Discrete Mathematics 339 (2016) 2785–2792

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On a packing problem of Alon and Yuster

A.V. Kostochka[a,b], A. McConvey[a], D. Yager[a,*]

[a] Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
[b] Sobolev Institute of Mathematics, Novosibirsk, Russia

ARTICLE INFO

Article history:
Received 6 October 2015
Received in revised form 16 March 2016
Accepted 19 May 2016
Available online 14 June 2016

Keywords:
Graph packing
Maximum degree
Edge sum

ABSTRACT

Two graphs G_1 and G_2, each on n vertices, *pack* if there exists a bijection f from $V(G_1)$ onto $V(G_2)$ such that $uv \in E(G_1)$ only if $f(u)f(v) \notin E(G_2)$. In 2014, Alon and Yuster proved that, for sufficiently large n, if $|E(G_1)| < n - \delta(G_2)$ and $\Delta(G_2) \leq \sqrt{n}/200$, then G_1 and G_2 pack. In this paper, we characterize the pairs of graphs for which the theorem of Alon and Yuster is sharp. We also prove the stronger result that for sufficiently large n, if $|E(G_1)| \leq n$, $\Delta(G_2) \leq \sqrt{n}/60$, and $\Delta(G_1) + \delta(G_2) \leq n - 1$, then G_1 and G_2 pack whenever there is a vertex $v_1 \in V(G_1)$ such that $d(v_1) = \Delta(G_1)$ and $\alpha(G_1 - N[v_1]) \geq \delta(G_2)$.

1. Introduction

Throughout this paper, the maximum degree and minimum degree of a vertex in a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The size of a largest independent set in G is denoted by $\alpha(G)$.

Two graphs G_1 and G_2 with $|V(G_2)| = |V(G_1)|$ *pack* if there is a bijection $f : V(G_1) \to V(G_2)$ such that if $uv \in E(G_1)$, then $f(u)f(v) \notin E(G_2)$. In other words, graphs G_1 and G_2 pack if G_1 is a subgraph of the complement of G_2. Important results on graph packing were obtained in 1978 by Bollobás and Eldridge [2] and by Sauer and Spencer [6]. In particular, they proved that if two n-vertex graphs together contain at most $\frac{3}{2}n - 2$ edges, they are guaranteed to pack.

Theorem 1.1 ([2,6]). *Let G_1 and G_2 be two n-vertex graphs. If*

$$|E(G_1)| + |E(G_2)| \leq \frac{3}{2}n - 2, \quad (1)$$

then G_1 and G_2 pack.

Restriction (1) cannot be relaxed in view of the pair $\{G_1, G_2\}$ where G_1 is an n-vertex star and G_2 has no isolated vertices. Furthermore, Bollobás and Eldridge showed that if neither graph contains a star on n vertices, then (1) can be relaxed significantly.

Theorem 1.2 ([2]). *Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1), \Delta(G_2) \leq n - 2$ and $|E(G_1)| + |E(G_2)| \leq 2n - 3$, then either G_1 and G_2 pack, or $\{G_1, G_2\}$ is one of the following 7 pairs: $\{2K_2, \overline{K_1 \cup K_3}\}$, $\{\overline{K_2} \cup K_3, K_2 \cup K_3\}$, $\{3K_2, \overline{K_2} \cup K_4\}$, $\{\overline{K_3} \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K_3} \cup K_4\}$, $\{\overline{K_4} \cup K_4, K_2 \cup 2K_3\}$, $\{\overline{K_5} \cup K_4, 3K_3\}$.*

* Corresponding author.
E-mail addresses: kostochk@math.uiuc.edu (A.V. Kostochka), mcconve2@illinois.edu (A. McConvey), yager2@illinois.edu (D. Yager).

http://dx.doi.org/10.1016/j.disc.2016.05.023

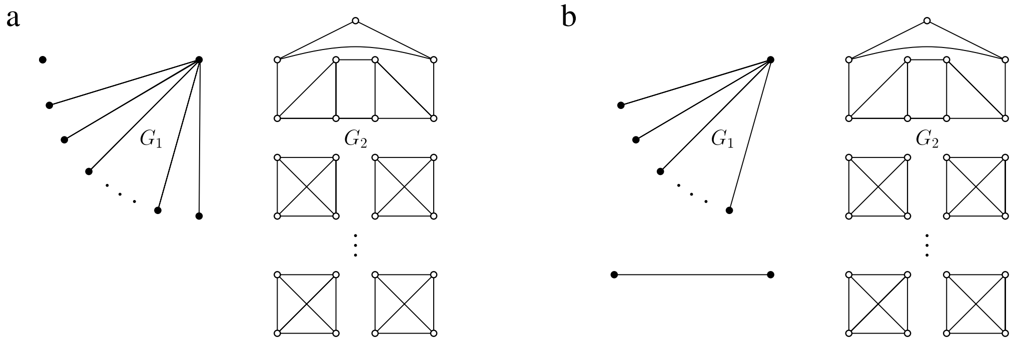

Fig. 1. Sharpness examples for Theorem 1.4 [1].

The restriction $2n-3$ in Theorem 1.2 is again sharp, since the cycle C_n does not pack with $K_{1,n-2} \cup K_1$ and, together, they have $2n-2$ edges. In a sense, Theorems 1.1 and 1.2 describe global properties of the graphs, since there are no restrictions on how the edges are arranged in the graph. On the other hand, the following result of Sauer and Spencer shows that two graphs even with many more edges will pack if their maximum degrees are not too large.

Theorem 1.3 (*[6]*)**.** *Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then G_1 and G_2 pack.*

Recently, Alon and Yuster [1] considered packing a graph with few edges with a graph of bounded maximum degree.

Theorem 1.4 (*[1]*)**.** *For all n sufficiently large, let G_1 and G_2 be n-vertex graphs such that $|E(G_1)| \leq n - \delta(G_2) - 1$ and $\Delta(G_2) \leq \sqrt{n}/200$. Then G_1 and G_2 pack.*

Alon and Yuster phrased their theorem in the language of Turán numbers. The *Turán number* $\mathrm{ex}(n, G)$ of a graph G is the maximum number of edges in an n-vertex graph that does not contain a subgraph isomorphic to G. A result of Ore [5] from 1961 shows that $\mathrm{ex}(n, C_n) = \binom{n-1}{2} + 1$ and that for $n \geq 5$ the only graph with n vertices and $\binom{n-1}{2} + 1$ edges that does not contain a C_n is K_n minus a star with $n-2$ edges [5]. In this language, Theorem 1.4 is the following stronger version of Ore's result.

Theorem 1.5 (*[1]*)**.** *For all n sufficiently large, if G is a graph of order n with no isolated vertices and $\Delta(G) \leq \sqrt{n}/200$, then $\mathrm{ex}(n, G) = \binom{n-1}{2} + \delta(G) - 1$.*

Theorem 1.4 has the additional property that, unlike Ore's result, there are different sharpness examples. In particular, the following two examples are provided in [1], though we rephrase them in the language of graph packing. First, let G_1 be a star with $n-2$ edges and an additional vertex, that is $G_1 = K_{1,n-2} \cup K_1$. Let G_2 be a graph on n vertices in which all vertices but one have degree 3, the last vertex has degree 2 and the neighbors of this vertex are adjacent. Then G_1 has $n - \delta(G_2)$ edges, but the two graphs do not pack (Fig. 1(a)). Alternatively, if G_1 is the disjoint union of a star with $n-3$ vertices and an edge and G_2 remains unchanged, then G_1 and G_2 still do not pack (Fig. 1(b)).

In Fig. 1(a), $\Delta(G_1) + \delta(G_2) \geq n$, so G_1 and G_2 cannot pack since there is no suitable vertex in G_2 to which we might map the vertex of maximum degree in G_1. In Fig. 1(b), $\Delta(G_1) + \delta(G_2) = n - 1$, so a potential packing could (and must) map the vertex of maximum degree in G_1 to the vertex of degree 2 in G_2. However, such an attempt will eventually fail to be a packing because no set of vertices could be mapped to the neighborhood of the degree 2 vertex.

With this observation, we can obtain a larger set of sharpness examples for Theorem 1.4. For example, fix constants n and d with n much larger than d. Let G_2 be a d-regular graph on n vertices consisting of a disjoint union of cliques. Let G_1 be the disjoint union of $d-1$ edges, together with a star containing $n - 2(d-1) - 1$ edges (Fig. 2(a), here $d = 6$). In fact, as long as there is no independent set of size d among the vertices in G_1 not in the star, we can create still more examples, e.g. Fig. 2(b).

The main result of this paper shows that if there is such an independent set of size $\delta(G_2)$, then G_1 and G_2 will pack even if G_1 contains as many as n edges.

Theorem 1.6. *For n sufficiently large ($n \geq 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \leq \sqrt{n}/60$, $|E(G_1)| \leq n$, and $\Delta(G_1) + \delta(G_2) \leq n - 1$. If there is a vertex $v_1 \in V(G_1)$ such that*

$$d(v_1) = \Delta(G_1) \quad \text{and} \quad \alpha(G_1 - N[v_1]) \geq \delta(G_2), \tag{2}$$

then G_1 and G_2 pack.

Our theorem shows that if we are able to appropriately place the vertex of maximum degree in the sparse graph, then the remainder of the graph can also be placed. In fact, Theorem 1.6 is a generalization of Theorem 1.4. Indeed, if $|E(G_1)| \leq n - \delta(G_2) - 1$, then $\Delta(G_1) + \delta(G_2) \leq n - 1$. Also, if $v_1 \in V(G_1)$ with $d(v_1) = \Delta(G_1)$, then $G - N[v_1]$ contains

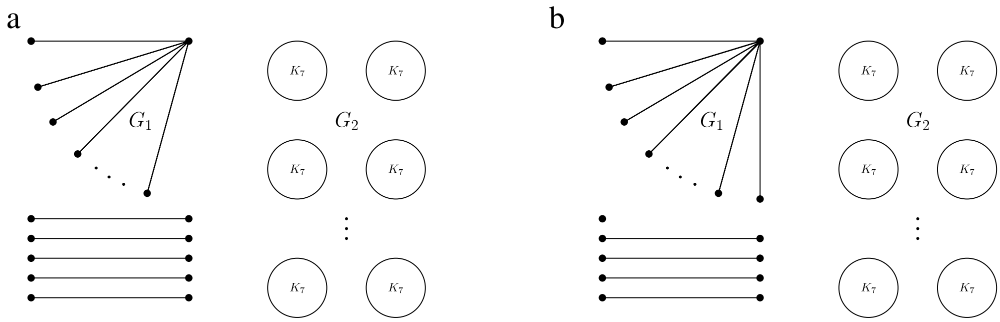

Fig. 2. Additional sharpness examples for Theorem 1.4.

$n - d(v_1) - 1$ vertices and $n - d(v_1) - \delta(G_2) - 1$ edges. Hence, $G - N[v_1]$ contains at least $\delta(G_2)$ components and an independent set of size at least $\delta(G_2)$.

We also adapt the methods used in the proof of Theorem 1.6 to characterize the sharpness examples for Theorem 1.4.

Corollary 1.7. *For n sufficiently large ($n \geq 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \leq \sqrt{n}/60$, $|E(G_1)| \leq n - \delta(G_2)$. Then,*

1. *G_1 and G_2 pack, or*
2. *$\Delta(G_1) + \delta(G_2) = n$, or*
3. *G_1 has exactly $n - \delta(G_2)$ edges and exactly one vertex of degree greater than 1. Moreover, for each $w \in V(G_2)$ with $d(w) = \delta(G_2)$, the neighborhood of w induces a clique.*

The remainder of the paper is organized as follows. In the next section, we provide some notation and preliminary results that will be used in the later sections. Section 3 introduces the framework of the proof and includes several lemmas that will be used in the proof of Theorem 1.6. In Section 4, we prove Theorem 1.6 by providing a packing of G_1 and G_2 in a 4-stage process. Finally, in Section 5, we prove Corollary 1.7 which describes the sharpness examples of Theorem 1.4.

2. Notation and preliminary results

We mainly use standard notation. All logarithms are base e. For distinct vertices x, y in a graph G, by $\|x, y\|$ we denote the number of edges in G connecting x with y. For $W \subseteq V(G)$, by $N(W)$ we denote the set of vertices in G that have a neighbor in W, and let $N[W] = W \cup N(W)$. If $W = \{v\}$, then we write $N(v)$ and $N[v]$ instead of $N(\{v\})$ and $N[\{v\}]$, respectively. For a positive integer d, a graph G is *d-degenerate* if every subgraph G' of G has a vertex of degree (in G') at most d. A *degenerate ordering* $v_1, \ldots, v_n$ of a graph G on n vertices is defined inductively. Let $G_1 = G$ and define v_1 to be a vertex of minimum degree in G_1. For $i \in \{1, \ldots, n-1\}$, let $G_{i+1} = G_i - v_i$ and define v_{i+1} be a vertex of minimum degree in G_{i+1}. A *greedy ordering* of $V(G)$ is defined similarly, with the only difference that we always choose a vertex of the maximum (and not minimum) degree.

We will use the following result from [3] on packing a d-degenerate graph with a graph with a small maximum degree.

Theorem 2.1 ([3]). *Let $d \geq 2$. Let G_1 be a d-degenerate graph of order n and maximum degree Δ_1 and G_2 a graph of order n and maximum degree at most Δ_2. If $40\Delta_1 \log \Delta_2 < n$ and $40d\Delta_2 < n$, then G_1 and G_2 pack.*

We use Theorem 2.1 only for $d = \left\lceil \sqrt{2n} \right\rceil - 1$. The proof of it uses the following lemma that also will be helpful for us.

Lemma 2.2 ([3]). *Fix $\Delta \geq 90$ (hence $\Delta \geq 20 \log \Delta$) and let $m = \left\lceil \frac{\Delta}{\log \Delta} \right\rceil$. Let G be a graph with maximum degree at most Δ. Then, for every $V' \subseteq V(G)$, there exists a partition $\left(V^{(1)}, \ldots, V^{(m)}\right)$ of V' such that for each vertex v of G, the neighborhood $N(v)$ has the following properties:*

1. *for each i, $|N(v) \cap V^{(i)}| \leq 5 \log \Delta$,*
2. *for each i_1 and i_2, $|N(v) \cap (V^{(i_1)} \cup V^{(i_2)})| \leq 8.7 \log \Delta$, and*
3. *for each i_1, i_2, and i_3, $|N(v) \cap (V^{(i_1)} \cup V^{(i_2)} \cup V^{(i_3)})| \leq 12.3 \log \Delta$.*

Throughout this paper, we will consider two n-vertex graphs G_1 and G_2 that satisfy the conditions of Theorem 1.6. For $i \in \{1, 2\}$, we let $V_i = V(G_i)$ and $E_i = E(G_i)$. Similarly, let Δ_i denote the maximum degree of G_i and δ_i denote the minimum degree of G_i. We will construct the packing $f : V_1 \to V_2$ iteratively. For subsets $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$, we say that $f' : W_1 \to W_2$ is a *partial packing* of G_1 and G_2 if f' is a packing of $G_1[W_1]$ and $G_2[W_2]$. Throughout the proof, we will have a partial packing f of G_1 and G_2 and enlarge the domain of f at each step.

3. Setup

We will construct a packing $f : V_1 \to V_2$ in four stages. In the first two stages, we consider each vertex $v \in V_1$ of large degree and for each such vertex, we find a permissible vertex in V_2 for its image. Then, we use a technique of Alon and Yuster in [1] to find a set $X \subseteq V_1$ such that an assignment $f(X) = N(f(v))$ keeps f a partial packing. Lemmas 3.1 and 3.2 show that, for each vertex $v \in V_1$ with large degree, we can find a permissible set X to map to $N(f(v))$. Lemma 2.2 will guarantee $N(f(v))$ is evenly distributed and we will then use a method similar to [3] to construct the packing of the remaining vertices.

First, observe that

$$G_1 \text{ is } d\text{-degenerate for } d = \left\lceil \sqrt{2n} \right\rceil - 1. \tag{3}$$

Indeed, if there is a subgraph $H \subseteq G_1$ such that $\delta(H) > d$, then $\delta(H) \geq d + 1$ and $|V(H)| > d + 2$. So

$$2|E(H)| = \sum_{v \in H} d(v) \geq |V(H)| \cdot \delta(H) \geq (d + 2)(d + 1) > 2n,$$

a contradiction to $|E(G_1)| \leq n$. Thus, (3) holds.

Since $\Delta_2 \leq \frac{\sqrt{n}}{60}$, we obtain $40d\Delta_2 < 40\sqrt{2n}\frac{\sqrt{n}}{60} < n$. If also $40\Delta_1 \log \Delta_2 < n$, then G_1 and G_2 pack by Theorem 2.1. Thus we assume that $40\Delta_1 \log \Delta_2 \geq n$. Then, since $\Delta_2 < \sqrt{n}$,

$$\Delta_1 > \frac{n}{20 \log n}. \tag{4}$$

Let $V_1 = \{v_1, \ldots, v_n\}$ and $d(v_1) \geq \cdots \geq d(v_n)$. We also may assume that (2) holds. Let $k \in [n]$ be the largest integer such that $d(v_k) \geq \frac{n}{50 \log n}$. Since $e_1 \leq n$, we have $2n \geq \sum_{i=1}^{k} d(v_i) \geq k \left(\frac{n}{50 \log n}\right)$ and so $k \leq 100 \log n$.

Lemma 3.1. *G_1 has an independent set $B_1 \subseteq V_1 - N[v_1]$ with $|B_1| = \delta_2$. Moreover, if $k > 1$, then such a set B_1 can be chosen so that each vertex in it has degree at most 2 in G_1.*

Proof. By (2), G_1 has an independent set $B_1 \subseteq V_1 - N[v_1]$ such that $|B_1| \geq \delta(G_2)$. This proves the first part. If $k \geq 2$, then $d(v_2) \geq \frac{n}{50 \log n}$.

The subgraph $G' = G_1[V_1 - v_1 - v_2]$ has $n - 2$ vertices and at most $n - d(v_1) - d(v_2) + \|v_1, v_2\|$ edges. Then G' has at least $d(v_1) + d(v_2) - 2 - \|v_1, v_2\|$ tree components and therefore contains an independent set of size at least $d(v_1) + d(v_2) - 2 - \|v_1, v_2\|$. Moreover, we form this independent set using only vertices of degree at most one in G'. Let B'_1 denote the set of these vertices that are contained in $V_1 - N[v_1] - v_2$. By the above, $|B'_1| \geq d(v_2) - 2$. Since $n \geq 10^9$ and $d(v_2) \geq \frac{n}{50 \log n}$,

$$|B'_1| \geq \frac{n}{50 \log n} - 2 \geq \delta_2.$$

Since each vertex in B'_1 has degree at most 1 in $G_1 - v_1 - v_2$ and $B_1 \cap N[v_1] = \emptyset$, every vertex in B'_1 has degree at most 2 in G_1. So we let B_1 be a subset of B'_1 of cardinality δ_2. $\square$

When $k \geq 2$, we also wish to find, for each $i \in \{2, \ldots, k\}$, an independent set $B_i \subseteq V_1 - N[v_i]$ such that we can map the vertices of B_i to the neighborhood of $f(v_i)$.

Lemma 3.2. *Let $k \geq 2$ and B_1 satisfy Lemma 3.1. There exist disjoint sets $B_2, \ldots, B_k$ such that*

(a) *$|B_i| \geq \Delta_2$ for each $i \in \{2, \ldots, k\}$,*
(b) *$B_j \cap B_i = \emptyset$ for all $j \neq i$,*
(c) *each vertex in $\bigcup_{i=1}^{k} B_i$ has degree at most 2 in G_1,*
(d) *the set $\bigcup_{i=1}^{k} B_i$ is independent in G_1,*
(e) *each vertex in $V_1 - v_1$ is adjacent in G_1 to at most one vertex in $\bigcup_{j=2}^{k} B_j$.*

Proof. Let $W \subseteq V_1$ be the set of all vertices reachable in $G_1 - v_1$ from $\{v_2, \ldots, v_k\}$. In particular, $\{v_2, \ldots, v_k\} \subseteq W$. By definition, $G_1[W]$ has at least $|W| - (k - 1)$ edges. Let $X = V_1 - W - v_1$. Then $|X| = n - 1 - |W|$ and, since G_1 has at most n edges, $|E(G_1[X])| \leq n - [|W| - (k - 1)] - d(v_1)$. Therefore, the number of tree components in $G_1[X]$ is at least $d(v_1) - k$. We form an independent set B by choosing one leaf or isolated vertex from each tree component in $G[X]$ and then removing all vertices in $N[B_1]$. Since each vertex in B_1 has degree at most 2 by Lemma 3.1, we have

$$|B| \geq (d(v_1) - k) - 3\delta_2. \tag{5}$$

Suppose that

$$d(v_1) - k - 3\delta_2 \geq (k - 1)\Delta_2. \tag{6}$$

Then by (5), B can be partitioned into $k-1$ disjoint sets $B_2, \ldots, B_k$, each of size at least Δ_2. Since all vertices $u \in B$ are leaves or isolated vertices in distinct components of $G_1 - v_1$, the claims (c) and (e) of the lemma hold. Since the sets $B_2, \ldots, B_k$ are formed by partitioning an independent set that is disjoint from $N[B_1]$, claims (b) and (d) also hold. So, to prove the lemma, it is enough to check that (6) holds. Now,

$$\begin{aligned} (d(v_1) - k) - 3\delta_2 &\geq (k-1)\Delta_2 && \text{if} \\ (d(v_1) - k) - 3\Delta_2 &\geq (k-1)\Delta_2 && \text{if} \\ d(v_1) - 1 - 3\Delta_2 &\geq (k-1)(\Delta_2 + 1) && \text{if} \\ d(v_1) &\geq (k+2)(\Delta_2+1) - 2. \end{aligned}$$

Since $k \leq 100 \log n$, $d(v_1) \geq n/(20 \log n)$, and $\Delta_2 \leq \sqrt{n}/60$, the last inequality follows from

$$\frac{n}{20 \log n} \geq (100 \log n + 2)\left(\frac{\sqrt{n}}{60} + 1\right),$$

which holds for $n \geq 10^9$. This proves (6) and thus the lemma. □

4. Proof of Theorem 1.6

Let $B_1, B_2, \ldots, B_k$ be as stipulated in Lemmas 3.1 and 3.2. Note that by 3.2(c), $\{v_1, \ldots, v_k\} \cap (B_1 \cup \cdots \cup B_k) = \emptyset$. Let $m = \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil$ and $(V^{(1)}, \ldots, V^{(m)})$ be a partition of V_2 with the properties guaranteed by Lemma 2.2. Order the parts of the partition so that $|V^{(1)}| \geq \cdots \geq |V^{(m)}|$. For $i = 1, \ldots, k$, let $\hat{V}^{(i)} = \bigcup_{j=1}^{i} V^{(j)}$. We will construct a packing $f : V_1 \to V_2$ in 4 stages. At each step in the proof, we ensure that f remains a partial packing.

Stage 1. Let $w_1 \in V_2$ be a vertex of minimum degree in G_2. Define $f(v_1) = w_1$. For each $w' \in N_{G_2}(w_1)$, we can choose an element $u \in B_1$ and assign $f(u) = w'$. In this way, all neighbors of w_1 are matched and, since $B_1 \cup \{v_1\}$ is an independent set, after this assignment f remains a partial packing.

Stage 2. If $k = 1$, then proceed to Stage 3. Otherwise, we will iteratively match $v_2, \ldots, v_k$ with vertices of G_2. During iteration i, we will match v_i to some vertex $f(v_i)$ in $V^{(1)}$. We will then proceed to match an unmatched subset of B_i to $N_{G_2}(f(v_i))$. Notice that after iteration i, the function f will remain a partial packing, the only matched vertices of V_1 will be $v_1, \ldots, v_i$, and vertices from $\bigcup_{j=1}^{i} B_j$, and at most $i(\Delta_2 + 1)$ vertices of G_1 (and, respectively, G_2) will be matched.

Consider the ith iteration. At this point, we have matched vertices $v_1, \ldots, v_{i-1} \in V_1$ to vertices $w_1, \ldots, w_{i-1} \in V_2$, respectively. Since $w_2, \ldots, w_{i-1}$ were chosen to be in $V^{(1)}$ and w_1 may also have been in $V^{(1)}$, at most $i - 1 < k$ of these vertices are in $V^{(1)}$. The only other matched vertices in G_2 are in $\bigcup_{j=1}^{i-1} N_{G_2}(w_j)$. By Lemma 2.2, $|N_{G_2}(w_j) \cap V^{(1)}| \leq 5 \log \Delta_2$. So there are at most $k(1 + 5 \log \Delta_2)$ vertices in $V^{(1)}$ that have already been matched. There are at least $\left\lceil \frac{n}{m} \right\rceil - k(1 + 5 \log \Delta_2)$ remaining vertices in $V^{(1)}$. From these remaining vertices, we will choose a vertex w_i such that after assigning $f(v_i) = w_i$, the function f remains a partial packing. If a vertex $x \in N(v_i)$ is already matched, then either $x \in \{v_1, \ldots, v_{i-1}\}$ or $x \in \bigcup_{j=1}^{i-1} B_j$. However, for each $j < i$, $N(f(v_j))$ is already matched, so v_i will not be matched to a neighbor of $f(v_j)$. Further, by Lemma 3.2, no vertex adjacent to $\{v_1, \ldots, v_{i-1}\}$ was chosen to be in B_j. So any available choice for w_i will allow f to remain a partial packing. Since there were $\left\lceil \frac{n}{m} \right\rceil - k(1 + 5 \log \Delta_2) > 0$ vertices to choose from, there is a permissible choice of w_i.

To complete the iteration, we must map some subset of B_i to the unmatched neighbors of w_i. However, by Lemma 3.2(d), B_j and B_i were chosen to be disjoint for each j, so no vertex in B_i is already matched. Further, if sending a vertex $x \in B_i$ to an unmatched vertex $y \in N(w_i)$ causes f to no longer be a partial packing, then x has a neighbor u such that $y \in N(f(u))$. Notice that if this is the case, then $u \notin \{v_1, \ldots, v_{i-1}\}$, since y is unmatched and $N(f(\{v_1, \ldots, v_{i-1}\}))$ contains only matched vertices. Therefore, if there is an $x \in B_i$ such that sending x to a vertex $y \in N_{G_2}(w_i)$ forces f to not be a partial packing, then $x \in \bigcup_{j=1}^{i-1} N(B_j)$. Again by Lemma 3.2, B_i does not contain any such vertex, so any vertex $x \in B_i$ can be mapped to any unmatched vertex in $N_{G_2}(w_i)$. By Lemma 3.2, $|B_i| \geq \Delta_2$, so we can match a subset of B_i to the neighborhood of w_i.

Stage 3. Let $W_1 \subset V_1$ be the set of vertices that have been matched before the start of Stage 3 and let V_2' be their matches. Recall that G_1 is d-degenerate for some $d \leq \sqrt{2n}$. Since $G_1[V_1 - W_1] \subseteq G_1$, it must also be d-degenerate. We will define disjoint subsets $W_2, \ldots, W_m$ with the goal of sending W_i into $\hat{V}^{(i)}$ for each i.

Let $X_1 = Y_1 = \emptyset$ and $z := \left\lceil \frac{n}{15m} \right\rceil \leq \left\lceil \frac{n \log \Delta_2}{15 \Delta_2} \right\rceil \leq 2\sqrt{n} \log n$. We now inductively construct sets X_i, Y_i and W_i for $i \in 2, \ldots, m$. Let $\hat{X}_i = \bigcup_{j=1}^{i} X_j$, $\hat{Y}_i = \bigcup_{j=1}^{i} Y_j$, and $\hat{W}_i = \bigcup_{j=1}^{i} W_j$ and then consider a greedy ordering of $V_1 - \hat{W}_{i-1}$. Define X_i to be the first z vertices in this ordering, so $|\hat{X}_i| = (i-1)z$. Add to Y_i any vertex in $y \in V_1 - \hat{W}_{i-1} - X_i$ such that y has at least $4d$ neighbors in $\{\hat{W}_{i-1} \cup X_i \cup Y_i\} - W_1$. Continue to add vertices to Y_i until every remaining vertex has at most $4d$ neighbors in $\hat{W}_{i-1} \cup X_i \cup Y_i$. Finally, let $W_i = X_i \cup Y_i$.

We next show that $|\hat{W}_i|$ is not too large. We have $e(G_1[\hat{W}_i - W_1]) \geq 4d|\hat{Y}_i|$, since each vertex in $\hat{Y}_i$ has at least $4d$ edges to previously matched vertices and at most $k+1$ of them are incident to vertices mapped in Stage 1. However, since $G[\hat{W}_i - W_1]$

is d-degenerate and has $|\hat{X}_i|+|\hat{Y}_i|$ vertices, it has less than $(|\hat{X}_i|+|\hat{Y}_i|)d$ edges. This implies that $4d|\hat{Y}_i|-(k+1) < d(|\hat{X}_i|+|\hat{Y}_i|)$. Since $d \geq 1$, solving for $|\hat{Y}_i|$ yields $|\hat{Y}_i| < \frac{(i-1)z}{3} + \frac{1}{3}(k+1)$. Finally, since $|\hat{W}_i| = |W_1| + |\hat{X}_i| + |\hat{Y}_i|$ and $W_1 \leq k(\Delta_2+1)$, we have

$$
\begin{aligned}
|\hat{W}_i| &< \frac{4(i-1)}{3}z + k\left(\Delta_2 + \frac{4}{3}\right) + \frac{1}{3} \\
&\leq \frac{4(i-1)}{3}\left\lceil \frac{n}{15m} \right\rceil + k\left(\Delta_2 + \frac{4}{3}\right) + \frac{1}{3} \\
&\leq \frac{4(i-1)}{3}\frac{n}{15m} + \frac{4(i-1)}{3} + k\left(\Delta_2 + \frac{4}{3}\right) + \frac{1}{3} \\
&\leq \frac{4(i-1)}{3}\frac{n}{15m} + \frac{4i}{3} + k\left(\Delta_2 + \frac{4}{3}\right) - 1 \\
&\leq \left(4(i-1) + \frac{60im}{n} + \frac{km(45\Delta_2+60)}{n}\right)\frac{n}{45m} - 1 \\
&\leq \left(4 + \frac{60m}{n} + \frac{km(45\Delta_2+60)}{n}\right)\frac{in}{45m} - 1 \\
&\leq \left(4 + \frac{60}{n}\left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil + \frac{k(45\Delta_2+60)\left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil}{n}\right)\frac{in}{45m} - 1.
\end{aligned} \tag{7}
$$

Finally, recall that $\Delta_2/\log(\Delta_2) \leq \sqrt{n}/(60\log(\sqrt{n}/60))$ and $k \leq 100\log n$. We can substitute these upper bounds into (7) and calculate that for $n \geq 10^9$,

$$
4 + \frac{60}{n}\left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil + \frac{k(45\Delta_2+60)\left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil}{n} < 9. \tag{8}
$$

Therefore, by (7) and (8),

$$
|\hat{W}_i| < \frac{in}{5m}. \tag{9}
$$

Now, we place W_i in $\hat{V}^{(i)}$ for each $i \in \{2, \ldots, m\}$. Consider a degenerate ordering of the vertices in W_i. We pack the vertices into $V^{(i)}$ in this order. Suppose it is the turn of vertex w to be packed. In particular, we have placed at most $|\hat{W}_i|$ vertices so far, so there are at least $\frac{in}{m} - |\hat{W}_i| \geq \frac{4in}{5m}$ free vertices left in $\hat{V}^{(i)}$. Suppose we send w to some unmatched vertex $v \in \hat{V}^{(i)}$. If w has a neighbor w' already matched to a neighbor of v, then f is not a partial packing. We show that the number of such bad vertices v is at most $\frac{4in}{5m}$.

Let w' be a matched neighbor of w. Then either $w' \in W_1$ or $w' \in \hat{X}_j \cup \hat{Y}_j$ for some j. If $w' \in \{v_1, \ldots, v_k\}$, by Stage 1 and 2, all neighbors of the images of $\{v_1, \ldots, v_k\}$ are already matched and are therefore not adjacent to v. On the other hand, by Lemma 3.2, w is only adjacent to at most one vertex of $W_1 - \{v_1, \ldots, v_k\}$ (since vertices in B_i were chosen from distinct components of $V_1 - v_1$).

Next, since the vertices of W_i are placed using a degenerate ordering of W_i and each vertex in W_i has fewer than $4d$ vertices in $\hat{W}_{i-1} - W_1$, vertex w has at most $5d$ neighbors in $\hat{W}_i - W_1$. We conclude that w has at most $1 + 5\sqrt{2n} \leq 8\sqrt{n}$ previously matched neighbors adjacent to unmatched neighbors in V_2. Further, by Lemma 2.2 the image of each of these neighbors has at most $5i\log\Delta_2$ neighbors in $\hat{V}^{(i)}$. Thus, there are at most $40i\sqrt{n}\log\Delta_2$ choices for v that cause f to not be a partial packing. Since we have $|W_i| > \frac{4in}{5m}$ vertices to choose from and $\frac{4n}{5m} - 40\sqrt{n}\log\Delta_2 > 0$, there is a vertex to which we can send w.

Stage 4. We now place the remaining vertices, i.e. those in $V_1 - \hat{W}_m$. Consider a degenerate ordering of $V_1 - \hat{W}_m$ and place these vertices in the reverse order. Suppose it is the turn of vertex w to be packed. Then, there is some unmatched vertex $v \in V_2$. We show that either we can send w to v or that there is another previously matched vertex $w' \in V_1 - \hat{W}_m$ such that w can be matched to the image of w', let us call it $v' \in V_2$, and w' can be matched to v.

Notice that for any $x \in N(w)$, we are unable to match w to an unmatched vertex $v \in V_2$ that is a neighbor of the image of x. Let us call such vertices *red/blue neighbors*, since they can be reached from w via a 2-edge path with the first edge being $wx \in E_1$ (i.e. red) and the second edge being $f(x)v \in E_2$ (i.e. blue). As in Stage 3, we notice that when it is the turn of w to be packed, it has at most $4d$ neighbors in $\hat{W}_m - W_1$ and, since we are placing $V_1 - \hat{V}_m$ in the reverse of a degenerate order, it has at most d neighbors in $V_1 - \hat{V}_m$ previously matched during Stage 4. In total, w has at most $5d$ previously matched neighbors in $V_1 - W_1$. By Lemma 3.2, w has at most 1 neighbor in $W_1 - \{v_1, \ldots, v_k\}$. The vertex w may be adjacent to many vertices in $\{v_1, \ldots, v_k\}$ but, by Stage 1 and Stage 2, the images of $\{v_1, \ldots, v_k\}$ have no unmatched neighbors so no red/blue neighbors may arise from these vertices. We conclude that, apart from $\{v_1, \dots, v_k\}$, the vertex w has at most $2+5d \le 8\sqrt{n}$ previously matched neighbors. The image of each of these neighbors has at most Δ_2 blue neighbors, so there are at most $8\sqrt{n}\Delta_2 \le \frac{8n}{60}$ red/blue neighbors of w.

On the other hand, for each $v_i \in \{v_1, \dots, v_k\}$, the neighbors of $f(v_i)$ are matched to vertices in W_1. So v has no neighbors in $f(W_1)$. We now count the number of vertices $x \in V_1$ such that x has a neighbor in V_1 matched to a neighbor of v in V_2. We call this set of vertices the *blue/red neighbors* of v. In particular, we only concern ourselves with blue/red neighbors x such that $x \notin \hat{W}_m$. We will use the method used in [4] to bound the number of such neighbors.

Let $\mathrm{br}(v)$ be the number of blue/red neighbors in $V_1 - \hat{W}_m$ and let $n_i = |N_{G_2}(v) \cap V^{(i)}|$. Recall that, in Stage 3, we considered a greedy ordering of $V_1 - \hat{W}_{i-1}$. Let D_i be the maximum degree of a vertex in $G[V_1 - W_{i-1}]$. In particular, if $x \in X_j$ is matched to a vertex in $\hat{V}^{(j)}$ for $j \ge 2$, then x has at most D_j neighbors in $V_1 - \hat{W}_i$ and at least D_{j+1} such neighbors. This implies $|X_2|D_3 + \cdots + |X_{m-1}|D_m = z(D_3 + \cdots + D_m) < n$, since there are at most n edges in G_1.

Further, we know that if a vertex $v_i \in V_1$ has more than $n/(50\log n)$ neighbors in G_1, then not only was it matched in Stage 1, but all vertices of $N_{G_2}(f(v_i))$ are matched in Stage 1 as well. So if a vertex x is matched to a neighbor of v, then $d(x) \le n/(100\log n)$. In particular,

$$\begin{aligned}
\mathrm{br}(v) &\le n_1\frac{n}{50\log n} + n_2\frac{n}{50\log n} + \sum_{k=3}^{m} n_k D_k \\
&\le \frac{n}{50\log n}(n_1+n_2) + (D_3+\cdots+D_m)5\log\Delta_2 \\
&\le \frac{n}{50\log n}(8.7\log\Delta_2) + \frac{n}{z}5\log\Delta_2 \\
&\le \frac{4.35n}{50} + (15m)5\log\Delta_2 \\
&\le \frac{4.35n}{50} + 75\left\lceil\frac{\Delta_2}{\log\Delta_2}\right\rceil\log\Delta_2 \\
&\le \frac{4.35n}{50} + 75\Delta_2 + 75\log\Delta_2 \\
&\le \frac{4.35n}{50} + \frac{5\sqrt{n}}{4} + 75\log\left(\sqrt{n}/60\right) < \frac{n}{10}.
\end{aligned}$$

We know that there are fewer than $\frac{n}{10}$ blue/red neighbors of v, at most $\frac{8n}{60}$ red/blue neighbors of w, and at most $\frac{n}{5}$ vertices in $\hat{W}_m$. This means that either we can send w to v and maintain that f is a partial packing or there is a vertex w' in $V_1 - \hat{W}_m$ placed on a vertex $v' \in V_2$ such that w' is not a blue/red neighbor of v and also that v' is not a red/blue neighbor of w. This implies that we can send v to w' and w to v' and maintain that f is a partial packing. Repeating this process for each unmatched vertex in V_1 yields a packing of G_1 and G_2.

5. Proof of Corollary 1.7

Let G_1 and G_2 be graphs such that $|E_1| \le n-\delta_2$ and $\Delta_2 \le \sqrt{n}/60$. We will show that if G_1 and G_2 do not satisfy conclusion (2) nor conclusion (3) of Corollary 1.7, then they pack. Let $v_1 \in V_1$ be a vertex of maximum degree in G_1. If $\Delta_1 = n-\delta_2$, then part 2 of the theorem holds and the proof is complete. So we assume that $\Delta_1 \le n-\delta_2-1$.

Let $v_1, \dots, v_n$ be an ordering of $V(G_1)$ such that $d(v_1) \ge \cdots \ge d(v_n)$ and let $X \subseteq E(G_1 - v_1)$ be the set of edges incident to $N(v_1)$. The subgraph $G_1 - N[v_1]$ has $n - d(v_1) - 1$ vertices and $|E_1| - d(v_1) - |X|$ edges. In particular, the number of tree components in $G_1 - N[v_1]$ is at least

$$(n - |E_1|) + |X| - 1. \tag{10}$$

By Theorem 1.6, if there exists an independent set S in $G_1 - N[v_1]$ of size δ_2, then G_1 and G_2 pack. We form an independent set S by taking one vertex from each component in $G_1 - N[v_1]$.

If $|E_1| \le n-\delta_2-1$ or $|X| \ge 1$, then by (10) there are at least δ_2 tree components in $G_1 - N[v_1]$ and so $|S| \ge \delta_2$. Therefore, we assume that $|E_1| = n-\delta_2$ and $|X| = 0$. Hence, the number of tree components in $G_1 - N[v_1]$ is exactly $\delta_2 - 1$. Moreover, if any tree component contains at least three vertices, two vertices from the same component could be selected to be in S and we would obtain an independent set of size δ_2. Finally, we assume that $G_1 - N[v_1]$ contains no other components, as otherwise we could add an additional vertex to S and obtain an independent set of size δ_2. After these assumptions, we are now in the case that G_1 is a forest with exactly $n-\delta_2$ edges and v_1 is the only vertex with degree greater than 1. Further, since $d(v_1) \le n-1-\delta_2$, some component of $G_1 - N[v_1]$ contains an edge.

We finally show that if conclusion (3) of Corollary 1.7 is not satisfied, then G_1 and G_2 pack. In this case, there is a vertex w_1 of degree δ_2 such that $N(w_1)$ does not induce a clique. By our assumptions on G_1, we can find an independent set S of size $\delta_2 - 1$ by selecting one vertex from each component. Recall that $G_1 - N[v_1]$ has some component that contains exactly one edge and let x be the vertex in that component not chosen to be in S. Let $B_1' \subseteq V_1 - N[v_1]$ be the set of vertices obtained by adding x to S and note that $G[B_1']$ contains exactly δ_2 vertices and exactly one edge.

We can now construct a packing of G_1 and G_2 almost exactly as we did in the proof of Theorem 1.6. In *Stage* 1, we define $f(v_1) = w_1$ and wish to map the set B_1' to the neighborhood of w_1 so that f remains a partial packing. Since $N(w_1)$ does not induce a clique and B_1' contains only one edge, such a mapping is possible. Now, since v_1 is the only vertex in G_1 with degree greater than 1, we proceed directly to *Stage* 3. However, *Stage* 3 and *Stage* 4 follow exactly as they did in Section 4, resulting in the desired packing of G_1 and G_2.

Acknowledgments

We thank both referees for helpful comments. The first author's Research was supported in part by NSF grant DMS-1266016 and by grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research. The second and third authors gratefully acknowledge support from the Campus Research Board, University of Illinois.

References

[1] N. Alon, R. Yuster, The turán number of sparse spanning graphs, J. Combin. Theory Ser. B 103 (3) (2013) 337–343.
[2] B. Bollobás, S.E. Eldridge, Packings of graphs and applications to computational complexity, J. Combin. Theory Ser. B 25 (2) (1978) 105–124.
[3] B. Bollobás, A. Kostochka, K. Nakprasit, Packing *d*-degenerate graphs, J. Combin. Theory Ser. B 98 (1) (2008) 85–94.
[4] A.V. Kostochka, G. Yu, Packing of graphs with small product of sizes, J. Combin. Theory Ser. B 98 (6) (2008) 1411–1415.
[5] O. Ore, Arc coverings of graphs, Ann. Mat. Pura Appl. 55 (1) (1961) 315–321.
[6] N. Sauer, J. Spencer, Edge disjoint placement of graphs, J. Combin. Theory Ser. B 25 (3) (1978) 295–302.