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DP-colorings of graphs with high chromatic number

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ABSTRACT

DP-coloring is a generalization of list coloring introduced recently by Dvořák and Postle (2015). We prove that for every n-vertex graph G whose chromatic number $\chi(G)$ is "close" to n, the DP-chromatic number of G equals $\chi(G)$. "Close" here means $\chi(G) \geq n - O(\sqrt{n})$, and we also show that this lower bound is best possible (up to the constant factor in front of $\sqrt{n}$), in contrast to the case of list coloring.

1. Introduction

We use standard notation. In particular, $\mathbb{N}$ denotes the set of all nonnegative integers. For a set S, $\mathrm{Pow}(S)$ denotes the power set of S, i.e., the set of all subsets of S. All graphs considered here are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex and the edge sets of G, respectively. For a set $U \subseteq V(G)$, $G[U]$ is the subgraph of G induced by U. Let $G - U := G[V(G) \setminus U]$, and for $u \in V(G)$, let $G - u := G - \{u\}$. For $U_1, U_2 \subseteq V(G)$, let $E_G(U_1, U_2) \subseteq E(G)$ denote the set of all edges in G with one endpoint in U_1 and the other one in U_2. For $u \in V(G)$, $N_G(u) \subset V(G)$ denotes the set of all neighbors of u, and $\deg_G(u) := |N_G(u)|$ is the *degree* of u in G. We use $\delta(G)$ to denote the *minimum degree* of G, i.e., $\delta(G) := \min_{u \in V(G)} \deg_G(u)$. For $U \subseteq V(G)$, let $N_G(U) := \bigcup_{u \in U} N_G(u)$. To simplify notation, we write $N_G(u_1, \ldots, u_k)$ instead of $N_G(\{u_1, \ldots, u_k\})$. A set $I \subseteq V(G)$ is *independent* if $I \cap N_G(I) = \varnothing$, i.e., if $uv \notin E(G)$ for all $u, v \in I$. We denote the family of all independent sets in a graph G by $\mathcal{I}(G)$. The complete k-vertex graph is denoted by K_k.

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1.1. The Noel–Reed–Wu theorem for list coloring

Recall that a *proper coloring* of a graph G is a function $f : V(G) \to Y$, where Y is a set of *colors*, such that $f(u) \neq f(v)$ for every edge $uv \in E(G)$. The smallest $k \in \mathbb{N}$ such that there exists a proper coloring $f : V(G) \to Y$ with $|Y| = k$ is called the *chromatic number* of G and is denoted by $\chi(G)$.

List coloring was introduced independently by Vizing [10] and Erdős, Rubin, and Taylor [5]. A *list assignment* for a graph G is a function $L : V(G) \to \mathrm{Pow}(Y)$, where Y is a set. For each $u \in V(G)$, the set $L(u)$ is called the *list* of u, and its elements are the *colors available* for u. A proper coloring $f : V(G) \to Y$ is called an *L-coloring* if $f(u) \in L(u)$ for each $u \in V(G)$. The *list chromatic number* $\chi_\ell(G)$ of G is the smallest $k \in \mathbb{N}$ such that G is L-colorable for each list assignment L with $|L(u)| \geq k$ for all $u \in V(G)$. It is an immediate consequence of the definition that $\chi_\ell(G) \geq \chi(G)$ for every graph G.

It is well-known (see, e.g., [5,10]) that the list chromatic number of a graph can significantly exceed its ordinary chromatic number. Moreover, there exist 2-colorable graphs with arbitrarily large list chromatic numbers. On the other hand, Noel, Reed, and Wu [6] established the following result, which was conjectured by Ohba [7, Conjecture 1.3]:

Theorem 1.1 (*Noel–Reed–Wu [6]*). *Let G be an n-vertex graph with $\chi(G) \geq (n-1)/2$. Then $\chi_\ell(G) = \chi(G)$.*

The following construction was first studied by Ohba [7] and Enomoto, Ohba, Ota, and Sakamoto [4]. For a graph G and $s \in \mathbb{N}$, let $\mathbf{J}(G, s)$ denote the *join* of G and a copy of K_s, i.e., the graph obtained from G by adding s new vertices that are adjacent to every vertex in $V(G)$ and to each other. It is clear from the definition that for all G and s, $\chi(\mathbf{J}(G, s)) = \chi(G) + s$. Moreover, we have $\chi_\ell(\mathbf{J}(G, s)) \leq \chi_\ell(G) + s$; however, this inequality can be strict. Indeed, Theorem 1.1 implies that for every graph G and every $s \geq |V(G)| - 2\chi(G) - 1$,

$$\chi_\ell(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s)),$$

even if $\chi_\ell(G)$ is much larger than $\chi(G)$. In view of this observation, it is interesting to consider the following parameter:

$$Z_\ell(G) := \min\{s \in \mathbb{N} \,:\, \chi_\ell(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s))\}, \tag{1.1}$$

i.e., the smallest $s \in \mathbb{N}$ such that the list and the ordinary chromatic numbers of $\mathbf{J}(G, s)$ coincide. The parameter $Z_\ell(G)$ was explicitly defined by Enomoto, Ohba, Ota, and Sakamoto in [4, page 65] (they denoted it $\psi(G)$). Recently, Kim, Park, and Zhu (personal communication, 2016) obtained new lower bounds on $Z_\ell(K_{2,n})$, $Z_\ell(K_{n,n})$, and $Z_\ell(K_{n,n,n})$. One can also consider, for $n \in \mathbb{N}$,

$$Z_\ell(n) := \max\{Z_\ell(G) \,:\, |V(G)| = n\}. \tag{1.2}$$

The parameter $Z_\ell(n)$ is closely related to the Noel–Reed–Wu Theorem, since, by definition, there exists a graph G on $n + Z_\ell(n) - 1$ vertices whose ordinary chromatic number is at least $Z_\ell(n)$ and whose list and ordinary chromatic numbers are distinct. The finiteness of $Z_\ell(n)$ for all $n \in \mathbb{N}$ was first established by Ohba [7, Theorem 1.3]. Theorem 1.1 yields an upper bound $Z_\ell(n) \leq n - 5$ for all $n \geq 5$; on the other hand, a result of Enomoto, Ohba, Ota, and Sakamoto [4, Proposition 6] implies that $Z_\ell(n) \geq n - O(\sqrt{n})$.

1.2. DP-colorings and the results of this paper

The goal of this note is to study analogs of $Z_\ell(G)$ and $Z_\ell(n)$ for the generalization of list coloring that was recently introduced by Dvořák and Postle [3], which we call *DP-coloring*. Dvořák and Postle invented DP-coloring to attack an open problem on list coloring of planar graphs with no cycles of certain lengths.

Definition 1.2. Let G be a graph. A *cover* of G is a pair (L, H), where H is a graph and $L : V(G) \to \mathrm{Pow}(V(H))$ is a function, with the following properties:

- the sets $L(u)$, $u \in V(G)$, form a partition of $V(H)$;
- if $u, v \in V(G)$ and $L(v) \cap N_H(L(u)) \neq \varnothing$, then $v \in \{u\} \cup N_G(u)$;

– each of the graphs $H[L(u)]$, $u \in V(G)$, is complete;
– if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (not necessarily perfect and possibly empty).

Definition 1.3. Let G be a graph and let (L, H) be a cover of G. An (L, H)-*coloring* of G is an independent set $I \in \mathcal{I}(H)$ of size $|V(G)|$. Equivalently, $I \in \mathcal{I}(H)$ is an (L, H)-coloring of G if $|I \cap L(u)| = 1$ for all $u \in V(G)$.

Remark 1.4. Suppose that G is a graph, (L, H) is a cover of G, and G' is a subgraph of G. In such situations, we will allow a slight abuse of terminology and speak of (L, H)-colorings of G' (even though, strictly speaking, (L, H) is not a cover of G').

The *DP-chromatic number* $\chi_{DP}(G)$ of a graph G is the smallest $k \in \mathbb{N}$ such that G is (L, H)-colorable for each cover (L, H) with $|L(u)| \geq k$ for all $u \in V(G)$.

To show that DP-colorings indeed generalize list colorings, consider a graph G and a list assignment L for G. Define a graph H as follows: Let $V(H) := \{(u, c) : u \in V(G) \text{ and } c \in L(u)\}$ and let

$$(u_1, c_1)(u_2, c_2) \in E(H) :\Longleftrightarrow (u_1 = u_2 \text{ and } c_1 \neq c_2) \text{ or } (u_1u_2 \in E(G) \text{ and } c_1 = c_2).$$

For $u \in V(G)$, let $\hat{L}(u) := \{(u, c) : c \in L(u)\}$. Then $(\hat{L}, H)$ is a cover of G, and there is a one-to-one correspondence between L-colorings and $(\hat{L}, H)$-colorings of G. Indeed, if f is an L-coloring of G, then the set $I_f := \{(u, f(u)) : u \in V(G)\}$ is an $(\hat{L}, H)$-coloring of G. Conversely, given an $(\hat{L}, H)$-coloring I of G, we can define an L-coloring f_I of G by the property $(u, f_I(u)) \in I$ for all $u \in V(G)$. Thus, list colorings form a subclass of DP-colorings. In particular, $\chi_{DP}(G) \geq \chi_\ell(G)$ for each graph G.

Some upper bounds on list-chromatic numbers hold for DP-chromatic numbers as well. For example, $\chi_{DP}(G) \leq d + 1$ for any d-degenerate graph G. Dvořák and Postle [3] pointed out that Thomassen's bounds [8,9] on the list chromatic numbers of planar graphs hold also for their DP-chromatic numbers; in particular, $\chi_{DP}(G) \leq 5$ for every planar graph G. On the other hand, there are also some striking differences between DP- and list coloring. For instance, even cycles are 2-list-colorable, while their DP-chromatic number is 3; in particular, the orientation theorems of Alon–Tarsi [1] and the Bondy–Boppana–Siegel Lemma (see [1]) do not extend to DP-coloring (see [2] for further examples of differences between list and DP-coloring).

By analogy with (1.1) and (1.2), we consider the parameters

$$Z_{DP}(G) := \min\{s \in \mathbb{N} : \chi_{DP}(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s))\},$$

and

$$Z_{DP}(n) := \max\{Z_{DP}(G) : |V(G)| = n\}.$$

Our main result asserts that for every graph G, $Z_{DP}(G)$ is finite:

Theorem 1.5. *Let G be a graph with n vertices, m edges, and chromatic number k. Then $Z_{DP}(G) \leq 3m$. Moreover, if $\delta(G) \geq k - 1$, then*

$$Z_{DP}(G) \leq 3m - \frac{3}{2}(k - 1)n.$$

Corollary 1.6. *For all $n \in \mathbb{N}$, $Z_{DP}(n) \leq 3n^2/2$.*

Note that the upper bound on $Z_{DP}(n)$ given by Corollary 1.6 is quadratic in n, in contrast to the linear upper bound on $Z_\ell(n)$ implied by Theorem 1.1. Our second result shows that the order of magnitude of $Z_{DP}(n)$ is indeed quadratic:

Theorem 1.7. *For all $n \in \mathbb{N}$, $Z_{DP}(n) \geq n^2/4 - O(n)$.*

Corollary 1.6 and Theorem 1.7 also yield the following analog of Theorem 1.1 for DP-coloring:

Corollary 1.8. *For $n \in \mathbb{N}$, let $r(n)$ denote the minimum $r \in \mathbb{N}$ such that for every n-vertex graph G with $\chi(G) \geq r$, we have $\chi_{DP}(G) = \chi(G)$. Then*

$$n - r(n) = \Theta(\sqrt{n}).$$

We prove Theorem 1.5 in Section 2 and Theorem 1.7 in Section 3. The derivation of Corollary 1.8 from Corollary 1.6 and Theorem 1.7 is straightforward; for completeness, we include it at the end of Section 3.

2. Proof of Theorem 1.5

For a graph G and a finite set A disjoint from $V(G)$, let $\mathbf{J}(G, A)$ denote the graph with vertex set $V(G) \cup A$ obtained from G be adding all edges with at least one endpoint in A (i.e., $\mathbf{J}(G, A)$ is a concrete representative of the isomorphism type of $\mathbf{J}(G, |A|)$).

First we prove the following more technical version of Theorem 1.5:

Theorem 2.1. *Let G be a k-colorable graph. Let A be a finite set disjoint from $V(G)$ and let (L, H) be a cover of $\mathbf{J}(G, A)$ such that for all $a \in A$, $|L(a)| \geq |A| + k$. Suppose that*

$$|A| \geq \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) + |A| - |L(v)| + 1,\ 0\}. \tag{2.1}$$

Then $\mathbf{J}(G, A)$ is (L, H)-colorable.

Proof. For a graph G, a set A disjoint from $V(G)$, a cover (L, H) of $\mathbf{J}(G, A)$, and a vertex $v \in V(G)$, let

$$\sigma(G, A, L, H, v) := \max\{\deg_G(v) + |A| - |L(v)| + 1,\ 0\}$$

and

$$\sigma(G, A, L, H) := \sum_{v \in V(G)} \sigma(G, A, L, H, v).$$

Assume, towards a contradiction, that a tuple (k, G, A, L, H) forms a counterexample which minimizes k, then $|V(G)|$, and then $|A|$. For brevity, we will use the following shortcuts:

$$\sigma(v) := \sigma(G, A, L, H, v); \quad \sigma := \sigma(G, A, L, H).$$

Thus, (2.1) is equivalent to

$$|A| \geq \frac{3\sigma}{2}.$$

Note that $|V(G)|$ and $|A|$ are both positive. Indeed, if $V(G) = \varnothing$, then $\mathbf{J}(G, A)$ is just a clique with vertex set A, so its DP-chromatic number is $|A|$. If, on the other hand, $A = \varnothing$, then (2.1) implies that $|L(v)| \geq \deg_G(v) + 1$ for all $v \in V(G)$, so an (L, H)-coloring of G can be constructed greedily. Furthermore, $\chi(G) = k$, since otherwise we could have used the same (G, A, L, H) with a smaller value of k.

Claim 2.1.1. *For every $v \in V(G)$, the graph $\mathbf{J}(G - v, A)$ is (L, H)-colorable.*

Proof. Consider any $v_0 \in V(G)$ and let $G' := G - v_0$. For all $v \in V(G')$, $\deg_{G'}(v) \leq \deg_G(v)$, and thus $\sigma(G', A, L, H, v) \leq \sigma(v)$. Therefore,

$$\frac{3}{2}\sigma(G', A, L, H) \leq \frac{3\sigma}{2} \leq |A|.$$

By the minimality of $|V(G)|$, the conclusion of Theorem 2.1 holds for (k, G', A, L, H), i.e., $\mathbf{J}(G', A)$ is (L, H)-colorable, as claimed. □

Corollary 2.1.2. *For every $v \in V(G)$,*

$$\sigma(v) = \deg_G(v) + |A| - |L(v)| + 1 > 0.$$

Proof. Suppose that for some $v_0 \in V(G)$,

$$\deg_G(v_0) + |A| - |L(v_0)| + 1 \leq 0,$$

i.e.,

$$|L(v_0)| \geq \deg_G(v_0) + |A| + 1.$$

Using Claim 2.1.1, fix any (L, H)-coloring I of $\mathbf{J}(G - v_0, A)$. Since v_0 still has at least

$$|L(v_0)| - (\deg_G(v_0) + |A|) \geq 1$$

available colors, I can be extended to an (L, H)-coloring of $\mathbf{J}(G, A)$ greedily; a contradiction. □

Claim 2.1.3. *For every $v \in V(G)$ and $x \in \bigcup_{a\in A} L(a)$, there is $y \in L(v)$ such that $xy \in E(H)$.*

Proof. Suppose that for some $a_0 \in A$, $x_0 \in L(a_0)$, and $v_0 \in V(G)$, we have $L(v_0) \cap N_H(x_0) = \varnothing$. Let $A' := A \setminus \{a_0\}$, and for every $w \in V(G) \cup A'$, let $L'(w) := L(w) \setminus N_H(x_0)$. Note that for all $a \in A'$, $|L'(a)| \geq |A'| + k$, and for all $v \in V(G)$, $\sigma(G, A', L', H, v) \leq \sigma(v)$. Moreover, by the choice of x_0, $|L'(v_0)| = |L(v_0)|$, which, due to Corollary 2.1.2, yields $\sigma(G, A', L', H, v_0) \leq \sigma(v_0) - 1$. This implies $\sigma(G, A', L', H) \leq \sigma - 1$, and thus

$$\frac{3}{2}\sigma(G, A', L', H) \leq \frac{3(\sigma - 1)}{2} \leq |A| - \frac{3}{2} < |A'|.$$

By the minimality of $|A|$, the conclusion of Theorem 2.1 holds for (k, G, A', L', H), i.e., the graph $\mathbf{J}(G, A')$ is (L', H)-colorable. By the definition of L', for any (L', H)-coloring I of $\mathbf{J}(G, A')$, $I \cup \{x_0\}$ is an (L, H)-coloring of $\mathbf{J}(G, A)$. This is a contradiction. □

Corollary 2.1.4. $k \geq 2$.

Proof. Let $v \in V(G)$ and consider any $a \in A$. Since, by Claim 2.1.3, each $x \in L(a)$ has a neighbor in $L(v)$, we have

$$|L(v)| \geq |L(a)| \geq |A| + k.$$

Using Corollary 2.1.2, we obtain

$$0 \leq \deg_G(v) + |A| - |L(v)| \leq \deg_G(v) - k,$$

i.e., $\deg_G(v) \geq k$. Since $V(G) \neq \varnothing$, $k \geq 1$, which implies $\deg_G(v) \geq 1$. But then $\chi(G) \geq 2$, as desired. □

Claim 2.1.5. *H does not contain a walk of the form $x_0 - y_0 - x_1 - y_1 - x_2$, where*

- *$x_0, x_1, x_2 \in \bigcup_{a\in A} L(a)$;*
- *$y_0, y_1 \in \bigcup_{v\in V(G)} L(v)$;*
- *$x_0 \neq x_1 \neq x_2$ and $y_0 \neq y_1$ (but it is possible that $x_0 = x_2$);*
- *the set $\{x_0, x_1, x_2\}$ is independent in H.*

Proof. Suppose that such a walk exists and let $a_0, a_1, a_2 \in A$ and $v_0, v_1 \in V(G)$ be such that $x_0 \in L(a_0)$, $y_0 \in L(v_0)$, $x_1 \in L(a_1)$, $y_1 \in L(v_1)$, and $x_2 \in L(a_2)$. Let $A' := A \setminus \{a_0, a_1, a_2\}$, and for every $w \in V(G) \cup A'$, let $L'(w) := L(w) \setminus N_H(x_0, x_1, x_2)$. Since $\{x_0, x_1, x_2\}$ is an independent set, for all $a \in A'$, $|L'(a)| \geq |A'| + k$, while for all $v \in V(G)$, $\sigma(G, A', L', H, v) \leq \sigma(v)$. Moreover, since for each $i \in \{0, 1\}$, the set $\{x_0, x_1, x_2\}$ contains two distinct neighbors of y_i, we have $\sigma(G, A', L', H, v_i) \leq \sigma(v_i) - 1$. Therefore, $\sigma(G, A', L', H) \leq \sigma - 2$, and thus

$$\frac{3}{2}\sigma(G, A', L', H) \leq \frac{3(\sigma - 2)}{2} \leq |A| - 3 \leq |A'|.$$

By the minimality of $|A|$, the conclusion of Theorem 2.1 holds for (k, G, A', L', H), i.e., the graph $\mathbf{J}(G, A')$ is (L', H)-colorable. By the definition of L', for any (L', H)-coloring I of $\mathbf{J}(G, A')$, $I \cup \{x_0, x_1, x_2\}$ is an (L, H)-coloring of $\mathbf{J}(G, A)$. This is a contradiction. □

Due to Corollary 2.1.4, we can choose a pair of disjoint independent sets $U_0, U_1 \subset V(G)$ such that $\chi(G - U_0) = \chi(G - U_1) = k - 1$. Choose arbitrary elements $a_1 \in A$ and $x_1 \in L(a_1)$. By Claim 2.1.3, for each $u \in U_0 \cup U_1$, there is a unique element $y(u) \in L(u)$ adjacent to x_1 in H (the uniqueness of $y(u)$ follows from the definition of a cover). Let

$$I_0 := \{y(u) : u \in U_0\} \quad \text{and} \quad I_1 := \{y(u) : u \in U_1\}.$$

Since U_0 and U_1 are independent sets in G, I_0 and I_1 are independent sets in H.

Claim 2.1.6. *There exists an element $a_0 \in A \setminus \{a_1\}$ such that $L(a_0) \cap N_H(I_0) \nsubseteq N_H(x_1)$.*

Proof. Assume that for all $a \in A \setminus \{a_1\}$, we have $L(a) \cap N_H(I_0) \subseteq N_H(x_1)$. Let $G' := G - U_0$, and for each $w \in V(G') \cup A$, let $L'(w) := L(w) \setminus N_H(I_0)$. By the definition of I_0, $L'(a_1) = L(a_1) \setminus \{x_1\}$, so

$$|L'(a_1)| = |L(a_1)| - 1 \geq |A| + (k - 1).$$

On the other hand, by our assumption, for each $a \in A \setminus \{a_1\}$, we have

$$|L'(a)| = |L(a) \setminus N_H(I_0)| \geq |L(a) \setminus N_H(x_1)| \geq |L(a)| - 1 \geq |A| + (k - 1).$$

Since for all $v \in V(G)$, $\sigma(G', A, L', H, v) \leq \sigma(v)$, the minimality of k implies the conclusion of Theorem 2.1 for $(k - 1, G', A, L', H)$; in other words, the graph $\mathbf{J}(G', A)$ is (L', H)-colorable. By the definition of L', for any (L', H)-coloring I of $\mathbf{J}(G', A)$, $I \cup I_0$ is an (L, H)-coloring of $\mathbf{J}(G, A)$; this is a contradiction. □

Using Claim 2.1.6, fix some $a_0 \in A \setminus \{a_1\}$ satisfying $L(a_0) \cap N_H(I_0) \nsubseteq N_H(x_1)$, and choose any

$$x_0 \in (L(a_0) \cap N_H(I_0)) \setminus N_H(x_1).$$

Since $x_0 \in N_H(I_0)$, we can also choose $y_0 \in I_0$ so that $x_0 y_0 \in E(H)$.

Claim 2.1.7. $x_0 \notin N_H(I_1)$.

Proof. If there is $y_1 \in I_1$ such that $x_0 y_1 \in E(H)$, then $x_0 - y_0 - x_1 - y_1 - x_0$ is a walk in H whose existence is ruled out by Claim 2.1.5. □

Claim 2.1.8. *There is an element $a_2 \in A \setminus \{a_0, a_1\}$ such that $L(a_2) \cap N_H(I_1) \nsubseteq N_H(x_0, x_1)$.*

Proof. The proof is almost identical to the proof of Claim 2.1.6. Assume that for all $a \in A \setminus \{a_0, a_1\}$, we have $L(a) \cap N_H(I_1) \subseteq N_H(x_0, x_1)$. Let $G' := G - U_1$, $A' := A \setminus \{a_0\}$, and for each $w \in V(G') \cup A'$, let $L'(w) := L(w) \setminus N_H(\{x_0\} \cup I_1)$. By the definition of I_1, $L(a_1) \cap N_H(I_1) = \{x_1\}$, so

$$|L'(a_1)| \geq |L(a_1)| - 2 \geq |A| + k - 2 = |A'| + (k - 1).$$

On the other hand, by our assumption, for each $a \in A \setminus \{a_0, a_1\}$, we have

$$|L'(a)| \geq |L(a) \setminus N_H(x_0, x_1)| \geq |L(a)| - 2 \geq |A| + k - 2 = |A'| + (k - 1).$$

Since for all $v \in V(G)$, $\sigma(G', A', L', H, v) \leq \sigma(v)$, the minimality of k implies the conclusion of Theorem 2.1 for $(k - 1, G', A', L', H)$; in other words, the graph $\mathbf{J}(G', A')$ is (L', H)-colorable. By the definition of L', for any (L', H)-coloring I of $\mathbf{J}(G', A)$, $I \cup \{x_0\} \cup I_1$ is an (L, H)-coloring of $\mathbf{J}(G, A)$. This is a contradiction. □

Now we are ready to finish the proof of Theorem 2.1. Fix some $a_2 \in A \setminus \{a_0, a_1\}$ satisfying $L(a_2) \cap N_H(I_1) \nsubseteq N_H(x_0, x_1)$, and choose any

$$x_2 \in (L(a_2) \cap N_H(I_1)) \setminus N_H(x_0, x_1).$$

Since $x_2 \in N_H(I_1)$, there is $y_1 \in I_1$ such that $x_2 y_1 \in E(H)$. Then $x_0 - y_0 - x_1 - y_1 - x_2$ is a walk in H contradicting the conclusion of Claim 2.1.5. ■

Now it is easy to derive Theorem 1.5. Indeed, let G be a graph with n vertices, m edges, and chromatic number k, let A be a finite set disjoint from $V(G)$, and let (L, H) be a cover of $\mathbf{J}(G, A)$ such that for all $v \in V(G)$ and $a \in A$, $|L(v)| = |L(a)| = \chi(\mathbf{J}(G, A)) = |A| + k$. Note that

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - |L(v)| + |A| + 1,\ 0\} = \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1,\ 0\}.$$

If $|A| \geq 3m$, then

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1,\ 0\} \leq \frac{3}{2} \sum_{v \in V(G)} \deg_G(v) = 3m \leq |A|,$$

so Theorem 2.1 implies that $\mathbf{J}(G, A)$ is (L, H)-colorable, and hence $Z_{DP}(G) \leq 3m$. Moreover, if $\delta(G) \geq k - 1$, then

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1,\ 0\} = \frac{3}{2} \sum_{v \in V(G)} (\deg_G(v) - k + 1) = 3m - \frac{3}{2}(k-1)n,$$

so $Z_{DP}(G) \leq 3m - \frac{3}{2}(k-1)n$, as desired. Finally, Corollary 1.6 follows from Theorem 1.5 and the fact that an n-vertex graph can have at most $\binom{n}{2} \leq n^2/2$ edges.

3. Proof of Theorem 1.7

We will prove the following precise version of Theorem 1.7:

Theorem 3.1. *For all even $n \in \mathbb{N}$, $Z_{DP}(n) \geq n^2/4 - n$.*

Proof. Let $n \in \mathbb{N}$ be even and let $k := n/2 - 1$. Note that $n^2/4 - n = k^2 - 1$. Thus, it is enough to exhibit an n-vertex bipartite graph G and a cover (L, H) of $\mathbf{J}(G, k^2 - 2)$ such that $|L(u)| = k^2$ for all $u \in V(\mathbf{J}(G, k^2 - 2))$, yet $\mathbf{J}(G, k^2 - 2)$ is not (L, H)-colorable.

Let $G \cong K_{n/2,n/2}$ be an n-vertex complete bipartite graph with parts $X = \{x, x_0, \ldots, x_{k-1}\}$ and $Y = \{y, y_0, \ldots, y_{k-1}\}$, where the indices $0, \ldots, k-1$ are viewed as elements of the additive group $\mathbb{Z}_k$ of integers modulo k. Let A be a set of size $k^2 - 2$ disjoint from $X \cup Y$. For each $u \in X \cup Y \cup A$, let $L(u) := \{u\} \times \mathbb{Z}_k \times \mathbb{Z}_k$. Let H be the graph with vertex set $(X \cup Y \cup A) \times \mathbb{Z}_k \times \mathbb{Z}_k$ in which the following pairs of vertices are adjacent:

- (u, i, j) and (u, i', j') for all $u \in X \cup Y \cup A$ and $i, j, i', j' \in \mathbb{Z}_k$ such that $(i, j) \neq (i', j')$;
- (u, i, j) and (v, i, j) for all $u \in \{x, y\} \cup A$, $v \in N_{\mathbf{J}(G,A)}(u)$, and $i, j \in \mathbb{Z}_k$;
- (x_s, i, j) and $(y_t, i + s, j + t)$ for all $s, t, i, j \in \mathbb{Z}_k$.

It is easy to see that (L, H) is a cover of $\mathbf{J}(G, A)$. We claim that $\mathbf{J}(G, A)$ is not (L, H)-colorable. Indeed, suppose that I is an (L, H)-coloring of $\mathbf{J}(G, A)$. For each $u \in X \cup Y \cup A$, let $i(u)$ and $j(u)$ be the unique elements of $\mathbb{Z}_k$ such that $(u, i(u), j(u)) \in I$. By the construction of H and since I is an independent set, we have

$$(i(u), j(u)) \neq (i(a), j(a))$$

for all $u \in X \cup Y$ and $a \in A$. Since all the $k^2 - 2$ pairs $(i(a), j(a))$ for $a \in A$ are pairwise distinct, $(i(u), j(u))$ can take at most 2 distinct values as u is ranging over $X \cup Y$. One of those 2 values is $(i(y), j(y))$, and if $u \in X$, then

$$(i(u), j(u)) \neq (i(y), j(y)),$$

so the value of $(i(u), j(u))$ must be the same for all $u \in X$; let us denote it by (i, j). Similarly, the value of $(i(u), j(u))$ is the same for all $u \in Y$, and we denote it by (i', j').

It remains to notice that the vertices $(x_{i'-i}, i, j)$ and $(y_{j'-j}, i', j')$ are adjacent in H, so I is not an independent set. ■

Now we can prove Corollary 1.8:

Proof of Corollary 1.8. First, suppose that G is an n-vertex graph with $\chi(G) = r$ that maximizes the difference $\chi_{DP}(G) - \chi(G)$. Adding edges to G if necessary, we may arrange G to be a complete r-partite graph. Assuming $2r > n$, at least $2r - n$ of the parts must be of size 1, i.e., G is of the form $\mathbf{J}(G', 2r - n)$ for some $2(n - r)$-vertex graph G'. By Corollary 1.6, we have $\chi_{DP}(G) = \chi(G)$ as long as $2r - n \geq 6(n - r)^2$, which holds for all $r \geq n - (1/\sqrt{6} - o(1))\sqrt{n}$. This establishes the upper bound $r(n) \leq n - \Omega(\sqrt{n})$.

On the other hand, due to Theorem 1.7, for each n, we can find a graph G with s vertices, where $s \leq (2 + o(1))\sqrt{n}$, such that $\chi_{DP}(\mathbf{J}(G, n - s)) > \chi(\mathbf{J}(G, n - s))$. Since $\mathbf{J}(G, n - s)$ is an n-vertex graph, we get

$$r(n) > \chi(\mathbf{J}(G, n - s)) = \chi(G) + n - s \geq n - (2 + o(1))\sqrt{n} = n - O(\sqrt{n}). \quad \blacksquare$$

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