COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **37** (3) (2017) 481–494
DOI: 10.1007/s00493-015-3262-0

CYCLES IN TRIANGLE-FREE GRAPHS OF LARGE CHROMATIC NUMBER*

ALEXANDR KOSTOCHKA†, BENNY SUDAKOV‡, JACQUES VERSTRAËTE§

Received April 17, 2014
Revised March 18, 2015
Online First May 10, 2016

More than twenty years ago Erdős conjectured [4] that a triangle-free graph G of chromatic number $k\geq k_0(\varepsilon)$ contains cycles of at least $k^{2-\varepsilon}$ different lengths as $k\to\infty$. In this paper, we prove the stronger fact that every triangle-free graph G of chromatic number $k\geq k_0(\varepsilon)$ contains cycles of $\frac{1}{64}(1-\varepsilon)k^2\log\frac{k}{4}$ consecutive lengths, and a cycle of length at least $\frac{1}{4}(1-\varepsilon)k^2\log k$. As there exist triangle-free graphs of chromatic number k with at most roughly $4k^2\log k$ vertices for large k, these results are tight up to a constant factor. We also give new lower bounds on the circumference and the number of different cycle lengths for k-chromatic graphs in other monotone classes, in particular, for K_r-free graphs and graphs without odd cycles C_{2s+1}.

1. Introduction

It is well known that every k-chromatic graph has a cycle of length at least k for $k\geq 3$. In 1991, Gyárfás [6] proved a stronger statement, namely, the conjecture by Bollobás and Erdős that every graph of chromatic number $k\geq 3$ contains cycles of at least $\lfloor\frac{1}{2}(k-1)\rfloor$ odd lengths. This is best possible in view of any graph whose blocks are complete graphs of order k. Mihók and Schiermeyer [9] proved a similar result for even cycles: every graph G

Mathematics Subject Classification (2000): 05C15, 05C35, 05C38

* The authors thank Institute Mittag-Leffler (Djursholm, Sweden) for the hospitality and creative environment.

† Research supported in part by NSF grant DMS-1266016 and by Grant NSh.1939.2014.1 of the President of Russia for Leading Scientific Schools.

‡ Research supported in part by SNSF grant 200021-149111 and by a USA-Israel BSF grant.

§ Research supported by NSF Grant DMS-1362650.

of chromatic number $k \ge 3$ contains cycles of at least $\lfloor \frac{k}{2} \rfloor - 1$ even lengths. A consequence of the main result in [14] is that a graph of chromatic number $k \ge 3$ contains cycles of $\lfloor \frac{1}{2}(k-1) \rfloor$ consecutive lengths. Erdős [4] made the following conjecture:

Conjecture 1. *For every $\varepsilon > 0$, there exists $k_0(\varepsilon)$ such that for $k \ge k_0(\varepsilon)$, every triangle-free k-chromatic graph contains more than $k^{2-\varepsilon}$ odd cycles of different lengths.*

The second and third authors proved [12] that if G is a graph of average degree k and girth at least five, then G contains cycles of $\Omega(k^2)$ consecutive even lengths, and in [13] it was shown that if an n-vertex graph of independence number at most $\frac{n}{k}$ is triangle-free, then it contains cycles of $\Omega(k^2 \log k)$ consecutive lengths.

1.1. Main result

In this paper, we prove Conjecture 1 in the following stronger form:

Theorem 2. *For all $\varepsilon > 0$, there exists $k_0(\varepsilon)$ such that for $k \ge k_0(\varepsilon)$, every triangle-free k-chromatic graph G contains a cycle of length at least $\frac{1}{4}(1-\varepsilon)k^2 \log k$ as well as cycles of at least $\frac{1}{64}(1-\varepsilon)k^2 \log \frac{k}{4}$ consecutive lengths.*

Kim [8] was the first to construct a triangle-free graph with chromatic number k and $\Theta(k^2 \log k)$ vertices. Bohman and Keevash [2] and Fiz Pontiveros, Griffiths and Morris [5] independently constructed k-chromatic triangle-free graphs with at most $(4+o(1))k^2 \log k$ vertices as $k \to \infty$, refining the earlier construction of Kim [8]. These constructions show that the bound in Theorem 2 is tight up to a constant factor.

1.2. Monotone properties

Theorem 2 is a special case of a more general theorem on monotone properties. A graph property is called *monotone* if it holds for all subgraphs of a graph which has this property, i.e., is preserved under deletion of edges and vertices. Throughout this section, let $n_{\mathcal{P}}(k)$ denote the smallest possible order of a k-chromatic graph in a monotone property $\mathcal{P}$.

Definition 1.1. Let $\alpha \ge 1$ and let $f\colon [3,\infty) \to \mathbb{R}^+$. Then f is α-bounded if f is non-decreasing and whenever $y \ge x \ge 3$, $y^\alpha f(x) \ge x^\alpha f(y)$.

For instance, any polynomial with positive coefficients is α-bounded for some $\alpha \geq 1$. We stress that an α-bounded function is required to be a non-decreasing positive real-valued function with domain $[3,\infty)$.

Theorem 3. *For all $\varepsilon > 0$ and $\alpha, m \geq 1$, there exists $k_1 = k_1(\varepsilon,\alpha,m)$ such that the following holds. If $\mathcal{P}$ is a monotone property of graphs with $n_{\mathcal{P}}(k) \geq f(k)$ for $k \geq m$ and some α-bounded function f, then for $k \geq k_1$, every k-chromatic graph $G \in \mathcal{P}$ contains (i) a cycle of length at least $(1-\varepsilon)f(k)$ and (ii) cycles of at least $(1-\varepsilon)f(\frac{k}{4})$ consecutive lengths.*

If $n_{\mathcal{P}}(k)$ itself is α-bounded for some α, then we obtain from Theorem 3 a tight result that a k-chromatic graph in $\mathcal{P}$ contains a cycle of length asymptotic to $n_{\mathcal{P}}(k)$ as $k \to \infty$. But proving that $n_{\mathcal{P}}(k)$ is α-bounded for some α is probably difficult for many properties, and in the case $\mathcal{P}$ is the property of F-free graphs, perhaps is as difficult as obtaining asymptotic formulas for certain Ramsey numbers. Even in the case of the property of triangle-free graphs, we have seen $n_{\mathcal{P}}(k)$ is known only up to a constant factor. We remark that the proofs of Theorems 2 and 3 yield rather large values for $k_0(\varepsilon)$ and $k_1(\varepsilon,\alpha,m)$; we have not optimized them.

1.3. An application: K_r-free graphs

As an example of an application of Theorem 3, we consider the property $\mathcal{P}$ of K_r-free graphs. A lower bound for the quantity $n_{\mathcal{P}}(k)$ can be obtained by combining upper bounds for Ramsey numbers together with a lemma on colorings obtained by removing maximum independent sets – see Section 5. In particular, we shall obtain the following from Theorem 3:

Theorem 4. *If $r,k \geq 3$ and G is a k-chromatic K_{r+1}-free graph, then G contains cycles of $\Omega(k^{\frac{r}{r-1}})$ consecutive lengths as $k \to \infty$.*

Theorem 4 is derived from upper bounds on the Ramsey numbers $r(K_r,K_t)$ combined with Theorem 3. In general, if for a graph F one has $r(F,K_t) = O(t^a(\log t)^{-b})$ for some $a > 1$ and $b > 0$, then any k-chromatic F-free graph has cycles of

$$\Omega\left(k^{\frac{a}{a-1}}(\log k)^{\frac{b}{a-1}}\right)$$

consecutive lengths. We omit the technical details, since the ideas of the proof are identical to those used for Theorem 4. These technical details are presented in the proof of Theorem 2 in Section 4, where F is a triangle (in which case $a = 2$ and $b = 1$), and the same ideas can be used to slightly improve the bound in Theorem 4 by logarithmic factors using the bounds on $r(K_s,K_t)$ by Ajtai, Komlós and Szemerédi [1]. Similarly, if C_ℓ denotes the cycle of length ℓ, then it is known that $r(C_{2s+1},K_t)=O(t^{1+1/s}(\log t)^{-1/s})$ – see [11]. This in turn provides cycles of $\Omega(k^{s+1}\log k)$ consecutive lengths in any C_{2s+1}-free k-chromatic graph, extending Theorem 2.

Notation and terminology. For a graph G, let $c(G)$ denote the length of a longest cycle in G and $\chi(G)$ the chromatic number of G. Furthermore, $|G|$ is the number of edges in G. If $F\subset G$ and $S\subset V(G)$, let $G[F]$ and $G[S]$ respectively denote the subgraphs of G induced by $V(F)$ and S. A chord of a cycle C in a graph is an edge of the graph joining two nonconsecutive vertices on the cycle. All logarithms in this paper are with the natural base.

Organization. In the next section, we present the lemmas which will be used to prove Theorem 3 in Section 3. Then in Section 4, we apply Theorem 3 to obtain the proof of Theorem 2. Theorem 4 is proved in Section 5.

2. Lemmas

2.1. Vertex cuts in k-critical graphs

When a small vertex cut is removed from a k-critical graph, all the resulting components still have relatively high chromatic number:

Lemma 5. *Let G be a k-critical graph and let S be a vertex cut of G. Then for any component H of $G-S$, $\chi(H)\geq k-|S|$.*

Proof. If $|S|+\chi(H)\leq k-1$, then a $(k-1)$-coloring of $G-H$ (existing by the criticality of G) can be extended to a $(k-1)$-coloring of G. ∎

2.2. Nearly 3-connected subgraphs

Our second lemma finds an almost 3-connected subgraph with high chromatic number in a graph with high chromatic number.

Lemma 6. *Let $k\geq 4$. For every k-chromatic graph G, there is a graph G^* and an edge $e^*\in E(G^*)$ such that*

(a) *$G^*-e^*\subset G$ and $\chi(G^*-e^*)\geq k-1$.*
(b) *G^* is 3-connected.*
(c) *$c(G^*)\leq c(G)$.*

Proof. Let G' be a k-critical subgraph of G. Then G' is 2-connected. If G' is 3-connected, then the lemma holds for $G^*=G'$ with any $e\in E(G')$ as e^*. So suppose G' is not 3-connected. Among all separating sets S in G' of size 2 and components F of $G'-S$, choose a pair (S,F) with the minimum $|V(F)|$. If $S=\{u,v\}$, then we let G^* be induced by $V(F)\cup S$ plus the edge $e^*=uv$. We claim $\chi(G^*-e^*)\geq k-1$. Since G' is k-critical, $G'-V(F)$ has a $(k-1)$-coloring $\varphi: V(G')\backslash V(F)\to\{1,2,\dots,k-1\}$. By renaming colors we can make $\varphi(u)=k-1$ and $\varphi(v)\in\{1,k-1\}$. Suppose for a contradiction that there is a coloring $\varphi^*: V(G^*-e^*)\to\{1,2,\dots,k-2\}$ of G^*-e^*. If $\varphi(v)=k-1$, then we let $\varphi'(x)=\varphi(x)$ if $x\in V(G')-V(F)$ and $\varphi'(x)=\varphi^*(x)$ if $x\in V(F)$, and this φ' is a proper $(k-1)$-coloring of G', a contradiction. Otherwise $\varphi(v)=1$. Then we change the names of colors in φ^* so that $\varphi^*(v)=1$ and again let $\varphi'(x)=\varphi(x)$ if $x\in V(G')-V(F)$ and $\varphi'(x)=\varphi^*(x)$ if $x\in V(F)$. Again we have a proper $(k-1)$-coloring of G'. This contradiction proves (a).

To prove (b), if G^* has a separating set S' with $|S'|=2$, then, since $uv\in E(G^*)$, it is also a separating set in G and at least one component of $G'-S'$ is strictly contained in F. This contradicts the choice of F and S.

For (c), let C be a cycle in G^* with $|C|=c(G^*)$. If $e^*\notin E(C)$, then C is also a cycle in G, and thus $c(G)\geq|C|=c(G^*)$. If $e^*\in E(C)$ and $G^*\neq G'$, then we obtain a longer cycle C in G' by replacing e^* with a uv-path in $G'-V(F)$ – note such a path exists since G' is 2-connected. This proves (c). ∎

2.3. Finding cycles of consecutive lengths

In this subsection we show how to go from longest cycles in graphs to cycles of many consecutive lengths. We will need the following result from [14], which is also implicit in the paper of Bondy and Simonovits [3]:

Lemma 7 (Lemma 2 in [14]). *Let H be a graph consisting of a cycle with a chord. Let (A,B) be a nontrivial partition of $V(H)$. Then H contains A,B-paths of every positive length less than $|H|$, unless H is bipartite with bipartition (A,B).*

The proof of the following lemma is similar to proofs given in [12,14].

Lemma 8. *Let $k\geq 4$ and $\mathcal{Q}$ be a monotone class of graphs. Let $h(k,\mathcal{Q})$ denote the smallest possible length of a longest cycle in any k-chromatic graph in $\mathcal{Q}$. Then every $4k$-chromatic graph in $\mathcal{Q}$ contains cycles of at least $h(k,\mathcal{Q})$ consecutive lengths.*

Proof. Let F be a connected subgraph of $G\in\mathcal{Q}$ with $\chi(F)\geq 4k$, and let T be a breadth-first search tree in F. Let L_i be the set of vertices at distance exactly i from the root of T in F. Then for some i, graph $H=F[L_i]$ has chromatic number at least $2k$. Let U be a breadth-first search tree in a component of H with chromatic number at least $2k$ and let M_s be the set of vertices at distance exactly s from the root of U in H. Then for some s, graph $J=H[M_s]$ has chromatic number at least k. Let J' be a k-critical subgraph of J.

Let P be a longest path in J'. Suppose the ends of P are x and y. Since J' is k-critical, it is 2-connected and has minimum degree at least $k-1 \geq 3$; thus each of x and y has at least three neighbors on P. By definition, $|P| \geq c(J') - 1 \geq h(k, \mathcal{Q}) - 1$. Moreover, if $|P| = h(k, \mathcal{Q}) - 1$, then each longest cycle in the 2-connected J' is hamiltonian, thus in this case we may choose P as a part of this cycle, so that $xy \in E(G)$.

Define $P' = P - x$ if P has an even length and $xy \notin E(G)$, and let $P' = P$ otherwise. In both cases, P' is a path with at least one chord, and $|P'| \geq h(k, \mathcal{Q}) - 1$. By the definition of M_i, the tree U contains a path Q of even length connecting endpoints of P', which is internally disjoint from P'. Then the cycle $C = Q \cup P'$ has at least one chord. Moreover, either $|C|$ is odd or it has a chord xy that together with Q forms an odd cycle. Also $|C| \geq |P'| + 2 \geq h(k, \mathcal{Q}) + 1$. Let $\ell := |C|$ and $H' = G[C]$. By construction, $V(H') \subset L_i$. Let T' be a minimal subtree of T whose set of leaves is $V(H')$. Then T' branches at its root. Let A be the set of leaves in some branch of T', and let $B = V(H') \backslash A$. Since H' has an odd cycle, it is not bipartite and therefore by Lemma 7, there exist paths $P_1, P_2, \ldots, P_{\ell-1} \subset H'$ such that for all $j = 1, \ldots, \ell-1$, P_j has length j and one end of P_j is in A and one end of P_j is in B. Also for each j, the ends of P_j are joined by a path Q_j of length $2r$, where r is the height of T' and Q_j and P_j are internally disjoint. Therefore, $P_j \cup Q_j$ is a cycle of length $2r + j$ for $j = 1, 2, \ldots, \ell - 1$, as required. ∎

2.4. Lemmas on α-bounded functions

The following technical lemma is required for the proof of Theorem 3.

Lemma 9. *Let $\alpha, x_0 \geq 1$, and let f be α-bounded. Then the function*

$$g(x) = \frac{xf(x)}{x + f(x_0)}.$$

is $(\alpha+1)$-bounded, $g(x) \leq x$ for $x \in [3, x_0]$, and $g(x) \leq f(x)$ for all $x \in [3, \infty)$.

Proof. By definition, $g(x) \leq f(x)$ for $x \in [3, \infty)$ and $g(x) \leq x$ for $x \in [3, x_0]$. Also, since f is non-decreasing and positive on $[3, \infty)$, g is non-decreasing on $[3,\infty)$. It remains to check that g is $(\alpha+1)$-bounded. For $y \geq x \geq 3$, using that $y^\alpha f(x) \geq x^\alpha f(y)$, we find

$$y^{\alpha+1} g(x) = \frac{y^{\alpha+1} x f(x)}{x + f(x_0)} \geq \frac{x^{\alpha+1} y f(y)}{x + f(x_0)} \geq x^{\alpha+1} g(y).$$

Therefore g is $(\alpha+1)$-bounded. ∎

We shall also need the following for the proof of Theorem 2.

Lemma 10. *Let $c > 0$ and $f(x) = cx^2 \log x$ for $x \geq 3$. Then f is 3-bounded.*

Proof. The function f is positive and non-decreasing, so one only has to check $y^3 f(x) \geq x^3 f(y)$ whenever $y \geq x \geq 3$. This follows from $y \log x \geq x \log y$ for $y \geq x \geq 3$, since the function $\frac{y}{\log y}$ is increasing for $y \geq 3$. ∎

3. Proof of Theorem 3

It is enough to prove Theorem 3 for all $\varepsilon < \frac{1}{2}$. Let $\beta = \alpha + 1$, $\eta = \frac{\varepsilon}{2}$ and $x_0 = \max\{2m, (\frac{8\beta}{\eta})^{\beta+1}\}$. Define $k_1 = k_1(\varepsilon, \alpha, m) = \frac{8}{\varepsilon} f(x_0)$. Let $g = \frac{x f(x)}{x + f(x_0)}$ be the β-bounded function in Lemma 9. We prove the following claim:

Claim. *For $k \geq 3$, every k-chromatic graph $G \in \mathcal{P}$ has a cycle of length at least $(1-\eta) g(k)$.*

Once this claim is proved, Theorem 3(i) follows since for $k \geq \frac{1}{4} k_1 \geq \frac{2}{\varepsilon} f(x_0)$,

$$(1-\eta) g(k) = (1 - \tfrac{\varepsilon}{2}) \frac{k f(k)}{k + f(x_0)} \geq (1 - \tfrac{\varepsilon}{2}) \frac{k f(k)}{k + \frac{\varepsilon}{2} k} \geq (1-\varepsilon) f(k),$$

as required. Also, if $k \geq k_1$, then by Lemma 8 every k-chromatic graph in $\mathcal{P}$ contains cycles of at least $(1-\eta) g(\frac{k}{4}) \geq (1-\varepsilon) f(\frac{k}{4})$ consecutive lengths, which gives Theorem 3(ii). This shows Theorem 3 follows from the claim with $k_1(\varepsilon, \alpha, m) = \frac{8}{\varepsilon} f(x_0)$.

We prove the claim by induction on k. For $3 \leq k \leq x_0$, $g(k) \leq k$ from Lemma 9. In that case G, contains a k-critical subgraph which has minimum degree at least k–1 and therefore also a cycle of length at least k. This proves the claim for $3 \leq k \leq x_0$. Now suppose $k > x_0$. Let G^* be the graph obtained from G in Lemma 6. By Lemma 6(c), it is sufficient to show that G^* has a cycle of length at least $(1-\eta) g(k)$. Let C be a longest cycle in $G^* - e^*$. By induction, $|C| \geq (1-\eta) g(k-1)$. Let $G_1 = G^*[C]$ and $\chi_1 = \chi(G_1)$, and let $G_2 = G^* - V(G_1) - e^*$ and $\chi_2 := \chi(G_2)$. Note that $\chi_2 \geq k - 1 - \chi_1$. Take C' to be a longest cycle in G_2. Let L be a minimum vertex set covering all paths from C to C'. Either L separates $C'-L$ from $C-L$ or $L=V(C')$. Let $|L|=\ell$. By Menger's Theorem, G^* has ℓ vertex-disjoint paths $P_1, P_2, \ldots, P_\ell$ between C and C' – note $\ell \geq 3$, as G^* is 3-connected. Let $H = \bigcup_{i=1}^{\ell} P_i \cup C \cup C'$. We find a cycle $C^* \subset H$ with

$$|C^*| \geq \frac{\ell-1}{\ell}|C| + \frac{1}{2}|C'|. \tag{1}$$

To see this, first note that two of the paths, say P_i and P_j, contain ends at distance at most $\frac{1}{\ell}|C|$ on C, and now $P_i \cup P_j \cup C \cup C'$ contains a cycle C^* of length at least

$$\frac{\ell-1}{\ell}|C| + \frac{1}{2}|C'| + |P_i| + |P_j| \geq \frac{\ell-1}{\ell}|C| + \frac{1}{2}|C'|.$$

At the same time, H contains a cycle C^{**} with

$$|C^{**}| \geq \frac{2}{3}(|C| + |C'|). \tag{2}$$

This follows from the fact that there exist three cycles that together cover every edge of $P_1 \cup P_2 \cup P_3 \cup C \cup C'$ exactly twice, namely, each of the three cycles contains two of the paths P_1, P_2, P_3 as well as both endpoints of the third path. Therefore the sum of the lengths of those cycles is at least $2|C|+2|C'|$, so one of those cycles has length at least $\frac{2}{3}(|C|+|C'|)$. Now we complete the proof of the claim in three cases. In each case we find a cycle of length at least $(1-\eta)g(k)$.

Case 1. $\chi_1 \geq (1-\frac{\eta}{\beta})k$. Then $\chi_1 \geq (1-\eta)k \geq \frac{k}{2} \geq \frac{x_0}{2} \geq m \geq 3$, which implies $n_{\mathcal{P}}(\chi_1) \geq f(\chi_1)$. Recall $\alpha \geq 1$ and $\beta = \alpha + 1 > 1$. Since $\eta \leq \frac{1}{2} < \beta$, we have $(1-\frac{\eta}{\beta})^\alpha \geq 1 - \frac{\eta\alpha}{\beta} \geq 1-\eta$. Since f is α-bounded,

$$|C| \geq n_{\mathcal{P}}(\chi_1) \geq f(\chi_1) \geq (1-\tfrac{\eta}{\beta})^\alpha f(k) \geq (1-\eta)f(k) \geq (1-\eta)g(k).$$

Case 2. $\chi_1 < (1-\frac{\eta}{\beta})k$ and $\chi_2 \geq (1-\frac{1}{4\beta})k$. Since $\chi_2 \geq 3$ and g is β-bounded,

$$g(\chi_2) \geq (1-\tfrac{1}{4\beta})^\beta g(k) \geq \tfrac{3}{4}g(k) \quad \text{and} \quad g(k-1) \geq (\tfrac{k-1}{k})^\beta g(k) \geq (1-\tfrac{\beta}{k})g(k).$$

Since, by induction, $|C| \geq (1-\eta)g(k-1)$ and $|C'| \geq (1-\eta)g(\chi_2)$, we obtain from (2):

$$\begin{aligned}|C^{**}| &\geq \tfrac{2}{3}(1-\eta)g(k-1) + \tfrac{2}{3}(1-\eta)g(\chi_2)\\ &\geq \tfrac{2}{3}(1-\eta)(1-\tfrac{\beta}{k})g(k) + \tfrac{1}{2}(1-\eta)g(k).\end{aligned}$$

Since $k > x_0 \geq (\frac{8\beta}{\eta})^{\beta+1} > \frac{8\beta}{\eta}$, we have $\frac{\beta}{k} < \eta/8 < \frac{1}{4}$ and therefore

$$|C^{**}| > (1-\eta)g(k) \cdot (\tfrac{2}{3}(1-\eta) + \tfrac{1}{2}) > (1-\eta)g(k).$$

Case 3. $\chi_1 < (1-\frac{\eta}{\beta})k$ and $\chi_2 < (1-\frac{1}{4\beta})k$. Then since $k > x_0 > \frac{8\beta}{\eta}$ (as we explained above),

$$\chi_2 \geq k-1-\chi_1 > k-1-(1-\tfrac{\eta}{\beta})k = \tfrac{\eta}{\beta}k-1 > \tfrac{\eta}{2\beta}k-1+\tfrac{\eta}{2\beta}k > \tfrac{\eta}{2\beta}k \geq 3.$$

Since every χ_2-chromatic graph contains a cycle of length at least χ_2, we have that $|C'| \geq \chi_2$. If $L = V(C')$, then $\ell = |C'| \geq \chi_2 > \frac{\eta}{2\beta}k$. Otherwise, by Lemma 5, $\chi_2 \geq k - \min\{\chi_1, \ell\} - 1$. Since $k > x_0$, $k > \frac{8\beta}{\eta}$ and $\eta < \frac{1}{2}$. Therefore

$$\ell > k-\chi_2-1 > k-(1-\tfrac{1}{4\beta})k-1 = \tfrac{k}{4\beta}-1 > \tfrac{\eta}{8\beta}k-1+\tfrac{\eta}{8\beta}k > \tfrac{\eta}{8\beta}k.$$

In all cases we have verified $\ell > \frac{\eta}{8\beta}k$. Since g is β-bounded,

$$g(\chi_2) \geq g(\tfrac{\eta}{2\beta}k) \geq (\tfrac{\eta}{2\beta})^{\beta} g(k) \quad \text{and} \quad g(k-1) \geq (\tfrac{k-1}{k})^{\beta} g(k).$$

By (1), and recalling $|C| \geq (1-\eta)g(k-1)$ and $|C'| \geq g(\chi_2)$,

$$\begin{aligned}|C^*| &\geq \tfrac{\ell-1}{\ell}(1-\eta)g(k-1) + \tfrac{1}{2}g(\chi_2) \\ &\geq (1-\eta)g(k)\cdot(\tfrac{\ell-1}{\ell}(\tfrac{k-1}{k})^{\beta} + \tfrac{1}{2}(\tfrac{\eta}{2\beta})^{\beta}) \\ &\geq (1-\eta)g(k)\cdot(1-\tfrac{\beta}{k}-\tfrac{1}{\ell}+\tfrac{1}{2}(\tfrac{\eta}{2\beta})^{\beta}).\end{aligned}$$

Since $k > x_0 \geq (\frac{8\beta}{\eta})^{\beta+1}$ and $\ell > \frac{\eta}{8\beta}k$ and $\beta > 1$,

$$\tfrac{\beta}{k} < \beta\cdot(\tfrac{\eta}{8\beta})^{\beta+1} \leq \tfrac{\eta}{8}\cdot(\tfrac{\eta}{2\beta})^{\beta} < \tfrac{1}{8}\cdot(\tfrac{\eta}{2\beta})^{\beta} \quad \text{and} \quad \tfrac{1}{\ell} < \tfrac{8\beta}{\eta}\cdot\tfrac{1}{k} < (\tfrac{\eta}{8\beta})^{\beta} \leq \tfrac{1}{4}\cdot(\tfrac{\eta}{2\beta})^{\beta}.$$

This shows

$$|C^*| > (1-\eta)g(k)\cdot(1-\tfrac{1}{8}(\tfrac{\eta}{2\beta})^{\beta}-\tfrac{1}{4}(\tfrac{\eta}{2\beta})^{\beta}+\tfrac{1}{2}(\tfrac{\eta}{2\beta})^{\beta}) \geq (1-\eta)g(k).$$

This completes the proof of Theorem 3. ∎

4. Proof of Theorem 2

Throughout this section, $0 < \delta < 1$. For the proof of Theorem 2, we use Theorem 3 with a specific choice of the α-bounded function $f(x)$, namely

$$f(x) = \frac{1}{4}(1-\delta)x^2 \log x.$$

4.1. The number of vertices in k-chromatic triangle-free graphs

Let $\alpha(G)$ denote the independence number of a graph G. Shearer [10] showed that for each $d>1$ and every n-vertex triangle-free graph G with average degree d,

$$\alpha(G) \geq \frac{d\log d - d + 1}{(d-1)^2}n \geq \frac{\log d - 1}{d}n.$$

Since function $h(d)=\frac{\log d-1}{d}$ is decreasing for $d>e$, we obtain the following.

Lemma 11. *Let $d>e$ and let G be an n-vertex triangle-free graph of average degree at most d. Then*

$$\alpha(G) > n\frac{\log(\frac{d}{e})}{d}. \tag{3}$$

This implies the following simple fact.

Lemma 12. *Let $\varphi(x)=(\frac{1}{2}x\log ex)^{1/2}$. If G is an n-vertex triangle-free graph and $n\geq e^{2e^3}$, then $\alpha(G)\geq\varphi(n)$.*

Proof. If $n \geq e^{2e^3}$, then $\varphi(n) > e(en)^{1/2}$. Suppose $\alpha(G) < \varphi(n)$. Since G is triangle-free, the neighborhood of each vertex is an independent set, so $\Delta(G)\leq\alpha(G)<\varphi(n)$. Since $e(en)^{1/2}<\varphi(n)<(\frac{1}{2}n\log en)^{1/2}$, by (3),

$$\varphi(n) > \alpha(G) > n\frac{\log(\frac{\varphi(n)}{e})}{\varphi(n)} > n\frac{\log(en)^{1/2}}{\varphi(n)} \geq \varphi(n),$$

a contradiction. ∎

To find a lower bound on the number of vertices of a triangle-free k-chromatic graph, we require a lemma of Jensen and Toft [7] (they took $s=2$, but their proof works for each positive integer s):

Lemma 13 ([7], Problem 7.3). *Let $s\geq1$ and let $\psi\colon [s,\infty)\to(0,\infty)$ be a positive continuous non-decreasing function. Let $\mathcal{P}$ be a monotone class of graphs such that $\alpha(G)\geq\psi(|V(G)|)$ for every $G\in\mathcal{P}$ with $|V(G)|\geq s$. Then for every such G with $|V(G)|=n\geq s$,*

$$\chi(G) \leq s + \int_s^n \frac{1}{\psi(x)}dx.$$

Proposition 14. *For $0<\delta<1$, there exists $k_2(\delta)$ such that whenever $k\geq k_2(\delta)$, every k-chromatic triangle-free graph has at least $\frac{1}{4}(1-\delta)k^2\log k$ vertices.*

Proof. For $x \geq 2$, define $\gamma(x) = \frac{1}{2} - \frac{1}{2\log ex}$. Then $0 < \gamma(x) < 1/2$ for $x \geq 2$. Let G be a k-chromatic triangle-free n-vertex graph. We apply the preceding lemma with $\psi(x) = \varphi(x) = (\frac{1}{2}x\log x)^{1/2}$ supplied by Lemma 12. It is enough to prove the lemma only for $\delta < 0.04$. So we assume below that

$$0 < \delta < 0.04 \quad \text{and let} \quad s = \left\lceil e^{4/\delta} \right\rceil. \tag{4}$$

Since $\gamma(s) \leq \gamma(x)$ for $x \geq s$, if $n \geq s$, then Lemma 13 gives:

$$\begin{aligned}\chi(G) &\leq s + \sqrt{2}\int_s^n (x\log ex)^{-1/2}dx = s + \frac{\sqrt{2}}{\gamma(s)}\int_s^n (x\log ex)^{-1/2}\gamma(s)dx \\ &\leq s + \frac{\sqrt{2}}{\gamma(s)}\int_s^n (x\log ex)^{-1/2}\gamma(x)dx.\end{aligned}$$

An antiderivative for the integrand is exactly $x^{1/2}(\log ex)^{-1/2}$, and therefore

$$k = \chi(G) \leq s + \frac{\sqrt{2}}{\gamma(s)} n^{1/2}(\log en)^{-1/2}. \tag{5}$$

Moving s from the right-hand side to the lefthand side of (5), multiplying both parts by $\gamma(s)/\sqrt{2}$, and taking squares, we get

$$\frac{1}{2}\gamma(s)^2(k-s)^2 \leq \frac{n}{\log(en)}. \tag{6}$$

For any $n, m \geq 1$, the inequality $n/\log(en) \geq m$ implies $n \geq m\log(em)$. Therefore, (6) yields

$$n \geq \frac{1}{2}\gamma(s)^2(k-s)^2\log\Big(\frac{e}{2}\gamma(s)^2(k-s)^2\Big) > \gamma(s)^2(k-s)^2\log\Big(\gamma(s)(k-s)\Big). \tag{7}$$

By (4) and the definition of $\gamma(x)$,

$$\gamma(s) \geq \gamma(e^{4/\delta}) = \frac{1}{2} - \frac{1}{2\log e^{4/\delta}} = \frac{1}{2}\Big(1 - \frac{\delta}{4}\Big) > 0.49.$$

From this and (7) we derive

$$n > \frac{1}{4}\Big(1 - \frac{\delta}{4}\Big)^2(k-s)^2\log\Big(0.49(k-s)\Big). \tag{8}$$

Let $k_2 = k_2(\delta) := e^{10/\delta}$. By (4), $k_2 > e^{5/\delta}s > e^{100}s$. Then for $k \geq k_2(\delta)$, $k - s \geq k - k_2/e^{100} > 0.99k$, and thus (8) yields

$$\begin{aligned} n &> \frac{1}{4}\Big(1 - \frac{\delta}{4}\Big)^2(k^2 - 2ks)\log\Big(\frac{k}{e}\Big) \\ &= \left(\frac{1-\delta}{4}k^2\log k\right)\frac{(1-\delta/4)^2}{1-\delta}\left(\frac{k-2s}{k}\right)\left(\frac{\log\frac{k}{e}}{\log k}\right).\end{aligned} \tag{9}$$

Let $F(k,s,\delta)=\frac{(1-\delta/4)^2}{1-\delta}\left(\frac{k-2s}{k}\right)\left(\frac{\log\frac{k}{e}}{\log k}\right)$ (it forms a part of the right hand side in (9)). We will finish the proof by showing that $F(k,s,\delta)\geq 1$.

Observe that for $k\geq k_2$, $\frac{\log k/e}{\log k}=1-\frac{1}{\log k_2}\geq 1-\frac{\delta}{10}$ and

$$\frac{k-2s}{k}\geq 1-\frac{2s}{k_2}\geq 1-\frac{3e^{4/\delta}}{e^{10/\delta}}\geq 1-\frac{3}{e^{6/\delta}}>1-3\left(\frac{\delta}{6}\right)^2>1-\frac{\delta}{10}.$$

Thus, since $\frac{(1-\delta/4)^2}{1-\delta}>1+\frac{\delta}{2}$,

$$F(k,s,\delta)\geq\left(1+\frac{\delta}{2}\right)\left(1-\frac{\delta}{10}\right)\left(1-\frac{\delta}{10}\right)>1,$$

as claimed. ∎

Proof of Theorem 2. We will derive Theorem 2 from Theorem 3. By Proposition 14, with $\delta=\frac{1}{2}\varepsilon$, for $k\geq m:=k_2(\delta)$, every triangle-free k-chromatic graph G has at least $\frac{1}{4}(1-\delta)k^2\log k$ vertices. Then $f(x)=\frac{1}{4}(1-\delta)x^2\log x$ is 3-bounded, by Lemma 10. By Theorem 3, (with $\alpha=3$ and $\varepsilon=\delta$), G contains a cycle of length at least $(1-\delta)f(k)$ as well as cycles of at least $(1-\delta)f(\frac{k}{4})$ consecutive lengths, provided $k\geq k_0(\varepsilon):=k_1(\delta,\alpha,m)$. Since

$$(1-\delta)f(k)=\frac{1}{4}(1-\delta)^2k^2\log k\geq\frac{1}{4}(1-2\delta)k^2\log k=\frac{1}{4}(1-\varepsilon)k^2\log k$$

a cycle of length at least $\frac{1}{4}(1-\varepsilon)k^2\log k$ is obtained when $k\geq k_0(\varepsilon)$, as required for Theorem 2. Similarly, we obtain cycles of at least $(1-\delta)f(\frac{k}{4})\geq\frac{1}{64}(1-\varepsilon)k^2\log\frac{k}{4}$ consecutive lengths. This completes the proof. ∎

5. Proof of Theorem 4

Lemma 15. *Let $r\geq 3$ and G be a K_r-free n-vertex graph. Then $\alpha(G)\geq n^{1/(r-1)}-1$.*

Proof. If $r\geq 3$ and $n\leq 2^{r-1}$, then $n^{1/(r-1)}-1\leq 1$, so the claim holds. Let $n>2^{r-1}$. If $r=3$, either $\Delta(G)\geq n^{1/2}$ or the graph is greedily $\lfloor n^{1/2}\rfloor+1$-colorable. Since vertex neighborhoods in a triangle-free graph are independent sets, either case gives an independent set of size at least $n^{1/2}-1$. For $r>3$, either the graph has a vertex v of degree at least $d\geq n^{(r-2)/(r-1)}$, or the graph is $n^{1-1/(r-1)}+1$-colorable. In the latter case, the largest color class is an independent set of size at least $n^{1/(r-1)}-1$. In the former case, since the neighborhood of v induces a K_{r-1}-free graph, by induction it contains an independent set of size at least $d^{1/(r-2)}-1\geq n^{1/(r-1)}-1$, as required. ∎

Lemma 16. *Let $r \geq 3$ and G be a K_r-free n-vertex graph. Then*

$$\chi(G) < 4n^{1-1/(r-1)}.$$

Proof. The function $f(x) = \max\{1, x^{1/(r-1)} - 1\}$ is positive continuous and non-decreasing. Since each nontrivial graph has an independent set of size 1, by Lemmas 13 and 15,

$$\chi(G) \leq 1 + \int_1^n \frac{1}{f(x)}\mathrm{d}x \leq 1 + \int_1^n \frac{2}{x^{1/(r-1)}}\mathrm{d}x < 4n^{1-1/(r-1)}. \qquad \blacksquare$$

Proof of Theorem 4. Let G be an n-vertex K_{r+1}-free graph with $\chi(G) = k \geq 3$. By Lemma 16, $k < 4n^{1-\frac{1}{r}}$, so $n \geq (\frac{k}{4})^{\frac{r}{r-1}} := f(k)$. Since f is $\frac{r}{r-1}$-bounded, the proof is complete by Theorem 3 with $\mathcal{P}$ being the property of K_{r+1}-free graphs. ∎

6. Concluding remarks

• In this paper, we have shown that the length of a longest cycle and the length of a longest interval of lengths of cycles in k-chromatic graphs G are large when G lacks certain subgraphs. In particular, when G has no triangles, this yields a proof of Conjecture 1 (in a stronger form). We believe that the following holds.

Conjecture 17. *Let G be a k-chromatic triangle-free graph and let n_k be the minimum number of vertices in a k-chromatic triangle-free graph. Then G contains a cycle of length at least $n_k - o(n_k)$.*

• If Shearer's bound [10] is asymptotically tight, i.e., $n_k \sim \frac{1}{4}k^2 \log k$, then the lower bound on the length of the longest cycle in any k-chromatic triangle-free graph in Theorem 2 would be asymptotically tight.

Acknowledgment. We thank the referees for the helpful comments.

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Alexandr Kostochka

Department of Mathematics
University of Illinois
at Urbana–Champaign
IL, USA

and

Sobolev Institute of Mathematics
Novosibirsk 630090, Russia
kostochk@math.uiuc.edu

Benny Sudakov

Department of Mathematics
ETH
8092 Zurich
benjamin.sudakov@math.ethz.ch

Jacques Verstraëte

Department of Mathematics
University of California
San Diego, CA, USA
jverstra@math.ucsd.edu.