On Coloring of Sparse Graphs

Alexandr Kostochka[1,2,⋆] and Matthew Yancey[1]

[1] University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA
[2] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

Abstract. Graph coloring has numerous applications and is a well-known NP-complete problem. The goal of this paper is to survey recent results of the authors on coloring and improper coloring of sparse graphs and to point out some polynomial-time algorithms for coloring (not necessarily optimal) of graphs with bounded maximum average degree.

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1 Introduction

A *proper k-coloring*, or simply *k-coloring*, of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$ such that for each $uv \in E$, $f(u) \neq f(v)$. A graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number*, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable. The problem of graph coloring with few colors has long been associated with resource assignment. One of the often cited related problems is the problem of assigning radio frequencies to a network of radio towers. The problem is known to be difficult: Determining if a graph is k-colorable when $k \geq 3$ was in Karp's [11] original list of 21 NP-complete problems. Furthermore, it is an NP-complete problem to even color a graph G with $1000\chi(G)$ colors [43]. This situation leads to constructing efficient algorithms for approximate coloring of graphs in special classes and to studying extremal problems on colorings.

What is said, relates also to improper colorings. A *d-improper k-coloring* of a graph G is a vertex coloring of G using k colors such that the graph induced by every color class has maximum degree at most d. By definition, a proper coloring of a graph is exactly a 0-improper coloring. So, a d-improper coloring is a relaxation of a proper coloring. Havet and Sereni [9] describe applications of d-improper k-colorings to frequency assignment problems. But improper colorings are even more complicated than the ordinary coloring. While it is easy to check whether a given graph is 2-colorable, Corrěa, Havet, and Sereni [10] proved that even the problem of existence of a 1-improper 2-coloring in the class of planar graphs is NP-complete.

It is natural that a graph G with a high *average degree*, $2\frac{|E(G)|}{|V(G)|}$, is harder to color with few colors. So, a low average degree of a graph may indicate that it is

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easier to color this graph. However, it could be that although the whole graph has low average degree, some its part is much denser and cannot be colored. A graph parameter controlling such anomalies is the *maximum average degree*, $mad(G) = \max_{H \subseteq G} 2\frac{|E(H)|}{|V(H)|}$. Kurek and Rucin'ski [36] called graphs with low maximum average degree *globally sparse.* In this paper, we describe some recent results of the authors (some of them are joint with O. Borodin and B. Lidický) on coloring and improper coloring of sparse graphs. These results imply polynomial-time algorithms for coloring globally sparse graphs with few colors.

One of the key notions in graph coloring is the one of critical graphs. A graph G is (d-improperly) k-*critical* if G is not (d-improperly) $(k-1)$-colorable, but every proper subgraph of G is (d-improperly) $(k-1)$-colorable. Critical graphs were first defined and used by Dirac [12–14] in 1951-52. A reason to study k-critical graphs is that every k-chromatic graph (i.e. graph with chromatic number k) contains a k-critical subgraph and k-critical graphs have more restricted structure. For example, k-critical graphs are 2-connected and $(k-1)$-edge-connected, which implies that every k-chromatic graph contains a 2-connected and $(k-1)$-edge-connected subgraph. The only 1-critical graph is K_1, and the only 2-critical graph is K_2. The only 3-critical graphs are the odd cycles. There are no k-critical graphs with $k+1$ vertices. For every $k \geq 4$ and every $n \geq k+2$, there exists a k-critical n-vertex graph.

One of the basic questions on k-critical graphs is: What is the minimum number $f_k(n)$ of edges in a k-critical graph with n vertices? This question was first asked by Dirac [16] in 1957 and then was reiterated by Gallai [22] in 1963, Ore [37] in 1967 and others [27, 28, 42]. More generally, we may ask about $f_{k,d}(n)$ — the minimum number of edges in a d-improperly k-critical graph with n vertices.

In this paper, we discuss results towards this problem and some applications of these results.

2 Gallai's Conjecture

Since the minimum degree of any k-critical graph is at least $k-1$,

$$f_k(n) \geq \frac{k-1}{2}n \tag{1}$$

for all $n \geq k$, $n \neq k+1$. Equality is achieved for $n = k$ and for $k = 3$ and n odd. Brooks' Theorem [11] implies that for $k \geq 4$ and $n \geq k+2$, the inequality in (1) is strict. In 1957, in order to to evaluate chromatic number of graphs embedded into fixed surfaces, Dirac [16] proved that for $k \geq 4$ and $n \geq k+2$,

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}. \tag{2}$$

The bound is tight for $n = 2k-1$ and yields $f_k(2k-1) = k^2 - k - 1$. Later, Kostochka and Stiebitz [30] improved (2) to

$$f_k(n) \geq \frac{k-1}{2}n + k - 3 \tag{3}$$

when $n \neq 2k-1, k$. This yields $f_k(2k) = k^2 - 3$ and $f_k(3k-2) = \frac{3k(k-1)}{2} - 2$.

Gallai [22] has found the values of $f_k(n)$ for $n \leq 2k-1$ and proved the following general bound for $k \geq 4$ and $n \geq k+2$:

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n. \tag{4}$$

For large n, the bound is much stronger than bounds (2) and (3). Based on his description of k-critical graphs with only one vertex of degree greater than $k-1$, Gallai [21] conjectured the following.

Conjecture 1 (Gallai [21]). *If $k \geq 4$ and $n = 1(\mod k-1)$, then*

$$f_k(n) = \frac{(k^2-k-2)n - k(k-3)}{2(k-1)}.$$

Ore [37] observed that Hajós' construction implies

$$f_k(n+k-1) \leq f_k(n) + \frac{(k-2)(k+1)}{2} = f_k(n) + (k-1)(k - \frac{2}{k-1})/2, \tag{5}$$

which yields that the limit $\phi_k := \lim_{n\to\infty} \frac{f_k(n)}{n}$ exists and satisfies

$$\phi_k \leq \frac{k}{2} - \frac{1}{k-1}. \tag{6}$$

Ore conjectured that his equation (5) is actually an equality:

Conjecture 2 (Ore [37]). *If $k \geq 4$, and $n \geq k+2$, then*

$$f_k(n+k-1) = f_k(n) + (k-1)(k - \frac{2}{k-1})/2.$$

Note that Conjecture 1 is equivalent to Conjecture 2 for $n = 1(\mod k-1)$. Some lower bounds on $f_k(n)$ were obtained in [16, 35, 21, 30, 31, 20]. Recently, the authors [32] proved Conjecture 1 in full.

Theorem 3 ([32]). *If $k \geq 4$ and G is k-critical, then $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil$. In other words, if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then*

$$f_k(n) \geq F(k,n) := \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil.$$

The result also confirms Conjecture 2 in the following cases: (a) $k=4$ and every $n \geq 6$, (b) $k=5$ and $n \equiv 2(\mod 4)$, and (c) every $k \geq 5$ and $n \equiv 1(\mod k-1)$. By examining known values of $f_k(n)$ when $n \leq 2k$, it follows that $f_k(n) - F(k,n) \leq k^2/8$.

It is known that there are infinitely many k-*extremal* graphs, i.e. the k-critical graphs G such that $|E(G)| = \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$. In particular, every graph in the family of so called k-*Ore graphs* is k-extremal. Very recently, the authors managed to describe all k-extremal graphs.

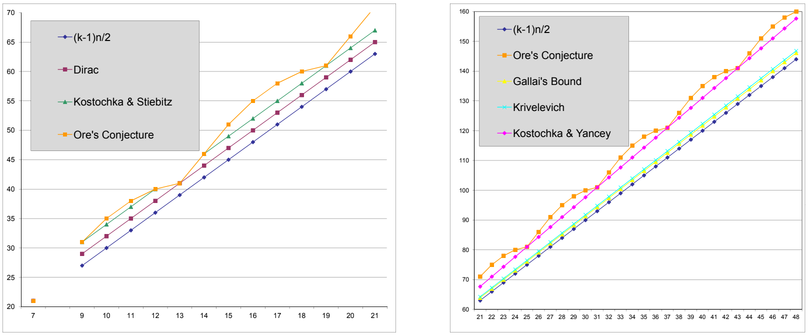

Fig. 1. Comparison of the bounds when $k = 7$

Theorem 4. *Let $k \geq 4$ and G be a k-critical graph. Then G is k-extremal if and only if it is a k-Ore graph. Moreover, if G is not a k-Ore graph, then $|E(G)| \geq \frac{(k^2-k-2)|V(G)|-y_k}{2(k-1)}$, where $y_k = \max\{2k-6, k^2-5k+2\}$.*

The message of Theorem 4 is that although for every $k \geq 4$ there are infinitely many k-extremal graphs, they all have a simple structure. In particular, every k-extremal graph distinct from K_k has a separating set of size 2. The theorem gives a slightly better approximation for $f_k(n)$ and adds new cases where we now know the exact values of $f_k(n)$: They are now known for (a) $k \in \{4, 5\}$ and every $n \geq k+2$, (b) all $k \geq 6$ and $n \equiv 1 (\mod k-1)$, (c) $k = 6$ and $n \equiv 0, 2 (\mod 5)$, and (d) $k = 7$ and $n \equiv 2 (\mod 6)$.

This value of y_k in Theorem 4 is best possible in the sense that for every $k \geq 4$, there exists an infinite family of 3-connected k-critical graphs with $|E(G)| = \frac{(k^2-k-2)|V(G)|-y_k}{2(k-1)}$. By (5), if we construct an n_0-vertex k-critical graph for which our lower bound on $f_k(n_0)$ is exact, then the bound on $f_k(n)$ is exact for every n of the form $n_0 + s(k-1)$. So, we only need to construct

- a 4-critical 6-vertex graph with $\lceil 9\frac{2}{3} \rceil = 10$ edges,
- a 4-critical 8-vertex graph with $\lceil 13 \rceil = 13$ edges,
- a 5-critical 10-vertex graph with $\lceil 22 \rceil = 22$ edges,
- a 5-critical 7-vertex graph with $\lceil 15\frac{1}{4} \rceil = 16$ edges,
- a 5-critical 8-vertex graph with $\lceil 17\frac{3}{4} \rceil = 18$ edges,
- a 6-critical 10-vertex graph with $\lceil 27\frac{1}{5} \rceil = 28$ edges,
- a 6-critical 12-vertex graph with $\lceil 32\frac{4}{5} \rceil = 33$ edges, and
- a 7-critical 14-vertex graph with $\lceil 45\frac{1}{3} \rceil = 46$ edges.

These graphs are presented in Figure 2.

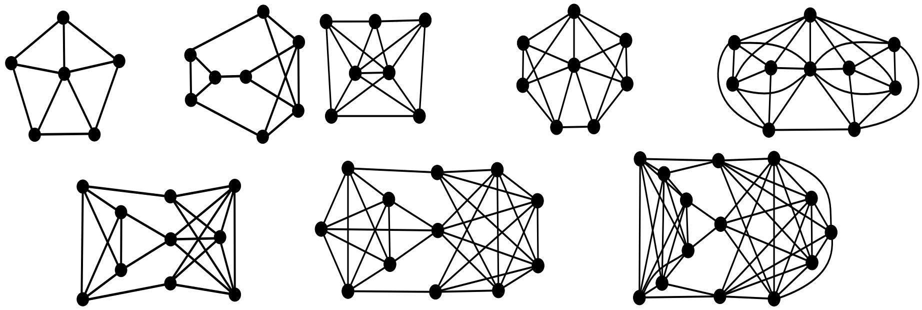

Fig. 2. Minimal k-critical graphs

3 Applications

3.1 Algorithms

Since the proof of Theorem 3 is constructive, it yields the following algorithmic counterpart.

Theorem 5. *If $k \geq 4$, then every n-vertex graph G with $|E(H)| < \frac{(k+1)(k-2)|V(H)|-k(k-3)}{2(k-1)}$ for each $H \subseteq G$ can be $(k-1)$-colored in $O(k^{3.5}n^{6.5}\log(n))$ time.*

The bound on complexity is not best possible. In particular, the algorithm finds maximum flows in auxiliary networks many times, so if one uses the Orlin's bounds on the running time of max-flow problems, the bounds will be improved.

Since every k-Ore graph distinct from the complete graph K_k contains a separating set of size 2 and decomposes into two k-Ore graphs of a smaller order, there is a simple algorithm that in time $O(n^5)$ checks whether a given n-vertex graph is a k-Ore graph. Together with an analog of the algorithm in Theorem 5 that uses the proof of Theorem 4 instead of Theorem 3, this yields the following

Theorem 6. *Let $k \geq 4$ and $y_k = \max\{2k-6, k^2-5k+2\}$. Then there exists a polynomial-time algorithm that for every n-vertex graph G with $|E(H)| < \frac{(k+1)(k-2)|V(H)|-y_k}{2(k-1)}$ for each $H \subseteq G$ either finds a $(k-1)$-coloring of G or finds a subgraph of G that is a k-Ore graph.*

3.2 Local and Global Graph Properties

Krivelevich [35] presented nice applications of his lower bounds on $f_k(n)$ and related graph parameters to finding complex graphs whose small subgraphs are simple. Two his bounds can be improved using Theorem 3 as follows.

Let $f(\sqrt{n}, 3, n)$ denote the maximum chromatic number over n-vertex graphs in which every $\sqrt{n}$-vertex subgraph is 3-colorable. Krivelevich proved that for every fixed $\epsilon > 0$ and sufficiently large n,

$$f(\sqrt{n}, 3, n) \geq n^{6/31-\epsilon}. \tag{7}$$

To show this, he applied his result that every 4-critical t-vertex graph with odd girth at least 7 has at least $31t/19$ edges. If instead of this, we simply use our bound on $f_4(n)$, then repeating almost word by word Krivelevich's proof of his Theorem 4 (choosing $p = n^{-0.8-\epsilon'}$), we get that for every fixed ϵ and sufficiently large n,

$$f(\sqrt{n}, 3, n) \geq n^{1/5-\epsilon}. \tag{8}$$

Another result of Krivelevich is:

Theorem 7 ([35]). *There exists $C > 0$ such that for every $s \geq 5$ there exists a graph G_s with at least $C\left(\frac{s}{\ln s}\right)^{\frac{33}{14}}$ vertices and independence number less than s such that the independence number of each 20-vertex subgraph at least 5.*

He used the fact that for every $m \leq 20$ and every m-vertex 5-critical graph H,

$$\frac{|E(H)| - 1}{m-2} \geq \frac{\lceil 17m/8 \rceil - 1}{m-2} \geq \frac{33}{14}.$$

From Theorem 3 we instead get

$$\frac{|E(H)| - 1}{m-2} \geq \frac{\left\lceil \frac{9m-5}{4} \right\rceil - 1}{m-2} \geq \frac{43}{18}.$$

Then repeating the argument in [35] we can replace $\frac{33}{14}$ in the statement of Theorem 7 with $\frac{43}{18}$.

3.3 Coloring Planar Graphs

Since planar graphs are sparse, Theorem 3 helps proving results on 3-coloring of planar graphs with restrictions on the structure. The case $k = 4$ of Theorem 3 is as follows:

Theorem 8. *If G is a 4-critical n-vertex graph then $|E(G)| \geq \frac{5n-2}{3}$.*

In [33], the authors gave a 3-page proof of Theorem 8. And this allows to give a half-page proof of the classical Grötzsch's Theorem [24] that *all triangle-free planar graphs are 3-colorable.* The original proof of Grötzsch's Theorem is somewhat sophisticated. The subsequent simpler proofs (see, e.g. [39] and references therein) are still not too simple. The proof below from [33] is the shortest so far:

Proof of Grötzsch's Theorem: Let G be a plane graph with the smallest $|E(G)| + |V(G)|$ for which the theorem does not hold. Then G is 4-critical. Suppose G has n vertices, e edges and f faces.

CASE 1: G has no 4-faces. Then $5f \leq 2e$ and so $f \leq 2e/5$. From this and Euler's Formula $n - e + f = 2$, we obtain $n - 3e/5 \geq 2$, i.e., $e \leq \frac{5n-10}{3}$, a contradiction to Theorem 8.

CASE 2: G has a 4-face (x, y, z, u). Since G has no triangles, $xz, yu \notin E(G)$. If the graph G_{xz} obtained from G by gluing x with z has no triangles, then by the minimality of G, G_{xz} is 3-colorable, and so G also is 3-colorable. Thus G has an x, z-path (x, v, w, z) of length 3. Since G itself has no triangles, $\{y, u\} \cap \{v, w\} = \emptyset$ and there are no edges between $\{y, u\}$ and $\{v, w\}$. But then G has no y, u-path of length 3, since such a path must cross the path (x, v, w, z). Thus the graph G_{yu} obtained from G by gluing y with u has no triangles, and so, by the minimality of G, is 3-colorable. Then G also is 3-colorable, a contradiction. □

Borodin, Lidický and the authors [8] used Theorem 8 to present simple proofs for some 3-coloring results on planar graphs, in particular, for the Grünbaum-Axenov Theorem that every planar graph with at most 3 triangles is 3-colorable. This theorem is sharp in the sense that there are infinitely many plane 4-critical graphs with exactly four triangles. Moreover, Thomas and Walls [38] constructed infinitely many such graphs without 4-faces.

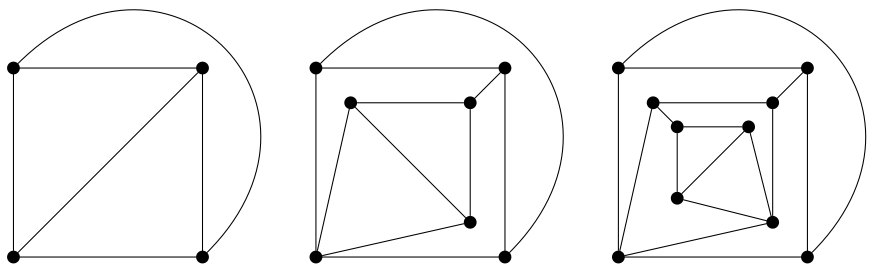

Fig. 3. Some 4-critical graphs from the family described by Thomas and Walls [38]

Very recently, Borodin, Lidický and the authors using Theorem 4 described all plane 4-critical graphs with exactly four triangles and no 4-faces. Using Theorem 8 they also proved:

Theorem 9. *Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree at most 4. Then H is 3-colorable.*

The theorem sharpens the similar result by Jensen and Thomassen [26] where the degree of v was at most 3 and is sharp in the sense that there are infinitely many plane triangle-free graphs that are obtained from a 4-critical graph by deleting a vertex of degree 5.

4 Improper 2-Colorings

As it was mentioned in the introduction, even the problem whether a given planar graph is 1-improperly 2-colorable is NP-complete. This motivates the study of improper 2-colorings for globally sparse graphs. A (j,k)-*coloring* of a graph G is a 2-coloring of $V(G)$ such that every vertex of Color 1 has at most j neighbors of Color 1 and every vertex of Color 2 has at most k neighbors of its color. By definition, a (d,d)-coloring is simply a d-improper 2-coloring. Esperet, Montassier, Ochem and Pinlou [19] proved that for every $j \geq 0$ and $k \geq 1$, the problem of verifying whether a given planar graph has a (j,k)-coloring is NP-complete. It is somewhat nonstandard but convenient to call a graph (j,k)-*critical* if it is not (j,k)-colorable but after deleting any edge or vertex it becomes (j,k)-colorable.

Glebov and Zambalaeva [23] proved that every planar graph G with girth, $g(G)$, at least 16 is $(0,1)$-colorable. Then Borodin and Ivanova [1] proved that every graph G with $mad(G) < \frac{7}{3}$ is $(0,1)$-colorable. Since $mad(G) \leq \frac{2g(G)}{g(G)-2}$ for every planar graph G with girth $g(G)$, this implies that every planar graph G with $g(G) \geq 14$ is $(0,1)$-colorable. Borodin and the first author [6] proved that every graph G with $mad(G) < \frac{12}{5}$ is $(0,1)$-colorable, and this is sharp. This also implies that every planar graph G with $g(G) \geq 12$ is $(0,1)$-colorable. Dorbec, Kaiser, Montassier, and Raspaud [18] have constructed a $(0,1)$-critical graph with girth 9.

Kurek and Ruciński [36] studied improper colorings within the more general framework of *vertex Ramsey probems*. We say that $G \to (H_1,\ldots,H_k)$ if for every coloring $\phi : V(G) \to [k]$, there exists an i such that the subgraph of G induced by the vertices of Color i contains H_i. By definition, a graph G is $(K_{1,j}, K_{1,k})$-vertex Ramsey exactly when G has no (j,k)-coloring. Kurek and Ruciński [36] considered the extremal function $m_{cr}(H_1,\ldots,H_k) = inf\{mad(F) : F \to (H_1,\ldots,H_k)\}$. In particular, Kurek and Ruciński showed that $8/3 \leq m_{cr}(K_{1,2}, K_{1,2}) \leq 14/5$. Ruciński offered 400,000 PLZ cash prize for the exact value of $m_{cr}(K_{1,2}, K_{1,2})$. Recently, Borodin and the authors solved this problem.

Theorem 10 ([9]). *If G is a $(1,1)$-critical graph, then* $5|E(G)| > 7|V(G)|$.

The result is sharp in the sense that there are infinitely many $(1,1)$-critical graphs with $5|E(G)| = 7|V(G)| + 1$. One such graph is present in Fig. 4.

The proof of the result is algorithmic and yields a polynomial-time algorithm that finds a $(1,1)$-coloring for every graph G with $mad(G) \leq \frac{14}{5}$. In a standard manner, the theorem yields that every planar graph with girth at least 7 is $(1,1)$-colorable. It also refines a result by Borodin and Ivanova [2]: They showed that every graph G with girth at least 7 and $mad(G) < \frac{14}{5}$ can be partitioned into two subsets such that every connected monochromatic subgraph has at most two edges. Our result shows that each component contains at most 1 edge.

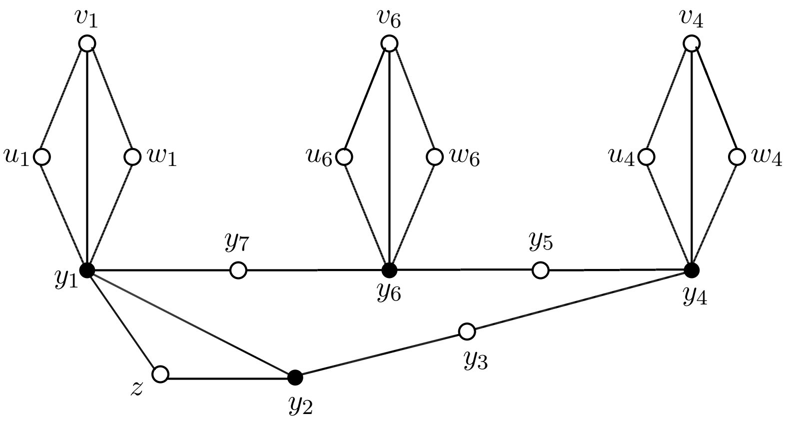

Fig. 4. A $(1,1)$-critical graph G with $5|E(G)| = 7|V(G)| + 1$

For general j and k, the problem of (j,k)-coloring of planar and globally sparse graphs was considered in [3–5]. Borodin and the first author [7] proved that if $k \geq 2j + 2$, then every graph with maximum average degree at most $2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$ is (j,k)-colorable. On the other hand, they constructed graphs with the maximum average degree arbitrarily close to $2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$ (from above) that are not (j,k)-colorable. Note that most likely if $j \leq k < 2j + 2$, then the answer differs from that in the case $k \geq 2j + 2$. In particular, the answer for $(1,1)$-colorings in Theorem 10 differs from it.

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