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Sharpening an Ore-type version of the Corrádi–Hajnal theorem

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Abstract Corrádi and Hajnal (Acta Math Acad Sci Hung 14:423–439, 1963) proved that for all $k \geq 1$ and $n \geq 3k$, every (simple) graph G on n vertices with minimum degree $\delta(G) \geq 2k$ contains k disjoint cycles. The degree bound is sharp. Enomoto and Wang proved the following Ore-type refinement of the Corrádi–Hajnal theorem: For all $k \geq 1$ and $n \geq 3k$, every graph G on n vertices contains k disjoint cycles, provided that $d(x)+d(y) \geq 4k-1$ for all distinct nonadjacent vertices x, y. Very recently, it was refined for $k \geq 3$ and $n \geq 3k+1$: If G is a graph on n vertices such that $d(x) + d(y) \geq 4k - 3$ for all distinct nonadjacent vertices x, y, then G has k vertex-disjoint cycles if and only if the independence number $\alpha(G) \leq n - 2k$ and G is not one of two small exceptions in the case $k = 3$. But the most difficult case, $n = 3k$, was not handled. In this case, there are more exceptional graphs, the statement is more sophisticated, and some of the proofs do not work. In this paper we resolve

Dedicated to the memory of Rudolf Halin.

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this difficult case and obtain the full picture of extremal graphs for the Ore-type version of the Corrádi–Hajnal theorem. Since any k disjoint cycles in a $3k$-vertex graph G must be 3-cycles, the existence of such k cycles is equivalent to the existence of an equitable k-coloring of the complement of G. Our proof uses the language of equitable colorings, and our result can be also considered as an Ore-type version of a partial case of the Chen–Lih–Wu Conjecture on equitable colorings.

Keywords Disjoint cycles · Equitable coloring · Minimum degree

Mathematics Subject Classification 05C15 · 05C35 · 05C40

1 Introduction

For a graph $G = (V, E)$, let $|G| = |V|$, $\|G\| = |E|$, $\delta(G)$ and $\Delta(G)$ be the minimum and the maximum degrees of G, and $\alpha(G)$ be the independence number of G. Let $\overline{G}$ denote the complement of G. For disjoint graphs G and H, let $G \cup H$ be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ and let $G \vee H$ denote $G \cup H$ together with all edges from $V(G)$ to $V(H)$.

In 1963, Corrádi and Hajnal proved a conjecture of Erdős by showing the following:

Theorem 1 [5] *Let $k \in \mathbb{Z}^+$. Every graph G with* (i) $|G| \geq 3k$ *and* (ii) $\delta(G) \geq 2k$ *contains k disjoint cycles.*

Both hypotheses (i) and (ii) in the theorem are sharp. In particular, if a graph G has k disjoint cycles, then $\alpha(G) \leq |G| - 2k$, since for any independent set I, every cycle contains at least two vertices of $G - I$. So, the graph $H := \overline{K_{k+1}} \vee K_{2k-1}$ (see Fig. 1) satisfies (i) and $\delta(H) = 2k - 1$, but H does not have k disjoint cycles, because $\alpha(H) = k + 1 > |H| - 2k$. One of the results in [12] is the following refinement of Theorem 1.

Theorem 2 [12] *Let $k \geq 2$ be fixed. Every graph G with* (i) $|G| \geq 3k$ *and* (ii′) $\delta(G) \geq 2k-1$ *contains k disjoint cycles if and only if*

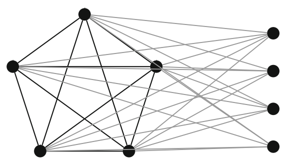
$$\alpha(G) \leq |G| - 2k \tag{1.1}$$

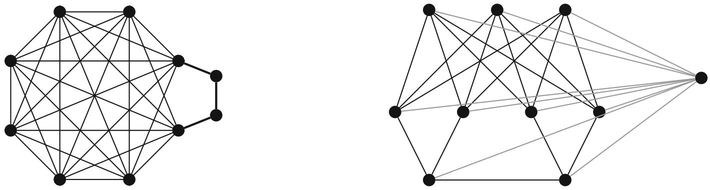
and

$$\text{if } k \text{ is odd and } |G| = 3k, \text{ then } G \neq 2K_k \vee \overline{K_k}; \text{ and if } k = 2 \text{ then } G \text{ is not a wheel.} \tag{1.2}$$

Theorem 2 was used in [13] to solve Dirac's problem of characterizing the (2k-1)-connected multigraphs with no k disjoint cycles. Enomoto [6] and Wang [21] generalized the Corrádi–Hajnal theorem in terms of the minimum Ore-degree $\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$:

Fig. 1 $\overline{K_{k+1}} \vee K_{2k-1}, k = 3$





(a) $\mathbf{U_1}$ (b) $\mathbf{U_2}$

Fig. 2 Two extremal examples for Theorem 3

Theorem 3 [6,21] *Let $k \in \mathbb{Z}^+$. Every graph G with* (i) $|G| \geq 3k$ *and*

$$\sigma_2(G) \geq 4k - 1 \tag{1.3}$$

contains k disjoint cycles.

It is natural to try to describe the extremal graphs in Theorem 3. Two such examples are in Fig. 2.

In [12], such graphs with at least $3k + 1$ vertices are described:

Theorem 4 [12] *Let $k \in \mathbb{Z}^+$ with $k \geq 3$. Every graph G with $|G| \geq 3k + 1$ satisfying* (1.1) *and*

$$\sigma_2(G) \geq 4k - 3 \tag{1.4}$$

contains k disjoint cycles, unless $k = 3$ and $G \in \{\mathbf{U}_1, \mathbf{U}_2\}$.

The goal of this paper is to handle the unsolved (and most difficult) case $|G| = 3k$. Since any $3k$ disjoint cycles in a $3k$-vertex graph G are triangles, the vertex sets of these triangles form color classes of an equitable k-coloring of $\overline{G}$. Recall that a vertex coloring of G is *equitable* if any two color classes differ in size by at most one. Equitable colorings and their generalizations have applications in Operation Research and Scheduling Theory (see e.g. [3,20]).

The fundamental result on equitable colorings is due to Hajnal and Szemerédi [7]:

Theorem 5 [7] *For every positive integer r, each graph G with $\Delta(G) \leq r$ has an equitable $(r + 1)$-coloring.*

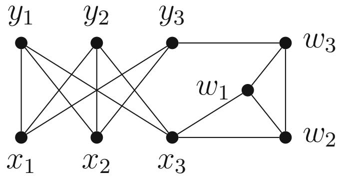
This result has interesting applications in extremal combinatorial and probabilistic problems, see e.g. [1,2,19].

In order to state Ore-type results in the language of equitable colorings, we use the notion of *Ore-degree*, $\theta(xy)$, of an edge xy. The Ore-degree of an edge is the sum the degrees of its endpoints; that is, $\theta(xy) = d(x) + d(y)$, whenever xy is an edge. By definition, the Ore-degree of an edge xy is two greater than the degree of the vertex xy in the line graph of G. We let the *Ore-degree* of a graph G be $\theta(G) = \max_{xy \in E(G)} \theta(xy)$. So for a $3k$-vertex graph G, the condition $\sigma_2(\overline{G}) \geq 4k - a$ is equivalent to $\theta(G) \leq 2k + a - 2$. By definition, $\theta(G) \leq 2\Delta(G)$. So the next Ore-type result refines the Hajnal–Szemerédi theorem.

Theorem 6 [8] *Every graph G with $\theta(G) < 2k$ has an equitable k-coloring.*

Chen et al. [4] conjectured that the Hajnal–Szemerédi Theorem can be refined in another direction:

Fig. 3 Graph **X**, from Example 9



Conjecture 7 [4] *Let G be a connected graph with $\Delta(G) = k$. Then G has no equitable k-coloring if and only if either* (1) $G = K_{k+1}$, *or* (2) $k = 2$ *and G is an odd cycle, or* (3) k *is odd and $G = K_{k,k}$.*

This conjecture is mainly open. Some partial results can be found in [4,10,11,16,22,23]. In particular, we will use the following known result, combining Theorem 37 from [10] and Theorem 9 from [11].

Theorem 8 [10,11] *Let G be a graph with $|G| = ks$ and $\chi(G), \Delta(G) \leq k$ that has no equitable k-coloring. If either $s \leq 4$ or $k \leq 4$ then k is odd, $K_{k,k} \subseteq G$, and $G - K_{k,k}$ is k-equitable. In particular, if $s = 3$ then $G = K_{k,k} + K_k$.*

The main result of this paper can be considered an Ore-type version of Theorem 8 for the case $s \leq 3$. Before stating it, we need to consider some extremal examples.

For disjoint sets X and Y, let $K(X)$ denote the complete graph with vertex set X, and $K(X, Y)$ denote the complete bipartite graph with parts X and Y. The graph $K(X, Y)$ is often denoted as $K_{|X|,|Y|}$.

Example 9 Let $Q := K(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$, $K = K(\{w_1, w_2, w_3\})$, and

$$\mathbf{X} = Q - x_3y_3 + K + x_3w_1 + x_3w_2 + y_3w_3. \tag{1.5}$$

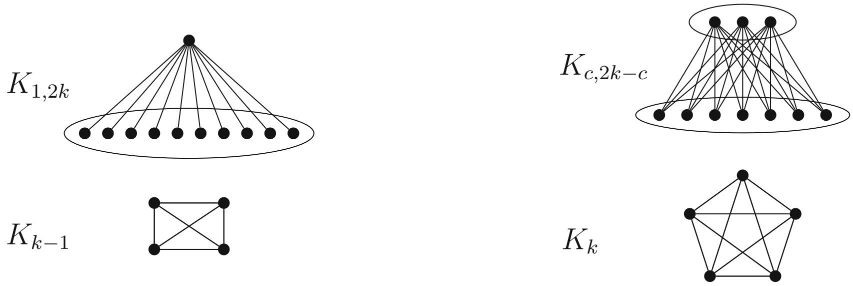
(See Fig. 3.) Then $|\mathbf{X}| = 9 = 3 \cdot 3$, $\chi(\mathbf{X}) = 3$, and $\theta(\mathbf{X}) = 2 \cdot 3 + 1$, but $\mathbf{X}$ has no equitable 3-coloring: Any 3-coloring f gives distinct colors to K and satisfies $f(x_3) = f(w_3) \neq f(y_3)$. So if f is an equitable 3-coloring of $\mathbf{X}$ then it is also an equitable 3-coloring of Q, contradicting that f is a proper coloring. Also, we will later make use of this observation:

$$\mathbf{X} \simeq Q - x_3y_3 - x_3y_2 + K + x_3w_1 + x_3w_2 + y_3w_3 + y_2w_3. \tag{1.6}$$

Example 10 Let $k \geq 2$, and $\mathbf{Y} = \mathbf{Y_k} = K_{1,2k} + K_{k-1}$. (See Fig. 4a.) Then $|\mathbf{Y}| = 3k$, $\chi(\mathbf{Y}) \leq k$, and $\theta(\mathbf{Y}) = 2k + 1$, but $\mathbf{Y}$ has no equitable k-coloring: for any k-coloring the class of the vertex r with $d(r) = 2k$ contains at most one vertex from K_{k-1}.

Example 11 For $k \geq 2$ and odd $c \leq k$, let $V = B_1 \cup B_2 = C_1 \cup C_2 \cup B_2$, where C_1, C_2, B_2 are disjoint, $|C_1| = c$, $|C_2| = 2k - c$, and $|B_2| = k$. Set $\mathbf{Z_{c,k}} = Q + K$, where $Q = K(C_1, C_2)$ and $K = K(B_2)$. (See Fig. 4b.) Then $|\mathbf{Z_{c,k}}| = 3k$, $\chi(\mathbf{Z_{c,k}}) = k$, and $\theta(\mathbf{Z_{c,k}}) = 2k$, but $\mathbf{Z_{c,k}}$ has no equitable k-coloring. Indeed, each class of an equitable coloring of $\mathbf{Z_{c,k}}$ must contain one vertex of K and two vertices from the same part of Q. As c and $2k - c$ are odd, this is impossible.

In particular, our results describe extremal examples for Theorem 6 when $n \leq 3k$. It is enough to consider the case of n divisible by k, as when $n \equiv r \pmod{k}$ for some $1 \leq r \leq$



(a) $\mathbf{Y_k}$, $k = 5$ **(b)** $\mathbf{Z_{c,k}}$, $k = 5$, $c = 3$

Fig. 4 Examples 10 and 11

$k-1$, we can consider the graph formed by adding a disjoint copy of K_{k-r} to G. If $s = 1$ then G has k vertices and trivially has an equitable k-coloring. Our first result, Theorem 12 (which has a simple proof) handles the case $s = 2$.

Theorem 12 *Let G be a graph satisfying $|G| = 2k$,*

(H1) $\chi(G) \leq k$ *and*
(H2) $\theta(G) \leq 2k + 1$.

If G has no equitable k-coloring then $K_{c,2k-c} \subseteq G$ for some odd $c \in [k]$.

Our main result is

Theorem 13 *Let G be a graph satisfying $|G| = 3k$,*

(H1) $\chi(G) \leq k$ *and*
(H2) $\theta(G) \leq 2k + 1$.

If G has no equitable k-coloring then $G \in \{\mathbf{X}, \mathbf{Y_k}\}$ or $\mathbf{Z_{c,k}} \subseteq G$ for some odd c.

A relevant question is: Which graphs G satisfying (H2) (i.e., $\theta(G) \leq 2k + 1$) do not satisfy (H1) (i.e., have $\chi(G) \geq k + 1$)? This question was resolved. First, Kierstead and Kostochka [9] showed that for $k \geq 6$ every such graph contains K_{k+1}, and Rabern [18] extended the result to $k = 5$. Then Kostochka, Rabern and Stiebitz [15] proved that for $k = 4$ every such graph contains K_5 or the graph O_5 in Fig. 5 (left). Finally, very recently Kierstead and Rabern [14] and independently Postle [17] described the infinite family of all 4-critical graphs G with $\theta(G) \leq 7$. Only one of them distinct from K_4 has at most 9 vertices, namely 7 vertices. This graph O_4 is on the right of Fig. 5.

Theorems 4 and 13 together describe all graphs G with $\sigma_2(G) \geq 4k-3$ that do not have k disjoint cycles. In the next section we prove Theorem 12, and the rest of the paper is devoted to the proof of Theorem 13. Namely, in Sect. 3 we set up the proof and prove simple properties of a minimum counterexample G to the theorem. In particular, in Lemma 15 we prove that this G has no complete k-vertex subgraphs. In Sect. 4 we prove that G has a *nearly equitable* coloring, i.e. a proper k-coloring in which one color class has size 2, one has size 4, and every other color class has size 3. In the next two sections we study the properties of nearly equitable colorings of G with additional properties, *normal* colorings and *optimal* colorings. Based on these properties, in Sect. 7 we show that G has a nearly equitable coloring with even more good properties. In Sect. 8 we derive many properties of so called *solo vertices*. And in Sect. 9 we finish the proof by finding a complete k-vertex subgraph in G contradicting Lemma 15 mentioned above.

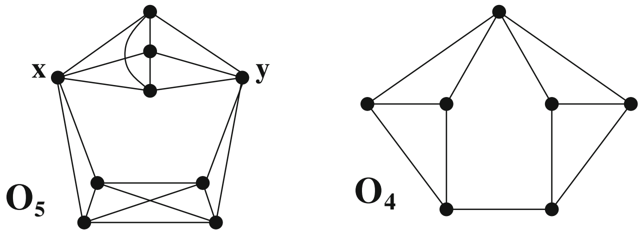


Fig. 5 Graphs O_5 (on the *left*) and O_4 (on the *right*)

Apart from standard notation, we will use the following. For a graph $G = (V, E)$ and sets $X, Y \subseteq V$, let $E(X) := E_G(X) = E(G[X])$ and let $E(X, Y) := E_G(X, Y)$ be the set of edges with one end in X and one end in Y. Define $\|X, Y\| := |E(X, Y)| + |E(X \cap Y)|$, so the edges in $G[X \cap Y]$ are counted twice, and $\|X\| := |E(X)|$. For a vertex $v \in V$, we write $N(v)$ for the set of vertices adjacent to v and $N[v]$ for $\{v\} \cup N(v)$. We often write $\|v, X\|$ for $\|\{v\}, X\|$, $X - v$ for $X \setminus \{v\}$ and $X + x$ for $X \cup \{v\}$. For an edge $e = xy \in E$, $\|e, X\|$ and $\|xy, X\|$ are equivalent to $\|\{x, y\}, X\|$. A k-coloring of G is a partition $\mathcal{V}$ of V into k independent sets. We may express this partition as a function $f : V \to [k]$.

2 Proof of Theorem 12

Assume G has no equitable k-coloring. Since $|G| = 2k$, this means $\overline{G}$ has no perfect matching. Since $|G|$ is even this yields that each matching in $\overline{G}$ does not cover at least two vertices. So, by Berge–Tutte's formula, there is a set $T \subseteq V(G)$ such that $\overline{G} - T$ has at least $|T| + 2$ odd components. Let $|T| = t$.

For a contradiction, it suffices to assume that (H2) holds and prove that (H1) fails or $K_{c,k-c} \subseteq G$ for some odd $c \leq k$. Let X and Y be the two smallest odd components of $\overline{G} - T$, $x \in X$ and $y \in Y$. Then

$$2k + 1 \geq \theta(G) \geq d(x) + d(y) \geq (2k - t - |X|) + (2k - t - |Y|),$$

so

$$|V(G) - T| \leq |X| + |Y| + t + 1. \tag{2.1}$$

This implies that if $t = 0$, then $V(G) = X \cup Y$ and $K_{c,2k-c} \subseteq G$, where $c = |X|$ is odd. So assume $t > 0$. Then, since there are at least t odd components other than X and Y in $\overline{G} - T$, none of these odd components has order 3 or greater. By the choice of X and Y, this also yields $|X| = |Y| = 1$. Hence, with (2.1),

$$t \geq \lceil (2k - 1 - |X| - |Y|)/2 \rceil = k - 1,$$

and $\chi(G) \geq \omega(G) = \alpha(\overline{G}) \geq t + 2 \geq k + 1$.

3 Setup and preliminaries for the proof of Theorem 13

Suppose $G = (V, E)$ is a counterexample to Theorem 13 with k minimum, and subject to this $\|G\|$ is minimum. So $|G| = 3k$, G satisfies (H1–H2), $G \notin \{\mathbf{X}, \mathbf{Y}\}$, $\mathbf{Z_{c,k}} \not\subseteq G$ for any odd c, and

$$G \text{ has no equitable } k\text{-coloring, but } G - e \text{ has an equitable } k\text{-coloring for all } e \in E. \quad (3.1)$$

By the minimality of k,

$$\text{Theorem 13 holds} \quad \text{for all } k' \in [1, \ldots, k-1]. \quad (3.2)$$

Call a vertex v *high* if $d(v) \geq k+1$, and *low* otherwise. For a subset W of $V(G)$, let $H(W)$ denote the set of high vertices in W and $L(W) = W \setminus H(W)$ denote the set of low vertices. An edge is *high* if it has a high end vertex. By (H2), $H(V)$ is independent; so a high edge also has a low vertex.

Lemma 14 *$k < \Delta(G) \leq 2k-2$. In particular, $k \geq 3$.*

Proof By Theorem 8, if $\Delta(G) \leq k$ then k is odd and $\mathbf{Z_{k,k}} \subseteq G$, a contradiction. Suppose $d(v) = d := \Delta(G) \geq 2k-1$ for some $v \in V$. As every neighbor of v has positive degree, $\theta(G) \leq 2k+1$ implies $d \leq 2k$. Let $X = N(v)$ and $Y = V(G) \setminus N[v]$. If Y is a clique then G contains $\mathbf{Y_k}$ or $\mathbf{Z_{1,k}}$; else choose distinct nonadjacent vertices $y_1, y_2 \in Y$ with $\|\{y_1, y_2\}, X\|$ maximum. Let $V_1 = \{v, y_1, y_2\}$ be one color class.

If $d = 2k$ then X is independent and $\|X, Y\| = 0$. Since $G - \{v, y_1, y_2\} \subseteq K_{k-3} + \overline{K}_{2k}$, it has an equitable $(k-1)$-coloring. Thus G has an equitable k-coloring, contradicting (3.1). So $d = 2k-1$. If $k = 2$ then X is independent by (H1), contradicting (3.1). Thus $k \geq 3$.

Since $\theta(G) \leq 2k+1$, each $x \in X$ has at most one neighbor in $V - v$. So $M := E(X)$ is a matching, the vertices of Y are not adjacent to vertices saturated by M, and $\|X, Y\| \leq d - 2t$, where $t = |M|$. Say $M = \{e_i : i \in [t]\}$. Order the vertices in $Y - y_1 - y_2$ so that $\|y_3, X\| \geq \cdots \geq \|y_k, X\|$.

Note that $\|y_3, X\| \leq k$, and if equality holds then $d(y_3) = d$: If not then $\|y_3, Y\| \leq d - (k+1) = k-2$; so there is $y \in Y - y_3$ with $yy_3 \notin E$. Thus $\|\{y_1, y_2\}, X\| \geq \|\{y_3, y\}, X\| \geq k$, so $\|X, Y\| \geq 2k > d$, a contradiction. Thus $|X \setminus N(y_3)| \geq k-1 \geq 2$. Then there exist distinct nonadjacent vertices $x_1, x_2 \in X \setminus N(y_3)$: if not, $X \setminus N(y_3) = K_2$, $\|y_3, X\| = k$, $d(y_3) = d$, and $V \setminus N[y_3] = K_3 = K_k$, so $\mathbf{Z_{1,k}} \subseteq G$.

Using that M is a matching, choose x_1 and x_2 to be in distinct edges of M if possible; that is, label X and M so that for each $j \leq \min\{2, t\}$, $x_j \in e_j$.

Let $V_2 = \{x_1, x_2, y_3\}$ be the second color class. Put $X_3 = X \setminus \{x_1, x_2\}$. If $k = 3$ then X_3 is independent, and we are done. So assume $k \geq 4$.

We recursively construct color classes $V_i = \{y_{i+1}, x_{2i-3}, x_{2i-2}\}$ for $i \in \{3, \ldots, k-1\}$. Suppose we have chosen $V_1, \ldots, V_{i-1}$, and set $X_i := N(v) \setminus \{x_1, \ldots, x_{2i-4}\}$. By our choice of labels in $Y \setminus \{y_1, y_2\}$, $\|y_{i+1}, X\| \leq \left\lfloor \frac{\|Y, X\|}{i-1} \right\rfloor \leq \left\lfloor \frac{2k-2t-1}{i-1} \right\rfloor$. Also $|X_i| = 2(k-i)+3$, so

$$\begin{aligned}
|X_i - N(y_{i+1})| &\geq |X_i| - \|y_{i+1}, X\| \geq 2(k-i) + 3 - \left\lfloor \frac{2k-2t-1}{i-1} \right\rfloor \\
&= \left\lceil 3 + 2(k-i)\left(1 - \frac{1}{i-1}\right) - \frac{2i-2t-1}{i-1} \right\rceil \qquad (*) \\
&\geq \left\lceil 3 + (k-i) - \frac{2i-1}{i-1} \right\rceil \geq \left\lceil 3 + 1 - \frac{5}{2} \right\rceil = 2.
\end{aligned}$$

Note that if $|X_i - N(y_{i+1})| = 2$, the starred line shows $i > t$. Now we select distinct, nonadjacent x_{2i-3}, x_{2i-2} in $X_i \setminus N(y_{i+1})$. If we can choose $x_{2i-3} \in e_i$, we do so. More precisely: using that $V(M) \subseteq X \setminus N(y_i)$, if $i \leq t$ and $e_i \cap X_i \neq \emptyset$, we choose $x_{2i-3} \in e_i$; then, since $|X_i - N(y_{i+1})| \geq 3$, we select $x_{2i-2} \in X_i \setminus (e_i \cup N(y_{i+1}))$. Suppose $i > t$, or $e_i \cap X_i = \emptyset$. If $|X_i \setminus N(y_{i+1})| = 2$, since $i > t$ and by our choice of $V_1, \ldots, V_{i-1}$, the two vertices of $X_i \setminus N(y_{i+1})$ are nonadjacent. Otherwise, since M is a matching, we let x_{2i-3}, x_{2i-2} be any two distinct, nonadjacent vertices in $X_i \setminus N(y_{i+1})$. Finally, let $V_k := X_k$ be the last color class. Since $|M| \leq k-1$, V_k is independent. □

Lemma 15 $\omega(G) \leq k-1$.

Proof Suppose K is a k-clique in G, and set $H = G - K$. As $\mathbf{Z_{c,k}} \nsubseteq G$ for any odd c, $K_{c,2k-c} \nsubseteq H$ for any odd c. Since $\theta(G) \leq 2k+1$,

$$\|xy, H\| \leq 3 \quad \text{for all } x, y \in K. \tag{3.3}$$

By Theorem 12, H has an equitable k-coloring f.

First suppose

$$K \nsubseteq N(U) \quad \text{for all classes } U \text{ of } f, \tag{3.4}$$

and note that, by Lemma 14,

$$\text{no vertex } x \in K \text{ has neighbors in all classes of } f. \tag{3.5}$$

Extend f to an equitable k-coloring f' of G by first greedily adding vertices of K into distinct classes of f starting with the vertex x with $\|x, H\|$ maximum. By (3.5) and (3.3) the process will not get stuck before the last vertex $z \in K$. If z cannot be greedily added to the last remaining class W, (3.3) implies W is the only class z is adjacent to. By (3.4) there is $y \in K \setminus N(W)$. Move y to W and z to the former class of y to finish. As this contradicts (3.1), (3.4) fails.

Say $K \subseteq N(Z)$ for some class $Z = \{z, z'\}$ of f. Put $H^+ = H + zz'$. Then $d_{H^+}(z) \leq d_G(z)$ and $d_{H^+}(z') \leq d_G(z')$. So $\theta(H^+) \leq 2k+1$. Suppose H^+ has no equitable k-coloring. Since $\chi(G) \leq k$, $\chi(H^+) \leq k$, so, by Theorem 12, $Q := K_{c,2k-c} \subseteq H^+$ for some odd $c \leq k$, and $zz' \in E(Q)$. Say $d_Q(z') = c$. Note each vertex of $\{z, z'\}$ has a neighbor in K because $\chi(G) \leq k$, and, by Lemma 14, $3 \leq c$. Then there exist $x \in K$ and $y \in V(H)$ with $xz, yz' \in E$. Since $G \neq \mathbf{X}$, $k \geq 4$. Since $\theta(G) \leq 2k+1$,

$$4k+2 \geq \theta(xz) + \theta(yz') \geq \|Z, K\| + k + (2k-c-1) + (2k-1) \geq 6k-2-c.$$

So $2k-4 \leq c \leq k$. As c is odd and $k \geq 4$, this is a contradiction. Thus H^+ has an equitable k-coloring f'.

Since (3.4) fails, there is a class Y of f' such that $K \subseteq N(Y)$. As $zz' \in E(H^+)$, $Y \neq Z$. As $\left\|K, H^+\right\| \leq k+1$, and $\chi(G) \leq k$, there are vertices $u \in K$ and $z'' \in V(H)$ with (say) $Y = \{z, z''\}$, $N(z) \cap K = K - u$, $uz', uz'' \in E$, and $N(K) = \{z, z', z''\}$. If $H^* := H^+ + zz''$ has an equitable coloring then it satisfies (3.4), and we are done. Otherwise, $Q := K_{c,2k-c} \subseteq H^*$ for some odd $c \leq k$, with $zz'' \in E(Q)$. By Lemma 14, $3 \leq c$. If $k = 3$ then $G = \mathbf{X}$ by (1.6). Else, for $w \in N_Q(z) \setminus \{z', z''\}$,

$$2k+1 \geq \theta(zw) \geq \|z, K\| + \theta_{H^*}(wz) - 2 \geq k-1+2k-2 = (2k+1)+(k-4),$$

so $k = 4$ and z', z'' are in one part Q' of Q. Since $d(u) = k+1$, $d(z'), d(z'') \leq k$, so $|Q'| = 5$. But now for $x \in V(K) - u$, $d(z) + d(u) \geq 6 + 4 = 2k+2$, a contradiction. □

Lemma 16 $k \geq 4$.

Proof For a contradiction, suppose $k \leq 3$. By Lemma 14, $k = 3$ and $\Delta(G) = 4$. Let $d(v) = 4$, $N = N(v)$, $G' = G - N[v]$, and $V(G') = N'$. By Lemma 15,

$$\omega(G) \leq 2. \tag{3.6}$$

So N is independent and, since $|G'| = 4$, G' is bipartite. Also $\theta(G) \le 2k+1$ implies

$$\left\|x, N'\right\| \le 2 \quad \text{for all } x \in N \tag{3.7}$$

and $\left\|N, N'\right\| \le 8$.

Suppose $d_{G'}(w) = 3$ for some $w \in N'$. Then $\|w, N\| \le 1$ because $\Delta(G) = 4$, and $N(w) \cap N(w') = \emptyset$ for all $w' \in N' - w$ by (3.6). Because $\|N, N'\| \le 8$, $\left\|w', N\right\| \le 2$ for some $w' \in N' - w$. Choose $x_1, x_2 \in N \smallsetminus N(w')$, including the neighbor of w if it exists. Then $\{\{w', x_1, x_2\}, N - x_1 - x_2 + w, N' - w - w' + v\}$ is an equitable 3-coloring of G.

Otherwise $\Delta(G') \le 2$, so N' has an equitable 2-coloring.

$$\text{If } Y \text{ is a class of an equitable 2-coloring of } N', \text{ then } N(x) \cap Y \neq \emptyset \text{ for all } x \in N: \tag{3.8}$$

else $\{(N' \smallsetminus Y) + v, Y + x, N - x\}$ is an equitable 3-coloring of G. Let $N' = \{y_1, y_2, y_3, y_4\}$ and $x \in N$. As N' has an equitable 2-coloring g, (3.7) and (3.8) imply $\left\|x, N'\right\| = 2$. Say $N(x) = \{y_1, y_2\}$. By (3.6) $y_1 y_2 \notin E$, so (3.8) implies $y_3 y_4 \in E$, and, by (3.6) again, $N(y_3) \cap N(y_4) = \emptyset$. Assume $\|y_3, N\| \ge \|y_4, N\|$. If $\|y_3, N\| \le 2$, then there exist disjoint 2-sets $X_1, X_2 \subseteq N$ with $N(y_3) \cap N \subseteq X_1$ and $N(y_4) \cap N \subseteq X_2$. So $\{\{v, y_1, y_2\}, X_1 + y_4, X_2 + y_3\}$ is an equitable 3-coloring of G. Otherwise, $N(y_3) \cap N = N - x$, and $N(y_4) \cap N = \emptyset$. Say $g(y_1) = g(y_3)$. By (3.8), $N(y_2) = N$ and, by (3.6), $N(y_1) \cap N = \{x\}$. By (3.6), $y_2 y_3 \notin E$, so if $x' \in N - x$, then $\{\{v, y_2, y_3\}, \{x, x', y_4\}, N - x - x' + y_1\}\}$ is an equitable 3-coloring of G. □

4 Nearly equitably colorings

Recall that a coloring of G is *nearly equitable* if one color class has size 2, one color class has size 4, and all other color classes have size 3.

Lemma 17 *G admits a nearly-equitable k-coloring.*

Proof Suppose not. By Lemma 14, $\Delta(G) \ge k+1$. Let x be a vertex with $d(x) \ge k+1$ and let $y \in N(x)$. By (3.1), $G - xy$ has an equitable k-coloring f with $f(x) = f(y)$. Let $\mathcal{C}$ be the set of color classes of f, and $X = \{x, y, z\} \in \mathcal{C}$. Choose xy and f so that $d(z)$ is minimum. If x (or y) has no neighbor in some class $W \in \mathcal{C} - X$ then moving it to W yields a nearly equitable k-coloring; so assume not. As y is low, $d(y) = k$, and $d(x) = k+1$. Furthermore,

$$y \text{ has exactly one neighbor in every class,} \tag{4.1}$$

and

$$x \text{ has exactly two neighbors in one class, and exactly one neighbor in every other class.} \tag{4.2}$$

For $W \in \mathcal{C} - X$, let $G_W := G[W \cup X]$. If G_W is bipartite, then its parts form an equitable or nearly equitable 2-coloring unless $G_W = K_{1,5}$. However, by (4.1) and (4.2), $\Delta(G_W) \le 3$, so $G_W \neq K_{1,5}$; thus if G_W is bipartite, it has an equitable or nearly equitable 2-coloring. If G_W has an equitable or nearly equitable 2-coloring, then G has an equitable or nearly equitable k-coloring. Thus G_W contains an odd cycle C_W that contains xy. Assume C_W is picked so that $|C_W|$ is as small as possible. Let $\mathcal{C}_1 = \{W \in \mathcal{C} - X : |C_W| = 3\}$ and $\mathcal{C}_2 = \mathcal{C} - X \smallsetminus \mathcal{C}_2$. For $W \in \mathcal{C}_1$, let $C_W = x v_W y x$. If v_W is movable to some class U then moving y to W and

v_W to U yields a nearly equitable k-coloring. As $v_W \in N(x)$, it is low. Thus v_W has two neighbors in X and one neighbor in each class of $\mathcal{C} - X - W$. In particular,

$$\text{if } W \in \mathcal{C}_1, \quad \text{then } v_W z \notin E. \tag{4.3}$$

For $W \in \mathcal{C}_2$, let $C_W = x x_W z y_W y x$, where $x_W, y_W \in W$. Since $W \in \mathcal{C}_2$, $G_W - z$ is triangle free, and since $\alpha(G_W - z) \geq |W| = 3$, $G_W - z$ contains no C_5. Then $G_W - z$ contains no odd cycle, so it is bipartite. Since $\Delta(G_W - z) \leq 3$, $G_W - z \neq K_{1,4}$, so $G_W - z$ can be partitioned into independent sets of size 2 and 3. If z is movable, we can move z to create a color class of size 4, and partition $G_W - z$ into one color class of size three and one of size four, providing a nearly equitable coloring of G. So z is not movable. Thus,

$$\text{if } |\mathcal{C}_2| \neq 0, \quad \text{then } |\mathcal{C}_1| + 2|\mathcal{C}_2| \leq d(z) \leq k+1. \tag{4.4}$$

If there are distinct $W, W' \in \mathcal{C}_1$ with $v_W v_{W'} \notin E$, then using (4.2), choose notation so that $\|x, W\| = 1$. By (4.1) and (4.3), moving x to W, y to W', and both v_W and $v_{W'}$ to X yields an equitable k-coloring. So $Q := \{v_W : W \in \mathcal{C}_1\} \cup \{x, y\}$ is a clique. By Lemma 15, $|Q| \leq k-1$. So $|\mathcal{C}_1| \leq k-3$, and $|\mathcal{C}_2| \geq 2$; by (4.4) $d(z) = k+1$. Consider distinct $W, W' \in \mathcal{C}_2$. Using (4.2) choose notation so that $\|x, W\| = 1$. By (4.1), switching x and x_W yields an equitable k-coloring of $G - z x_W$, with color class $\{z, x_W, y\}$. As $d(y) < d(z)$, this contradicts the choice of f. $\square$

5 Normal colorings

Fix a nearly equitable k-coloring $f := \{V_1, \ldots, V_k\}$, where $V^- = V_1$ and $V^+ = V_k$. As our proof progresses we will put more and more stringent conditions on f.

Construct an auxiliary digraph $\mathcal{H} := \mathcal{H}(G, f)$ as follows. The vertices of $\mathcal{H}$ are the color classes $V_1, \ldots, V_k$. A directed edge $V'V''$ belongs to $E(\mathcal{H})$ if some vertex $x \in V'$ has no neighbors in V''. In this case we say that x *is movable to* V" and that x *witnesses* the edge $V'V''$. Call a color class V_i of f *accessible* if V^- is reachable from V_i in the digraph $\mathcal{H}$. A vertex $v \in V_i$ is *movable* if it is movable to some accessible class; otherwise it is unmovable. Let $M = M(f)$ be the set of movable vertices and $\overline{M} = \overline{M(f)}$ be the set of unmovable vertices. By definition, V^- is accessible. Let $\mathcal{A} := \mathcal{A}(f)$ denote the family of accessible classes, $\mathcal{B}$ denote the family of inaccessible classes, $A := \bigcup \mathcal{A}$, and $B := \bigcup \mathcal{B} = V - A$. If $V_k \in \mathcal{A}$ then switching witnesses along a path from V^+ to V^- yields an equitable r-coloring; so $V^+ \in \mathcal{B}$. Let $a := |\mathcal{A}|$ and $b := |\mathcal{B}| = ks - a$. Then $|A| = as - 1$ and $|B| = bs + 1$.

An *in-tree* is a digraph T with a *root* $r \in V(T)$ such that every $v \in V(T)$ has a unique vr-walk. So the undirected graph underlying T is acyclic. A vertex $v \in T$ is a *leaf* if $d^-(v) = 0$. Fix a spanning in-tree $\mathcal{F} \subseteq \mathcal{H}[\mathcal{A}]$ with the most leaves possible. Write $W\mathcal{F}$ for the unique W, V^--path in $\mathcal{F}$, and let w_x be the witness for its first edge. Let $\mathcal{D} \subseteq \mathcal{H}[\mathcal{A}]$ be the spanning graph with $UW \in E(\mathcal{D})$ if and only if $UW \in E(\mathcal{H})$ and $U \notin W\mathcal{F}$.

A class $Z \in \mathcal{A}$ is *terminal* if there is a UV^--path in $\mathcal{H} - Z$ for every $U \in \mathcal{A} - Z$. For example, any leaf of $\mathcal{F}$ is terminal. Class V^- is terminal if and only if $a = 1$. Let $\mathcal{A}' = \mathcal{A}'(f)$ be the set of terminal classes, $A' := \bigcup \mathcal{A}'$ and $a' := |\mathcal{A}'|$.

A *nearly equitable* k-coloring is *normal* if

(C1) among nearly equitable k-colorings a is maximum, and
(C2) if $a \geq 3$, then $\mathcal{F}$ has at least two in-leaves.

Lemma 18 *There exists a normal coloring.*

Proof Suppose f is a nearly equitable k-coloring with a maximum. If $a \leq 2$, (C2) is vacuously true, so we may suppose $a \geq 3$. If $\mathcal{F}$ has at least two leaves then we are done; else $\mathcal{F}$ is a dipath with leaf Z and last edge UV^- witnessed by w. As $a \geq 3$, $U \neq Z$. Shifting w to V^- yields a normal k-coloring with in-leaves $V^- + w$ and Z. □

Fix a normal coloring f. A vertex $y \in B$ is *good* if $G[B - y]$ has an equitable b-coloring; else y is *bad*. A major goal of this section is to show that every vertex in B is good.

Lemma 19 $a = a(f) \geq 2$.

Proof Assume $a = 1$ for all nearly equitable k-colorings of G, and choose one with

$$d(v) + d(v') \text{ minimal,} \tag{5.1}$$

where $V^- = \{v, v'\}$. Say $d(v) \leq d(v')$. By Lemma 14, $d(v') \leq 2k-2$. As $N(V^-) = V - V^-$, $d(v) + d(v') \geq 3k - 2 + |N(v) \cap N(v')|$.

Case 1: $N(v) \cap N(v') = \emptyset$. If $\|v, V^+\| = \|v', V^+\| = 2$, then coloring v resp. v' with its non-neighbors in V^+ yields an equitable k-coloring. Therefore we suppose $\left\|u, V^+\right\| \geq 3$ for some $u \in V^-$. Pick $Y \in \mathcal{B}$ with $\|u, Y\|$ minimum. If $\|u, Y\| = 0$ then moving u to Y and $x \in N(u) \cap V^+$ to V^- yields a nearly equitable k-coloring with $a \geq 2$: any vertex $N(u) \cap V^+ - x$ is movable to the new small class $V^- - u + x$. Else, since $d(u) \leq 2k-2 = 2b$, $\|u, Y\| = 1$ and $d(u) \geq k + 1$. Switching u with $y \in N(u) \cap Y$ yields a nearly equitable coloring, contradicting (5.1) since $d(y) \leq (2k + 1) - d(u) \leq k$.

Case 2: $N(v) \cap N(v') \neq \emptyset$. Then $d(v) \geq k + 1$ and $d(v') \geq k + 2$. Put $G' = G[B]$. Then $\chi(G') \leq b$. Since $\theta(G) \leq 2k + 1$, $\Delta(G') \leq 2k + 1 - d(v) - 1 \leq b$. If $S \subseteq V$ with $|S| = 2k$ then there is $x \in N(v') \cap S$, and $d_{G'}(x) \leq b - 1$. So $K_{b,b} \nsubseteq G'$. Pick $w \in N(v') \setminus N(v)$. Theorem 8 implies $G' - w$ has an equitable b-coloring $\mathcal{Y}$. As $\left\|v', B - w\right\| < 2b$, some class $Y \in \mathcal{Y}$ satisfies $\left\|v', Y\right\| \leq 1$. Move w to $V^- - v'$ and v' to Y; if v' has a neighbor $y \in Y$ then move y to a class X in which it has no neighbors; X exists as $d(y) \leq k - 1$. This yields an equitable k-coloring, or a nearly equitable k-coloring, contradicting (3.1) or (5.1) since $d(w) < d(v')$. □

An edge xy with $x \in X \in \mathcal{A}$ and $y \in B$ is *solo* if $\|y, X\| = 1$; else it is nonsolo. If xy is solo then *x and y are solo neighbors* of each other. For $x \in A$ and $y \in B$ let S_x denote the set of solo neighbors of x in B and S^y denote the set of solo neighbors of y in A.

Lemma 20 *Let $z \in Z \in \mathcal{A}$, $y \in S_z$, and g be an equitable b-coloring of $G[B - y]$. Then*

0. *if $\mathcal{P}$ is a W, V^--path in $\mathcal{H} - Z$ and w witnesses $WW' \in E(\mathcal{P})$ then $\|z, W - w\| \geq 1$.*

If (a) *the nonsolo neighbors of y are unmovable* (*as when $\|y, A\| = a$ and y does not have nonsolo neighbors*) *or* (b) *$Z \in \mathcal{A}'$ then*

1. *z is unmovable*;
2. *If, in addition,* (c) *$\|z, A\| \leq a - 1$, then z has no movable neighbor $w \in W \in \mathcal{A}$.*

Proof In all cases, we will contradict (3.1) by constructing an equitable a-coloring h of $A + y$, since then $g \cup h$ is an equitable k-coloring of G.

(0) If not, shift witnesses along $\mathcal{P}$, move z to W, and move y to Z to obtain an equitable a-coloring h of $A + y$.

(1) Suppose (a) or (b) holds and z is movable to $U \in \mathcal{A}$. Pick U and a U, V^--path $\mathcal{P}$ in $\mathcal{H}$. By (0), $Z \in \mathcal{P}$; in particular, there is no Z, V^- path in $\mathcal{H}$ where z is the witness to the

first edge. Then (b) fails, so (a) holds; say x witnesses $XZ \in \mathcal{P}$. By (0) applied to x, x is not a solo neighbor of y; by (a) applied to x, x is not a neighbor of y at all. We move z to U, then shift witnesses along $\mathcal{P}$, noting that the witness from Z is not z; then we move y to $Z - z + x$ to complete an equitable a-coloring of $A + y$.

(2) Suppose (a) or (b) holds; further suppose (c) holds and $wz \in E$ with w movable to $U \in A$. Note by (1) and (c), z has precisely one neighbor in every class of $\mathcal{A} - Z$. Pick a U, V^--path $\mathcal{P}$ in $\mathcal{H}$ so that $Z \notin \mathcal{P}$ if (b) holds. Subject to this, choose $w, W, U, \mathcal{P}$ so that $|\mathcal{P}|$ is minimum. Suppose $W \in \mathcal{P}$. By the minimality of $|\mathcal{P}|$, w does not witness the out-edge of W on $\mathcal{P}$, and, if w' witnesses the in-edge to W on $\mathcal{P}$, then $zw' \notin E$, because otherwise w' is preferable to w by the minimality of $|\mathcal{P}|$. If $Z \in \mathcal{P}$, then let z' witness the in-edge to Z on $\mathcal{P}$. In this case, $yz' \notin E$, because $Z \in \mathcal{P}$ implies that (b) fails for Z, so (a) holds, and (0) implies yz' is not solo, so (a) implies that $yz' \notin Z$. By (0), we also have that z does not witness the out-edge of Z on $\mathcal{P}$. Therefore, switching witnesses on $\mathcal{P}$, and moving w to U, z to W and y to Z yields an equitable a-coloring of $A + y$. □

Lemma 21 *Every color class in $\mathcal{A}$ contains at most one unmovable vertex.*

Proof Suppose $Z \in \mathcal{A}$ has two unmovable vertices z_1 and z_2. If $Z \neq V^-$ then let $Z = \{z_1, z_2, z_3\}$. Let $B_0 = B + z_1 + z_2$ and $A_0 = A - z_1 - z_2$. Since z_3 (if it exists) is the witness for the first edge ZZ' of $\mathcal{P}_0 := Z\mathcal{F}$, shifting witnesses on $\mathcal{P}_0$ yields an equitable $(a-1)$-coloring f_0 of $G[A_0]$. Thus $G' := G[B_0]$ has no equitable $(b+1)$-coloring, but $g := f|B_0$ is a nearly equitable $(b+1)$-coloring. As each $v \in B_0$ is unmovable,

$$\text{(a) } d(v) \geq a - 1 + d_{G'}(v) + \|v, z_3\|, \quad \text{and} \quad \text{(b) } \theta(G') \leq 2b + 3. \tag{5.2}$$

By Lemma 19, $b + 1 < a + b = k$. As G' has no equitable $(b+1)$-coloring, our choice of k minimum in the setup implies $G' \in \{\mathbf{X}, \mathbf{Y_{b+1}}\}$ or $G' \supseteq \mathbf{Z_{b+1,c}}$ for some odd c. Now consider several cases, always assuming all previous cases fail for all choices of Z.

Case 0: $G' = \mathbf{X}$. Use the notation of Example 9. In this case, $b = 2$. For every $u \in N_{G'}(x_3)$, $d_{G'}(u) + d_{G'}(x_3) = 7$, so $\|\{u, x_3\}, A_0\| = 2a - 2$ and $uz_3 \notin E$. One of X or Y is contained in V^+ and if $X \subseteq V^+$, then, for some $i \in [2]$, $y_i \in B$, but y_i is movable to Z, a contradiction. So $Y \subseteq V^+$. Since $\{w_1, w_2\} \supseteq V^+ \setminus Y$, we can assume $\{w_1\} = V^+ \setminus Y$. If $Z = V^-$, then let $Z' := \{w_1, y_1\}$, and, otherwise, let $Z' := \{w_1, y_1, z_3\}$. In either case, let

$$f' := f|(A - Z) \cup \{Z', \{w_2, y_2, y_3\}, \{x_1, x_2, x_3, w_3\}\}.$$

so f' is a nearly equitable k-coloring of G and $Z' \in \mathcal{A}(f')$. Let $u \in Z' \setminus Z$, so $u \in \{w_1, y_1\}$ and $u \in N(x_3)$. With respect to the original coloring f, every vertex in $N(u) \cap A_0$ is solo and every nonsolo neighbor of u in A is unmovable, so, since u is good, Lemma 20(1) implies that every vertex in $N(u) \cap A_0$ is unmovable, and u is not adjacent to a witness of an in-edge of $Z \in \mathcal{H}[\mathcal{A}]$. This implies $\mathcal{A}(f') \supseteq \mathcal{A}(f) - Z$. Therefore, because y_2 is movable to Z', $a(f') > a(f)$ which contradicts (C1).

Case 1: $G' = K_{1,2b+2} + K_b$. Let $K = K_b$ and $r \in B_0$ with $d_{G'}(r) = 2b + 2$. Then $d_{G'}(w) = b - 1$ for all $w \in K$. As r is not contained in an independent 3-set in G', $r \in Z - z_3$. By (5.2)(a), $d(r) \geq a + 2b + 1$ and $d(v) \geq a$ for every $v \in N_{G'}(r)$. Since $\theta(G) \leq 2k + 1$, these bounds are sharp. Let $y \in N(r) \cap B$. Then $\|y, A\| = a$, and so $\|y, B_0 - r\| = 0$. Thus ry is solo. Also y is good. Let $u \in N(r) \cap A$. Lemma 20(2) implies all neighbors of r are unmovable. So $\|u, B_0\| \leq 2$, and witnesses of edges of $\mathcal{P}_0$ are not adjacent to r. Replace u with r in f_0 to obtain a new equitable $(a-1)$-coloring of $G[A_0]$.

Finally, as $\|u, B_0 - r\| \leq 1$, $\Delta(G' - r + u) \leq b$. By Theorem 6, $G' - r + u$ has an equitable $(b+1)$-coloring, contradicting (3.1).

Case 2: $G' \supseteq K_{c,2b+2-c} + K_{b+1}$ for some odd $c \in [b+1]$. Use the notation of Example 11, but with $V = B_0 = B_1 \cup B_2$, and $c \in [2b+1]$. As

$$\text{the clique } B_2 \text{ has one vertex in every class of } g, \tag{5.3}$$

assume $z_2 \in B_2$. Then $z_1 \in B_1$. Say $z_1 \in C_1$. Also by (5.3), every class Y of g of size three has precisely one vertex in B_2, so Y has two vertices in B_1; since those vertices are nonadjacent, Y has two vertices in either C_1 or C_2. Then each of C_1 and C_2 has an even number of vertices from the classes in g other than V^+ and $\{z_1, z_2\}$. By (5.3), V^+ has one vertex in B_2 and three in B_1; since c is odd, and the vertices of V^+ in B_1 are all in the same part, $V^+ \setminus B_2 \subseteq C_2$.

Case 2.1: $c \geq 3$. Then $C_1 - z_1 \neq \emptyset$. Let $y_1 \in C_1 - z_1$ and $y_2 \in V^+ \setminus B_2 \subseteq C_2$. Then

$$\begin{aligned} d(y_1) &= \|y_1, A \cup (B_1 - z_1) \cup (B_2 - z_2)\| \geq a + |C_2| + \|y_1, B_2 - z_2\|; \\ d(y_2) &= \|y_2, (A \setminus Z) \cup B_1 \cup (B_2 + z_3)\| \geq a - 1 + |C_1| + \|y_2, B_2 + z_3\|; \quad \text{and} \\ d(z_1) &= \|z_1, (A \setminus Z) \cup B_1 \cup B_2\| \geq a - 1 + |C_2| + \|z_1, B_2 \cup C_1\|. \end{aligned}$$

So $\theta(y_1 y_2) = 2k+1$, $\|y_1, B_2 - z_2\| = \|y_2, B_2 + z_3\| = 0$ and $\|y_2, A\| = a$. Also $\theta(z_1 y_2) \geq 2k$ and $\|z_1, B_2 \cup C_1\| \leq 1$. Let $Y = \{y_1, y_1', w\}$ be the class in $\mathcal{B}$ containing y_1, with $y_1' \in C_1$ and $w \in B_2$; and let $y_2' \in C_2 \cap V^+ - y_2$. Note $\left\|y_1', B_2 - z_2\right\| = \|y_1, B_2 - z_2\| = 0$ and $\left\|y_2', B_2 + z_3\right\| = \|y_2, B_2 + z_3\| = 0$ Let $w' \in V^+ \cap B_2$. Move y_2 to $Z - z_1$, z_1 to Y, and if $z_1 w \in E$ then switch w and w'. This yields a new nearly equitable k-coloring f' with y_2' movable to $Z - z_1 + y_2$. Since $y_2 \in V^+$ it is good. As $\|y_2, A\| = a$, Lemma 20 implies the neighbors in A of y_2 are unmovable. Therefore, all of the in-neighbors of Z in $\mathcal{H}(G, f)[\mathcal{A}(f)]$ are in-neighbors of $Z - z_1 + y_2$ in $\mathcal{H}(G, f')$. Furthermore, since z_1 is unmoveable in f and y_2 is unmoveable in f', the out-neighbors of Z in $\mathcal{H}(G, f)[\mathcal{A}(f)]$ are all out-neighbors of $Z - z_1 + y_2$ in $\mathcal{H}(G, f')$. Hence, $a(f) < a(f')$, contradicting (C1).

Case 2.2: $c = 1$. Then $C_1 = \{z_1\}$ and $|C_2| = 2b+1$. So

$$d(z_1) \geq a + 2b, \tag{5.4}$$
$$d(y) \leq a + 1 \quad \text{for all } y \in N(z_1). \tag{5.5}$$

For any $y \in B_2$, $d(y) \geq k - 1$, so since $\theta(G) \leq 2k+1$:

$$d(y) \leq k + 2 \quad \text{for all } y \in B_2. \tag{5.6}$$

Because Case 1 does not hold, $\|z_1, B\| = 2b + 1$. We now prove the following:

Claim 21.1 If some $y \in Y \in \mathcal{B}$ is bad then $b = 2$, $d(z_1) = a + 2b$, $Y \neq V^+$, and the unique $u \in B_2 \cap Y$ is high and satisfies $\|u, B\| \geq 3$. In particular, there are at most two bad vertices.

Proof of Claim 21.1. Suppose $G_y := G[B - y] = G' - \{z_1, z_2, y\}$ has no equitable b-coloring. Then $y \notin V^+$; so $Y \neq V^+$ and $b \geq 2$. By (5.5) and (5.6), $\Delta(G_y) \leq \Delta(G[B]) \leq b+2$, and $d_{G_y}(y') \leq 1$ for all $y' \in C_2$. Recall $\theta(G[B]) \leq 2b+1$, so $\theta(G_y) \leq 2b+1$. By the choice of k minimum in the setup, $G_y \in \{\mathbf{X}, \mathbf{Y_b}\}$, or $\mathbf{Z_{c,b}} \subseteq G_y$ for some odd c. Since $d_{G_y}(y') \leq 1$ for all $y' \in C_2 - y$, this implies $\Delta(G_y) \geq 2b$ or there are at least $b+1$ vertices $v \in B - y$ with $d_{G_y}(v) \geq b - 1$. So $b = 2$, $d_{Gy}(y') = 1$ for some $y' \in C_2$, and there is $u \in B_2 - y$ such that $\left\|u, G_y\right\| \geq 3$. As $\theta(y' z_1) \leq 2k+1$, (5.4) implies $d(z_1) = a + 2b$. As

$|Y - y| = 2$, $u \in Y \cap B_2$, so both vertices of $Y - u$ are in C_2. Since $b = 2$, $\mathcal{B} = \{Y, V^+\}$. Then u is not bad, since $\Delta(G[B - u]) \leq 2$. So if any vertex v is bad, then $v \in Y - u$.

Case 2.2.0: Every $X \in \mathcal{A}$ has a unmovable vertex v_X with $\|v_X, B\| \geq 2b+1$. By Lemma 19, $a \geq 2$. For all $T \in \mathcal{A} - V^-$, let $T = \{u_T, v_T, w_T\}$, where w_T witnesses the edge of $\mathcal{F}$ leaving T. Since $d(v_T) \geq (a-1) + 2b + 1 = k + b$, the set $D = \{v_T : T \in \mathcal{A}\}$ is independent. Let $v = v_{V^-}$ and $V^- = \{v, v'\}$. Since v_T is unmovable and D is independent, $v_T v' \in E$. Hence $D - v \subseteq N(v')$; so v' is unmovable. Use V^- for Z, so $v = z_1$ and $v' = z_2$. Then

$$k - 1 \leq \left\|v', A\right\| + b \leq d(v') \leq 2k + 1 - d(v_X) \leq k - b + 1, \tag{5.7}$$

so $b \in \{1, 2\}$. It follows that we can choose a leaf X of $\mathcal{F}$ so that $\|v', X\| = 1$: If $\mathcal{F}$ has only one leaf X then by (C2), $a = 2$, by Lemma 16, $b = 2$, and $\left\|v', X\right\| = 1$ because equality holds in (5.7). Otherwise, $\mathcal{F}$ has two leaves T and X and (say) $\left\|v', X\right\| = 1$. Switch v' and v_X to obtain $Z' = \{v, v_X\}$, $X' = \{v', u_X, w_X\}$, and a new nearly equitable k-coloring f'. For all $U \in \mathcal{A} - X - Z$, v_U witnesses that $UZ' \in \mathcal{H}(f')$, and w_X witnesses the edge from X' to V^- in $\mathcal{H}(f')$ or the edge from X' to the successor of X on $X\mathcal{F}$ in $\mathcal{H}(f')$. So f' is normal. Since both vertices in Z' are high, all vertices in B are low, so Claim 21.1 implies every vertex in B is good.

If $a = 2$ then by Lemma 16, $b = 2$. Also $\left\|v', B\right\| = 2$ and $E(A) = \{v'v_X, vu_X\}$. Moving w_X to Z' in f' shows that $B \subseteq N(v') \cup N(u_X)$: otherwise, we move a vertex $y \in B$ to $\{v', u_x\}$, and equitably color $B - y$, since y is good. Then $d(u_x) + d(v) \geq 2(1 + |B \setminus N(v')|) = 12$, contradicting $\theta(vu_X) \leq 9$. So $a \geq 3$ and by (C2) there is a leaf $T \neq X$. As v_T is movable to Z', $\|B, T\| \geq 3b+1+\|v_T, B\| \geq 5b+2$. If $\left\|v', T\right\| = 1$ then by symmetry $\|B, X\| \geq 5b+2$. Else, by (5.7), $\|v', B\| = d(v') - \|v', A\| \leq (k-b+1) - a = 1$. Let $y \in B$. Since $X' \in \mathcal{A}(f')$ and y is good, y has a neighbor in both X and X', so

$$\|B, X\| \geq |B| + \|v_X, B\| - \left\|v', B\right\| \geq (3b+1) + (2b+1) - 1 \geq 4b + 2.$$

Regardless, $\|B, T \cup X\| > 9b + 3$. So there exists $y \in B$ with $\|y, A\| \geq 4 + a - 2 = a + 2$. As f' is a nearly equitable coloring of A, and y is good, $yz \in E$ for some $z \in Z'$, and this gives the contradiction $\theta(yz) \geq k + b + a + 2 = 2k + 2$.

Case 2.2.1: $\|y, A\| = a$ for all $y \in C_2$. First suppose (*) for every $X \in \mathcal{A}$ and $y \in C_2$ the unique $x \in S^y \cap X$ is unmovable. If $X \in \mathcal{A}$ has a unique unmovable vertex v_X then $\|v_X, B\| \geq 2b+1$. Else X has two unmovable vertices. Using X for Z, yields some unmovable v_X with $\|v_X, B\| \geq 2b + 1$. Regardless, Case 2.2.0 holds. So (*) fails.

Pick $X \in \mathcal{A}$ and $y \in C_2$ with $x_3 \in S^y \cap X$ movable, and $|X\mathcal{F}|$ maximum. By Lemma 20(1), y is bad. By Claim 21.1, $\mathcal{B}$ has the form $\{U, V^+\}$, where $U = \{u, y, y'\}$, $w, w' \in V^+ \cap C_2$, $u \in B_2$, $\left\|u, V^+\right\| \geq 3$, u high, and all vertices in V^+ are good. By (5.5), $\|y', B\| \leq 1$, and we can label so $w'y' \notin E$. By Lemma 20(1), each $v \in C_2 \cap V^+$ is adjacent to an unmovable $x_v \in X$. If $x_w \neq x_{w'}$ then X is a candidate for Z, and either x_w or $x_{w'}$ is adjacent to y, i.e. $x_3 \in \{x_w, x_{w'}\}$. But this contradicts the fact that x_3 is unmovable. So, since $\|C_2 \cap V^+\| = 3$, $d(x_w) \geq (a-1) + 3 + \|x_w, u\| = k + \|x_w, u\|$. Since $\theta(G) \leq 2k+1$, $ux_w \notin E$. If $x_w y' \in E$, switch x_w and y'. Since the only neighbor of y in X is x_3, and the only neighbor of y' and w' in X is x_w, this yields a nearly equitable k-coloring f' with w' movable to $X - x_w + y'$. By maximality of $|X\mathcal{F}|$, y' is not adjacent to any witness of an edge $TX \in \mathcal{F}$. So $a(f') > a(f)$, contradicting (C1). If $x_w y' \notin E$, then move x_w to U and w to $X - x_w$. This yields a nearly equitable k-coloring f'' with w' movable to $X - x_w + w$. Again, by maximality of $|X\mathcal{F}|$, w is not adjacent to any witness of an edge $TX \in \mathcal{F}$, so $a(f'') > a(f)$, contradicting (C1).

Case 2.2.2: $\|w, A\| = a$ for some $w \in C_2$. If possible, choose w to be good. By $\theta(z_1 w) \leq 2k + 1$ and not Case 2.2.1, there exists a vertex in C_2 with degree at least $a + 1$, so $\|z_1, A\| =$

$a-1$. If w is bad, then by Claim 21.1, $b=2$ and there exists a good $y \in C_2 \cap V^+$ with $\|y, B\| \geq 1$. As $\theta(z_1 y) \leq 2k+1$, $\|y, A\| \leq a$. But then we would have chosen y instead of w, so w is good. As $z_1 \in S^w$, $wz_2 \notin E$.

By Lemma 20, the unique $w_X \in N(w) \cap X$ is unmovable for every $X \in \mathcal{A}$, and the unique $z_X \in N(z_1) \cap X$ is unmovable for every $X \in \mathcal{A} - Z$. If $X \in \mathcal{A}$ has two unmovable vertices, then by Case 2.2, one of them has $2b+1$ neighbors in B. Since Case 2.2.0 fails, there is $X \in \mathcal{A}$ with a unique unmovable vertex $v_X = z_X = w_X$. Since $\theta(G) \leq 2k+1$, $d(v_X), d(w) \leq a+1$. If $y \in N(z_2) \cap C_2$ is good, then since (5.5) implies that $|N(y) \cap X| = 1$, Lemma 20(1) implies that $yv_X \in E$.

Consider f_0, the equitable k-coloring of $G[A_0]$ defined in the beginning of this proof, obtained by shifting witnesses along $Z\mathcal{F}$ starting with z_3. As unmovable vertices remained in their color classes, v_X still is the unique neighbor of z_1 and w in the new X. Replacing v_X with z_1 in f_0 yields an equitable $(a-1)$-coloring f_1 of $G[A_0 + z_1 - v_X]$. Suppose $v_X z_2 \notin E$. Since $d(v_X) = a+1$ and v_X is unmovable, $\|v_X, B\| \leq 2$. Since $|V^+ \cap C_2| = 3$, we can choose $y \in (V^+ \cap C_2) \setminus N(v_X)$. Because y is good and $yz_X \notin E$, $yz_2 \notin E$, and there is an equitable b-coloring g of $B - y$, so $f_1 \cup g + \{v_X, z_2, y\}$ is an equitable k-coloring, contradicting (3.1). Otherwise, $v_X z_2 \in E$. Then $\|v_X, B - w\| = 0$. As w is good there is an equitable b-coloring g of $B - w$. Let $y \in V^+ \setminus N[w]$, and g' be the result of replacing y with v_X in g. As y is good and $v_X y \notin E$, $yz_2 \notin E$. So $f_1 \cup g' + \{z_2, w, y\}$ contradicts (3.1).

Case 2.2.3: There does not exist $y \in C_2$ such that $\|y, A\| = a$. That is, $\|y, A\| = a+1$ for all $y \in C_2$.

$$\text{For each } y \in C_2 \text{ there is } T \in \mathcal{A} \text{ with } N(y) \cap (A - T) \subseteq S^y. \tag{5.8}$$

Also

$$\|z_1, A\| = a - 1, \tag{5.9}$$
$$\|z_1, B\| = 2b + 1, \tag{5.10}$$
$$\|C_2, B\| = 0, \tag{5.11}$$

and

$$\text{every vertex in } B \text{ is good.} \tag{5.12}$$

Let $X \in \mathcal{A}' - Z$. As z_1 is unmovable, (5.9) implies it has a unique neighbor $v_X \in X$, and

$$d(v_X) \leq a + 1. \tag{5.13}$$

Suppose $x \in X$ and $y, y' \in S_x \cap C_2$ are distinct, and note $yy' \notin E$. By Lemma 20(1), x is unmovable. If x is low then $\|x, B\| \leq b+1$, and, by symmetry in B, we may assume that $N(x) \cap V^+ = \{y, y'\}$ and so switching x with y and y', and switching witnesses on a X, V^--path in $\mathcal{F}$ contradicts (3.1). So

$$\text{if } x \in X \text{ is low it has at most one solo neighbor in } C_2. \tag{5.14}$$

Suppose $\mathcal{A} = \{V^-, X\}$. By Lemma 16, $b \geq 2$. Assume $V^- = \{z_1, z_2\}$, as otherwise moving z_3 to V^- yields this. By (5.13), $\|v_X, B_0 - z_1\| \leq 2 \leq b$. For any $y \in B_1$, $d(y) \geq a+b-1$, so, since $\theta(G) \leq 2k+1$, z_2 is unmovable and $N(z_2) \supseteq B_1$, $\|z_2, C_2\| \leq 3$. Using this and (5.11), $G[B_0 - z_1 + v_X]$ has an equitable $(b+1)$-coloring, and by (5.9), $X - v_X + z_1$ is independent, contradicting (3.1). So $a \geq 3$, and $\mathcal{F}$ has two leaves.

An unmovable vertex $x \in A$ is *big* if $\|x, B\| \geq 2b+1$, and *small* if $\|x, B\| \leq 2b$. By Case 2.2,

$$\text{no class has two small vertices.} \tag{5.15}$$

Suppose z_1 and z_2 are big. Then $|N(z_1) \cap N(z_2) \cap C_2| \geq b+1$. Let $y, y' \in N(z_1) \cap N(z_2) \cap C_2$. Each $x \in X \cap N(\{y, y'\})$ is solo by (5.8). By Lemma 20 each $v \in N_A[x]$ is unmovable; so $x \in N(\{z_1, z_2\})$. As z_1 and z_2 are high, x is low. By (5.14) $|S_x \cap C_2| \leq 1 < b+1$. So X contains at least two distinct low solo vertices x and x'. Lemma 20(1) implies x and x' are unmovable. So $\|x, B\|, \left\|x', B\right\| \leq b+1$. Thus x and x' are small, contradicting (5.15). So

$$\text{no class has two big vertices.} \tag{5.16}$$

For a class $U \in \mathcal{A}$, let $S(U) := \{v \in C_2 : \|v, U\| = 1\}$. Over all color classes in $\mathcal{A}$ with two unmovable vertices, pick Z, with $S(Z) \neq \emptyset$ if possible; subject to this, choose Z to be a leaf if possible; and subject to these, choose $|S(Z)|$ maximum. Suppose $S(Z) = \emptyset$ or Z is not a leaf. By (5.8) there is a leaf X with $S(X) \geq \frac{1}{2}|C_2| \geq b+1$. By (5.13) and (5.14), $|S_{v_X} \cap C_2| \leq 1$. So there is a solo vertex $x \in X - v_X$. By Lemma 20(1), the solo vertices in X are unmovable. Because we did not choose X for Z, both vertices in $X - x$ are movable. So $S_x = S(X)$. Say v_X is movable to $W \in \mathcal{A}$.

As X is a leaf, $X \notin \mathcal{P} := W\mathcal{F}$. If $Z \in \mathcal{P}$, let u witness $UZ \in \mathcal{P}$. Consider any $y \in C_2$. By (5.8), $y \in S_x \cup S_{z_1}$. Suppose $y \in S_{z_1}$. If $uy \notin E$ or u is undefined, then moving y to $Z - z_1$, z_1 to $X - v_X$, v_X to W, and shifting witnesses along $\mathcal{P}$ contradicts (3.1). So $uy \in E$. By Lemma 20(1), uy is not solo. By (5.8), $y \in S_x$. Thus $C_2 \subseteq S_x$. So x is big. Since $\theta(G) \leq 2k+1$, $xz_1 \notin E$. Now $X \in \mathcal{A}'$, $y \in S_x$ for some $y \in C_2$, and $\|x, A\| \leq a-1$, so Lemma 20(2) implies $xz_2 \in E$. Since $\theta(G) \leq 2k+1$, $d(z_2) \leq a+1$, and so $\|z_2, C_2\| \leq 2-b \leq 1$. Let $V^+ = \{y_0, y_1, y_2, y^*\}$, where $y^* \in B_2$ and $N(z_2) \cap V^+ \subseteq \{y_0, y^*\}$. Shifting vertices starting with z_3 (if z_3 exists) on $Z\mathcal{F}$, and recoloring $X, Z - z_3, V^+$ as $X - x + y_0, \{z_2, y_1, y_2\}, \{z_1, x, y^*\}$ contradicts (3.1). So $S(Z) \neq \emptyset$ and Z is a leaf.

Let $X = \{v_X, x_2, x_3\} \neq Z$ be a leaf, where x_3 witnesses an edge of $\mathcal{F}$. Put $H = G[X \cup Z \cup V^+]$. Since $S_{z_1} = S(Z) \neq \emptyset$, (5.9) and Lemma 20(2) imply that v_X is unmovable. By (3.1),

$$\text{if some } v \in V(H) \text{ is movable to } \mathcal{A} - X - Z \text{ then } H - v \text{ has no equitable 3-coloring.} \tag{5.17}$$

By (5.16), z_2 is small, so $|C_2 \setminus N(z_2)| \geq b+1 \geq 2$. Using (5.11), choose $V^+ = \{y_1, y_2, y_3, y^*\}$ so that $y^* \in B_2$ and $y_1, y_2 \in C_2 \setminus N(z_2)$. Since v_X is unmovable, (5.13) implies that $\|v_X, B \cup \{z_2, z_3\}\| \leq d(v_X) - (a-1) \leq 2$. As z_3 witnesses an edge of $\mathcal{F}$, (5.17) implies $\{\{x_2, x_3, z_1\}, \{z_2, y_1, y_2\}, \{y_3, y^*, v_X\}\}$ is not a coloring of $H - z_3$. So $\|v_X, \{y_3, y^*\}\| \geq 1$ and $v_X y_i \notin E$ for some $i \in [2]$. Also $\{\{x_2, x_3, z_1\}, \{z_2, v_X, y_i\}, V^+ - y_i\}$ is not a coloring. So $v_X z_2 \in E$, $v_X z_3 \notin E$ and $\|v_X, B\| = 1$. In particular, $v_X y_1, v_X y_2 \notin E$.

Suppose x_2 is unmovable. By Case 2.2, $\mathbf{Z_{1,b+1}} \subseteq G[X \cup B - x_3]$. Since $\|v_X, B\| \leq 1$, $B = V^+$ and x_2 is big. So $\|x_2, A\| = a-1$ and $\|x_2, B\| = 3$. If x_2 is not solo, then for every $y \in N(x_2) \cap B$, $N(y) \cap X = \{x_2, x_3\}$ and $\|y, Z\| = 1$, so since z_3 is movable, by Lemma 20(1), $yz_3 \notin E$. Let $\tilde{y} \in B \setminus N(x_2)$, so $N_H(z_3) \subseteq \{x_2, x_3, \tilde{y}\}$. Let $H' := H - x_3$ and $e \in E(H')$. If $e = wz_3$, then $w \in \{x_2, \tilde{y}\}$, and $d_{H'}(w) \leq 4$, so $\theta(H') \leq 7$. If e is not incident to z_3, then both ends have at least $a-2$ neighbors in $V(G - H')$, so $\theta(H') \leq 7$. Since for every $w \in Y - \tilde{y} + z_3$, $d_{H'}(w) \leq 2$, $\Delta(H') \leq 4$, $\chi(H') \leq 3$, the maximality of k and Lemma 16 imply that there exists an equitable 3-coloring of H', contradicting (5.17). Now assume x_2 is solo. Since $\|x_2, A\| = a-1$, Lemma 20(2) implies that x_2 has an unmovable neighbor in

Z. Since $\theta(G) \leq 2k+1$, $x_2z_1 \notin E$ and so $x_2z_2 \in E$. For each color class $T \notin \{V^+, Z\}$, $\|y^*z_2, T\| \geq 2$ and each $y \in V^+$ satisfies $\|yz_1, T\| \geq 2$. Let $Q = z_1v_Xz_2x_2$. Note Q induces P_4. By inspection, $d_H(z_1) = 4 = d_H(x_2)$, $d_H(z_2) = 3 = d_H(v_X)$, and $\left\|V^+, \{x_3, z_3\}\right\| \leq 5$. Say $d_H(z_3) \leq d_H(x_3)$. Let $H' = H - x_3$. Then $\Delta(H') \leq 4$, $\theta(H') \leq 7$, $\chi(H') \leq 3$, and $d_{H'}(z_3) \leq 2$. Since H' contains an induced P_4, and $d_{H'}(z_3) \leq 2$, by (3.2), H' has a nearly equitable 3-coloring. An analogous argument works if $d_{H'}(x_3) \leq d_{H'}(z_3)$. So x_2 is movable. By Lemma 20(1), for $j \in \{1, 2\}$, $\left\|y_j, X\right\| = 2$, so $\{x_2, x_3\} \subseteq N(y_j)$. Also $y_jz_3 \notin E$ by (5.8) Let $i \in \{2, 3\}$. By (5.17), $\{\{v_X, z_3, y_1\}, \{z_1, z_2, x_i\}, V^+ - y_1\}$ is not a coloring of $H - x_{5-i}$. So $x_iz_2 \in E$.

Now suppose $v_Xy^* \in E$. Then by (5.13), $v_Xy_3 \notin E$. Because v_X is the only unmovable vertex in X, then $y_3x_2, y_3x_3 \in E$ by Lemma 20(1). By (5.8), $\{z_2, z_3, y_3\}$ is an independent set. For $i \in \{2, 3\}$, consider coloring $\{\{z_2, z_3, y_3\}, \{x_i, z_1, y^*\}, \{v_X, y_1, y_2\}\}$. Since x_{5-i} is movable, (5.17) implies this is not a proper coloring, so by (5.9) and (5.10), $y^*x_i \in E$. But now

$$d(y^*) + d(z_2) \geq (a + 2 + b - 1) + (a + 1 + b) = 2k + 2,$$

contradicting $\theta(G) \leq 2k+1$. Therefore $v_Xy^* \notin E$, and so $v_Xy_3 \in E$. Now by Lemma 20(1), $y^*x_2, y^*x_3 \in E$. Then $d(y^*) + d(z_2) \geq (a+1+b-1) + (a+1+b) = 2k+1$; so equality holds, and in particular $z_2y_3 \notin E$. Now $\{\{z_2, y_2, y_3\}, \{v_X, y_1, y^*\}, \{z_1, x_2, x_3\}\}$ is a proper equitable coloring of $H - z_3$, contradicting (5.17). □

If $T \in \mathcal{A}$ and $T \cap \overline{M} \neq \emptyset$, let $T = \{u_T, m_T, w_T\}$, where $u_T \in \overline{M}$.

Lemma 22 *Every $y \in B$ is good.*

Proof Suppose not. Say $G_0 := G[B - y_0]$ has no equitable b-coloring. Then $b \geq 2$. Also $|B - y_0| = 3b$, $\chi(G[B]) \leq b$, and, as every $y \in B$ is unmovable, $\theta(G[B]) \leq 2b + 1$. So (3.2) implies $G_0 \in \{\mathbf{X}, \mathbf{Y_b}\}$ or $\mathbf{Z_{c,b}} \subseteq G_0$ for some odd c. For any $y, y' \in E(G_0)$, if $\left\|yy', B\right\| = 2b+1$ then define yy', y and y' to be *B-heavy*. If $\|y, B\| > b$ then y is *B-high*. If y is B-heavy then $\|y, A\| = a$, and so y has a solo neighbor v in every class $X \in \mathcal{A}$. If y is good then Lemmas 20(1) and 21 imply v is the unique unmovable vertex $u_X \in X$. Suppose there exists $Y' \subseteq V(G_0)$ such that $|Y| = b+2$ and every vertex in Y' is both B-heavy and good, and furthermore there exists $y \in Y'$ that is B-high. Given some $X \in \mathcal{A}'$, every vertex in Y' is adjacent to an unmovable u_X, so $d(u_X) \geq (a-1) + (b+2) = k+1$. Since y is B-high, $d(y) \geq a+b+1 = k+1$. Then $\theta(u_Xy) \geq 2k+2$, contradicting $\theta(G) \leq 2k+1$. So:

$$\text{if } b+2 \text{ vertices are good and } B\text{-heavy, then none of them is } B\text{-high,} \tag{5.18}$$

Consider several cases, always assuming previous cases fail for all bad $y_0 \in B$.

Case 1: $G_0 = \mathbf{X}$. Then $\Delta(G[B]) = 4$. Using the notation of Example 9, x_3 is B-high and all five vertices in $N_{G_0}[x_3]$ are B-heavy. By (5.18), there is a bad $v \in N_{G_0}[x_3]$. By $\Theta(G) \leq 2k+1$, no vertex in $N_{G_0}[x_3]$ is adjacent to y_0, and, since the neighbors in B of y_0 are high, $\|y_0, B\| \leq 3$. so $\Delta(G[B - v]) \leq 3$, and $G[B - v]$ does not contain $\mathbf{Y_3}$ or $\mathbf{Z_{1,3}}$, so $\delta(G[B - v]) \geq 3$. Furthermore, since y_0 is high, $N(y_0)$ is independent; thus $N(y_0) \cap B = \{x_1, x_2, w_3\}$. So, the B-high vertices x_1, x_2, x_3, w_3 are good and B-heavy; by inspection, w_1 is B-heavy and good. This contradicts (5.18).

Case 2: $G_0 = \mathbf{Y_b}$. Let y be the vertex with degree $2b$ in G_0. Then the class of f containing y is $\{y, y_0, w\}$, where $w \in K_{b-1}$. So $V^+ \subseteq N(y)$. Since $\|N[y], B - y\| = 0$, the vertices of

$N(y)$ are all good; by inspection, also y is good. But the vertices of $N[y]$ are B-heavy and y is B-high, contradicting (5.18).

Case 3: $G_0 \supseteq \mathbf{Z_{c,b}}$, for some odd $c \leq b$. Recall $M = \{v \in A : v \text{ is movable}\}$ and $\overline{M} = A \smallsetminus M$, and use the notation of Example 11 with $V = B - y_0$.

Case 3.1: $a = 2$. Then $x \in A$ is movable if and only if it has no neighbors in A. Thus an unmovable vertex has an unmovable neighbor. By Lemma 21, $|M| \geq 3$. So $\|A\| \leq 1$, and $\{S, A \smallsetminus S\}$ is an equitable coloring for any 2-set $S \subseteq A$ with $|S \cap \overline{M}|, |(A - S) \cap \overline{M}| \leq 1$. Thus (C1) implies every $w \in B$ satisfies $\|w, M\| \geq 3$ or $\left\|w, \overline{M}\right\| \geq 2$. Let $e \in E(Q)$. Then $\theta(e) \geq 2b + \|e, A\|$. Since $\theta(G) \leq 2k + 1$, e has an end w_0 with $\|w_0, A\| = 2$; say $N(w_0) \cap A = \{u_1, u_2\}$. So $u_1u_2 \in E$ and $u_1, u_2 \in \overline{M}$. Set $R = \{w \in B : \|w, M\| \geq 3\}$ and $P = \{w \in B : \left\|w, \overline{M}\right\| \geq 2\}$. As $\theta(u_1u_2) \leq 2k + 1$, $|P| \leq b + 1$. Let $v \in M$. Then $2b \leq |R| \leq d(v)$. Thus there is $y_2 \in R \cap B_1$. Then $d(y_2) \geq 3 + c$. Since $2b + 3 + c \leq \theta(vy_2) \leq 2k + 1$ and c is odd, $c = 1$, and $y_2 \in C_2$. Let $C_1 = \{y_1\}$. Then $y_1 \in P$, and $d(y_1) \geq 2b + 1$. By Lemma 15, there is $w^* \in R \cap B_2$. As $d(w^*) \geq b + 2$, $\theta(G) \leq 2k + 1$ implies $|R| \leq d(v) \leq b + 3$. So $|P| \geq 2b - 2$ and $d(u_1) \geq 2b - 1$. Since $\theta(G) \leq 2k + 1$, $4b \leq \theta(u_1y_1) \leq 2k + 1$. Thus $b = 2$, and by Lemma 15, P is independent. So $y_1 \in P$ implies $C_2 \subseteq R$ and, since $\theta(G) \leq 2k + 1$, $N(C_2) = M + y_1$ and $d(v) \leq 5$. If there is $y \in P \cap R$ then $|R| \geq 5$ and $d(y) \geq 5$, contradicting $\theta(vy) \leq 9$. Else $w^*u_i \notin E$ for some $i \in [2]$. If $|P| = 3$ then $\{\{u_i, w^*, y_2\}, C_2 - y_2 + u_{3-i}, M, P\}$ contradicts (3.1). Else $|R| = 5$, and the coloring $\{\{u_i, w^*, y_2\}, M - v + u_{3-i}, P + v, R - w^* - y_2\}$ contradicts (3.1).

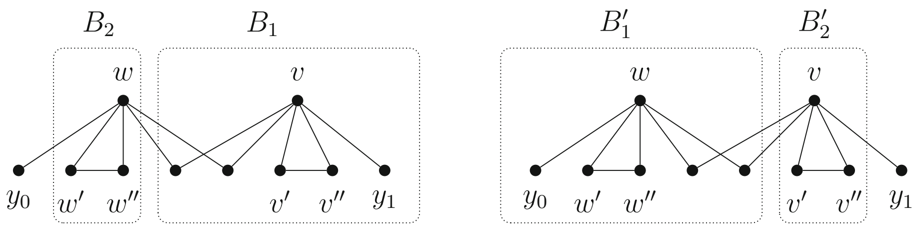
Case 3.2: There is a bad $y_1 \in B_1$. Say $G[B - y_1] \supseteq Q' + K' := K(C_1', C_2') + K(B_2')$. Set $B_0 = B_1 + y_0$. Then each $v \in V^+$ is good, $V^+ \smallsetminus B_2 \subseteq C_i$ and $V^+ \smallsetminus B_2' \subseteq C_{i'}'$ for some $i, i' \in [2]$. By (C2) and $a \geq 3$, there are distinct $Z_1, Z_2 \in \mathcal{A}'$. For distinct $v_1, v_2 \in B_2$,

$$2k + 1 \geq \theta(v_1v_2) \geq 2(a - 2) + \|v_1v_2, Z_1 \cup Z_2\| + 2(b - 1) \geq 2k - 6 + \|v_1v_2, Z_1 \cup Z_2\|.$$

So there exists $Z^* = \{z, z^*, z'\} \in \{Z_1, Z_2\}$ and $v^* \in \{v_1, v_2\}$ such that $z^*, z' \in M$ and $z^*v^* \notin E$. Shifting witnesses on $Z^*\mathcal{F}$, starting with z', yields an equitable $(a - 1)$-coloring $\mathcal{A}^*$ of $A - z - z^*$.

Case 3.2.1: $b = 2$. Say $\mathcal{B} = \{Y, V^+\}$. Then $Q = K_{1,3}$, $C_2 = V^+ \smallsetminus B_2$, and $C_1 = \{y_1\}$. So $Y = \{y_0, y_1, y_2\}$, where $y_2 \in B_2$. Since Case 2 fails, $\Delta(G[B]) \leq 3$. As y_1 is bad, $\left\|\{y_0, y_2\}, V^+\right\| \geq 4$, and $\left\|y^*, V^+\right\| = 3$ for some y^*, where $Y = \{y_1, y^*, y'\}$. So y_1 and y^* are high. Thus each $v \in V^+$ satisfies $\|v, B\| \leq 2$ and $\|y', B\| \leq 2$. So V^+ has the form $\{v_1, v_2, v_3, v'\}$, where $1 \leq \left\|v', B\right\| \leq 2$, and $\|v_i, B\| = 2$ for $i \in [3]$. Thus $\|v_i, A\| = a$. As v_i is good, Lemma 20(1) and Lemma 21 imply that $z \in N(v_i) \cap A = \overline{M}$. So $d(z) \geq a + 2 + \|z, \{y_1, y^*\}\|$, and $\{z, y_1, y^*\}$ is independent. If $v'y' \notin E$, then we can label so that $y'v_1 \notin E$ and let $\mathcal{B}^* := \{\{y', v', v_1\}, \{v_2, v_3, z^*\}\}$, otherwise we can reselect Z^* and z^* if necessary so that $\|z^*, \{v', y'\}\| \leq 1$, which implies that there exists an equitable 2-coloring $\mathcal{B}^*$ of $V^+ + y' + z^*$. In either case, $\mathcal{A}^* \cup \mathcal{B}^* + \{z, y_1, y_2\}$ contradicts (3.1).

Case 3.2.2: $B_2 = B_2'$ and $b \geq 3$. Then $V(Q \cap Q') = B_0 - y_0 - y_1$. As Q and Q' are connected, so is $Q \cup Q'$. If $O \subseteq Q \cup Q'$ is an odd cycle then $y_0 \in V(O)$, and $V(O) - y_0 := v_1 \dots y_{2r} \subseteq V(Q)$. So $v_1v_{2r} \in E$ and $\theta(v_1v_{2r}) = 2a+2b+2$, contradicting $\theta(G) \leq 2k+1$. Thus $Q \cup Q'$ is bipartite. Since it has bad vertices, it is complete. So $\theta_{Q \cup Q'}(e) = 2b + 1$ for every $e \in E(B_0)$, and every $w \in B_0$ satisfies $\|w, A\| = a$ and $\|w, B_2\| = 0$. Let $\{D_1, D_2\}$ be the unique 2-coloring of $Q \cup Q'$, where $|D_1|$ is odd. Consider any $w_1 \in D_1$. Then w_1 is good, so $y_0, y_1 \in D_2$. By Lemmas 20 and 21, $N(w_1) \cap A = \overline{M}$. Let $z \in Z^* \cap \overline{M}$. Then $D_1 \subseteq N(z)$, and $\theta(zw_1) \geq 2a - 1 + 2b + 1 + \|z, D_2\|$. Thus $\|z, D_2\| \leq 1$. If $\|z, D_2\| = 0$



(a) $B - y_0 = B_1 \cup B_2$ **(b)** $B - y_1 = B'_1 \cup B'_2$

Fig. 6 $G[B]$ in Case 3.2.3, maybe missing the edge y_0y_1

then (*) $|D_2 \setminus N(z)| \geq 2$. Else there is $w_2 \in N(z) \cap D_2$. Then $\theta(zw_2) \geq 2a - 1 + 2|D_1|$. So $|D_1| \leq b + 1$, $|D_2| \geq b \geq 3$, and again (*) holds. So there are distinct $y', y'' \in D_2 \setminus N(z)$. Let $B^* = B_0 + z^* - y' - y''$. Then $D_1 + z^*$ and $D_2 - y' - y''$ are even independent sets, and $N(B^*) \cap B_2 = N(z^*) \cap B_2 \neq B_2$. So B^* has an equitable b-coloring $\mathcal{B}^*$. Thus the coloring $\mathcal{A}^* \cup \mathcal{B}^* + \{z, y', y''\}$ contradicts (3.1).

Case 3.2.3: $B_2 \neq B'_2$ and $b \geq 3$. Let $w \in B_2 \cap B'_1$. As $|B_2| \geq 3$ and $\|w, B'_2\| \leq 1$, there is $w' \in B_2 \cap B'_1 - w$. As $\|ww', B'_2\| \leq 1$, $B_2 \subseteq B'_1$. Thus $b = 3$. Now there are $i \in [2]$ and distinct $w', w'' \in C'_i \cap B_2$. Then $\theta(w'w'') \geq 2(a + 1 + |C'_{3-i}|)$, and $|C'_{3-i}| = 1$. Say $C'_1 = \{w\}$. Similarly, $C_1 = \{v\}$, where $B'_2 = \{v, v', v''\} \subseteq B_1$. (See Fig. 6.) So all vertices of $B - \{y_0, y_1\}$ are B-heavy and good, and w is B-high, contradicting (5.18).

Case 3.3: Every $y \in B_1$ is good. There is $i \in [2]$ with $\|w, A\| = a$ for all $w \in C_i$ and $\|w, A\| \leq a+1$ for all $w \in C_{3-i}$. We set $|C_i| = c$, for some odd $c \in [2b-1]$. By Lemma 20(1) and Lemma 21, $C_i \subseteq N(x)$ for all $x \in \overline{M}$ and $|S_z \cap C_{3-i}| \geq |C_{3-i}|/2$ for some $z \in \overline{M}$ with $z \in Z \in \{Z_1, Z_2\} \subseteq \mathcal{A}'$. Suppose $|C_i| \geq |C_{3-i}|$. Let $z' \in \overline{M} - z$ with $z' \in Z' \in \mathcal{A}'$ and $w \in C_i$. If $c = 2b-1$, then, for any $y' \in N(z) \cap C_{3-i}$, $\theta(zy') \geq a-1+2b+a+c > 2k+1$, so $2b-c \geq 3$. If $C_{3-i} \subseteq N(z')$ then $\theta(z'w) \geq a-1+2b+a+2b-c \geq 2k+2$, contradicting $\theta(G) \leq 2k+1$. So there is $y' \in C_{3-i} \setminus N(z')$. By Lemma 20(1), $\|y', Z'\| = 2$ and $y'z \in E$. Now

$$2k + 1 \geq \theta(zy') \geq a - 1 + |C_i| + |C_{3-i}|/2 + a + 1 + |C_i| \geq 2k + |C_i| - |C_{3-i}|/2,$$

another contradiction. So $|C_i| < |C_{3-i}|$. Say $i = 1$. For $y \in C_1$,

$$2k + 1 \geq \theta(zy) \geq a - 1 + |C_1| + |C_2|/2 + a + |C_2| \geq 2k - 1 + |C_2|/2.$$

So $|C_2| = 3$, $|C_1| = 1$, and $b = 2$. Let $\mathcal{B} = \{W, V^+\}$ and $C_1 = \{w\}$. Then $C_2 = V^+ \setminus B_2$ and $d(w) \geq a + 3$. Also $d(z) \geq a - 1 + |C_1| + |C_2|/2$. As $wz \in E$, $d(z) = a + 2$ and $d(w) = a + 3$. So z has exactly two neighbors $v_1, v_2 \in V^+$, and $v_1, v_2 \in S_z$ by the choice of z. Switching witnesses on $Z\mathcal{F}$, and switching z with v_1 and v_2 yields an equitable k-coloring. □

For $w \in W \in \mathcal{A}$ and $i \in [3]$, let $B_i(w) := \{y \in N(w) \cap B : \|y, W\| = i\}$ and let $B_0(w) := B \backslash N(w)$. Let $b_i(w) := |B_i(w)|$ for $i \in \{0, 1, 2, 3\}$.

Corollary 23 *For every* $X = \{x, x', x''\} \in \mathcal{A}'$ *with* $b_1(x) > 0$, $B_0(x) \cup B_3(x) \subseteq N(x') \cap N(x'')$.

Proof By definition, $B_3(x) \subseteq N(x') \cap N(x'')$. Since $b_1(x) > 0$, by Lemmas 22 and 20(b), x is unmovable, and by Lemma 21, x' and x'' are both movable. By Lemmas 22 and 20(b) again, every vertex of $B_0(x)$ is adjacent to both x' and x''. $\square$

Lemma 24 *Every solo $x \in X \in \mathcal{A}'$ satisfies $\|x, B\| \leq 2b$.*

Proof Suppose $\|x, B\| \geq 2b + 1$, and let $y \in S_x$. Since $\theta(xy) \leq 2k + 1$, Lemmas 20 and 22 imply $a + 2b \leq d(x) \leq a + 2b + 1$. First suppose $d(x) = a + 2b + 1$. Consider any $w \in N(x) \cap B$. Then $\theta(xw) \leq 2k+1$ implies $\|w, A\| = a$. Thus $S^w = N(w) \cap A = \overline{M}$. So for unmovable $u_Z \in Z \in \mathcal{A}$, $d(u_Z) \geq a - 1 + \|x, B\| \geq k + 1$. Thus the set $\{u_Z : U \in \mathcal{A}\}$ is independent. By Lemma 21, the unique vertex $v \in V^- - u_{V^-}$ is movable; say v is movable to $U \in \mathcal{A}$. Since u_U is not movable to V^-, it is adjacent to u_{V^-}, a contradiction.

So $d(x) = a + 2b$, $\|x, A\| = a - 1$ and $\|w, A\| \leq a + 1$ for every $w \in N(x) \cap B$. As $X \in \mathcal{A}'$, Lemmas 20 and 22 imply $N[x] \cap A = \overline{M}$. Some $W \in \mathcal{B}$ satisfies $\|x, W\| \geq 3$; set $W' = N(x) \cap W$. Each $w \in W'$ has at most one neighbor in $\{x_1, x_2\} := X - x$. Thus $\left\|x_i, W' - w'\right\| = 0$, for some $i \in [2]$ and $w' \in W'$. Say x_i is movable to $U \in \mathcal{A}$, and $x_U \in N(x) \cap U$. Then

$$\|x_U, B \cup \{x_1, x_2\}\| \leq 2k + 1 - d(x) - \|x_U, A - X + x\| \leq a + 1 - a - 1 = 2. \quad (5.19)$$

If $x_U x_{3-i} \notin E$ then switch x and x_U. As $N[x] \cap A = \overline{M}$, this yields a new normal k-coloring f' with $X' := X - x + x_U \in \mathcal{A}'(f')$. By (5.19), some $w \in W'$ is not adjacent to x_U. By Lemmas 20 and 22, $\left\|w, X'\right\| \geq 2$, a contradiction.

Else $x_U x_{3-i} \in E$. By (5.19), $\|x_U, W\| \leq 1$. So there is $w \in W$ with $\{w, x_U, x_i\}$ independent. Shift witnesses, starting with x_{3-i}, on an X, V^--path in $\mathcal{H}$. This does not affect neighbors of x since they are unmovable. Now switch x with x_U, move w to $X - x - x_{3-i} + x_U$, and equitably b-color $B - w$. This yields an equitable k-coloring of G. $\square$

Lemma 25 *If $x \in X \in \mathcal{A}'$, $y \in S_x$, $y' \in N(x) \cap B - y$ and $\left\|y', X\right\| \leq 2$, then $yy' \in E$.*

Proof If not, there exist $y \in S_x$ and $y' \in N(x) \cap B - N[y]$ with $\left\|y', X\right\| \leq 2$. Choose such a pair with $\|y, B\|$ maximum. By Lemmas 20 and 22, x is unmovable; so $\|x, A - X\| \geq a - 1$. Put $A^* = A - x + y$, $X^* = X - x + y$ and $B^* = B - y$. By Lemma 22, $G[B^*]$ has an equitable b-coloring $\mathcal{B}^*$; say $y' \in Y \in \mathcal{B}^*$. Then $\mathcal{A}^* := \mathcal{A} - X + X^*$ is an equitable a-coloring of A^*. By Lemma 24, $\|x, B\| \leq 2b$. So $\|x, W\| \leq 1$ for some $W \in \mathcal{B}^*$; consider any such W.

Since x is unmovable and $X \in \mathcal{A}'$, if $\mathcal{B}^+$ is an equitable b-coloring of $B^* + x$, then $f^+ := \mathcal{A}^* \cup \mathcal{B}^+$ is a normal k-coloring with $X^* \in \mathcal{A}(f^+)$. As y is unmovable in f and $yy' \notin E$, $\left\|y', X^*\right\| \geq 2$, a contradiction. So $B^* + x$ has no equitable b-coloring. Thus x has a neighbor in every class of $\mathcal{B}^* - W$. In particular, $N(x) \cap W = \{w\}$. Then $\|w, A - X + x\| \geq a$, and w (like x) has a neighbor in every class of $\mathcal{B}^* - W$.

For $x_0 \in X - x$, $G[A - X + x_0]$ has an equitable $(a - 1)$-coloring obtained by shifting witnesses, starting with x_0, on an X, V^--path in $\mathcal{H}$. If $G[B^* + x - u]$ has an equitable b-coloring, where $u \in B^*$, then (3.1) implies $X^* + u - x_0$ is not independent. Thus w is not movable to X^*, and $\left\|w, Y - y'\right\|, \left\|x, Y - y'\right\| \geq 1$ where $y' \in Y \in \mathcal{B}^*$. So $d(w) \geq \|w, A - X + x\| + \|w, B^*\| + \|w, X^*\| \geq k$ and $d(x) \geq k + 1$. Since $\theta(G) \leq 2k + 1$, $d(x) = k + 1$, $d(w) = k$, $\|x, B\| = b + 2$, $\|x, A\| = a - 1$, and $\|w, X^*\| = 1$. So $wy \in E$, $w \in S_x$, $\|w, A\| = a$, $\|w, B\| = b$, and $\|w, Y\| = 1$. Thus $wy' \notin E$.

As $\theta(xy) \leq 2k + 1$, $\|y, B\| \leq b$. So any $w' \in S := N(x) \cap B \smallsetminus Y$ can play the role of y. By the maximality of $\|y, B\|$, $\left\|w', B\right\| = b$ and $\left\|w', A\right\| = a$ for all $w' \in S$. By Lemmas 20, 21 and 22, $N(w') \cap A = \overline{M}$ for each $w' \in S$, and $N(x) \cap A = \overline{M} - x$. Let $u_Z \in Z \cap \overline{M}$ for $Z \in \mathcal{A}$. By Lemma 15, $\omega(G) < k$. Since S is a clique, there are distinct $Z, Z' \in \mathcal{A} - X$

with $u_Z u_{Z'} \notin E$. First, we note $\|u_Z, Z\| \geq 2$ by Lemmas 20(0), 21, and 22. Since $u_Z x \in E$, and $\theta(G) \leq 2k+1$, $d(u_Z) \leq k$, so $\|u_Z, A\| = a$ and $\|u_Z, B\| = b = |S|$. In particular, $u_Z y' \notin E$. Then switching x and u_Z yields a normal k-coloring in which y' has a movable, solo neighbor in a terminal class, a contradiction. □

6 Optimal colorings

A normal k-coloring f of G is *optimal* if

(C3) among normal k-colorings, $|H(B)|$ is minimum, and
(C4) subject to (C3), a' is maximum.

Let f be optimal.

Lemma 26 *If $y \in H(B)$ then $S^y \cap A' = \emptyset$.*

Proof Suppose $y \in H(B)$, $X \in \mathcal{A}'$ and $x \in S^y \cap X$. We will obtain a contradiction by showing that either G has a normal coloring with $|H(B)|$ smaller or $\omega(G) = k$.

By Lemmas 20 and 22, x is unmovable and $G[B-y]$ has an equitable b-coloring $\mathcal{B}^*$. Thus if $G[B+x-y]$ has an equitable b-coloring then putting y in $X-x$ yields a normal k-coloring with fewer high vertices in B, contradicting (C3). Thus $\|x, Y\| \geq 1$ for all $Y \in \mathcal{B}^*$. Because $xy \in E$ and y is high, $k \leq d(x)$; but by the above, $d(x) \geq (a-1)+b+1$, so indeed x has precisely one neighbor in every class of $\mathcal{B}^*$. By Lemma 20, $N[x] \cap A = \overline{M}$ and $d(y) = k+1$. Suppose there exists $y' \in N(x) \cap B - y$ in class $Y' \in \mathcal{B}^*$ that is movable to class $Y'' \in \mathcal{B}^*$. Then moving y' to Y'' and x to $Y' - y'$ yields an equitable b-coloring of $G[B+x-y]$. Thus each $y' \in N(x) \cap B - y$ satisfies $\left\|y', B-y\right\| \geq b-1$.

Let $W = B \cap N(x) \cap N(y)$ and $W' = B \cap N(x) \smallsetminus N[y]$. Let $w \in W$; then w is low. So $\|w, A\| = a$ and $\|w, B\| = b$. Thus $W + y \subseteq S_x$, and by Lemma 20, $S^w = N[w] \cap A = \overline{M}$. By Lemma 25, S_x is a clique. As $G[B]$ is b-colorable, $|W| \leq b-1$, so $|W'| \geq 1$. Consider any $w' \in W'$. As $w'y \notin E$, Lemma 25 implies $X \subseteq N(w')$. So $d(w') \geq (b-1)+3+(a-1) = k+1$. Let $X = \{x, x', x''\}$. Every $u \in B \smallsetminus N(x) + w'$ is adjacent to x' by Lemmas 20(1) and 21. Thus $2k+1 \geq \theta(x'w') \geq 2b+1+k+1$. So $a > b$; as $k \geq 4$, $a \geq 3$. Thus there is $Z \in \mathcal{A}' - X$. Then $u_Z \in S^{w'}$. So $W \cup W' \subseteq S_{u_Z}$ is a b-clique. As w' is high, $|W'| = 1$. Also Z, u_Z, w' can play the role of X, x, y. Thus there is a high w'' with $\left\|w'', W\right\| = b-1$ and $\left\|w'', Z\right\| = 3$. Indeed, we can choose $w'' = y$. So

$$N[x] = N[u_Z] = N[w] = \overline{M} \cup W + y + w' \quad \text{for all } w \in W. \tag{6.1}$$

Choose $T \in \mathcal{A} \smallsetminus \{X, Z\}$. By (6.1), $W \subseteq N(u_T)$ and $u_T x \in E$. Thus

$$k+1 \geq d(u_T) \geq a-3+|W| + \|u_T, X+y\| + \left\|u_T, Z+w'\right\|.$$

So $\|u_T, X+y\| + \left\|u_T, Z+w'\right\| \leq 5$. Say $\|u_T, X+y\| \leq 2$. Then there is $x' \in X-x$ with $\left\|u_T, X-x-x'\right\| = 0$. Suppose $u_T y \notin E$. Let x' be movable to $U \in \mathcal{A}' - X$; move x' to U, and switch witnesses along a UV^- path in $\mathcal{A} - X$; moving u_T and y to $X - x - x'$, and moving x to $T - u_T$ contradicts (3.1). So $u_T y \in E$ and $\|u_T, X+y\| \geq 2$. As y is high, $d(u_T) \leq k$, so $\left\|u_T, Z+w'\right\| \leq 2$. By an analogous argument, $u_T w' \in E$. Thus

$$N(u_T) \supseteq W \cup \{y, w', x, u_Z\} \quad \text{for every } U \in \mathcal{A} - X - Z. \tag{6.2}$$

Finally, switching u_Z with x yields a nearly equitable k-coloring f' of G. As the neighbors of x in A are unmovable and X is terminal, $A = A(f')$, $X - x + u_T$, $Z \in \mathcal{A}'(f')$, and $u_T \in S^y$.

Thus f' is normal. By (6.2) and Lemma 20, $N[u_T] = \overline{M} \cup W + y + w'$. Combining this with (6.1), shows $\overline{M} \cup W + y$ is a k-clique, contradicting Lemma 25. □

7 Almost all color classes in $\mathcal{A}$ are terminal

For $X \in \mathcal{A}$, let $\mathcal{T}(X)$ be the set of $U \in \mathcal{A} - X$ such that every U, V^--path in $\mathcal{H}$ contains X. Then $\mathcal{T}(X) = \emptyset$ if and only if $X \in \mathcal{A}'$, and if $X' \in \mathcal{T}(X)$ then $\mathcal{T}(X') \subsetneq \mathcal{T}(X)$. So $\mathcal{T}(X)$ contains a terminal class if X is nonterminal. Choose $X_0 \in \mathcal{A} \smallsetminus \mathcal{A}'$ so that $\mathcal{T}(X_0)$ is a minimum nonempty set. Then $\mathcal{T}(X_0) \subseteq \mathcal{A}'$. Set $\mathcal{A}'' = \mathcal{T}(X_0)$. As usual, set $A'' := \bigcup \mathcal{A}''$, and $a'' := |\mathcal{A}''|$. Then $1 \leq a'' \leq a'$, and if $a' = a - 1$, then $X_0 = V^-$ and $\mathcal{A}'' = \mathcal{A}'$. Also, $\|w, A\| \geq a - a'' - 1$ for every $w \in A''$.

Proposition 27 *If $a'' = a'$, then $a = a' + 1$.*

Proof Argue by contraposition. If $a' \leq a - 2$ then the set X_0 defined above the proposition differs from V^-, and $\mathcal{A}'' = \mathcal{T}(X_0) \subseteq \mathcal{A}'$. Let $\mathcal{P}$ be a minimum X_0, V^--path in $\mathcal{H}$, and let its last edge be UV^-. If there exists $W \neq U$ such that $WV^- \in E(\mathcal{H})$, then $W \notin V(\mathcal{P})$ by the minimality of $\mathcal{P}$. So $\mathcal{T}(W) \cap \mathcal{T}(X_0) = \emptyset$ and $\mathcal{T}(W) + W$ contains a terminal class. Thus $a'' < a'$. Else $\mathcal{A}' \subseteq \mathcal{T}(U) = \mathcal{A} - V^- - U$. Shifting a witness w of UV^- to V^- yields a normal k-coloring f' with small class $U - w$, $A(f) = A(f')$ and $\mathcal{A}'(f') = \mathcal{A}'(f) + (V^- + w)$, preserving (C3) and contradicting (C4). □

Lemma 28 *If $b \leq a' - 1$ then $|L(B)| \leq b + 1$. Moreover, if $|L(B)| = b + 1$ then $d(y) = k$ for all $y \in L(B)$, $G[L(B)]$ is the disjoint union of cliques, and $a' = b + 1$. If in addition $a' = a - 1$, then $b \leq 2$.*

Proof Suppose $L = L(B)$, $b \leq a' - 1$ and $|L| \geq b + 1$. Let I be an inclusion maximal independent subset of L of size at least 2; it exists since $G[B]$ is b-colorable.

$$\text{All } y \in L \text{ satisfy } a + b \geq d(y) \geq a + a' - |S^y \cap A'| + \|y, B\|. \tag{7.1}$$

By Lemmas 21 and 20, each solo vertex in A' is the unique unmovable vertex in its class. By Theorem 25, $|S_x \cap I| \leq 1$ for all $x \in A'$. By maximality, $\|L \smallsetminus I, I\| \geq |L \smallsetminus I|$. Thus

$$\begin{aligned} a' &\geq \sum_{x \in A'} |S_x \cap I| = \sum_{y \in I} |S^y \cap A'| \geq \sum_{y \in I} (a' - b + ||y, B||) \geq |I|(a' - b) + \|L \smallsetminus I, I\| \\ &\geq |I|(a' - b - 1) + |L| = (|I| - 1)(a' - b - 1) + (a' - b - 1 + |L|) \geq a'. \end{aligned}$$

So all four inequalities in the chain are sharp. This yields (in order) $E(X, I)$ has a solo edge for all $X \in \mathcal{A}'$; y has a solo neighbor in A' and $d(y) = k$ for all $y \in I$; $\|w, B\| = \|w, I\| = 1$ for all $w \in L \smallsetminus I$; and $a' = b + 1$ and $|L| = b + 1$. As I can contain any pair of nonadjacent vertices in L, no $w \in L$ has two nonadjacent neighbors in L; so $G[L]$ is the disjoint union of cliques.

Finally, suppose $a' = a - 1$ and $b \geq 3$. Let C_1 and C_2 be components of $G[L]$ with $|C_1| \leq |C_2|$. For $i = 1, 2$, let $y_i \in V(C_i)$, and $x_i y_i \in E(C, A')$ be a solo edge, where $X_i = \{x_i, x_i', x_i''\} \in \mathcal{A}'$. By Lemma 25, each $y' \in B - C_i$ is adjacent to x_{3-i}' and x_{3-i}''. So x_i' and x_i'' are low, and

$$2b + 2 = a + b \geq d(x_i'), d(x_i'') \geq \left\|x_i', B\right\|, \left\|x_i', B\right\| \geq 3b + 1 - |C_i|.$$

Thus $b - 1 \leq |C_1| \leq (b + 1)/2$, $b = 3$, $|C_1| = 2 = |C_2|$, $8 = k \geq d(x_i'), d(x_i'') \geq 8$, and $\left\|x_i', A\right\| = 0 = \left\|x_i'', A\right\|$. Then switching x_1 with x_2 yields a nearly equitable coloring with a larger a, since $y_1 x_2 \notin E$. So $\left\|y_1, \{x_2, x_1', x_1''\}\right\| = 0$. □

When $b \geq a'$, we use the following analog of low vertices. A vertex $y \in B$ is *petite* if $d(y) \leq a + a' - 1$ or both $d(y) = a + a'$ and the following strengthening of inequality (7.1) holds: $\|y, A\| \geq a + a' + 1 - |S^y \cap A'|$. This inequality implies y has 3 neighbors in a terminal class or at least two neighbors in a nonterminal class of $\mathcal{A}$. If y is petite, modifying (7.1), yields

$$|S^y \cap A'| \geq \|y, B\| + 1; \tag{7.2}$$

so y is solo. For a subset C of B, let $L'(C)$ denote the set of the petite vertices in C and $H'(C) = C - L'(C)$. By (7.2) and Lemma 26,

$$L'(B) \subseteq L(B). \tag{7.3}$$

Lemma 29 $|L'(B)| \leq a'$.

Proof Suppose $|L'(B)| \geq a' + 1$ and let I be an inclusion maximal independent subset of $L'(B)$. By (7.2), the total number of solo neighbors in A' of vertices in I is at least

$$\sum_{y \in I} (1 + \|y, B\|) \geq |L'(B)| \geq a' + 1.$$

But A' has at most a' unmovable vertices, contradicting Lemma 20. □

If $\mathcal{T}(X) \neq \emptyset$ (i.e., X is not terminal), let $\mathcal{T}'(X)$ be a minimum nonempty subset $\mathcal{S} \subseteq \mathcal{T}(X)$ with no out-neighbors in $\mathcal{A} \smallsetminus (\mathcal{S} + X)$. Choose $X_0' \in \mathcal{A} \smallsetminus \mathcal{A}'$ such that $|\mathcal{T}'(X_0')|$ is minimum, and set $\mathcal{A}''' = \mathcal{T}'(X_0')$. As usual, set $A''' = \bigcup \mathcal{A}'''$, and $a''' = |\mathcal{A}'''|$. By definition, for all $z \in A'''$

$$\|z, A\| \geq a - a''' - 1. \tag{7.4}$$

Lemma 30 *Every $z \in A'''$ satisfies $\|z, B\| \leq \max\{b, 2b + 2 + a''' - a' - \beta\}$, where $\beta = 1$ if every vertex in $N(z) \cap B$ is petite or $\|z, A\| \geq a - a'''$; else $\beta = 0$.*

Proof Let $z \in Z \in \mathcal{A}'''$ and $B_1 = N(z) \cap B$. Suppose the lemma fails for z. Then $|B_1| \geq b+1$ and $|B_1| \geq 2b + 2 + a''' - a' - \beta + 1$. So $B_1 \neq \emptyset$. Also, every $y \in B_1$ is petite: if not

$$d(z) = \|z, A \cup B\| \geq (a - a''' - 1) + (2b + 2 + a''' - a') + 1 = 2k + 2 - a - a';$$

so every $y \in B_1$ is petite since $d(y) \leq \theta(zy) - d(z) \leq a + a' - 1$. By (7.2)

$$|S^y \cap A'| \geq 1 + \|y, B\|. \tag{7.5}$$

So by Lemma 26, $B_1 \subseteq L(B)$ and $|L(B)| \geq b + 1$. By Lemma 28, $a' \leq b + 1$. Let I be a largest independent subset of B_1. Counting the solo edges in $E(A', I)$ as in Lemma 28 and using (7.5) yields the contradiction:

$$a' \geq \sum_{z \in A'} |S_z \cap I| = \sum_{y \in I} |S^y \cap A'| \geq |I| + \|I, B\| \geq |B_1| \geq 2b + 2 + a''' - a' \geq a' + a'''.$$

□

Lemma 31 $a' \leq a''' + 1$.

Proof Suppose $a' \geq a''' + 2$ and let $Z \in \mathcal{A}'''$. Then Lemma 30 implies

$$\|v, B\| \leq \max\{b, 2b+2+a''' - a'\} \leq 2b \quad \text{for all } v \in Z. \tag{7.6}$$

Consider the discharging from B to Z, where each $y \in B$ sends $ch(y) = ||y, Z||^{-1}$ to each of its neighbors in Z. If $v \in Z$ has no solo neighbors in B, then $ch(v) \leq \|v, B\|/2 \leq b$. As Z gets charge $3b+1$, there is a solo edge $zy \in E(Z, B)$ with $z \in Z$ and $ch(z) \geq b+1$. For $i \in [3]$, let $c_i = |\{y \in N(z) \cap B : \|y, Z\| = i\}$. Then $\|Z - z, B\| \geq 2(3b+1-c_1) - c_2$. So $3b+1-c_1-c_2/2 \leq d(z') \leq 2b$ for some $z' \in Z - z$. Thus $c_1 + c_2 \geq b+1+c_2/2$. If $c_2 = 0$ then $|S_z| \geq b+1$, contradicting Lemma 25. Else $c_2 \geq 1$ and $c_1 + c_2 \geq b+2$. By Lemma 25, $\|y, B\| \geq c_1 + c_2 - 1$. Then $d(y) \geq a+b+1$, contradicting Lemma 26. □

Lemma 32 *If $a' = a''' + 1 \leq a - 2$ then $a' = 2$ and $a'' = 1$.*

Proof Suppose $a' = a''' + 1 \leq a - 2$. Then by Proposition 27, $1 \leq a'' < a'$; so $a' \geq 2$. By Lemma 31, $a' \leq a''' + 1$. So it suffices to show $a''' = 1$. As a and a' are invariants of optimal colorings, it suffices to show $a'''(f') = 1$ for some optimal coloring f'.

Let $\mathcal{A}'' = \mathcal{T}(X) \subseteq \mathcal{A}'$. Since $1 \leq a' - 1 = a''' \leq a'' = |\mathcal{T}(X)| \leq a' - 1$, there is exactly one $Z \in \mathcal{A}' - \mathcal{T}(X)$. Let $\mathcal{H}' = \mathcal{H}[\mathcal{A}] - (\mathcal{A}' + X)$. We first prove that

$$\text{for every } W \in V(\mathcal{H}'),\ V^- \text{ is reachable from } W \text{ in } \mathcal{H}'. \tag{7.7}$$

Suppose V^- is unreachable from $W \in V(\mathcal{H}')$ in $\mathcal{H}'$. As $W \notin \mathcal{A}' = \mathcal{T}(X) + Z$, there is a W, V^--path $\mathcal{P}$ in $\mathcal{H}'$ avoiding X. So $\mathcal{P} \cap \mathcal{T}(X) = \emptyset$. Thus $Z \in \mathcal{P}$ and $Z \notin \mathcal{T}(W)$. Similarly, there is a W, V^--path $\mathcal{Q}$ avoiding Z. Thus $\mathcal{Q} \cap (\mathcal{T}(X) + X) \neq \emptyset$, and so $X \in \mathcal{Q}$. Thus $\mathcal{T}(W) \cap \mathcal{T}(X) = \emptyset$. So $\mathcal{A}' \cap \mathcal{T}(W) = \emptyset$, contradicting $W \notin \mathcal{A}'$. This proves (7.7).

Pick a spanning in-tree $\mathcal{F}'$ of $\mathcal{H}'$ that is rooted at V^-, and whose leaf set $\mathcal{L}$ is maximum. Since $\mathcal{L} \cap \mathcal{A}' = \emptyset$, every leaf $L \in \mathcal{L}$ satisfies $\mathcal{T}(L) \cap \{X, Z\} \neq \emptyset$. Also by definition, $\mathcal{T}(L) \cap \mathcal{T}(L') = \emptyset$ for distinct $L, L' \in \mathcal{L}$. So $|\mathcal{L}| \leq 2$. If $|\mathcal{L}| = 2$, then we may assume $\mathcal{L} = \{X', Z'\}$, where $X \in \mathcal{T}(X')$ and $Z \in \mathcal{T}(Z')$. In this case, $a''' \leq \|\mathcal{T}(Z')\| = 1$ and lemma holds. So suppose $\mathcal{L} = \{W\}$. Then $\mathcal{F}'$ is a W, V^--path.

First, suppose $W = V^-$. Then $\mathcal{A} = \{V^-, Z, X\} \cup \mathcal{T}(X)$. If V^- is the only out-neighbor of Z, then $\mathcal{T}'(V^-) = \{Z\}$ and so $a''' = 1$. If Z has an out-neighbor $Z' \in \mathcal{A} - V^-$, then move a witness x of $XV^- \in E(\mathcal{H})$ to V^- to get a new coloring f'. In f', the class $V^- + x$ is terminal, because of Z'. If $Z' \notin \mathcal{T}(X)$ in f or is terminal in f', then $a'(f') > a'(f)$, a contradiction. Else $Z' \in \mathcal{T}(X)$ and is nonterminal in f'. Then $\mathcal{T}(Z') = \{Z\}$, f' is optimal, and $a'''(f') = 1$.

Now suppose $W \neq V^-$. Let W' be the penultimate vertex on the path $W\mathcal{F}'V^-$, and f' be the coloring obtained by moving a witness x of $W'V^-$ to V^-. If each of X and Z has an out-neighbor in $\mathcal{A} \setminus \mathcal{T}(X) - V^-$, then $\mathcal{A}'(f') = \mathcal{A}' + (V^- + x)$, a contradiction. If $X \notin \mathcal{T}(W)$ then $\mathcal{T}(W) = \{Z\}$ and $a''' = 1$. Otherwise $X \in \mathcal{T}(W)$ and Z has an out-neighbor in $\mathcal{T}(X) + V^-$. If $N^+(Z) = \{V^-\}$ then we can take $\mathcal{A}''' = \{Z\} \subseteq \mathcal{T}(V^-)$, and so $a''' = 1$. Else there is $U \in N^+(Z) \cap \mathcal{T}(X)$. Then $\mathcal{A}'(f') = \mathcal{A}' + (V^- + x) - U$ and in f' we can take $\mathcal{A}''' = \mathcal{T}'(U)$; so $a'''(f') = 1$. □

Lemma 33 *Suppose $a'' = 1$, $a' = 2$, $\mathcal{A}'' = \{W\}$ and $\mathcal{A}' = \{W, Z\}$. Then $\mathcal{H}[\mathcal{A}]$ has a W, V^--path $\mathcal{P} = WX_0 \ldots X_s V^-$ and a Z, V^--path $\mathcal{P}' = ZU_0 \ldots U_t V^-$ such that $V(\mathcal{P}) \cup V(\mathcal{P}') = \mathcal{A}$ and $V(\mathcal{P}) \cap V(\mathcal{P}') = \{V^-\}$. Moreover, each of W and Z has exactly one out-neighbor in $\mathcal{H}[\mathcal{A}]$.*

Proof Let $\mathcal{A}'' = \mathcal{T}(X_0)$. Since $Z \notin \mathcal{T}(X_0)$, $X_0 \neq V^-$. Then X_0 is the only out-neighbor of W in $\mathcal{A}$. Since $Z \in \mathcal{A}'$, $\mathcal{H}$ has a shortest W, V^--path $\mathcal{P} = W, X_0, \ldots, X_s = V^-$ avoiding Z. Since $Z \notin \mathcal{T}(X_0)$, $\mathcal{H}$ has a shortest Z, V^--path $\mathcal{P}' = Z, U_0, \ldots, U_t = V^-$

avoiding X_0. Choose such a shortest path with the most common edges with $\mathcal{P}$. If $\mathcal{C} = \mathcal{A} - (V(\mathcal{P}) \cup V(\mathcal{P}')) \neq \emptyset$, then $\mathcal{H}[\mathcal{A}]$ has a spanning in-tree with root V^- and a leaf in $\mathcal{C}$. But any such leaf is in $\mathcal{A}'$, a contradiction. Thus $V(\mathcal{P}) \cup V(\mathcal{P}') = \mathcal{A}$.

Suppose $X_i = U_j \neq V^-$ for some i and j. Then $X_{i+1}\mathcal{P}V^- = U_{j+1}\mathcal{P}'V^-$ by the choice of $\mathcal{P}'$. Then moving a witness from X_{s-1} to $X_s = V^-$, we obtain a coloring with more terminal classes, a contradiction. Thus $V(\mathcal{P}) \cap V(\mathcal{P}') = \{V^-\}$.

Moreover, observe that if $U_0 \neq V^-$ and Z has an out-neighbor $Z' \in \mathcal{A} - U_0$, then $U_0 \in \mathcal{A}'$, a contradiction. □

Lemma 34 $a' = a - 1$.

Proof By Lemmas 31, and 32, if $a' < a-1$, then $a'' = 1$ and $a' = 2$. By Lemma 33, there are $X_0 \in \mathcal{A} - \mathcal{A}' - V^-$, $U_0 \in \mathcal{A} - \mathcal{A}' - X_0$ and a labeling $\{W, Z\} = \mathcal{A}'$ such that $\mathcal{T}(X_0) = \{W\}$ and U_0 is the only outneighbor of Z in $\mathcal{H}[\mathcal{A}]$. In particular, if $U_0 \neq V^-$, then $\mathcal{T}(U_0) = \{Z\}$. Also, there are chordless paths $\mathcal{P} = WX_0 \ldots X_s V^-$ and a $\mathcal{P}' = ZU_0 \ldots U_t V^-$ such that $V(\mathcal{P}) \cup V(\mathcal{P}') = \mathcal{A}$ and $V(\mathcal{P}) \cap V(\mathcal{P}') = \{V^-\}$. Both $\mathcal{A}''' = \{W\}$ and $\mathcal{A}''' = \{Z\}$ work; so Lemma 30 applies to both W and Z. Let $Z = \{z, z', z''\}$, $U_0 \subseteq \{u, u', u''\}$ and $X_0 = \{x, x', x''\}$ with w'', x'', z'' being a witness of WX_0, X_0X_1, $ZU_0 \in E(\mathcal{F})$, respectively. Also if $U_0 = V^-$, then u'' does not exist; otherwise, let u'' be a witness of $U_0U_1 \in E(\mathcal{F})$.

Our first claim is that

$$\text{neither of } X_0 \cup W - x'' \quad \text{and} \quad U_0 \cup Z - u'' \text{ is independent.} \tag{7.8}$$

Indeed, if $X_0 \cup W - x''$ is independent, then $\|y, X_0 \cup W - x''\| \geq 4$ for all $y \in B$, since otherwise we can color equitably $X_0 \cup W - x'' + y$ with two colors, $B - y$ with b colors, and $A - X_0 - W + x''$ with $a-2$ colors. So $\|B, X_0 \cup W - x''\| \geq 4(3b+1) > 5(2b+1)$, and there is $s \in W \cup X_0 - x''$ with $\|s, B\| \geq 2b+2$. Assume $s \in W$ as else s can be swapped with w''. This contradicts Lemma 30.

Similarly, if $U_0 \neq V^-$, then $U_0 \cup Z - u''$ is not independent. Finally suppose $U_0 = V^-$, and $V^- \cup Z$ is independent. Then as above, $\|y, V^- \cup Z\| \leq 4$ for each $y \in B$ and $\|B, V^- \cup Z\| \geq 4(3b+1)$. Since $\|V^-, B\| \leq |V^-| \cdot |B| = 6b+2$, $\|B, Z\| \geq 6b+2$. So there exists $z \in Z$ with $\|z, B\| \geq 2b+1 = 2b+2+a''' - a'$. Then by Lemma 30, there exists some non-petite neighbor of z in B. Since every vertex in B has two neighbors in V^- or three in Z, the non-petite neighbor y of z in B has $d(y) > a + a' = a+2$. But now $d(z) + d(y) > 2b + 1 + a - 2 + a + 2 = 2k+1$, contradicting the degree conditions of G. This yields (7.8).

Each vertex $w^* \in W$ with a neighbor in X_0 is unmovable; by Lemma 21 such w^* is unique and $w^* \neq w''$. Say $w = w^*$. Similarly, let z be unmovable. Then w' and z' are movable to X_0 and U_0. Using (7.8), assume $wx, zu \in E(G)$. As $\|w, A\| = a - a''' = \|z, A\|$, Lemma 30 implies

$$\text{each of } w \text{ and } z \text{ has at most } 2b \text{ neighbors in } B. \tag{7.9}$$

Next we claim

$$\|W, Z\| \geq 4. \tag{7.10}$$

Indeed, as $WZ, ZW \notin E(\mathcal{H})$, if $\|W, Z\| \leq 3$, then $\|W, Z\| = 3$ and these three edges form a matching. In this case, by symmetry, we may assume $N(z') \cap W = \{w'\}$ and $N(w') \cap Z = \{z'\}$. Then switch w' with z'. Since Z and W are leaves in $\mathcal{F}$, V^- is reachable from every class in $\mathcal{A} - W - Z$ in the new coloring f^*. Also, X_0 and U_0 are out-neighbors of $W^* = W - w' + z'$

and so X_0 is a new terminal class in f^*, a contradiction to the maximality of $\mathcal{A}'$. This proves (7.10).

Case 1: Vertex w is not solo. By (7.9), $\|w, B\| \le 2b$ and by Lemma 30, $\|w', B\|, \|w'', B\| \le 2b+1$. As $\|W, B\| \ge 6b+2$, equality holds throughout and $\beta = 0$ in Lemma 30. Thus $\|w', B\|, \|w'', B\| \le a-2 = a-a'''-1$, so $\|\{w', w''\}, Z\| = 2$ and $\|w, Z\| = 2$, by (7.10). So $d(w) \ge 2b+a$. Therefore, since $\theta(G) \le 2k+1$, $\|y, Z\| = 1$ and $\|y, B\| = 0$ for all of the $2b$ vertices $y \in N(w) \cap B$, a contradiction to Lemma 25.

The proof of the case when z is not solo is analogous. So the remaining case is:

Case 2: Both w and z are solo. Then $B_1(w) \ne \emptyset$ and $B_1(z) \ne \emptyset$. Since each $y' \in B_0(w) \cup B_3(w)$ is adjacent to both w' and w'' by Corollary 23, Lemma 30 yields $b_0(w) + b_3(w) \le \|B, w'\| \le 2b+1$. So, $b_1(w) + b_2(w) \ge |B| - (2b+1) = b$. Similarly, $b_1(z) + b_2(z) \ge b$.

Case 2.1: $b_1(w)+b_2(w) \ge b+1$. By Lemma 25, $B_1(w) \cup B_2(w) \subseteq N[y]$ for each $y \in B_1(w)$. Fix $y \in B_1(w)$. By Lemma 26, $y \in L(B)$. So $k \le a + (b_1(w) + b_2(w) - 1) \le d(y) \le k$. Thus $b_0(w) + b_3(w) = \|w^*, B_0(w) + B_3(w)\| \ge 2b$ for both $w^* \in W - w$. Since $G[B]$ is b-colorable, there are $y_1, y_2 \in B_1(w) \cup B_2(w)$ with $y_1 y_2 \notin E(G)$. Then $y_1, y_2 \in B_2(w)$. Applying Lemma 25 to Z, yields $i \in [2]$ with $\|y_i, Z\| \ge 2$, and there is $w^* \in N(y_i) \cap W - w$. So $d(y_i) \ge (a+2)+1$, $d(w^*) \ge (a-2)+(2b+1)$, and $\theta(w^* y_i) \ge 2k+2$, a contradiction.

The proof of the case $b_1(z)+b_2(z) \ge b+1$ is exactly the same. So, since $b_1(w)+b_2(w) \ge b$ and $b_1(z) + b_2(z) \ge b$, the last subcase is:

Case 2.2: $b_1(w)+b_2(w) = b = b_1(z)+b_2(z)$. Then $b_0(w)+b_3(w) = 2b+1 = b_0(z)+b_3(z)$. Let $y \in (B_0(w) \cup B_3(w)) \setminus (B_1(z) \cup B_2(z))$. For both $w^* \in W - w$

$$2k+1 \ge \theta(w^* y) \ge \left\|w^*, A\right\| + \|y, A\| + 2b + 1 \ge (a-2) + (a+2) + 2b + 1 = 2k+1.$$

So all three inequalities in the chain are sharp. In particular, $\|w^*, Z\| = 1$ and $\|y, W \cup Z\| = 4$. Thus $y \in B_0(w) \cap B_0(z)$. Similarly, $\|z^*, W\| = 1$ for both $z^* \in Z - z$. If $z^* w^* \in E(G)$, then as in the proof of (7.10), switching w^* with z^* yields a coloring with more terminal classes, a contradiction. Thus, $\{wz', wz'', zw', zw''\} \subset E(G)$. As w and z are solo, they are unmovable. So $\|w, V - z\| \ge k$ and $\|z, V - w\| \ge k$. Thus $wz \notin E(G)$. Finally, obtain an equitable k-coloring by combining an equitable b-coloring of $B - y$ with $\{y, z, w\}$, $\{w', z', z''\}$, and shifting witnesses along $\mathcal{P}$ (starting from w''). $\square$

8 Properties of the set of solo vertices

Let f be an optimal coloring. Let $S_f \subseteq E$ be the set of solo edges xy with $x \in A'$ and $y \in B$. For any $W \subseteq V$, let $S_f(W)$ be the set of the *solo* vertices in W, i.e., vertices in W incident to a solo edge, and let $T_f(W) = W \setminus S(W)$. We will normally drop the subscript when the coloring is clear from the context. For every $x \in X \in A$ and $i \in \{0, 1, 2, 3\}$ let $B_i(x) = \{y \in B : \|y, X\| = i\}$ and $b_i(x) = |B_i(x)|$. Call a vertex x *free* if $\|x, A\| = 0$. For easier reference, we restate several important lemmas using this new notation.

(A.1) $a = a' + 1$ (Lemma 34);
(A.2) for every $y \in B$, $G[B - y]$ has an equitable b-coloring (Lemma 22);
(A.3) every $x \in S(A')$ is unmovable (Lemmas 20 and 22);
(A.4) for every $X = \{x, x', x''\} \in \mathcal{A}'$ with $b_1(x) > 0$, $B_0(x) \cup B_3(x) \subseteq N(x') \cap N(x'')$ (Corollary 23);
(A.5) for every $x \in S(A')$, $B_1(x)$ is a clique (Theorem 25);

(A.6) for every $x \in S(A')$, $y \in B_1(x)$ and $y' \in B_2(x)$, $yy' \in E$ (Theorem 25);
(A.7) every color class of $\mathcal{A}'$ contains at most one unmovable vertex (Lemma 21); and
(A.8) for every $y \in S(B)$, $d(y) \leq k$ (Lemma 26).

Proposition 35 *There are at most $b+1$ vertices y in B such that $d(y) < 2a-1$.*

Proof If $b \geq a'$, then using (A.1), $2a-1 = a+a' \leq k$. By Lemma 28, $|L'(B)| \leq a' \leq b$. By the definition of $L'(B)$, there are at most b vertices y in B with $d(y) \leq a+a'-1 = 2a-2$.

If $b \leq a'-1$, then by Lemma 29, $|L(B)| \leq b+1$. Also, $2a-1 \geq k+1$, so there exist at most $b+1$ vertices y in B with $d(y) < 2a-1$. □

Proposition 36 (a) *If $x \in A'$ and $\|x, B\| \geq 2b+1$, then $d(x) \leq 2b+2$.*
(b) *If $x \in S(A')$, then $b-1 \leq b_1(x) + b_2(x) \leq b+1$.*

Proof If $\|x, B\| \geq 2b+1 > b+1$ then, by Proposition 35, there exists $y \in N(x) \cap B$ such that $d(y) \geq 2a-1$. Together with $\theta(xy) \leq 2k+1$, this yields $d(x) \leq 2b+2$, proving (a).

Suppose $x \in S(A')$, where $x \in X \in \mathcal{A}'$, and $y \in B_1(x)$. By (a) and (A.4), $b_0(x)+b_3(x) \leq 2b+2$, so $b_1(x)+b_2(x) \geq b-1$. Finally, (A.5), (A.6) and (A.8), yield

$$b_1(x) + b_2(x) - 1 \leq \|y, B\| = d(y) - \|y, A\| \leq k - a = b.$$

□

Proposition 37 *Let $x \in X \in \mathcal{A}'$, $z \in A - X$, $xz \notin E$, and $A^* = A - x - z$. Then either*

1. *$N(z) \cup N(x) \supseteq B$, or*
2. *there is no equitable $(a-1)$-coloring of $G[A^*]$.*

In particular, if $\|X - x, A^*\| \leq 1$ and $z \notin V^-$, then (1) holds.

Proof If $xz \notin E$ and there exists $y \in B \smallsetminus (N(z) \cup N(x))$, then the class $\{x, z, y\}$ together with an equitable $(a-1)$-coloring of $G[A^*]$ and an equitable b-coloring of $G[B-y]$ (which exists by (A.2)) give an equitable coloring of G. For the second part, note that if $\|X - x, A^*\| \leq 1$ and $z \notin V^-$, then $G[A^*]$ has an equitable $(a-1)$-coloring. □

Lemma 38 *Let $X = \{x, x', x''\} \in \mathcal{A}'$ with $x \in S(X)$. Then $B_2(x) \subseteq S(B)$, and $N(x'), N(x'') \supseteq T(B)$.*

Proof The second part of the lemma follows from the first part and (A.4). For the first part, let $xy \in S$; then $y \in B_1(x)$. By (A.6), $B_2(x) \subseteq N(y)$. By (A.3), (A.8), and (A.5)

$$d(x) \geq a - 1 + b_1(x) + b_2(x) \quad \text{and} \quad k \geq d(y) \geq a + b_1(x) + b_2(x) - 1. \tag{8.1}$$

Assume the lemma fails, and pick $y' \in T(B) \cap B_2(x)$. As $\left\|y', U\right\| \geq 2$, (A.7) implies there is $u' \in N(y') \cap U \cap M$ for each class $U \in \mathcal{A}'$. Let $U = \{u, u', u''\}$, where $u \in \overline{M}$; set $M' = \{u' : U \in \mathcal{A}'\}$, $M'' = \{u'' : U \in \mathcal{A}'\}$ and $V^- = \{v, v'\}$, where $v \in \overline{M}$. By (A.4), $B_0(x) \cup B_3(x) + y' \subseteq N(x')$. By (A.6), $B_1(x) \subseteq N(y')$. Using Proposition 36,

$$d(x') \geq 3b + 1 - b_1(x) - b_2(x) + 1 \geq 2b + 1 \quad \text{and} \quad d(y') \geq 2a - 1 + b_1(x) \geq 2a.$$

Thus $\theta(x'y') = 2k+1$, $b_1(x) = 1$, $b_2(x) = b$, $b_0(x) + b_3(x) = 2b$ and $N(y') \cap B = \{y\}$. By (8.1), $\|y, A\| = a$ and $\|y, B\| = b$. By (A.3), $uy \in S$ for all $U \in \mathcal{A}'$. By (A.5) and (A.6),

$$N[y] = \overline{M} \cup B_1 \cup B_2(x). \tag{8.2}$$

We will obtain a contradiction to Lemma 15 by proving $\overline{M} \cup B_2(x) - y' + y$ is a k-clique. Consider any $U \in \mathcal{A}'$. Since $uy \in S$ and $u'y' \in E$, we have $B_1(u) \cup B_2(u) \subseteq N[y] \cap B = B_1(x) \cup B_2(x)$, $B_0(u) \cup B_3(u) \supseteq B_0(x) \cup B_3(x)$, $\|u', B\| \geq 2b + 1$, $\theta(u'y') = 2k + 1$, $N(u') \cap B_2(x) - y' = \emptyset$ and u' is free. Using (A.4)

$$B_2(x) - y' + y \subseteq B_1(u) \cup B_2(u) \subseteq N(u) \quad \text{for all } U \in \mathcal{A}'. \tag{8.3}$$

Consider any $y'' \in B_2(x) - y'$. As $y'y'' \notin E$, $y' \notin B_2(u)$ or $y'' \notin B_1(u)$. Anyway, $N(u'') \cap \{y', y''\} \neq \emptyset$. So $\|u'', B\| \geq b_0(x) + b_3(x) + 1 \geq 2b + 1$, and $\|u'', A\| \leq 1$. Consider $w \in \overline{M} \smallsetminus \{u, v\}$. By Proposition 37 and $\|\{u', u''\}, A\}\| \leq 1$, $uw \in E$. Thus

$$\overline{M} - v \quad \text{is a clique.} \tag{8.4}$$

If $vy'' \notin E$ then moving v' to some class $W \in \mathcal{A}'$, moving w' and y'' to the class of v, and equitably b-coloring $B - y''$ yields an equitable k-coloring. Thus $vy'' \in E$. Suppose $uv \notin E$. If $vu'' \in E$ then switch v with u''; else switch v with u'. Moving y to the class of v' and equitably b-coloring $G[B] - y$ yields an equitable k-coloring. So $uv \in E$ and

$$\overline{M} \cup B_2(x) \smallsetminus \{v, y'\} \subseteq N(v). \tag{8.5}$$

If there is $y'' \in B_2(x) \cap T(B) - y'$ then y'' plays the same role as y'. Thus $B_2(x) = \{y', y''\}$ and $B_2(x) - y'$ is a 1-clique. Otherwise for every $y'' \in B_2(x) \cap T(B) - y'$ there is $U \in \mathcal{A}'$ with $y'' \in B_1(u)$. By (8.3), any other $y^* \in B_2(x) - y'$ satisfies $y^* \in B_1(u) \cup B_2(u)$, so $y''y^*$. Thus $B_2(x) - y'$ is a k-clique. Combining this with (8.2), (8.3), (8.4), and (8.5) yields that $\overline{M} \cup B_2(x) - y' + y$ is a k-clique. □

Lemma 39 *$S(B)$ is a clique.*

Proof Suppose $y, y' \in S(B)$ and $yy' \notin E$. Then there are $X = \{x, x', x''\} \in \mathcal{A}'$ and $Z = \{z, z', z''\} \in \mathcal{A}'$ with $yx, y'z \in S$. As $yy' \notin E$, (A.5), (A.6), and (A.4) imply $X \neq Z$, $a' \geq 2$, $y' \in B_0(x) \cup B_3(x)$, $y \in B_0(z) \cup B_3(z)$, $N(y) \supseteq \{z', z''\}$ and $N(y') \supseteq \{x', x''\}$. Using Proposition 36(b), assume $b_1(z) + b_2(z) \geq b_1(x) + b_2(x) \geq b - 1$. Let $V^- = \{v, v'\}$.

Case 1: $b_1(x) + b_2(x) = b - 1$. Since $b_1(x) \geq 1$, $b \geq 2$. By Proposition 36(a), $N(x') = N(x'') = B_0(x) \cup B_3(x)$, so $d(x'), d(x'') = b_0(x) + b_3(x) = 2b + 2$, and x' and x'' are free.
Subcase 1.a: $b_1(z) + b_2(z) = b - 1$. Then $d(z'), d(z'') = b_0(z) + b_3(z) = 2b + 2$ and both z' and z'' are free. As $xz', y'z' \notin E$, Proposition 37 implies $xy' \in E$. Similarly, $zy \in E$. Since $d(y), d(y') \leq k$ by (A.8), and since $\|y, A\|, \|y', A\| \geq a + 2$ and $\|y, B\|, \|y', B\| \geq b - 2$, both y and y' have solo neighbors in every class in $\mathcal{A}' - X - Z$. As $yy' \notin E$, (A.5) implies $\mathcal{A}' = \{X, Z\}$ and $a = 3$. Since $z'y \in E$ and $\theta(z'y) \geq 2b + 2 + a + b = a + 3b + 2$, $b \leq a - 1 = 2$. So $b = 2$, $S(B) = \{y, y'\}$ and $T(B) = \{y_1, \ldots, y_5\}$. By Lemma 38, $N(y_i) \supseteq \{x', x'', z', z'', v_i\}$ for some $v_i \in V^-$ for all $i \in [5]$. By (H2), B is independent and both x and z have no neighbors in $T(B)$. Now $\{\{v, x', x''\}, \{v', z', z''\}, \{x, y_1, y_2\}, \{z, y_3, y_4\}, \{y, y', y_5\}\}$ is an equitable 5-coloring.
Subcase 1.b: $b_1(z) + b_2(z) \geq b$. Then $\|z', B\| \geq 2b + 1$. By (A.8), (A.5) and (A.6),

$$k \geq d(y') = \|y', B \cup (A \smallsetminus X) \cup \{x', x''\} + x\| \geq b - 1 + a - 1 + 2 + 0 \geq k,$$

$\|y', A \smallsetminus X\| = a - 1$, $\|y', U\| = 1$ for all $U \in \mathcal{A} - X$, and $xy' \notin E$. Say $vy' \in E$. Consider any class $U = \{u, u', u''\} \in \mathcal{A}' - X$ with $uy' \in S$. As x' and x'' are free and $v'y', u'y', u''y' \notin E$, Proposition 37 implies $v'x, u'x, u''x \in E$. Also $v^*x \in E$ for both

$v^* \in \{u, v\}$: else moving y' to the class of v^*, v^* to X, and x' to V^-, and equitably b-coloring $B - y'$ contradicts (3.1). As $k \geq 5$, this gives the contradiction

$$\theta(xz') = \|x, A \cup B\| + d(z') \geq 3(a-2) + 2 + b - 1 + 2b + 2 \geq 3k - 3 \geq 2k + 2.$$

Case 2: $b_1(x) + b_2(x) \geq b$. By (A.8), (A.5) and (A.6), $\|y, B\| = b - 1 = \|y', B\|$ and $\|y, A\| = a + 1 = \|y', A\|$. Thus $\|y, U\| = 1$ for all $U \in \mathcal{A} - X$ and $\left\|y', U\right\| = 1$ for all $U \in \mathcal{A} - Z$. As $yy' \notin E$, (A.1) and (A.5) imply $\mathcal{A}' = \{V^-, X, Z\}$. Also $b_1(x) + b_2(x) = b_1(z) + b_2(z) = b$ and $b_0(x) + b_3(x) = b_0(z) + b_3(z) = 2b + 1$. By Proposition 36(a), $\|u, A\| \leq 1$ for all $u \in \{x', x'', z', z''\}$. Also $y \in B_0(z)$ and $y' \in B_0(x)$.

Suppose $x'z' \in E$. Then $\left\|Z - z, X - x'\right\| \leq 1$ and $y \notin N(x') \cup N(z)$. By Proposition 37, $x'z \in E$. Thus $\left\|x', A\right\| \geq 2$, a contradiction. By similar arguments, $\|X - x, Z - z\| = 0$.

Suppose $\|x, Z - z\| \leq 1$. Then $\|X, Z - z\| \leq 1$. Again Proposition 37 implies $zx' \in E$. Similarly, $zx'' \in E$. Thus $\|\{x, Z - z\| = 2$ or $\|z, X - x\| = 2$. Say $\|z, X - x\| = 2$. Then $\|z, A\| \geq a$. By (A.3), x and z are unmovable. Say $vz \in E$.

Suppose $xz \notin E$. Then x has a movable neighbor (say) z' in Z. By Lemma 20, $\|x, A\| \geq a$. By Proposition 37, $B \subseteq N(x) \cup N(z)$. By Proposition 35, there is $w \in B$ with $d(w) \geq 2a - 1$. Let $u' \in \{x', z'\}$, where $u' = z'$ if and only if $zw \in E$. Then

$$4k + 2 \geq d(x) + d(z) + d(w) + d(u') \geq 2a + 3b + 1 + 2a - 1 + 2b + 2 \geq 4k + b + 2,$$

a contradiction. Thus $xz \in E$ and $\|z, X\| = 3$; as $d(y') = k$ and $\|z, B\| \geq b$, we have $d(z) = k + 1$, $\left\|z, V^-\right\| = 1$, $v'z \notin E$, $d(x) \leq k$ and $\|x, Z - z\| \leq 1$. By (C2), $v'z^* \notin E$ for some $z^* \in Z - z$; say $z^* = z'$. Then $\{v', z, z'\}$ and $\{v, x', x''\}$ are independent and $xy', z''y' \notin E$, so $xz'' \in E$ by Proposition 37. Now $v'z'' \notin E$. Switching z' and z'', yields $xz' \in E$ and $d(x) = k + 1$, contradicting (3.1) □

Lemma 40 *Every $x \in S(A')$ satisfies $b_1(x) + b_2(x) = b$.*

Proof Suppose the lemma fails for some $x \in \{x, x', x''\} = X \in \mathcal{A}'$ with $x \in S(A')$. By Lemma 38, $S(B) \supseteq B_1(x) \cup B_2(x)$, $S(B)$ is a clique, and by Lemma 39, $b_1(x) + b_2(x) \leq \chi(G[B]) \leq b$. Using Proposition 36, this implies $1 \leq b_1(x) + b_2(x) = b - 1$, $N(x') = N(x'') = B_0(x) \cup B_3(x)$, x' and x'' are free, and $b_2(x) = 0$.

Suppose there exists $y \in B_3(x)$. If $y \in T(B)$, then $\|y, A\| \geq 2a$, but the fact that $yx' \in E$ and $d(x') \geq 2b+2$ contradicts $\theta(G) \leq 2k+1$. Otherwise, $y \in S(B)$, $|S(B)| \geq b-1+1 = b$, and since $S(B)$ is a clique, $\|y, B\| \geq b - 1$. Since $\|y, A\| \geq a + 2$, $d(y) \geq a + b + 1$, contradicting (A.8). So $b_3(x) = 0$ and $b_0(x) = 2b + 2$. As $T(B) \subseteq N(x')$, $\|w, V^-\| = 1$, $\|w, U\| = 2$ and $\|w, B\| = 0$ for every $w \in T(B)$ and $U \in \mathcal{A}'$.

Suppose $|S(B)| = b - 1$. Then $|T(B)| = 2b + 2$ and $G[B] = K_{b-1} + \overline{K_{2b+2}}$. There exist distinct $y_1, y_2, y_3, y_4 \in T(B)$ and $v, v' \in V^-$ with $vy_1, vy_2 \in E$. Then $\{v', y_1, y_2\}$, $\{v, x', x''\}$ and $\{x, y_3, y_4\}$ are independent sets. Then $B \setminus \{y_1, y_2, y_3, y_4\}$ admits an equitable $(b-1)$-coloring, a contradiction. So $|S(B)| = b$.

Now there exist $y' \in S(B) \setminus B_1(x)$ and $Z = \{z, z', z''\} \in \mathcal{A}' - X$ with $zy' \in S$. Recall that $S(B)$ is a clique, $N(y') \supseteq \{x', x''\}$ and $d(y') \leq k$, so $N(y') \cap B = S(B) - y'$ and $\|y', A\| = a + 1$. Let $\{y_1, y_2, y_3\} \subseteq T(B)$ be a 3-set. Then $\|y_i, Z\| = 2$ for every $i \in [3]$. Since $B_2(z) \subseteq S(B)$, $zy_i \notin E$. So $\{x, y', y_1\}$, $\{z, y_2, y_3\}$, $V^- + x'$ and $\{z', z'', x''\}$ are independent. As $G[B] - \{y', y_1, y_2, y_3\}$ admits an equitable $(b-1)$-coloring, this completes the contradiction. □

Corollary 41 *Suppose $X = \{x, x', x''\} \in \mathcal{A}'$. If x is solo, then $G[S(B)] = K_b$, $N(x) \supseteq S(B)$, $N(x'), N(x'') \supseteq T(B)$ and $\|x', A\|, \|x'', A\| \leq 1$.*

9 Finding a clique on k vertices

Lemma 42 *If $W \subseteq A$ is a 5-set and there is an equitable $(a-2)$-coloring of $G[A \setminus W]$, then $G[W]$ contains an edge. In particular, $\|X, V^-\| \geq 1$ for every $X \in \mathcal{A}'$.*

Proof Suppose $W \subseteq A$ is an independent 5-set and that there is an equitable $(a-2)$-coloring of $G[A \setminus W]$. For every $y \in B$, $G[W+y]$ has no equitable 2-coloring by (A.2), which implies $\|y, W\| \geq 4$. So there exist distinct $w, w' \in W$ such that

$$\|\{w, w'\}, B\| \geq \lceil 8|B|/5 \rceil = 4b + 1 + \lceil (4b+3)/5 \rceil \geq 4b + 3.$$

Therefore using Proposition 36(a), we can assume that $\|w, B\| = 2b+2$, $\|w', B\| \geq 2b+1$ and $b \leq 2$. Proposition 36(a) further implies that $d(w), d(w') \leq 2b+2$, and $\|\{w, w'\}, A\| \leq 1$, and there is an optimal coloring f' such that $V^-(f') = \{w, w'\}$ and $X = W - \{w, w'\} \in \mathcal{A}'(f')$. Furthermore, since $|S_{f'}(B)| \leq b$ by Lemma 39, there exists $y \in T_{f'}(B)$ such that $wy \in E$, so $d(y) = 2a - 1$. Since $y \in T_{f'}(B)$, $\|y, Z\| \geq 2$ for every $Z \in \mathcal{A}'(f') - X$, so $4 + 2(a-2) \leq d(y) = 2a - 1$, a contradiction. □

Lemma 43 $a \geq 3$.

Proof Suppose $a = 2$. Since $k \geq 4$, $b \geq 2$. Let $\{v, v'\} = V^-$ and $\{x, x', x''\} = X \in \mathcal{A}'$. By Lemma 42, we can assume that $xv \in E$. Since X has at most one unmovable vertex, every edge in $E(G[A])$ is incident to x. We know that $xv' \notin E$, for otherwise, $\{x, x'\}, \{v, v', x''\}$ are both independent sets and both v and v' are unmovable in the new coloring. Therefore, $E(G[A]) = \{xv\}$. Let $X' = \{v', x', x''\}$. For any $y \in B$, there is no equitable 2-coloring of $G[A+y]$ by (A.2). Hence, either $N(y) \supseteq \{x, v\}$ or $N(y) \supseteq X'$.

Suppose there exists $w \in X'$ with $d(w) \geq 2b+2$. By the degree-sum condition, every vertex in $N(w)$ has degree precisely three, with neighborhood X' or $\{w, x, v\}$ and $d(w) = 2b+2$. Note that $|B - N(w)| = b - 1$. Since one of $\{x, v\}$ is low, and both are adjacent to every vertex in $B - N(w)$, there are at most two vertices in $N(w)$ whose neighborhood is $\{w, x, v\}$, so there are at least $(2b+2) - 2 \geq 4$ vertices in $N(w)$ whose neighborhood is X'. Let $\{y_1, \ldots, y_4\}$ be four such vertices. Now $\{x, y_1, y_2\}$, $\{v, y_3, y_4\}$ and X' are independent sets and we can equitably $(b-1)$-color, $B - \{y_1, \ldots, y_4\}$ by pairing each of the $b-1$ vertices of $B - N(w)$ with two vertices in $N(w) - \{y_1, \ldots, y_4\}$. Then every vertex in X' has degree at most $2b+1$.

Suppose there exists $y \in S(B)$. Since x is not movable, $xy \in S$ and, by Corollary 41, $G[S(B)] = K_b$. Since y is not adjacent to x', y must be adjacent to v. So $S(B) \cup \{x, v\}$ is a clique, which contradicts the fact that $\omega(G) \leq k-1$. Therefore, for every $y \in B$, $y \in T(B)$ and $|N(y) \cap \{x', x''\}| \geq 1$. Let $Y' = \{y \in B : N(y) \supseteq \{x, v\}\}$. Since $d(x) + d(v) \leq 2b+5$, $|Y'| \leq b+1$. For every vertex $y' \in B - Y'$, $N(y') \supseteq X'$. Therefore, we have the following contradiction

$$4b+3 \leq 5b+1 \leq 2|B| - |Y'| \leq \|\{x', x''\}, B\| \leq 4b+2.$$

□

Let $T'(B) := \{y \in T(B) : d(y) \geq 2a\}$.

Lemma 44 *If there exists an edge $xy \in G[T'(B)]$, then $a \leq b$.*

Proof Since $x, y \in T'(B)$, $4a \leq d(x) + d(y) \leq 2a + 2b + 1$ and the conclusion follows. □

Lemma 45 *Suppose that* $\{x, x', x''\} = X \in \mathcal{A}'$ *has no solo vertex. The following statements are true*:

(a) $\|X, A\| \geq 2$ *and if every vertex in* X *is movable, then* $\|X, A\| \geq 3$.
(b) *For every* $y \in B$, $|N(y) \cap X| = 2$, *so* $\{B_0(x), B_0(x'), B_0(x'')\}$ *is a partition of* B.
(c) *There are no edges with one endpoint in* $S(B)$ *and one endpoint in* $T(B)$.
(d) *There exists* $x^* \in X$ *with* $N(x^*) \cap T'(B) = \emptyset$.
(e) *For every* $y \in T(B)$, $d(y) \leq 2a$.
(f) *If* $G[T(B)]$ *contains an edge, then* $3 = a \leq b$, *and* X *contains an unmovable vertex.*

Proof We will first show that $\|X, A\| \geq 2$. If X has an unmovable vertex x, then this is clear, because in this case $\|x, A\| \geq a - 1 \geq 2$. Now suppose that every vertex in X is movable. Move the witness w_X along a path in $\mathcal{H}$ to V^-. The new coloring is optimal, since otherwise there is a class $Z \neq V^- + w_X$ in which all 3 vertices are adjacent to $X - w_X$, as claimed. Thus by Lemma 42 for the new coloring, each of the classes has a neighbor in $X - w_X$. So $\|X, A\| \geq \|X - w_X, A\| \geq a - 1 \geq 2$ with equality only if $a = 3$ and w_X is free. In this case, we can assume each class $V^- + w_X$ and $Z \in \mathcal{A}' - X$ has exactly one neighbor in $X - w_X$. Since each vertex in X is movable in the original coloring, these two neighbors are distinct. Then taking the neighbor of Z in X as w_X yields (a).

Assume that $\|x', B\| \leq \|x'', B\|$. Let $y \in B$. If $y \in S(B)$, then since $\|y, X\| \geq 2$, $d(y) \leq a + b$, $S(B)$ is a clique and $|S(B)| \geq b$, $N(y) \cap B = S(B) - y$, $\|y, T(B)\| = 0$, $\|y, A\| = a + 1$ and $\|y, X\| = 2$. If $y \in T(B)$, then $\|y, X\| = 3$ implies $d(y) \geq 2a$ and $d(x), d(x'), d(x'') \leq 2b + 1$, so $\|X, A\| \leq d(x) + d(x') + d(x'') - (2|B| + 1) = 0$, a contradiction to (a). This proves (b) and (c).

We will now prove (d) and (e). Let $y \in T'(B)$. By (b), we can label so that $x, x' \in N(y)$ and $x'' \notin N(y)$. If $d(y) = 2a$, suppose, for a contradiction to (d), that there exists $y' \in T'(B) \cap N(x'')$. If $d(y) \geq 2a+1$, then $\|x, B\|, \|x', B\| \leq 2b$, so $\|x'', B\| \geq 2b+2$, so we can let $y' \in N(x'') \cap T(B)$. In either case, $2d(y) + d(y') \geq 6a$, so $d(x) + d(x') + d(x'') \leq 6b+3$, since $\|y, X\| = 2$. This implies that $\|X, A\| \leq 1$, which is a contradiction to (a).

Suppose there exists $yy' \in E(G[T(B)])$, so $y, y' \in T'(B)$. By (d), there exists $x'' \in X$ such that $N(x'') \cap T'(B) = \emptyset$, so $d(x), d(x') \leq 2b+1$ and $\|x'', B\| \geq 2b$. Since $|S(B)| \leq b$, x'' is adjacent to a vertex in $T(B)$, so $d(x'') \leq 2b + 2$, hence

$$\|X, A\| \leq d(x) + d(x') + d(x'') - (6b + 2) \leq 2.$$

So $\|X, A\| = 2$ and there exists an unmovable vertex $x \in X$. Since $a' \leq \|x, A\| \leq \|X, A\| = 2$ and $a \geq 3$, it must be that $a = 3$. By Lemma 44, $b \geq a$ which proves (f). □

Lemma 46 *If there exists* $y \in T(B)$ *such that* $N(y) \supseteq V^-$, *then* $a \leq b + 1$. *In particular, if* $b = 1$, *then for every* $y \in T(B)$, $\|y, V^-\| = 1$.

Proof Suppose $V^- = \{v, v'\}$ and some $y \in T(B)$ is adjacent to both v and v' and that $a \geq b+2$. Since $\|V^-, B\| \geq 3b+2$, Lemma 42 implies that $d(v) + d(v') \geq (3b+2) + (a - 1) \geq 4b+3$. So we may assume $d(v) \geq 2b+2$. But $d(y) \geq 2a$ and so $d(y) + d(v) \geq 2k+2$. □

Lemma 47 $|S(B)| = b$.

Proof By Corollary 41, we are done when $S(B) \neq \emptyset$. So assume $S(B) = \emptyset$, which means that no class in $\mathcal{A}'$ contains a solo vertex. Let $X = \{x, x', x''\} \in \mathcal{A}'$. Let $x \in X$ be the unmovable vertex in X, if it exists and, in this case, assume $\|x', B\| \leq \|x'', B\|$. Otherwise, every vertex in X is movable and we label them so that $\|x, B\| \le \|x', B\| \le \|x'', B\|$. In either case $\|x, B\| \le 2b$, so $|B_0(x)| \ge b + 1$.

Suppose there exists an edge $yy' \in E(G[B])$. Since $B = T(B)$, Lemma 45(f) implies that $a = 3 \le b$ and x is unmovable. By Lemma 45(d), there exists x^* such that $N(x^*) \cap T'(B) = \emptyset$. If $x = x^*$, then $\{x', x''\} \subseteq N(y)$, and $\|x, B\| = 2b$, $\|x', B\| = \|x'', B\| = 2b + 1$ and both x' and x'' are free. If $x^* \ne x$, then $\|x, B\| \le 2b + 1 - (a - 1) = 2b - 1$, so it must be that $\|x', B\| = 2b + 1$, $\|x'', B\| = 2b + 2$, both x' and x'' are free and $x^* = x''$. In either case, $|B_0(x)|, |B_0(x')| \ge 3$, so there exist $y_1, y_2 \in B_0(x)$ and $y_3, y_4 \in B_0(x')$ such that $\{x, y_1, y_2\}$ and $\{x', y_3, y_4\}$ are independent 3-sets. Graph $G[B - y_1 - y_2 - y_3 - y_4]$ has an equitable $(b - 1)$-coloring, since $\Delta(G[B]) \le 1$ and $b \ge 3$. The independent set $V^- + x''$ completes an equitable coloring of G. So assume that B is an independent set.

We can assume $b_0(x') \le 1$, for otherwise, as in the previous case, we can form two independent 3-sets that contain x and x' and 4 vertices from B; an equitable $(a - 1)$-coloring of $A - X + x''$ (move x'' and switch witnesses); and an equitable $(b - 1)$-coloring of the remaining vertices in B. By the same reasoning, we can assume $b_0(x'') \le 1$. Then $\|x'', B\| \ge 3b$, so $3b \le 2b + 2$, hence $b \le 2$.

Suppose $b = 2$. Then $\|x', B\|, \|x'', B\| \le 2b + 2 = 6$, so $b_0(x') = b_0(x'') = 1$. Say $B_0(x') = \{y'\}$ and $B_0(x'') = \{y''\}$. Then $\|x', B\| = \|x'', B\| = 2b + 2$, so x' and x'' are free, and $\|x, B\| = 2$. Since $d(x') = 2b + 2$, $d(y') = d(y'') = 2a - 1$ and y' and y'' each have precisely one neighbor $v_{y'}$ resp. $v_{y''}$ in V^-. If $v_{y'} = v_{y''}$, we color $\{v_{y'}, x', x''\}$ and $V - v_{y'} + \{y', y''\}$. If $v_{y'} \ne v_{y''}$, we color $\{v_{y'}, x'', y''\}$ and $\{v_{y''}, x', y'\}$. In either case, we then color x with two non-neighbors in B, and the remaining uncolored vertices in B are an independent triple. Then we can assume $b = 1$.

Suppose $b = 1$ and $b_0(x') = b_0(x'') = 0$. Then $\|x', B\| = \|x'', B\| = 4 = 2b + 2$, so by Proposition 36(a), x' and x'' are free. Also, $\|x, B\| = 0$. By Lemma 46, every vertex in B has precisely one neighbor in V^-, so we can choose $y', y'' \in B$ that are both nonadjacent to some $v \in V^-$. Since $b_0(x) \ge b + 1 = 2$, we color $B - y' - y'' + x$, $\{v, y', y''\}$, and $V^- - v + X - x$. Together with the remaining color classes $\mathcal{A} - V^- - X$, this is an equitable k-coloring of G.

Now we can assume $b = 1$ and $b_0(x') = 1$, so $\|x', B\| = 3$. Since every vertex of B has degree at least $2a - 1$, $d(x), d(x'') \le 2b + 2 = 4$. Since $\|X, B\| = 2|B| = 8$, $\|\{x, x''\}, B\| = 5$, so $\|\{x, x''\}, A\| \le 3$. Since x' is movable, there exists an equitable coloring of $A - \{x, x''\}$; combine this with $\{x, x''\}$ and B to form a nearly-equitable coloring f' of G. In f', $\{x, x''\}$ is the small class, and $\|\{x, x''\}, B\| = 5$, so some vertex of B has two neighbors in the small class of f'. By Lemma 46, f' is not optimal. If $\{x, x''\}$ have only two neighbors to every class of $f' - B - \{x, x''\}$, then f' is optimal, so there exists a class Z of $f' - B - \{x, x''\}$ with $\|\{x, x''\}, Z\| = 3$. Since $\|\{x, x''\}, A\| \le 3$ and $a \ge 3$, there exists a class W in f', distinct from Z, $\{x, x''\}$, and B, such that $\|\{x, x''\}, W\| = 0$. This violates Lemma 42. □

Lemma 48 *Every color class in $\mathcal{A}'$ has an unmovable vertex.*

Proof Suppose that $\{x, x', x''\} = X \in \mathcal{A}'$ has no unmovable vertex (and so also no solo vertex). By Lemma 47, there exists $y \in S(B)$, so there exists $\{z, z', z''\} = Z \in \mathcal{A}'$ such that $yz \in S$. Assume that $\|x, B\| \le \|x', B\| \le \|x'', B\|$. This implies that $\|\{x', x''\}, B\| \ge 4b+2$. Since $\|x'', B\| \le 2b+2$, $\|x', B\| \ge 2b > b$, so both x' and x'' are adjacent to some $y' \in T(B)$. Hence $d(x'), d(x'') \le 2b + 2$ and $\|\{x', x''\}, A\| \le 2$. Since there is an equitable $(a - 1)$-coloring of $A - X + x$ and $a \ge 3$, Lemma 42 implies that $\|\{x', x''\}, A\| = 2 = a - 1$, so $d(x') = d(x'') = 2b + 2$, $\|\{x', x''\}, B\| = 4b + 2$ and $\|x, B\| = 2b$. So $\mathcal{A}' = \{X, Z\}$, and therefore, since X is terminal, there exists $z' \in Z - z$ that is movable to V^-. Note that, since every vertex in X is movable, for every vertex in X there is at least one class in $\{V^-, Z\}$ to which it is movable. Therefore, if we assume $N(z) \cap X = \{w\}$, then $V^- + w$, $Z - z + y$ and $X - w + z$ is an equitable a-coloring of $G[A + y]$. Therefore, because z is unmovable, z must have at least two neighbors $\{w, w'\} \subseteq X$. Let $w'' = X - \{w, w'\}$ and note that Lemma 42 implies $\|X, V^-\| \geq 1$, so since $\|\{w, w'\}, V^-\| = 0$, w'' has a neighbor in V^- and therefore does not have a neighbor in Z, so $Z - z' + w''$ is an independent set. Since $\|w'', A\| \geq 1$ and $d(w'') \leq 2b + 2$, $\|w'', B\| \leq 2b + 1 < |B|$, so there exists $y' \in B_0(w'')$. Note that $V^- + z'$, $Z - z' + w''$, $X - w''$ and $\mathcal{B}$ form an optimal coloring of G, because $\|V^- + z', X - w''\| = 0$ and $\|w'', X - w''\| = 0$. Therefore, because $N(y') \supseteq X - w''$, Lemma 46 implies that $b \geq 2$. If either w or w', say w, is adjacent to z', then $d(z') = |T(B)| + 1 = 2b + 2$ and, since $\|w, A\| \geq 2$, $d(w) \geq 2b + 2$, so $4b + 4 \leq 2a + 2b + 1$, which implies that $2b + 3 \leq 2a = 6$, and $b < 2$ a contradiction. Therefore, $V^- + z' + w + w'$ is an independent set, and this contradicts Lemma 42, because $\{z, z'', w''\}$ is also an independent set. □

Lemma 49 *There exists an optimal coloring f' such that $\mathcal{F}(f')$ is a star.*

Proof If $\mathcal{F}$ is not star, there exists $X \in \mathcal{A}'$ such that XV^- is not an edge in $\mathcal{F}$. Because $a' = a - 1$, there exists $Z \in \mathcal{A}'$ such that ZV^- is an edge in $\mathcal{F}$. Because Z is in $\mathcal{A}'$, there exists an X, V^--path $X, \ldots, W, V^-$ in $\mathcal{F}$ that avoids Z. Therefore, there exists another classes and $W \in \mathcal{A}' - X - Z$ such that ZV^- and WV^- are both edges in $\mathcal{F}$. Hence,

$$a \geq 4. \tag{9.1}$$

We let $X = \{x, x', x''\}$ with x unmovable. Since $XV^- \notin E(\mathcal{F})$,

$$\text{every vertex in } X \text{ has a neighbor in } V^-. \tag{9.2}$$

We make the following claims.

Claim 1 *For any $U \in \mathcal{A}'$, if $u \in U$ is unmovable and $\|U - u, A\| \geq 2$, then u is solo.*

Proof If $\|U - u, A\| \geq 2$, then $\|U - u, B\| \leq 4b + 2$ by Proposition 36. If u is not solo, $\|u, B\| \geq 2b$. So u is adjacent to some $y \in T(B)$, but this implies $d(u) + d(y) \geq 2b + a - 1 + 2a - 1$, which contradicts (9.1). □

Claim 1 and (9.2) imply

$$x \text{ is solo}. \tag{9.3}$$

By Corollary 41,

$$\|x', A\| = \|x'', A\| = \|x', V^-\| = \|x'', V^-\| = 1 \tag{9.4}$$

Claim 2 *For every $Z \in \mathcal{A}'$, $\|x, Z\| \leq 2$.*

Proof Suppose $\|x, Z\| = 3$ for some $Z \in \mathcal{A}'$. By Claim 1 and Lemma 48, we can assume that there exists $z \in Z$ such that z is solo, and by Corollary 41, $\|z, B\| \geq b$ and for any $u \in Z - z$, $N(u) \cap A = \{x\}$. Since $\|x, B\| \geq b$ by (9.3) and Corollary 41, and since $\|x, A\| \geq a + 1$ and x is adjacent to z, we have $d(z) \leq a + b$. Since $\|z, B\| \geq b$, we have that $\|z, A\| \leq a$. This implies that $\|z, U\| \leq 2$ for every $U \in \mathcal{A}'$. If we let $\{z', z''\} = Z - z$, then we can move z'' to V^-. Now $\{z, z'\}$ is the small class of a nearly equitable coloring. In this new coloring, using (9.4), the classes $V^- + z''$ and $\{x, x', x''\}$ are movable to $\{z, z'\}$. Furthermore, any class $U \in \mathcal{A}' - Z - X$ is still a class of the new coloring, and it is movable to $\{z, z'\}$ since the only neighbor of z' in A is x and z has at most two neighbors in U. This implies that, in the new coloring, every class of $\mathcal{A} - V^-$ is movable to V^-. □

Claim 3 *For every $u \in A' - X$, if x is not adjacent to u, then u is not movable to V^-.*

Proof Suppose there exists a vertex $z' \in A' - X$ such that $z' \in Z \in \mathcal{A}'$ is not adjacent to x and z' is movable to V^-. Form a new nearly equitable coloring by moving z' to V^- and x'' to $Z - z'$, which forms an independent set by (9.4). Note that $\{x, x'\}$ is the small class in this coloring and that z', and hence $V^- + z'$, is movable to $\{x, x'\}$. Clearly $Z - z' + x''$ is movable to $\{x, x'\}$. Every $U \in \mathcal{A}' - Z - X$ is a color class of the new coloring and, since $\|x', U\| = 0$ and $\|x, U\| \leq 2$ by Claim 2, U is movable to $\{x, x'\}$. This implies that, in the new coloring, every class of $\mathcal{A} - V^-$ is movable to V^-. □

By Claim 3, every vertex in $Z \cup W$ has a neighbor in A: either x or a vertex in V^-. Therefore, by Claim 1, there exist solo vertices $z \in Z$ and $w \in W$. Furthermore, by Corollary 41, each vertex in $Z \cup W - z - w$ has exactly one neighbor in $V^- + x$ and no neighbors in $A' - x$.

Note that since both z and w are solo, and hence unmovable, they both have neighbors in X. By (9.4), x is adjacent to both w and z. Furthermore, there exists $w' \in W - w$ and $z' \in Z - z$ that witness the edges WV^- and ZV^-, respectively. Claim 3 then implies x is adjacent to both w' and z'. This, with the fact that x is solo and unmovable, implies that $\|x, A\| \geq a + 1$. Since $\|x, B\|, \|w, B\|, \|z, B\| \geq b$ by Corollary 41, $\|w, A\|, \|z, A\| \leq 2a + 2b + 1 - (a + 1) - b - b = a$. Therefore, each of w and z has at most 2 neighbors in any class of $\mathcal{A}$. Let $\{z''\} = Z - z - z'$. The only neighbor of z'' in A is either x or a vertex in V^- by Corollary 41 and Claim 3. Moving z' to V^- then creates a coloring f' with small class $\{z, z''\}$. We have that z', x' and x'' are movable to $\{z, z''\}$. This implies that the classes $V^- + z'$ and X are both movable to $\{z, z''\}$. We also have that for any class $U \in \mathcal{A}' - X - Z$, z'' has no neighbors in $A' - X \supseteq U$ and z has at most 2 neighbors in U. This implies that every class of $\mathcal{A}(f')$ is movable to $\{z, z''\}$ and $\mathcal{F}(f')$ is a star. □

By Lemma 49, we will assume below that $\mathcal{F}$ is a star.

Lemma 50 *For every movable vertex $x' \in A'$, $\|x', A\| \leq 1$. Furthermore, for any distinct $X, Z \in \mathcal{A}$, with unmovable $x \in X$ and $z \in Z$, there is an equitable 2-coloring of $G' := G[V^- \cup (X - x) \cup (Z - z)]$.*

Proof Let $\{x, x', x''\} = X \in \mathcal{A}'$ with x unmovable and $\|x', B\| \leq \|x'', B\|$. If x is solo, then by Lemmas 38 and 40, and by Proposition 36(a), the conclusion holds for x' and x'', so assume that x is not solo and $\|x', A\| \geq 2$. By Proposition 36(a) and Lemma 43, $\|x, B\| \leq 2b$ and $\|x', B\| \leq 2b$. Since $\|X, B\| \geq 2(3b + 1) = 2(2b) + (2b + 2)$, this leaves $\|x'', B\| \geq 2b + 2$. By Proposition 36(a), $\|x'', B\| = 2b + 2$ and $\|x, B\| = \|x', B\| = 2b$. Since $d(x') = d(x'') = 2b+2$, for every $y' \in T(B)$, $\|y', B\| = 0$. Since $|B_0(x)|, |B_0(x')| = b+1 > |S(B)|$ and $|B_0(x'')| = b-1 < |S(B)|$, both $B_0(x)$ and $B_0(x')$ intersect $T(B)$ and at least one of $B_0(x)$ and $B_0(x')$ intersects $S(B)$. Therefore, using Lemma 45(c) we can select a 4-set $\{y_1, y_2, y_3, y_4\} \subseteq B_0(x) \cup B_0(x')$ such that $\{x, y_1, y_2\}$ and $\{x', y_3, y_4\}$ are independent sets and there exists $i \in [4]$ such that $y_i \in S(B)$. Therefore, there is a $(b-1)$-coloring of $B - y_1 - y_2 - y_3 - y_4$. Since x'' is movable, we can obtain an equitable k-coloring of G.

Recall $\mathcal{F}$ is a star, so for $v \in V^-$, $\|v, V(G')\| \leq 2$. As we just showed, every vertex in $(X - x) \cup (Z - z)$ has at most one neighbor in A. Then for every $uu' \in E(G')$, $d_{G'}(u) + d_{G'}(u') \leq 3 < 2(2) + 1$. Now the final sentence of the statement follows from Theorem 12. □

Lemma 51 *If $x \in A'$ is unmovable, then $N(x) \supseteq S(B)$.*

Proof The conclusion is true by Corollary 41 if x is solo, so assume that x is not solo and there exists $y \in S(B) \cap B_0(x)$. Let $z \in \{z, z', z''\} = Z \in \mathcal{A}' - X$ be such that $zy \in S$. Since

$y \in B_0(x)$ and $|S(B)| = b$, either $\|x, B\| \le b-1$, or there exists $y' \in N(x) \cap T(B)$. Since $\|x', B\|, \|x'', B\| \le 2b+2$, we have $\|x, B\| \ge 2b-2$. So we have such a y' unless $b = 1$, $\|x, B\| = 0$, and $N(x'), N(x'') \supseteq B$. Note that in this case, since $\|x', B\| = \|x'', B\| = 2b+2$, for every $y' \in T(B)$, $d(y') = 2a-1$, so $\|y', Z\| = 2$; by Corollary 41, $z', z'' \in N(y')$, so $y'z \notin E$. Therefore, we can label B as $\{y, y_1, y_2, y_3\}$ to have the independent sets $\{x, y, y_1\}$, $\{z, y_2, y_3\}$. Since, by Lemma 50, there is an equitable 2-coloring of $G[V^- \cup X - x \cup Z - z]$ we are done. So assume there exists $y' \in N(x) \cap T(B)$ which implies $d(x) \le 2b+2$. Then since $\|x, A\| \ge a-1 \ge 2$, $\|x, B\| \le 2b < |T(B)|$, so there exists $y'' \in T(B) \cap B_0(x)$.

First assume that $\|x', A\| \le 1$ and $\|x'', A\| = 0$. By Proposition 37, and the fact that xy, $z'y$ and $z''y$ are all not edges, xz' and xz'' must both be edges. Since $d(z'), d(z'') = 2b+2$, $d(y'') = 2a-1$, so $\|y'', Z\| = 2$ and $y''z$ is not an edge. Again by Proposition 37, $xy'' \notin E$ implies that $xz \in E$, so $\|x, A\| \ge a+1$. Now $\|x, A\| + \|x, B\| \le 2b+2$, $\|x, B\| \ge 2b-2$, and $a \ge 3$ imply that $\|x, B\| = 2b-2$ and $a = 3$. Since $y' \in N(x)$, $2b-2 = \|x, B\| \ge 1$. So, $b \ge 2$. But, $d(x) + d(z') = 4b+4$ implies that $6 \le 2b+2 \le 2a-1 = 5$, a contradiction.

So $\|\{x', x''\}, A\| \ge 2$, which by Proposition 36(a) implies $\|x, B\| = 2b$ and $d(x') = d(x'') = 2b+2$. Since $d(x) \le 2b+2$, we have that $a = 3$. Also, since both x' and x'' are adjacent to y, $a+b+2b+2 \le 2a+2b+1$, so $b \le a-1 = 2$. Since $d(x') = d(x'') = 2b+2$ and $N(x') \cup N(x'') \supseteq T(B)$, all vertices of $T(B)$ are isolated in B and $N(z) \cap T(B) = \emptyset$ by (A.6). Therefore, there exist $y_1, y_2 \in T(B) - y''$, and $\{x, y, y''\}$ and $\{z, y_1, y_2\}$ are independent sets. Since $y \in S(B)$, there is an equitable $(b-1)$-coloring of $B - y - y'' - y_1 - y_2$. By Lemma 50, there is also an equitable 2-coloring of $G[V^- \cup (X - x) \cup (Z - z)]$, which completes the proof. □

Lemma 52 *The set of unmovable vertices in A' forms a clique.*

Proof By Lemma 43, $a \ge 3$. Suppose there exist distinct, unmovable $x, z \in A'$ such that $xz \notin E$. Let $y \in S(B)$. We know $|S(B)| = b$, $S(B)$ is clique and $d(y) \le a+b$, so $\|y, A\| \le a+1$. Then y has a solo neighbor in all but at most one class of $\mathcal{A}$, so either yx or yz is in S. Assume $yx \in S$. Since x and z are not movable, there exist vertices $z' \in N(x) \cap Z$ and $x' \in N(z) \cap X$. With Lemma 50, this implies that $N(z') \cap A = \{x\}$ and $N(x') \cap A = \{z\}$. Let $\{x, x', x''\} = X \in \mathcal{A}'$ and $\{z, z', z''\} = Z \in \mathcal{A}'$ be the color classes of x and z, respectively, and let $\{v, v'\} = V^-$. If $xz'' \notin E$, then $\{z', v, v'\}$, $\{y, x', x''\}$, $\{x, z, z''\}$ are independent sets. These sets, together with an equitable k-coloring with the classes of $\mathcal{A} - V^- - X - Z$ and an equitable b-coloring of $B - y$, provides an equitable k-coloring of G. So we can assume that xz'' is an edge. Since $d(x') \ge 2b+2$, every vertex in $T(B)$ has degree exactly $2a-1$, with precisely two neighbors in every class of $\mathcal{A}'$ and no neighbors in B. Therefore, no vertex in $T(B)$ is adjacent to x. Since $\|z', A\| = \|z'', A\| = 1$ by Lemma 50, if z is not solo, then $\|z, B\| = 2b$, $\|z', B\| = \|z'', B\| = 2b+1$ and $a = 3$ by Proposition 36(a). Therefore, if z is solo or not solo, there exists $y'' \in B_0(z) \cap T(B)$. Since $\{v, v', z'\}$, $\{x', x'', z''\}$ and $\{x, z, y''\}$ are independent sets, we are done with an equitable b-coloring of $B - y''$. □

Lemma 53 *There exists $v \in V^-$ such that every unmovable vertex in A' is adjacent to v.*

Proof Let $\{v, v'\} = V^-$. Suppose the contrary, i.e., there exist unmovable vertices $x \in X \in \mathcal{A}'$ and $z \in Z \in \mathcal{A}'$ such that $xv \notin E$ and $zv' \notin E$. Since x and z are unmovable, $x \ne z$. Let $y \in S(B)$. Since $\|y, A\| \le a+1$,

$$y \text{ is adjacent to at most one vertex in } W := (X - x) \cup (Z - z). \tag{9.5}$$

Note there is no equitable 3-coloring of $V \cup X \cup Z \cup \{y\}$, since such a coloring could be extended to an equitable coloring of G. Call distinct $w_1, w_2 \in W$ a *good pair* if there is an equitable 2-coloring of $V^- \cup \{x, z, w_1, w_2\}$. Suppose that $\{w_1, w_2\}$ is a good pair and let $\{w_3, w_4\} = W - \{w_1, w_2\}$. Then $\{w_3, w_4, y\}$ is not an independent set, since otherwise we could combine it with an equitable 2-coloring of $V^- \cup \{x, z, w_1, w_2\}$ to create an equitable 3-coloring of $V \cup X \cup Z \cup \{y\}$. If $w_3w_4 \in E(G)$, by Lemma 50, both $\{w_1, w_3\}$ and $\{w_2, w_4\}$ are good pairs. Then neither $\{w_2, w_4, y\}$ nor $\{w_1, w_3, y\}$ is an independent set, lest we equitaly 3-color $V \cup X \cup Z \cup \{y\}$. This contradicts (9.5). So $w_3w_4 \notin E(G)$. Therefore,

$$\text{if } \{w_1, w_2\} \text{ is a good pair,} \quad \text{then } \|y, W - w_1 - w_2\| \geq 1. \tag{9.6}$$

Since $\mathcal{F}$ is a star, there exist vertices $x' \in X$ and $z' \in Z$ that are movable to V^-. Let $X = \{x, x', x''\}$ and $Z = \{z, z', z''\}$. Since $\{x', z'\}$ is a good pair, we can assume, by the symmetry of x'' and z'', (9.5) and (9.6), that x'' is the unique neighbor of y in W. So (9.6) implies that $\{z', x''\}$ is not a good pair. With Lemma 50, this implies that x and v are the unique neighbors in A of z' and x'', respectively. So $\{x', x''\}$ is a good pair and $\{y, z', z''\}$ is an independent set, a contradiction. □

Lemma 54 $\omega(G) \geq k$.

Proof Let $\{v, v'\} = V^-$. By Lemma 53, we can assume that every unmovable vertex in $\mathcal{A}'$ is adjacent to v. Recall that for every $y \in S(B)$, $\|y, A\| \leq a + 1$ and for every $y' \in T(B)$, $\|y', A\| \geq 2a - 1$. Therefore, since $a \geq 3$, if f' is an optimal coloring such that $B(f') = B(f)$, then $T_{f'}(B) = T_f(B)$ and $S_{f'}(B) = S_f(B)$.

By Corollary 41, Lemmas 48, 51 and 52, and (A.5), we only need to show that $N(v) \supseteq S(B)$. We will achieve this by showing that there exists an optimal coloring f' in which $\mathcal{F}(f')$ is a star and v is not movable and not in $V^-(f')$. The conclusion then follows from Lemma 51. By Lemma 43, Lemma 48, and Lemma 52, there exists a class $\{x, x', x''\} \in X \in \mathcal{A}'$ such that x is low and unmovable. Since $\mathcal{F}(f')$ is a star, one of x' or x'', say x', is movable to V^-. By the selection of v, v is not movable, and we are done unless there exists $\{z, z', z''\} = Z \in \mathcal{A}' - X - V^-$ such that no vertex in Z is movable to $\{x, x''\}$. So assume that this is the case. Since $N(x) \supseteq S(B)$, $\|x, B\| \geq b$, so $\|x, A\| \leq a$. So since $\|x'', A\| \leq 1$ by Lemma 50, we can assume that x is adjacent to z and z', and x'' is adjacent to z'' and there are no other edges in $G[Z + x + x'']$. Since $\|z'', A\| \leq 1$, x' is not adjacent to z''. Therefore we get the desired coloring by moving x'' instead of x' to V^-. □

The contradiction between this lemma and Lemma 15 completes the proof of Theorem 13.

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