

TURÁN PROBLEMS AND SHADOWS III: EXPANSIONS OF GRAPHS*

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Abstract. The expansion G^+ of a graph G is the 3-uniform hypergraph obtained from G by enlarging each edge of G with a new vertex disjoint from $V(G)$ such that distinct edges are enlarged by distinct vertices. Let $\text{ex}_3(n, F)$ denote the maximum number of edges in a 3-uniform hypergraph with n vertices not containing any copy of a 3-uniform hypergraph F . The study of $\text{ex}_3(n, G^+)$ includes some well-researched problems, including the case that F consists of k disjoint edges, G is a triangle, G is a path or cycle, and G is a tree. In this paper we initiate a broader study of the behavior of $\text{ex}_3(n, G^+)$. Specifically, we show $\text{ex}_3(n, K_{s,t}^+) = \Theta(n^{3-3/s})$ whenever $t > (s-1)!$ and $s \geq 3$. One of the main open problems is to determine for which graphs G the quantity $\text{ex}_3(n, G^+)$ is quadratic in n . We show that this occurs when G is any bipartite graph with Turán number $o(n^\varphi)$ where $\varphi = \frac{1+\sqrt{5}}{2}$, and in particular this shows $\text{ex}_3(n, G^+) = O(n^2)$ when G is the three-dimensional cube graph.

Key words. triple systems, Turán problems, expansions

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1. Introduction. An r -uniform hypergraph F , or simply r -graph, is a family of r -element subsets of a finite set. We associate an r -graph F with its edge set and call its vertex set $V(F)$. Given an r -graph F , let $\text{ex}_r(n, F)$ denote the maximum number of edges in an r -graph on n vertices that does not contain F . The *expansion* of a graph G is the 3-graph G^+ with edge set $\{e \cup \{v_e\} : e \in G\}$, where v_e are distinct vertices not in $V(G)$. By definition, the expansion of G has exactly $|G|$ edges. Note that Füredi and Jiang [10, 11] used a notion of expansion to r -graphs for general r , but this paper considers only 3-graphs.

Expansions include many important hypergraphs whose extremal functions have been investigated; for instance, the celebrated Erdős–Ko–Rado theorem [7] for 3-graphs is the case of expansions of a matching. A well-known result is that $\text{ex}_3(n, K_3^+) = \binom{n-1}{2}$ [5, 9, 18]. If a graph is not 3-colorable, then its expansion has positive Turán density; this case is fairly well understood [16, 19], so we focus on the case of expansions of 3-colorable graphs. It is easy to see that $\text{ex}_3(n, G^+) = \Omega(n^2)$ unless G is a star (the case that G is a star is interesting in itself, and for $G = P_2$ determining $\text{ex}_3(n, G^+)$ constituted a conjecture of Erdős and Sós [6] which was solved by Frankl [8]). The authors [13] had previously determined $\text{ex}_3(n, G^+)$ exactly (for large n) when G is a path or cycle of fixed length $k \geq 3$, thereby answering questions posed

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by Füredi, Jiang, and Seiver [12] and Füredi and Jiang [11]. The case when G is a forest is solved asymptotically in [14], thus settling a conjecture of Füredi [10]. The following straightforward result provides general bounds for $\text{ex}_3(n, G^+)$ in terms of the number of edges of G .

PROPOSITION 1.1. *If G is any graph with v vertices and $f \geq 4$ edges, then for some $a > 0$,*

$$an^{3-\frac{3v-9}{f-3}} \leq \text{ex}_3(n, G^+) \leq (n-1)\text{ex}_2(n, G) + (f+v-1)\binom{n}{2}.$$

The proof of Proposition 1.1 is given in section 3. Some key remarks are that $\text{ex}_3(n, G^+)$ is not quadratic in n if $f > 3v - 6$, and if G is not bipartite, then the upper bound in Proposition 1.1 is cubic in n . This suggests the question of identifying the graphs G for which $\text{ex}_3(n, G^+) = O(n^2)$, and in particular evaluation of $\text{ex}_3(n, G^+)$ for planar G .

1.1. Expansions of planar graphs. We give a straightforward proof of the following proposition, which is a special case of a more general result of Füredi [10] for a larger class of triple systems.

PROPOSITION 1.2. *Let G be a graph with tree-width at most two. Then $\text{ex}_3(n, G^+) = O(n^2)$.*

On the other hand, there are 3-colorable planar graphs G for which $\text{ex}_3(n, G^+)$ is not quadratic in n . To state this result, we need a definition. A proper k -coloring $\chi : V(G) \rightarrow \{1, \dots, k\}$ is *acyclic* if every pair of color classes induces a forest in G . We pose the following question.

QUESTION 1. *Does every planar graph G with an acyclic 3-coloring have $\text{ex}_3(n, G^+) = O(n^2)$?*

Let $g(n, k)$ denote the maximum number of edges in an n -vertex graph of girth larger than k .

PROPOSITION 1.3. *Let G be a planar graph such that in every proper 3-coloring of G , every pair of color classes induces a subgraph containing a cycle of length at most k . Then $\text{ex}_3(n, G^+) = \Omega(ng(n, k)) = \Omega(n^{2+\Theta(\frac{1}{k})})$.*

The last statement follows from the known fact that $g(n, k) \geq n^{1+\Theta(\frac{1}{k})}$. The octahedron graph O is an example of a planar graph where in every proper 3-coloring, each pair of color classes induces a cycle of length four, and so $\text{ex}_3(n, O^+) = \Omega(n^{5/2})$. Even wheels do not have acyclic 3-colorings, and we do not know whether their expansions have quadratic Turán numbers.

QUESTION 2. *Does every even wheel G have $\text{ex}_3(n, G^+) = O(n^2)$?*

1.2. Expansions of bipartite graphs. The behavior of $\text{ex}_3(n, G^+)$ when G is a dense bipartite graph is somewhat related to the behavior of $\text{ex}_2(n, G)$ according to Proposition 1.1. In particular, Proposition 1.1 shows for $t \geq s \geq 2$ and some constants $a, c > 0$ that

$$an^{3-\frac{3s+3t-9}{st-3}} \leq \text{ex}_3(n, K_{s,t}^+) \leq cn^{3-\frac{1}{s}}.$$

We show that both the upper and lower bounds can be improved to determine the order of magnitude of $\text{ex}_3(n, K_{s,t}^+)$ when good constructions of $K_{s,t}$ -free graphs are available (see Alon, Rónyai, and Szabó [2]).

THEOREM 1.4. *Fix $3 \leq s \leq t$. Then $\text{ex}_3(n, K_{s,t}^+) = O(n^{3-\frac{3}{s}})$ and, if $t > (s-1)! \geq 2$, then $\text{ex}_3(n, K_{s,t}^+) = \Theta(n^{3-\frac{3}{s}})$.*

The following closely related problem was recently investigated by Alon and Shikhelman [3]. For a graph F , let $g(n, F)$ denote the maximum number of triangles in an n -vertex graph that contains no copy of F as a subgraph. From a graph G achieving this maximum, we can form a 3-graph H with $V(H) = V(G)$ and H consists of the triangles in G . Then a copy K of F^+ in H would yield a copy of F in G as $\partial K \supset F$. Consequently, we have

$$g(n, F) \leq \text{ex}_3(n, F^+).$$

Alon and Shikhelman [3] independently proved that for fixed $3 \leq s \leq t$ and $t > (s-1)!$ we have $g(n, K_{s,t}) = \Theta(n^{3-3/s})$. Their lower bound construction is exactly the same as ours, though the proofs are different.

The case of $K_{3,t}$ is interesting since $\text{ex}_3(n, K_{3,t}^+) = O(n^2)$, and perhaps it is possible to determine a constant c such that $\text{ex}_3(n, K_{3,3}^+) \sim cn^2$, since the asymptotic behavior of $\text{ex}_2(n, K_{3,3})$ is known, due to a construction of Brown [4] and the upper bounds of Füredi [10]. In general, the following bounds hold for expansions of $K_{3,t}$.

THEOREM 1.5. *For fixed $r \geq 1$ and $t = 2r^2 + 1$, we have $(1 - o(1))\frac{t-1}{12}n^2 \leq \text{ex}_3(n, K_{3,t}^+) = O(n^2)$.*

The upper bound in this theorem is a special case of a general upper bound for all graphs G with $\sigma(G^+) = 3$ (see Theorem 1.7). Finally, we prove a general result that applies to expansions of a large class of bipartite graphs.

THEOREM 1.6. *Let G be a graph with $\text{ex}_2(n, G) = o(n^\phi)$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Then $\text{ex}_3(n, G^+) = O(n^2)$.*

Let \mathbb{Q} be the graph of the three-dimensional cube (with 8 vertices and 12 edges). Erdős and Simonovits (see [6]) proved $\text{ex}_2(n, \mathbb{Q}) = O(n^{1.6}) = o(n^\phi)$, so a corollary to Theorem 1.6 is that

$$\text{ex}_3(n, \mathbb{Q}^+) = \Theta(n^2).$$

Determining the growth rate of $\text{ex}_2(n, \mathbb{Q})$ is a longstanding open problem. Since it is known that for any graph G the 1-subdivision of G has Turán number $O(n^{3/2})$ —see Alon, Krivelevich, and Sudakov [1]—Theorem 1.6 also shows that for such graphs G , $\text{ex}_3(n, G^+) = \Theta(n^2)$. Erdős conjectured that $\text{ex}_2(n, G) = O(n^{3/2})$ for each 2-degenerate bipartite graph G . If this conjecture is true, then by Theorem 1.6, $\text{ex}_3(n, G^+) = O(n^2)$ for any 2-degenerate bipartite graph G .

1.3. Crosscuts. A set of vertices in a hypergraph containing exactly one vertex from every edge of a hypergraph is called a *crosscut* of the hypergraph, following Frankl and Füredi [9]. For a 3-uniform hypergraph F , let $\sigma(F)$ be the minimum size of a crosscut of F if it exists, i.e.,

$$\sigma(F) := \min\{|X| : \forall e \in F, |e \cap X| = 1\}$$

if such an X exists. Since the triple system consisting of all edges containing exactly one vertex from a set of size $\sigma(F) - 1$ does not contain F , we have

$$(1.1) \quad \text{ex}_3(n, F) \geq (\sigma(F) - 1 + o(1)) \binom{n}{2}.$$

An intriguing open question is, For which F an asymptotic equality is attained in (1.1)? Recall that a graph has tree-width at most two if and only if it has no subdivision of K_4 . Informally, these are subgraphs of a planar graph obtained by

starting with a triangle, and then picking some edge uv of the current graph, adding a new vertex w , and then adding the edges uw and vw .

QUESTION 3. *Is it true that*

$$(1.2) \quad \text{ex}_3(n, G^+) \sim (\sigma(G^+) - 1) \binom{n}{2}$$

for every graph G with tree-width two?

If G is a forest or a cycle, then (1.2) holds [13, 14] (corresponding results for $r > 3$ were given by Füredi [10]). If G is a graph with $\sigma(G^+) = 2$, then again (1.2) holds [14]. Proposition 1.1 and Theorem 1.4 give examples of graphs G with $\sigma(G^+) = 4$ and $\text{ex}_3(n, G^+)$ superquadratic in n . This leaves the case $\sigma(G^+) = 3$, and in this case, Theorem 1.5 shows that $\text{ex}_3(n, K_{3,t}^+)/n^2 \rightarrow \infty$ as $t \rightarrow \infty$, even though $\sigma(K_{3,t}^+) = 3$ for all $t \geq 3$. A quadratic upper bound for $\text{ex}_3(n, K_{3,t}^+)$ in Theorem 1.5 is a special case of the following theorem.

THEOREM 1.7. *For every G with $\sigma(G^+) = 3$, $\text{ex}_3(n, G^+) = O(n^2)$.*

2. Preliminaries. First we introduce some notation. A 3-graph is called a *triple system*. The edges will be written as unordered lists, for instance, xyz represents $\{x, y, z\}$. For a set X of vertices of a hypergraph H , let $H - X = \{e \in H : e \cap X = \emptyset\}$. If $X = \{x\}$, then we write $H - x$ instead of $H - X$. For a set S of two vertices in a 3-graph H , $N_H(S) = \{x \in V(H) : S \cup \{x\} \in H\}$. The *codegree* of a pair $S = \{x, y\}$ of vertices in a 3-graph H is $d_H(x, y) = |N_H(S)|$. The *shadow* of H is the graph $\partial H = \{xy : \exists e \in H, \{x, y\} \subset e\}$. The edges of ∂H will be called the *subedges* of H . As usual, for a graph G and $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G and $d_G(v) = |N_G(v)|$.

A 3-graph H is *d-full* if every subedge of H has codegree at least d .

Thus H is *d-full* if and only if the minimum nonzero codegree in H is at least d . The following lemma from [14] extends the well-known fact that any graph G has a subgraph of minimum degree at least d with at least $|G| - (d - 1)|V(G)|$ edges.

LEMMA 2.1. *For $d \geq 1$, every n -vertex 3-graph H has a $(d + 1)$ -full subgraph F with*

$$|F| \geq |H| - d|\partial H|.$$

Proof. A *d-sparse sequence* is a maximal sequence $e_1, e_2, \dots, e_m \in \partial H$ such that $d_H(e_1) \leq d$, and for all $i > 1$, e_i is contained in at most d edges of H which contain none of e_1, e_2, \dots, e_{i-1} . The 3-graph F obtained by deleting all edges of H containing at least one of the e_i is $(d + 1)$ -full. Since a *d-sparse sequence* has length at most $|\partial H|$, we have $|F| \geq |H| - d|\partial H|$. \square

3. Proofs of propositions. In this section we deliver the proofs of three propositions.

Proof of Proposition 1.1. The proof of the lower bound in Proposition 1.1 is via a random triple system. The idea is to take a random graph not containing a particular graph G and then observe that the triple system of triangles in the random graph does not contain G^+ . Consider the random graph on n vertices, whose edges are placed independently with probability p , to be chosen later. If X is the number of triangles and Y is the number of copies of G in the random graph, then

$$\mathbb{E}(X) = p^3 \binom{n}{3}, \quad \mathbb{E}(Y) \leq p^f n^v.$$

Therefore choosing $p = 0.1n^{-(v-3)/(f-3)}$, since $f \geq 4$, we find

$$\mathbb{E}(X - Y) \geq 0.0001n^{3-3(v-3)/(f-3)}.$$

Now let H be the triple system of vertex sets of triangles in the graph obtained by removing one edge from each copy of G in the random graph. Then $\mathbb{E}(|H|) \geq \mathbb{E}(X - Y)$, and $G^+ \not\subset H$. Select an H so that $|H| \geq 0.0001n^{3-3(v-3)/(f-3)}$. This proves the lower bound in Proposition 1.1 with $a = 0.0001$.

Now suppose G is a bipartite graph with f edges e_1, e_2, \dots, e_f and v vertices. If a triple system H on n vertices has more than $(n-1)\text{ex}_2(n, G) + (f+v-1)\binom{n}{2}$ triples, then by deleting at most $(f+v-1)\binom{n}{2}$ triples we arrive at a triple system $H' \subset H$ which is $(f+v)$ -full, by Lemma 2.1 and $|H'| > (n-1)\text{ex}_2(n, G)$. There exists $x \in V(H')$ such that more than $\text{ex}_2(n, G)$ triples of H' contain x . So the graph of all pairs $\{w, y\}$ such that $\{w, x, y\} \in H'$ contains G . Since every pair $\{w, y\}$ has codegree at least $f+v$, we find vertices $z_1, z_2, \dots, z_f \notin V(G)$ such that $e_i \cup \{z_i\} \in H'$ for all $i = 1, 2, \dots, f$, and this forms a copy of G^+ in H' . \square

Proof of Proposition 1.2. Let G be a graph of tree-width two. Then $G \subset F$, where F is a graph obtained from a triangle by repeatedly adding a new vertex and joining it to two adjacent vertices of the current graph. It is enough to show $\text{ex}_3(n, F^+) = O(n^2)$. Suppose F has v vertices and f edges. By definition, F has a vertex x of degree two such that the neighbors x' and x'' of x are adjacent. Then $F' := F - x$ has $v-1$ vertices and $f-2$ edges. Let H be an n -vertex triple system with more than $(v+f-1)\binom{n}{2}$ edges. By Lemma 2.1, H has a $(v+f)$ -full subgraph H' . We claim H' contains F^+ . Inductively, H' contains a copy H'' of the expansion of F' . By the definition of H' , $\{x', x''\}$ has codegree at least $v+f$ in H' . Therefore we may select a new vertex z that is not in H'' such that $\{z, x', x''\}$ is an edge of H' , and now F is embedded in H' by mapping x to z . \square

Proof of Proposition 1.3. Let G be a 3-colorable planar graph with the given conditions. To show $\text{ex}_3(n, G^+) = \Omega(ng(n, k))$, form a triple system H on n vertices as follows. Let F be a bipartite $\lfloor \frac{n}{2} \rfloor$ -vertex graph of girth $k+1$ with at least $\frac{1}{2}g(\lfloor \frac{n}{2} \rfloor, k)$ edges. Let U and V be the partite sets of F . Let X be a set of $\lfloor \frac{n}{2} \rfloor$ vertices disjoint from $U \cup V$. Then set $V(H) = U \cup V \cup X$ and let the edges of H consist of all triples $e \cup \{x\}$ such that $e \in F$ and $x \in X$. Then

$$|H| \geq |X| \cdot g\left(\left\lfloor \frac{n}{2} \right\rfloor, k\right) = \Omega(ng(n, k)).$$

Now ∂H has a natural 3-coloring given by U, V, X . If $G^+ \subset H$, then $G \subset \partial H$ and therefore G is properly colored, with color classes $V(G) \cap U$, $V(G) \cap V$, and $V(G) \cap X$. By the assumptions on G , $V(G) \cap (U \cup V)$ induces a subgraph of G which contains a cycle of length at most k . However, that cycle is then a subgraph of F , by the definition of H , which is a contradiction. Therefore $G^+ \not\subset H$. \square

4. Proof of Theorem 1.6. Suppose $\text{ex}_2(n, G) = o(n^\phi)$ and $|G| = k$, and H is an G^+ -free 3-graph with $|H| \geq (k+1)\binom{n}{2}$. By Lemma 2.1, H has a k -full subgraph H_1 with at least $n^2/3$ edges. If $G \subset \partial H_1$, then we can expand G to $G^+ \subset H_1$ using that H_1 is k -full. Therefore $|\partial H_1| \leq \text{ex}_2(n, G) = o(n^\phi)$. By Lemma 2.1, and since $|H_1| \geq \delta n^2$, H_1 has a nonempty $n^{2-\phi}$ -full subgraph H_2 if n is large enough. Let H_3 be obtained by removing all isolated vertices of H_2 and let $m = |V(H_3)|$. Since H_3 is $n^{2-\phi}$ -full, $m > n^{2-\phi}$. Since H_1 is G^+ -free, $H_3 \subset H_1$ is also G^+ -free, and therefore if $F = \partial H_3$, then $|V(F)| = |V(H_3)| = m$ and $|F| \leq \text{ex}_2(m, G) = o(m^\phi)$. So some vertex v of the graph $F = \partial H_3$ has degree $o(m^{\phi-1})$. Now the number

of edges of F between the vertices of $N_F(v)$ is at least the number of edges of H_3 containing v . Since H_3 is $n^{2-\phi}$ -full, there are at least $\frac{1}{2}n^{2-\phi}|N_F(v)|$ such edges. On the other hand, since the subgraph of F induced by $N_F(v)$ does not contain G , the number of such edges is $o(|N_F(v)|^\phi)$. It follows that $n^{2-\phi} = o(|N_F(v)|^{\phi-1})$. Since $|N_F(v)| = o(m^{\phi-1}) = o(n^{\phi-1})$, we get $2 - \phi < (\phi - 1)^2$, contradicting the fact that ϕ is the golden ratio. \square

5. Proof of Theorem 1.4. For the upper bound, we repeat the proof of Theorem 1.6 when $F = K_{s,t}$, using the bounds $\text{ex}_2(n, K_{s,t}) = O(n^{2-1/s})$ provided by the Kövari–Sós–Turán theorem [15], except at the stage of the proof where we use the bound on $\text{ex}_2(|N_G(v)|, F)$. We may now use

$$\text{ex}_2(|N_G(v)|, K_{s-1,t}) = O(|N_G(v)|^{2-1/(s-1)}),$$

for if the subgraph of G of edges between $N_G(v)$ contains $K_{s-1,t}$, then by adding v we see G contains $K_{s,t}$. A calculation gives $|H| = O(n^{3-3/s})$.

For the lower bound we must show that $\text{ex}_3(n, K_{s,t}^+) = \Omega(n^{3-3/s})$ if $t > (s - 1)!$. We will use the *projective norm graphs* defined by Alon, Rónyai, and Szabó [2]. Given a finite field \mathbb{F}_q and an integer $s \geq 2$, the norm is the map $N : \mathbb{F}_{q^{s-1}}^* \rightarrow \mathbb{F}_q^*$ given by $N(X) = X^{1+q+\dots+q^{s-2}}$. The norm is a (multiplicative) group homomorphism and is the identity map on elements of \mathbb{F}_q^* . This implies that for each $x \in \mathbb{F}_q^*$, the number of preimages of x is exactly

$$(5.1) \quad \frac{q^{s-1} - 1}{q - 1} = 1 + q + \dots + q^{s-2}.$$

DEFINITION 5.1. Let q be a prime power and $s \geq 2$ be an integer. The projective norm graph $PG(q, s)$ has vertex set $V = \mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$ and edge set

$$\{(A, b)(B, b) : N(A + B) = ab\}.$$

LEMMA 5.2. Fix an integer $s \geq 3$ and a prime power q . Let $x \in \mathbb{F}_q^*$, and $A, B \in \mathbb{F}_{q^{s-1}}$ with $A \neq B$. Then the number of $C \in \mathbb{F}_{q^{s-1}}$ with

$$(5.2) \quad N\left(\frac{A + C}{B + C}\right) = x$$

is at least q^{s-2} .

Proof. By (5.1) there exist distinct $X_1, \dots, X_{q^{s-2}+1} \in \mathbb{F}_{q^{s-1}}^*$ such that $N(X_i) = x$ for each i . As long as $X_i \neq 1$, define

$$C_i = \frac{BX_i - A}{1 - X_i}.$$

Then $(A + C_i)/(B + C_i) = X_i$, and $C_i \neq C_j$ for $i \neq j$ since $A \neq B$. \square

LEMMA 5.3. Fix an integer $s \geq 3$ and a prime power q . The number of triangles in $PG(q, s)$ is at least $(1 - o(1))q^{3s-3}/6$ as $q \rightarrow \infty$.

Proof. Pick a vertex (A, a) and then one of its neighbors (B, b) . The number of ways to do this is at least $q^{s-1}(q - 1)(q^{s-1} - 1)$. Let $x = a/b$ and apply Lemma 5.2 to obtain at least $q^{s-2} - 2$ distinct $C \notin \{-A, -B\}$ satisfying (5.2). For each such C , define

$$c = \frac{N(A + C)}{a} = \frac{N(B + C)}{b}.$$

Then (C, c) is adjacent to both (A, a) and (B, b) . Each triangle is counted six times in this way, and the result follows. \square

For appropriate n the n -vertex norm graphs $PG(q, s)$ (for fixed s and large q) have $\Theta(n^{2-1/s})$ edges and no $K_{s,t}$. By Lemma 5.3 the number of triangles in $PG(q, s)$ is $\Theta(n^{3-3/s})$. The hypergraph H whose edges are the vertex sets of triangles in $PG(q, s)$ is a 3-graph with $\Theta(n^{3-3/s})$ edges and no $K_{s,t}^+$. This completes the proof of Theorem 1.4. \square

6. Proof of Theorems 1.5 and 1.7.

We need the following result.

THEOREM 6.1. *Let F be a 3-uniform hypergraph with v vertices and $\text{ex}_3(n, F) < c \binom{n}{2}$. Then $\text{ex}_3(n, (\partial F)^+) < (c + v + |F|) \binom{n}{2}$.*

Proof. Suppose we have an n vertex 3-uniform hypergraph H with $|H| > (c + v + |H|) \binom{n}{2}$. Apply Lemma 2.1 to obtain a subhypergraph $H' \subset H$ that is $(v + |F|)$ -full with $|H'| > c \binom{n}{2}$. By definition, we may find a copy of $F \subset H'$ and hence a copy of $\partial F \subset \partial H'$. Because H' is $(v + |F|)$ -full, we may expand this copy of ∂F to a copy of $(\partial F)^+ \subset H' \subset H$ as desired. \square

Define H_t to be the 3-uniform hypergraph with vertex set $\{a, b, x_1, y_1, \dots, x_t, y_t\}$ and $2t$ edges $x_i y_i a$ and $x_i y_i b$ for all $i \in [t]$. It is convenient (though not necessary) for us to use the following theorem of the authors [17].

THEOREM 6.2 (see [17]). *For each $t \geq 2$, we have $\text{ex}_3(n, H_t) < t^4 \binom{n}{2}$.*

Now we prove the upper bound in Theorem 1.7. Suppose $\sigma(G^+) \leq 3$. This means that G has an independent set I and set R of edges such that I intersects each edge in $G - R$, and $|I| + |R| \leq 3$. It follows that G is a subgraph of one of the following graphs (cases (i) and (ii) correspond to $|I| = 1$, case (iii) corresponds to $|I| = 2$, and case (iv) corresponds to $|I| = 3$):

- (i) $K_4 - e$ together with a star centered at one of the degree 3 (in $K_4 - e$) vertices;
- (ii) two triangles sharing a vertex x and a star centered at x ;
- (iii) the graph obtained from $K_{2,t}$ by adding an edge joining two vertices in the part of size t ;
- (iv) $K_{3,t}$.

Now suppose we have a 3-uniform H with $|H| > cn^2$ for some $c > |G| + |V(G)|$. Applying Lemma 2.1, we find a c -full $H' \subset H$ with $|H'| > c \binom{n}{2}$. As in the proof of Theorem 6.1, it is enough to find G in $\partial H'$. Since $|H'| > c \binom{n}{2}$, the codegree of some pair $\{x, y\}$ is at least $c + 1$. Then the shadow of the set of triples in H' containing $\{x, y\}$ contains the graph of the form (i). Similarly, H' contains two edges sharing exactly one vertex, say x , and the shadow of the set of triples in H' containing x contains the graph of the form (ii). If G is of the form in (iii), we apply Theorems 6.1 and 6.2 and observe that $\partial H_t \supset G$. Finally, if $G \subseteq K_{3,t}$ then we apply Theorem 1.4.

For the lower bound in Theorem 1.5, we use a slight modification of the construction in Theorem 1.4. Set $s = 3$ and let $r|q - 1$. Let Q_r denote a subgroup of \mathbb{F}_q^* of order r . Define the graph $H = H_r(q)$ with $V(H) = \mathbb{F}_{q^2} \times \mathbb{F}_q^*/Q_r$, and two vertices (A, aQ_r) and (B, bQ_r) are adjacent in H if $N(A + B) \in abQ_r$. Then H has $n = (q^3 - q^2)/r$ vertices and each vertex has degree $q^2 - 1$. It also follows from [2] that H has no $K_{3,t}$, where $t = 2r^2 + 1$. Now we construct a 3-uniform hypergraph H' with $V(H') = V(H)$ and whose edges are the triangles of H . We must count the number of triangles in H to determine $|H'|$. For every choice of $(A, a), (B, b)$ in $\mathbb{F}_{q^2} \times \mathbb{F}_q^*$, the number of $(C, c) \in \mathbb{F}_{q^2} \times \mathbb{F}_q^*$ with $C \neq A, B$, $N(A + C) = ac$ and $N(B + C) = bc$ is at least $q - 2$ by (the proof of) Lemma 5.3. Consequently, the number of (C, c) such that $N(A + C) \in acQ_r$ and $N(B + C) \in bcQ_r$ is at least $r^2(q - 2)$. Since (C, c) satisfies these equations if and only if (C, cq) satisfies these equations

for all $q \in Q_r$ (i.e., the solutions come in equivalence classes of size r), the number of common neighbors of (A, aQ_r) and (B, bQ_r) is at least $r(q-2)$. The number of edges in H is at least $(1-o(1))q^5/2r$, so the number of triangles in H is at least $(1-o(1))q^6/6 = (1-o(1))(r^2/6)n^2$. \square

7. Concluding remarks. In this paper we studied $\text{ex}_3(n, G^+)$, where G is a 3-colorable graph. If G has tree-width two, then we believe $\text{ex}_3(n, G^+) \sim (\sigma(G^+) - 1) \binom{n}{2}$ (Question 3), and if a planar graph G has an acyclic 3-coloring, then we believe $\text{ex}_3(n, G^+) = O(n^2)$ (Question 1). In fact, we also do not know any nonplanar acyclically 3-colorable graph G with superquadratic $\text{ex}_3(n, G^+)$. We are also not able to prove or disprove $\text{ex}_3(n, G^+) = O(n^2)$ when G is an even wheel (Question 2). This is equivalent to showing that if F is an n -vertex graph with a superquadratic number of triangles, then F contains every even wheel with a bounded number of vertices.

A number of examples of 3-colorable G with superquadratic $\text{ex}_3(n, G^+)$ were given. In particular we determined the order of magnitude of $\text{ex}_3(n, K_{s,t}^+)$ when near-extremal constructions of $K_{s,t}$ -free bipartite graphs are known. One may ask for the asymptotic behavior of $\text{ex}_3(n, K_{3,t}^+)$ for each $t \geq 3$, since in that case we have shown $\text{ex}_3(n, K_{3,t}^+) = \Theta(n^2)$. Finally, we gave a general upper bound on $\text{ex}_3(n, G^+)$ when G is a bipartite graph and showed that if G has Turán number much smaller than n^φ , where φ is the golden ratio, then $\text{ex}_3(n, G^+) = O(n^2)$. Determining exactly when $\text{ex}_3(n, G^+)$ is quadratic in n remains an open problem for further research.

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