq 2 \log_{u/(u-k)} u$,
(iii) for every component H_j of $H_0 = H - \bigcup_{i=1}^{s} A_i$, the set $A_{i,j} = A_i \cap V(H_j)$ is contained in one connected component of $H[A_i]$.

Let $H_1, \ldots, H_\ell$ be the components of H_0 and let $A_0 = \bigcup_{i=1}^{s} A_i$. By (ii) and (10),

$$|A_0| \leq 2s \log_{\frac{u}{u-k}} u \leq 100s^2 \log_2 s(1 + \ln t). \tag{11}$$

Thus by (9),

$$\delta(H_0) \geq \frac{t}{1 + 16s \log_2 s} - 100s^2 \log_2 s(1 + \ln t) > k.$$

So, if $U_j = V(H_j)$ for $j = 1, \ldots, \ell$, then $|U_j| \geq 1 + k$ for all j and hence $\ell \leq \frac{u}{k+1} < 50s \log_2 s$. Note that if $G[U]$ is $100s^2 \log_2 s(1 + \ln t)$-connected (for example, if $U = V(G)$), then $\ell = 1$.

Let $\mathcal{U} = \{U_1, U_2, \ldots, U_\ell\}$. Let $G_0 = G - A_0$. Since G is $150s^2 \log s \ln t$-connected, by (11), the connectivity of G_0 is at least

$$\begin{aligned} &150s^2 \log s \ln t - 100s^2 \log_2 s(1 + \ln t) \\ &\quad = 50s^2 \log s \ln t - 100s^2 \log_2 s \geq 50s^2 \log_2 s \geq s\ell. \end{aligned}$$

Then by Lemma 11, G contains s vertex-disjoint $\mathcal{U}$-connecting families of paths $\mathcal{P}_1, \ldots, \mathcal{P}_s$. For $i = 1, \ldots, s$, let $W_i = \bigcup_{P \in \mathcal{P}_i} V(P)$, and let $W = \bigcup_{i=1}^{s} W_i$.

For $i = 1, \ldots, s$, let $B_i = A_i \cup W_i$. For every i, since W_i has a vertex in each U_j and is $\mathcal{U}$-connecting, and A_i dominates $V(H_0)$ and satisfies (iii),

$$G[B_i] \text{ is connected and } B_i \text{ dominates } V(G_0). \tag{12}$$

Also by (i), for all $1 \leq i < i' \leq s$, B_i has a neighbor in $B_{i'}$. It follows that we have found a $K^*_{s,z}$ minor in G, where

$$z = |U| - |U \cap W| - |A_0|.$$

By the definition of a $\mathcal{U}$-connecting family, $|U \cap W| = 2s(\ell - 1) \le 100s^2 \log_2 s$. So, since $|U| \ge t + 200s^2 \log s \ln t$, by (11),

$$z \ge t + 200s^2 \log_2 s \ln t - 100s^2 \log_2 s - 100s^2 \log_2 s(1 + \ln t) > t.$$

This proves the lemma. ■

5. The main argument

For the proof, we will use two classical results. The first is the following.

Lemma 14 (Mader [8]). *Every graph with average degree d contains a $\lceil d/4 \rceil$-connected subgraph.*

The second uses the notion of color-critical graphs. Recall that a graph G is k-critical, if its chromatic number is at least k, but after deleting any edge or vertex from G the resulting graph is $(k-1)$-colorable.

Theorem 15 (Gallai [2]). *Let $k \ge 3$ and G be a k-critical graph. If $|V(G)| \le 2k - 2$, then G has a spanning complete bipartite subgraph.*

Suppose now that Theorem 3 is proved for all $s' < s$ and $t > t_1(s')$. By Theorem 7, it is enough to consider $s \ge 4$. Let G_0 be a counterexample for s and some $t > t_1(s)$, which is minimal with respect to $|V(G)| + |E(G)|$. Then G_0 is color-critical, namely $(s+t)$-critical. Let $n_0 = |V(G_0)|$.

The next three lemmas repeat the corresponding lemmas in [4] but we present their simple proofs for the convenience of the reader.

Lemma 16. $n_0 \ge 2(s+t) - 1$.

Proof. Suppose not. Then by Theorem 15, $V(G_0)$ can be partitioned into nonempty V_1 and V_2 so that each vertex in V_1 is adjacent to each vertex in V_2. Suppose that $\chi(G_0[V_1]) = k_1$ and $\chi(G_0[V_2]) = k_2$. By definition, $k_1 + k_2 = s + t$. We may assume that $k_1 \le k_2$. Since the theorem holds for $s = 1$ and any t, $G_0[V_1]$ has a K_{1,k_1-1} minor. Since $t \ge t_1(s)$ and $t_1(s) > 3t_1(s-1)$, $k_2 - s + 1 > t_1(s-1)$. Thus, by the minimality of s, $G_0[V_2]$ has a K^*_{s-1,k_2-s+1} minor. But then using the edges of the complete bipartite subgraph we construct a K^*_{s,k_1+k_2-s} minor of G_0 from these two minors. ■

By $\alpha(G)$ we denote the independence number of the graph G.

Lemma 17 (Seymour). *Let k be a non-negative integer. If $v \in V(G_0)$ and $d(v) = s + t - 1 + k$, then $\alpha(G_0[N(v)]) \le k + 1$.*

Proof. Suppose that $v \in V(G_0)$, $d(v) = s + t - 1 + k$, and $G[N(v)]$ has an independent set I with $|I| = k + 2$. Let G' be obtained from G_0 by contracting all edges connecting v with vertices in I and let v^* be the new vertex, which is the result of these contractions. Since G' is a minor of G_0, it does not have a $K^*_{s,t}$ minor. Therefore, by the minimality of G_0, G' is $(t+s-1)$-colorable. Let f' be a proper $(t+s-1)$-coloring of G'. Let $f'(v^*) = \alpha$. Coloring f' naturally yields a proper $(t+s-1)$-coloring f of $G_0 - v$ in which the color of each $w \in I$ is α. But then at most $d(v) - |I| + 1 = s + t - 2$

colors are present on $N(v)$, and we have an admissible color for v, a contradiction to the definition of G_0. ■

Lemma 18. *Let k be a non-negative integer. If $v \in V(G_0)$ and $d(v) = s + t - 1 + k$, then there exists a subset $Y(v) \subseteq N(v)$ such that $\delta(G_0[Y(v) \cup \{v\}]) \geq \frac{t}{k+1}$*

Proof. Suppose that the lemma is not true for some $v \in V(G_0)$, and let $F_0 = G_0[N(v)]$. Then F_0 is d-degenerate for some $d < \frac{t}{k+1} - 1$. Therefore, F_0 is $(d+1)$-colorable and hence

$$\alpha(F_0) \geq \frac{s+t-1+k}{d+1} > \frac{t+3}{t/(k+1)} > k+1,$$

a contradiction to Lemma 17. ■

Lemma 19. *The connectivity of G_0 is at least $\lceil 150s^2 \log_2 s \ln t \rceil$.*

Proof. Suppose that the connectivity of G is $x \leq \lfloor 150s^2 \log_2 s \ln t \rfloor$. First we show that

$$t \geq 1200x(\sqrt{\ln x} + 200). \tag{13}$$

By (4),

$$1200x \cdot 200 \leq 1200 \cdot 150s^2 \log_2 s \ln t \cdot 200 \leq 4.5\frac{t}{s \log_2 s} \leq \frac{9t}{16}.$$

Thus to show (13), it is enough to prove that

$$t \geq \frac{16}{7}1200x\sqrt{\ln x}. \tag{14}$$

Since $t \geq x$ (for example by (4)), instead of (14) it is enough to prove that

$$t \geq \frac{16}{7}1200 \ln t \sqrt{\ln t},$$

which follows from (5).

Now let X be a separating set in G_0 of size x. Let V_1 be the vertex set of a component of $G_0 - X$ and $V_2 = V(G_0) - X - V_1$. Let $i \in \{1, 2\}$, and $G_i = G_0[V_i]$. Since G_0 is $(s+t)$-critical, $\delta(G_i) \geq \delta(G_0) - x \geq s + t - 1 - x$. By Lemma 14, G_i contains an $\frac{s+t-1-x}{4}$-connected subgraph $\widetilde{G}_i$. Let $W_i := V(\widetilde{G}_i)$. Since G_0 is x-connected, there are x vertex-disjoint paths $P_{1,i}, \ldots, P_{x,i}$ connecting X to W_i (if $W_i \cap X \neq \emptyset$, then some paths will have length 0). For $j = 1, \ldots, x$, let $p_{j,i}$ be the vertex in W_i that belongs to $P_{j,i}$. Since $\sqrt{\ln x} > 2$ by (13),

$$\frac{s+t-1-x}{4} \geq \frac{t-x}{4} \geq 300x(\sqrt{\ln x} + 200) - \frac{x}{4} \geq 300(\sqrt{\ln x} + 190)x.$$

So by Corollary 10, $\widetilde{G}_i$ has a K_x minor such that all vertices $p_{1,i}, \ldots, p_{x,i}$ are in distinct branch sets. Contracting each such branch set into a vertex and then contracting each of the paths $P_{1,i}, \ldots, P_{x,i}$ into a vertex, we construct the minor G'_{3-i} of G_0 which is obtained from $G_0 - V_i$ by adding all edges between the vertices in X.

Since G_0 has no $K^*_{s,t}$ minor, neither of G'_1 and G'_2 has a $K^*_{s,t}$ minor. The minimality of G_0 implies then that for $i = 1, 2$, G'_i has an $(s+t-1)$-coloring f_i. Since

$G'_1[X] = G'_2[X] = K_x$, in both of these colorings, the colors of all vertices in X are distinct, and we can change the names of the colors in f_2 so that $f_1 \cup f_2$ is an $(s+t-1)$-coloring of G_0, a contradiction. ■

Now we are ready to prove Theorem 2. By Theorem 6,

$$\sum_{v \in V(G_0)} d(v) < (t + 8s \log_2 s)(n_0 - s + 1). \tag{15}$$

Since G_0 is color-critical, $\delta(G_0) \geq t + s - 1$. Say that a vertex $v \in V(G_0)$ is *low* if $d(v) < t + 16s \log_2 s$, and let L be the set of low vertices in G_0. By (15),

$$|L|(t + s - 1) + (n_0 - |L|)(t + 16s \log_2 s) < (t + 8s \log_2 s)(n_0 - s + 1).$$

It follows that $|L|(16s \log_2 s - s + 1) > 8s \log_2 s\, n_0$ and hence

$$|L| > 0.5n_0. \tag{16}$$

By Lemma 19, G_0 is $\lceil 150s^2 \log_2 s \ln t \rceil$-connected. Recall that by Lemma 16, $n_0 \geq 2t + 2s - 1$. If $n_0 \leq 2.5t$, then G_0 with $U = V(G_0)$ satisfies the conditions of Lemma 13 and hence has a $K^*_{s,t}$ minor, a contradiction. So,

$$n_0 > 2.5t. \tag{17}$$

Thus for $s \geq 4$ by (16), $|L| > 0.5(2.5t) = 1.25t$.

By Lemma 18, for every $v \in L$, there exists a subset $Y(v) \subseteq N(v)$ such that $\delta(G_0[Y(v) \cup \{v\}]) \geq \frac{t}{16s \log_2 s}$. Let $v_1, \ldots, v_{|L|}$ be the vertices of L, and for $j = 1, \ldots, |L|$, let $Z_j = \bigcup_{i=1}^{j} (Y(v_i) \cup \{v_i\})$. By construction, for every $j \geq 2$, $\delta(G_0[Z_j]) \geq \frac{t}{16s \log_2 s}$ and $j \leq |Z_j| \leq |Z_{j-1}| + t + 16s \log_2 s + 1$. It follows that there exists j_0 such that

$$1.25t < |Z_{j_0}| \leq 2.25t + 16s \log_2 s + 1 \leq 2.5t. \tag{18}$$

Since by (4), $1.25t \geq t + 200s^2 \log s \ln t$, the graph G_0 with $U = Z_{j_0}$ satisfies the conditions of Lemma 13 and hence has a $K^*_{s,t}$ minor, a contradiction.

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