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Stability in the Erdős–Gallai Theorems on cycles and paths

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Dedicated to the memory of G.N. Kopylov

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ABSTRACT

The Erdős–Gallai Theorem states that for $k \geq 2$, every graph of average degree more than $k-2$ contains a k-vertex path. This result is a consequence of a stronger result of Kopylov: if k is odd, $k = 2t+1 \geq 5$, $n \geq (5t-3)/2$, and G is an n-vertex 2-connected graph with at least $h(n,k,t) := \binom{k-t}{2} + t(n-k+t)$ edges, then G contains a cycle of length at least k unless $G = H_{n,k,t} := K_n - E(K_{n-t})$.
In this paper we prove a stability version of the Erdős–Gallai Theorem: we show that for all $n \geq 3t > 3$, and $k \in \{2t+1, 2t+2\}$, every n-vertex 2-connected graph G with $e(G) > h(n,k,t-1)$ either contains a cycle of length at least k or contains a set of t vertices whose removal gives a star forest. In particular, if $k = 2t+1 \neq 7$, we show $G \subseteq H_{n,k,t}$. The lower bound $e(G) > h(n,k,t-1)$ in these results is tight

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and is smaller than Kopylov's bound $h(n, k, t)$ by a term of $n - t - O(1)$.

1. Introduction

A cornerstone of extremal combinatorics is the study of Turán-type problems for graphs. One of the fundamental questions in extremal graph theory is to determine the maximum number of edges in an n-vertex graph with no k-vertex path. According to [10], this problem was posed by Turán. A solution to the problem was obtained by Erdős and Gallai [7]:

Theorem 1.1 *(Erdős and Gallai [7]). Let G be an n-vertex graph with more than $\frac{1}{2}(k-2)n$ edges, $k \geq 2$. Then G contains a k-vertex path P_k.*

This result is best possible for n divisible by $k-1$, due to the n-vertex graph whose components are cliques of order $k-1$. To obtain Theorem 1.1, Erdős and Gallai observed that if H is an n-vertex graph without a k-vertex path P_k, then adding a new vertex and joining it to all other vertices we have a graph H' on $n+1$ vertices $e(H)+n$ edges and containing no cycle C_{k+1} or longer. Then Theorem 1.1 is a consequence of the following:

Theorem 1.2 *(Erdős and Gallai [7]). Let G be an n-vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges, $k \geq 3$. Then G contains a cycle of length at least k.*

This result is best possible for $n-1$ divisible by $k-2$, due to any n-vertex graph where each block is a clique of order $k-1$. Let $\mathrm{ex}(n, P_k)$ be the maximum number of edges in an n-vertex graph with no k-vertex path; Theorem 1.1 shows $\mathrm{ex}(n, P_k) \leq \frac{1}{2}(k-2)n$ with equality for n divisible by $k-1$. Several proofs and sharpenings of the Erdős–Gallai theorem were obtained by Woodall [16], Lewin [12], Faudree and Schelp [8,9] and Kopylov [11] – see [10] for further details. The strongest version was proved by Kopylov [11]. To describe his result, we require the following graphs. Suppose that $n \geq k$, $(k/2) > a \geq 1$. Define the n-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n-k+a$ and $|C| = k-2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$. Let

$$h(n,k,a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a).$$

Theorem 1.3 *(Kopylov [11]). Let $n \geq k \geq 5$ and $t = \lfloor \frac{k-1}{2} \rfloor$. If G is an n-vertex 2-connected graph with no cycle of length at least k, then*

$$e(G) \leq \max\{h(n,k,2), h(n,k,t)\} \tag{1}$$

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

In this paper, we prove a stability version of Theorems 1.1 and 1.3. A *star forest* is a vertex-disjoint union of stars.

Theorem 1.4. *Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t+1, 2t+2\}$. Let G be a 2-connected n-vertex graph containing no cycle of length at least k. Then $e(G) \leq h(n,k,t-1)$ unless*

(a) $k = 2t+1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$ or
(b) $k = 2t+2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most t.

This result is best possible in the following sense. Note that $H_{n,k,t-1}$ contains no cycle of length at least k, is not a subgraph of $H_{n,k,t}$, and $H_{n,2t+2,t-1} - A$ has a cycle for every $A \subseteq V(H_{n,2t+2,t-1})$ with $|A| = t$. Thus the claim of Theorem 1.4 does not hold for $G = H_{n,k,t-1}$. Therefore the condition $e(G) \leq h(n,k,t-1)$ in Theorem 1.4 is best possible. Since

$$h(n, 2t+2, t) = \binom{t}{2} + t(n-t) + 1 = h(n, 2t+1, t) + 1$$

and

$$h(n, 2t+2, t-1) = \binom{t}{2} + (t-1)(n-t) + 6 = h(n, 2t+1, t-1) + 3,$$

the difference between Kopylov's bound and the bound in Theorem 1.4 is

$$h(n,k,t) - h(n,k,t-1) = \begin{cases} n-t-3 & \text{if } k = 2t+1 \\ n-t-5 & \text{if } k = 2t+2. \end{cases} \tag{2}$$

It is interesting that for a fixed k, the difference in (2) divided by $h(n,k,t)$ does not tend to 0 when $n \to \infty$.

We will need to prove a more detailed version of Theorem 1.4. This version, Theorem 4.1, will yield the following cleaner claim for 3-connected graphs.

Corollary 1.5. *Let $k \geq 11$, $t = \lfloor \frac{k-1}{2} \rfloor$, and $n \geq \frac{3k}{2}$. If G is an n-vertex 3-connected graph with no cycle of length at least k, then $e(G) \leq h(n,k,t-1)$ unless $G \subseteq H_{n,k,t}$.*

In the same way that Theorem 1.2 implies Theorem 1.1, Theorem 1.4 applies to give a stability theorem for paths:

Theorem 1.6. *Let $t \geq 2$ and $n \geq 3t-1$ and $k \in \{2t, 2t+1\}$, and let G be a connected n-vertex graph containing no k-vertex path. Then $e(G) \leq h(n+1, k+1, t-1) - n$ unless*

(*a*) $k = 2t$, $k \neq 6$, *and* $G \subseteq H_{n,k,t-1}$ *or*
(*b*) $k = 2t+1$ *or* $k = 6$, *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* $t-1$.

Indeed, let G' be obtained from an n-vertex connected graph G with more than $h(n+1, k+1, t-1) - n$ edges by adding a vertex adjacent to all vertices in G. Then G' is 2-connected and G' has more than $h(n+1, k+1, t-1)$ edges. If G has no k-vertex path, then G' has no cycle of length at least $k+1$. By Theorem 1.4, G' satisfies (a) or (b) in Theorem 1.4, which means G satisfies (a) or (b) in Theorem 1.6. Repeating this argument, Corollary 1.5 implies the following.

Corollary 1.7. *Let $k \geq 11$, $t = \lfloor \frac{k-1}{2} \rfloor$, and $n \geq \frac{3k}{2}$. If G is an n-vertex 2-connected graph with no k-vertex paths, then $e(G) \leq h(n+1, k+1, t-1) - n$ unless $G \subseteq H_{n,k,t-1}$.*

Organization. The proof of Theorem 1.4 will use a number of classical results listed in Section 2 and some lemmas on contractions proved in Section 3. Then in Section 4 we describe several families of extremal graphs and state and prove a more technical Theorem 4.1, implying Theorem 1.4 for $k \geq 9$. Finally, in Section 5 we prove the analog of our technical Theorem 4.1 for $4 \leq k \leq 8$. In particular, we describe *all* 2-connected graphs with no cycles of length at least 6.

Notation. We use standard notation of graph theory. Given a simple graph $G = (V, E)$, the *neighborhood* of $v \in V$, i.e. the set of vertices adjacent to v, is denoted by $N_G(v)$ or $N(v)$ for short, and the *closed neighborhood* is $N[v] := N(v) \cup \{v\}$. The *degree* of vertex v is $d_G(v) := |N_G(v)|$. Given $A \subseteq V$ we also use $N_G(v, A)$ for $N(v) \cap A$, $d(v, A)$ for $|N(v) \cap A|$, and $N(A) := \bigcup_{v \in A} N(v) \backslash A$. For an edge xy in G, let $T_G(xy)$ denote the number of triangles containing xy and $T(G) := \min\{T_G(xy) : xy \in E\}$. The minimum degree of G is denoted by $\delta(G)$. For an edge xy in G, G/xy denotes the graph obtained from G by contracting xy. We frequently use $x * y$ for the new vertex. The length of the longest cycle in G is denoted by $c(G)$, and $e(G) := |E|$. Denote by K_n the complete n-vertex graph, and $K(A, B)$ the complete bipartite graph with parts A and B ($A \cap B = \emptyset$). Given vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the graph $G_1 + G_2$ has vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E(K(V_1, V_2))$. If G is a graph, then $\overline{G}$ denotes the complement of G and for a positive integer ℓ, ℓG denotes the graph consisting of ℓ components, each isomorphic to G. For disjoint sets $A, B \subseteq V(G)$, let $G(A, B)$ denote the bipartite graph with parts A and B consisting of all edges of G between A and B, and for $A \subseteq V(G)$, let $G[A]$ denote the subgraph induced by A.

2. Classical theorems

We require a number of theorems on long paths and cycles in dense graphs. The following is an extension to 2-connected graphs of the well-known fact that an n-vertex non-hamiltonian graph has at most $\binom{n-1}{2}+1$ edges:

Theorem 2.1 *(Erdős [6]). Let $d \geq 1$ and $n > 2d$ be integers, and*

$$\ell_{n,d} := \max\left\{ \binom{n-d}{2} + d^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

Then every n-vertex graph G with $\delta(G) \geq d$ and $e(G) > \ell_{n,d}$ is hamiltonian.

The bound on $\ell_{n,d}$ is sharp, due to the graphs $H_{n,n,2}$ and $H_{n,n,\lfloor (n-1)/2 \rfloor}$. Since $\delta(G) \geq 2$ for every 2-connected G, this has the following corollary.

Theorem 2.2 *(Erdős [6]). If $n \geq 5$ and G is an n-vertex 2-connected non-hamiltonian graph, then $e(G) \leq \binom{n-2}{2} + 4$, with equality only for $G = H_{n,n,2}$.*

It is well-known that every graph of minimum degree at least $d \geq 2$ contains a cycle of length at least $d+1$. A stronger statement was proved by Dirac for 2-connected graphs:

Theorem 2.3 *(Dirac [4]). If G is 2-connected then $c(G) \geq \min\{n, 2\delta\}$.*

This theorem was strengthened as follows by Kopylov [11], based on ideas of Pósa [14]:

Theorem 2.4 *(Kopylov [11]). If G is 2-connected, P is an x,y-path of ℓ vertices, then $c(G) \geq \min\{\ell, d(x,P) + d(y,P)\}$.*

Theorem 2.5 *(Chvátal [3]). Let $n \geq 3$ and G be an n-vertex graph with vertex degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. If G is not hamiltonian, then there is some $i < n/2$ such that $d_i \leq i$ and $d_{n-i} < n-i$.*

The *k-closure* of a graph G is the unique smallest graph H of order $n := |V(G)|$ such that $G \subseteq H$ and $d_H(u) + d_H(v) < k$ for all $uv \notin E(H)$. The k-closure of G is denoted by $Cl_k(G)$, and can be obtained from G by a recursive procedure which consists of joining nonadjacent vertices with degree-sum at least k.

Theorem 2.6 *(Bondy and Chvátal [1]). If $Cl_n(G)$ is hamiltonian, then so is G. Therefore if $Cl_n(G) = K_n$, $n \geq 3$, then G is hamiltonian.*

Concerning long paths between prescribed vertices in a graph, Lovász [13] showed that if G is a 2-connected graph in which every vertex other than u and v has degree at least k, then there is a u, v-path of length at least $k+1$. This result was strengthened by Enomoto. The following theorem immediately follows from Corollary 1 in [5]:

Theorem 2.7 *(Enomoto [5]). Let $5 \le s \le n$ and $\ell := 2(n-3)/(s-4)$. Suppose H is a 3-connected n-vertex graph with $d(x)+d(y) \ge s$ for all non-adjacent distinct $x, y \in V(H)$. Then for every distinct vertices x and y of H, there is an x, y-path of length at least $s-2$. Moreover, if for some distinct $x, y \in V(H)$, there is no x, y-path of length at least $s-1$, then either*

$$\overline{K_{s/2}} + \overline{K_{n-s/2}} \subseteq H \subseteq K_{s/2} + \overline{K_{n-s/2}}$$

or ℓ is an integer and

$$\overline{K_3} + \ell K_{s/2-2} \subseteq H \subseteq K_3 + \ell K_{s/2-2}.$$

A further strengthening of this result was given by Bondy and Jackson [2]. Finally, we require some results on cycles containing prescribed sets of edges. The following was proved by Pósa [15]:

Theorem 2.8 *(Pósa [15]). Let $n \ge 3$, $k < n$ and let G be an n-vertex graph such that*

$$d(u) + d(v) \ge n + k \qquad \text{for every non-edge } uv \text{ in } G. \tag{3}$$

Then for every linear forest F with k edges contained in G, the graph G has a hamiltonian cycle containing all edges of F.

The analog of Pósa's Theorem for bipartite graphs below is a simple corollary of Theorem 7.3 in [17].

Theorem 2.9 *(Zamani and West [17]). Let $s \ge 3$ and K be a subgraph of the complete bipartite graph $K_{s,s}$ with partite sets A and B such that for every $x \in A$ and $y \in B$ with $xy \notin E(K)$, $d(x) + d(y) \ge s + 1 + i$. Then for every linear forest $F \subseteq K$ with at most $2i$ edges, there is a hamiltonian cycle in K containing all edges of F.*

We will use only the following partial case of Theorem 2.9.

Corollary 2.10. *Let $s \ge 4$, $1 \le i \le 2$ and K be a subgraph of $K_{s,s}$ with at least $s^2 - s + 2 + i$ edges. If $F \subseteq K$ is a linear forest with at most $2i$ edges and at most two components, then K has a hamiltonian cycle containing all edges of F.*

3. Lemmas on contractions

An essential part of the proof of Theorem 1.4 is to analyze contractions of edges in graphs. Specifically, we shall start with a graph G and contract edges according to some basic rules. Let us mention that the extensive use of contractions to prove the Erdős–Gallai Theorem was introduced by Lewin [12]. In this section, we present some basic structural lemmas on contractions.

Lemma 3.1. *Let $n \geq 4$ and let G be an n-vertex 2-connected graph. Let $v \in V(G)$ and $W(v) := \{w \in N(v) : N[v] \not\subseteq N[w]\}$. If $W(v) \neq \emptyset$, then there is $w \in W(v)$ such that G/vw is 2-connected.*

Proof. Let $w \in W(v)$, $G_w = G/vw$. Recall that $v * w$ is the vertex in G_w obtained by contracting v with w. Since G is 2-connected, G_w is connected. If $x \neq v * w$ is a cut vertex in G_w, then it is a cut vertex in G, a contradiction. So, the only cut vertex in G_w can be $v * w$. Thus, if the lemma does not hold, then for every $w \in W(v)$, $v * w$ is the unique cut vertex in G_w. This means that for every $w \in W(v)$, $\{v, w\}$ is a separating set in G.

Choose $w \in W(v)$ so that to minimize the order of a minimum component in $G-v-w$. Let C be the vertex set of such a component in $G - v - w$ and $C' = V(G) \setminus (C \cup \{v, w\})$. Since G is 2-connected, v has a neighbor $u \in C$ and a neighbor $u' \in C'$. Since $uu' \notin E(G)$, $u \in W(v)$. But the vertex set of every component of $G - v - u$ not containing w is contained in C. This contradicts the choice of w. □

This lemma yields the following fact.

Lemma 3.2. *Let $n \geq 4$ and let G be an n-vertex 2-connected graph. For every $v \in V(G)$, there exists $w \in N(v)$ such that G/vw is 2-connected.*

Proof. If $W(v) \neq \emptyset$, this follows from Lemma 3.1. Suppose $W(v) = \emptyset$. This means $G[N(v)]$ is a clique. Then contracting any edge incident with v is equivalent to deleting v. Let $G' = G - v$. Since $d(v) \geq 2$ and $G[N(v)]$ is a clique, any cut vertex in G' is also a cut vertex in G. □

For an edge xy in a graph H, let $T_H(xy)$ denote the number of triangles containing xy. Let $T(H) := \min\{T_H(xy) : xy \in E(H)\}$. When we contract an edge uv in a graph H, the degree of every $x \in V(H) \setminus \{u, v\}$ either does not change or decreases by 1. Also the degree of $u * v$ in H/uv is at least $\max\{d_H(u), d_H(v)\} - 1$. Thus

$$\delta(H/uv) \geq \delta(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H). \tag{4}$$

Similarly,

$$T(H/uv) \geq T(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H). \tag{5}$$

Suppose we contract edges of a 2-connected graph one at a step, choosing always an edge xy so that

(i) the new graph is 2-connected and,
(ii) xy is in the fewest triangles;
(iii) the contracted edge xy is incident to a vertex of degree as small as possible up to (ii).

Lemma 3.3. *Let h be a positive integer. Suppose a 2-connected graph G is obtained from a 2-connected graph G' by contracting edge xy into $x * y$ using the above rules (i)–(iii). If G has at least h vertices of degree at most h, then either $G' = K_{h+2}$ or G' also has a vertex of degree at most h.*

Proof. Since G is 2-connected, $h \geq 2$. If G has a vertex of degree less than h, the lemma holds by (4). So, let A_j denote the set of vertices of degree exactly j in G, and assume $|A_h| \geq h$. Let $A'_h = A_h \setminus \{x * y\}$. Suppose the lemma does not hold. Then we have

$$\text{each } v \in A'_h \text{ has degree } h+1 \text{ in } G' \text{ and is adjacent to both, } x \text{ and } y \text{ in } G'. \tag{6}$$

Case 1: $|A'_h| \geq h$. Then by (6), xy belongs to at least h triangles in which the third vertex is in A_h. So by (iii) and the symmetry between x and y, we may assume $d_{G'}(x) = h + 1$. This in turn yields $N_{G'}(x) = A_h \cup \{y\}$. Since G' is 2-connected each $v \in A'_h$ is not a cut vertex. Even more, xv is not a cut edge. Indeed, y is a common neighbor of all neighbors of x so all neighbors of x must be in the same component as y in $G' - x - v$. It follows that

$$\text{for every } v \in A'_h,\ G'/vx \text{ is 2-connected.} \tag{7}$$

If $uv \notin E(G)$ for some $u, v \in A_h$, then by (7) and (ii), we would contract the edge xu and not xy. Thus $G'[A'_h \cup \{x, y\}] = K_{h+2}$ and so either $G' = K_{h+2}$ or y is a cut vertex in G', as claimed.

Case 2: $|A'_h| = h - 1$. Then $x * y \in A_h$. We obtain that $d_{G'}(x) = d_{G'}(y) = h + 1$ and $N_{G'}[x] = N_{G'}[y]$. So by (6), there is $z \in V(G)$ such that $N_{G'}[x] = N_{G'}[y] = A'_h \cup \{x, y, z\}$. Again (7) holds (for the same reason that $N_{G'}[x] \subseteq N_{G'}[y]$). Thus similarly $vu \in E(G')$ for every $v \in A'_h$ and every $u \in A'_h \cup \{z\}$. Hence $G'[A'_h \cup \{x, y, z\}] = K_{h+2}$ and either $G' = K_{h+2}$ or z is a cut vertex in G', as claimed. □

Lemma 3.4. *Suppose that G is a 2-connected graph and C is a longest cycle in it. Then no two consecutive vertices of C form a separating set.*

Proof. Indeed, if for some i the set $\{v_i, v_{i+1}\}$ is separating, then let H_1 and H_2 be two components of $G - \{v_i, v_{i+1}\}$ such that $V(C) \cap V(H_1) \neq \emptyset$. Then $V(C) \setminus \{v_i, v_{i+1}\} \subseteq V(H_1)$. Let $x \in V(H_2)$. Since G is 2-connected, it contains two paths from x to $\{v_i, v_{i+1}\}$ that share only x. Since $\{v_i, v_{i+1}\}$ separates $V(H_2)$ from the rest, these paths are fully contained in $V(H_2) \cup \{v_i, v_{i+1}\}$. So adding these paths to $C - v_iv_{i+1}$ creates a cycle longer than C, a contradiction. □

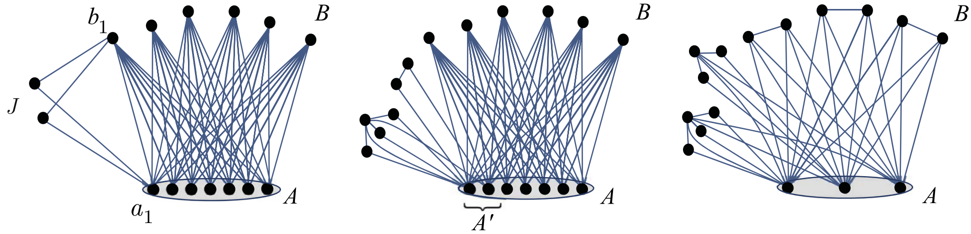

Fig. 1. Classes $\mathcal{G}_2(n,k)$, $\mathcal{G}_3(n,k)$ and $\mathcal{G}_4(n,10)$.

4. Proof of the main result, Theorem 1.4, for $k \geq 9$

In this section, we give a precise description of the extremal graphs for Theorem 1.4 for $k \geq 9$. The description for $k \leq 8$ is postponed to Section 5. For Theorem 1.4(a), when $k = 2t+1$ and $t \neq 3$, these are simply subgraphs of the graphs $H_{n,k,t}$: recall that $H_{n,k,a}$ has a partition into three sets A, B, C such that $|A| = a$, $|B| = n-k+a$ and $|C| = k-2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$. For Theorem 1.4(b), when $k = 2t+2$ or $k = 7$, the extremal graphs G contain a set A of size at most t such that $G - A$ is a star forest. In this case a more detailed description is required.

Classes $\mathcal{G}_i(n,k)$ for $i \leq 3$. Let $\mathcal{G}_1(n,k) := \{H_{n,k,t}\}$. Each $G \in \mathcal{G}_2(n,k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ and a pair $a_1 \in A$, $b_1 \in B$ such that $G[A] = K_t$, $G[B]$ is the empty graph, $G(A,B)$ is a complete bipartite graph and for every $c \in J$ one has $N(c) = \{a_1, b_1\}$. Every member of $G \in \mathcal{G}_3(n,k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ such that $G[A] = K_t$, $G(A,B)$ is a complete bipartite graph, and

- $G[J]$ has more than one component
- all components of $G[J]$ are stars with at least two vertices each
- there is a 2-element subset A' of A such that $N(J) \cap (A \cup B) = A'$
- for every component S of $G[J]$ with at least 3 vertices, all leaves of S are adjacent to the same vertex $a(S)$ in A'.

The class $\mathcal{G}_4(n,k)$ is empty unless $k = 10$. Each member of $\mathcal{G}_4(n,10)$ has a 3-vertex set A such that $G[A] = K_3$ and $G - A$ is a star forest such that if a component S of $G - A$ has more than two vertices then all its leaves are adjacent to the same vertex $a(S)$ in A. These classes are illustrated in Fig. 1.

Statement of main theorem. Having defined the classes $\mathcal{G}_i(n,k)$ for $i \leq 4$, we now state a theorem which implies Theorem 1.4 for $k \geq 9$ and shows that the extremal graphs are the graphs in the classes $\mathcal{G}_i(n,k)$:

Theorem 4.1 *(Main theorem). Let $k \geq 9$, $n \geq \frac{3k}{2}$ and $t = \lfloor \frac{k-1}{2} \rfloor$. Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then $e(G) \leq h(n,k,t-1)$ or G is* *a subgraph of a graph in $\mathcal{G}(n,k)$, where*

(1) *if k is odd, then $\mathcal{G}(n,k) := \mathcal{G}_1(n,k) = \{H_{n,k,t}\}$;*
(2) *if k is even and $k \neq 10$, then $\mathcal{G}(n,k) := \mathcal{G}_1(n,k) \cup \mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$;*
(3) *if $k = 10$, then $\mathcal{G}(n,k) := \mathcal{G}_1(n,10) \cup \mathcal{G}_2(n,10) \cup \mathcal{G}_3(n,10) \cup \mathcal{G}_4(n,10)$.*

We prove this theorem in this section. We also observe that if $k \geq 11$, then the only graph in the classes $\mathcal{G}_i(n,k)$ that is 3-connected is $H_{n,k,t}$. Therefore Theorem 4.1 implies Corollary 1.5.

The idea of the proof is to take a graph G satisfying the conditions of the theorem with $c(G) < k$, and to contract edges while preserving the average degree and 2-connectivity of G. A key fact is that if a graph contains a cycle of length at least k and is obtained from another graph by contracting edges, then that other graph also contains a cycle of length at least k. The process terminates with an m-vertex graph G_m such that G_m is 2-connected, $m \geq k$, and if $m > k$ then G_m has minimum degree at least $t-1$. If $m > k$, then we apply Theorem 2.7 to show that G_m is a dense subgraph of $H_{m,k,t}$. If $m = k$, then we apply Theorems 2.1, 2.2, 2.5, and 2.6 to show that G_m is a dense subgraph of $H_{k,k,t}$. Using this, we show that G_m contains a dense nice subgraph. Analyzing contractions, we then show that G itself contains a dense nice subgraph. Finally, we show that every dense n-vertex graph containing a dense nice subgraph but not containing a cycle of length at least k must be a subgraph of a graph in one of the classes described in Theorem 4.1.

4.1. Basic Procedure

Let k, n be positive integers with $n \geq k$. Let G be an n-vertex 2-connected graph with $c(G) < k$ and $e(G) \geq h(n,k,t-1)+1$. We denote G as G_n and run the following procedure.

Basic Procedure. At the beginning of each round, for some $j : k \leq j \leq n$, we have a j-vertex 2-connected graph G_j with $e(G_j) \geq h(j,k,t-1)+1$.

(R1) If $j = k$, then we stop.
(R2) If there is an edge xy with $T_{G_j}(xy) \leq t-2$ such that G_j/xy is 2-connected, choose one such edge so that
  (i) $T_{G_j}(xy)$ is minimum, and subject to this
  (ii) xy is incident to a vertex of minimum possible degree.
  Then obtain G_{j-1} by contracting xy.
(R3) If (R2) does not hold, $j \geq k+t-1$ and there is $uv \in E(G_j)$ such that $G_j - u - v$ has at least 3 components and one of the components, say H_1 is a K_{t-1}, then let $G_{j-t+1} = G_j - V(H_1)$.
(R4) If neither (R2) nor (R3) occurs, then we stop.

Remark 1. By construction, every obtained G_j is 2-connected and has $c(G_j) < k$. Let us check that

$$e(G_j) \geq h(j,k,t-1)+1 \tag{8}$$

for all $m \leq j \leq n$. For $j = n$, (8) holds by assumption. Suppose $j > m$ and (8) holds. If we apply (R2) to G_j, then the number of edges decreases by at most $t-1$, and $(h(j,k,t-1)+1)-(h(j-1,k,t-1)+1) = t-1$. If we apply (R3) to G_j, then the number of edges decreases by at most $\binom{t+1}{2}-1$, and $(h(j,k,t-1)+1)-(h(j-(t-1)),k,t-1)+1) = (t-1)^2$. But for $k \geq 9$, $(t-1)^2 \geq \binom{t+1}{2}-1$. Thus every step of the basic procedure preserves (8).

Let G_m denote the graph with which the procedure terminates.

Remark 2. Note that if the rule (R3) applies for some G_j, then the set $\{u,v\}$ is still separating in G_{j-t+1}, and $T_{G_{j-t+1}}(xy) \geq t-1$ for every edge xy such that G_{j-t+1}/xy is 2-connected. In particular, $\delta(G_{j-t+1}) \geq t$. So after any application of (R3), rule (R2) does not apply, and $\delta(G_m) \geq t$.

4.2. The structure of G_m

In the next two subsections, we prove Proposition 4.2 below, considering the cases $m = k$ and $m > k$ separately. Let F_4 be the graph obtained from $K_{3,6}$ by adding three independent edges in the part of size six. In this section we usually suppose that $n \geq 3t$, $t \geq 4$, although many steps work for smaller values as well.

Proposition 4.2. *The graph G_m satisfies the following properties:*

(1) $G_m \subseteq H_{m,k,t}$ *or*
(2) $m > k = 10$ *and* $G_m \supseteq F_4$.

4.2.1. The case $m = k$

If G_k is hamiltonian, then $c(G) \geq k$, a contradiction. So G_k is not hamiltonian.

By Theorem 2.5, for every non-hamiltonian n-vertex graph G with vertex degrees $d_1 \leq d_2 \leq \ldots \leq d_n$, we define

$$r(G) := \min\{i \ : d_i \leq i \ \textit{and} \ d_{n-i} < n-i\}.$$

Lemma 4.3. *Let $t \geq 4$, $n \geq 3t$. If the vertex degrees of G_k are $d_1 \leq d_2 \leq \ldots \leq d_k$, then $r(G_k) = t$.*

Proof for $k = 2t+2$. Note that $r(G_k) \leq t$ since $r(G) < n/2$ (see Theorem 2.5). Suppose $r := r(G_k) \leq t-1$. Then by Remark 2, rule (R3) never applied, and G_k was obtained from G by a sequence of $n-m$ edge contractions according to (R2). We may assume that for all $m \le j < n$, graph G_j was obtained from G_{j+1} by contracting edge $x_j y_j$. Then conditions for (R2) imply

$$T_{G_j}(x_{j-1}y_{j-1}) \le t-2 \quad \text{for every} \quad m+1 \le j \le n. \tag{9}$$

By Lemma 3.3, $\delta(G_{m+1}) \le r$. This together with (9) and (4) yield that for every $m < j \le n$,

$$\delta(G_j) \le r+j-m-1 \text{ and so } T_{G_j}(x_{j-1}y_{j-1}) \le \min\{r+j-m-2, t-2\}. \tag{10}$$

Contracting edge $x_{j-1}y_{j-1}$ in G_j, we lose $T_{G_j}(x_{j-1}y_{j-1})+1$ edges. Since $e(G) \ge h(n,k,t-1)+1$, by (5) we obtain

$$\begin{aligned} e(G_k) &\ge h(n,k,t-1)+1-\sum_{j=m+1}^{n} \min\{t-1, r+j-m-1\} \qquad (11)\\ &= \binom{t+3}{2} + (t-1)(n-t-3)+1-\sum_{j=m+1}^{n} \min\{t-1, r+j-m-1\} \\ &= \binom{t+3}{2} + (t-1)(n-t-3)+1-(t-1)(n-m) \\ &\quad + \sum_{j=m+1}^{n} \max\{0, m+t-r-j\} \\ &= \frac{3t^2+t+10}{2} + \sum_{j=m+1}^{n} \max\{0, 3t+2-r-j\}. \end{aligned}$$

Since $n \ge 3t$, $\{\max\{0, 3t+2-r-j\} : m+1 \le j \le n\} = \{0,1,2,\dots,t-1-r\}$. Therefore

$$e(G_k) \ge \frac{3t^2+t+10}{2} + \sum_{i=1}^{t-1-r} i = \frac{3t^2+t+10}{2} + \binom{t-r}{2}. \tag{12}$$

On the other hand, by the definition of r, G_m has at most r^2 edges incident with the r vertices of the smallest degrees and at most $\binom{m-r}{2}$ other edges. Thus $e(G_m) \le r^2 + \binom{2t+2-r}{2}$. Hence

$$\frac{3t^2+t+10}{2} + \binom{t-r}{2} \le r^2 + \binom{2t+2-r}{2}. \tag{13}$$

Expanding the binomial terms in (13) and regrouping we get

$$t(r-3) \le r^2-2r-4. \tag{14}$$

If $r = 3$, then the left hand side of (14) is 0 and the right hand side is -1, a contradiction. If $r \geq 4$, then dividing both sides of (14) by $r - 3$ we get $t \leq r + 1 - 1/(r - 3)$, which yields $r \geq t$, as claimed.

So suppose $r = 2$ and let v_1, v_2 be two vertices of degree 2 in G_k. Then by (12), the graph $H = G_k - v_1 - v_2$ has at least

$$\frac{3t^2 + t + 10}{2} + \binom{t-2}{2} - 2(2) = 2t^2 - 2t + 4$$

edges. So the complement of H has at most $t - 4$ edges and thus, for $u, w \in V(H)$:

$$d_H(u) + d_H(w) \geq 2(2t - 1) - (t - 4) - 1 = 3t + 1 = |V(H)| + t + 1.$$

Hence by Theorem 2.8,

$$\text{for each linear forest } F \subseteq H \text{ with } e(F) \leq t + 1,\ H \text{ has a spanning cycle containing } E(F). \tag{15}$$

If $N(v_i) = \{u_i, w_i\}$ for $i = 1, 2$ and $v_1v_2 \in E(G_k)$, say $u_1 = v_2$ and $u_2 = v_1$, then $w_1 \neq w_2$ since H is 2-connected. Thus by (15), graph $H' = H + w_1w_2$ has a spanning cycle containing w_1w_2, and this cycle yields a hamiltonian cycle in G_k, a contradiction. So $v_1v_2 \notin E(G_k)$. Similarly, if $N(v_1) \neq N(v_2)$, then by (15), graph $H'' = H + u_1w_1 + u_2w_2$ has a spanning cycle containing u_1w_1 and u_2w_2. Again this yields a hamiltonian cycle in G_k. Thus we may assume $N(v_1) = N(v_2) = \{u, w\}$. Let

$$H_0 = H + uw \text{ if } uw \notin E(G) \text{ and } H_0 = H \text{ otherwise.} \tag{16}$$

If $x_m * y_m \notin N[v_1] \cup N[v_2]$, then $T_{G_{m+1}}(x_my_m) \leq 1$ (since $T_{G_{m+1}}(v_1u_1) \leq 1$) and G_{m+1} contains vertices v_1 and v_2 of degree 2. So by Lemma 3.3 for $h = 2$, G_{m+2} also has a vertex of degree 2. Thus by (4) for $r = 2$ instead of (10) we have for every $m+2 \leq j \leq n$,

$$\delta(G_j) \leq \min\{j - m, t - 1\} \text{ and so } T_{G_j}(x_{j-1}y_{j-1}) \leq \min\{j - m - 1, t - 2\}. \tag{17}$$

Plugging (17) instead of (10) into (11) for $r = 2$, we will instead of (13) get the stronger inequality

$$\frac{3t^2 + t + 10}{2} + (t - 3) + \binom{t-2}{2} \leq 2^2 + \binom{2t + 2 - 2}{2}. \tag{18}$$

Thus instead of (14) we have for $r = 2$ the stronger inequality $t(2-3) + (t-3) \leq 2^2 - 4 - 4$, which does not hold. This contradiction implies $x_m * y_m \in N[v_1] \cup N[v_2]$. By symmetry we have two cases.

Case 1: $x_m * y_m = v_1$. As above, graph H_0 has a spanning cycle C containing uw. If

$$x_m u, y_m w \in E(G_{m+1}), \tag{19}$$

then C extends to a k-cycle in G_{m+1} by replacing uw with path u, x_m, y_m, w. A similar situation holds if

$$x_m w, y_m u \in E(G_{m+1}). \tag{20}$$

But by degree conditions each of x_m, y_m has a neighbor in $\{u, w\}$. By definition, each of u, w has a neighbor in $\{x_m, y_m\}$. So at least one of (19) and (20) holds.

Case 2: $x_m * y_m = u$. If $d_{G_{m+1}}(v_1) = d_{G_{m+1}}(v_2) = 2$, then as before we get (18) instead of (14) and get a contradiction. So by symmetry we may assume that v_1 is adjacent to both x_m and y_m in G_{m+1}. Since G_m is 2-connected, vertex w does not separate $\{v_1, v_2, u\}$ from the rest of the graph. Thus by symmetry we may assume that y_m has a neighbor $z \in V(G_{m+1}) \setminus \{x_m, v_1, v_2, w\}$. Again by (15), graph H_0 defined by (16) has a spanning cycle containing edges uw and uz, and again this cycle yields a k-cycle in G_{m+1} (using path w, v_1, x_m, y_m, z), a contradiction. □

Proof for $k = 2t + 1$. We repeat the argument for $k = 2t + 2$, but instead of (12) and (13), we get

$$\frac{3t^2 - t + 6}{2} + \binom{t-r}{2} \le e(G_k) \le r^2 + \binom{2t+1-r}{2}.$$

Expanding the binomial terms and regrouping, similarly to (14), we get

$$t(r-2) \le r^2 - r - 3.$$

The analysis of this inequality is simpler than that of (14): If $r = 2$, then the left hand side is 0 and the right hand side is -1, while if $r \ge 3$, then dividing both sides by $r - 2$ we get $t \le r + 1 - 1/(r-2)$, which yields $r \ge t$, as claimed. □

Lemma 4.4. *Under the conditions of Lemma 4.3, G_k is a subgraph of the graph $H_{k,k,t}$.*

Proof for $k = 2t + 2$. By Lemma 4.3, $r(G_k) = t$. Let G' be the k-closure of G_k and $d'_1 \le d'_2 \le \ldots \le d'_k$ be the vertex degrees in G'. By the definition of the k-closure,

$$d(u) + d(v) \le k - 1 \qquad \textit{for every non-edge } uv \textit{ in } G'. \tag{21}$$

Since $d'_i \ge d_i$ for every i and G' is also non-hamiltonian, $r(G') \ge r(G_k) = t$. Since $r(G') \le t$ from $r(G) < n/2$, $r(G') = t$. Let $V(G') = \{v_1, \ldots, v_k\}$ where $d_{G'}(v_i) = d'_i$ for all i. By the definition of $r(G')$, on the one hand $d'_t \le t$ and $d'_{k-t} \le k - t - 1 = t + 1$, on the other hand either $d'_{t-1} > t-1$ or $d'_{k-(t-1)} \geq k-(t-1) = t+3$. In any case, $d'_{t+3} \geq t$. Summarizing,

$$d'_{t+3} \geq t,\ d'_t \leq t \ \textit{ and } \ d'_{t+1} \leq d'_{t+2} \leq t+1. \tag{22}$$

Let $B = \{v_1, \dots, v_{t+2}\}$ and $A = V(G') \setminus B$. If $d'_{t+4} \leq t+2$, then

$$\sum_{i=1}^{k} d'_i \leq (t|B|+2) + (t+2)2 + (2t+1)(t-2) = 3t^2 + t + 4,$$

a contradiction to $e(G_k) \geq h(k,k,t-1)+1$. Thus $d'_{t+4} \geq t+3$, and by (21) and (22), $G'[A] = K_t$. In summary,

$$d'_{t+4} \geq t+3 \quad \textit{ and } \quad G'[A] = K_t. \tag{23}$$

Suppose that there are distinct $v_{i_1}, v_{i_2} \in B$ and distinct $v_{j_1}, v_{j_2} \in A$ such that $v_{i_1}v_{j_1}$ and $v_{i_2}v_{j_2}$ are non-edges in G'. Then by (21) and (22),

$$\begin{aligned}\sum_{i=1}^{2t+2} d'_i &\leq (2t+1)2 + t(|B|-2) + 2 + (2t+1)(|A|-2) \\ &= 4t + 2 + t^2 + 2 + 2t^2 - 3t - 2 = 3t^2 + t + 2.\end{aligned}$$

This contradicts $e(G_k) > h(k,k,t-1)$. So, some v_j is incident with all non-edges of G' connecting A with B.

Case 1: $j \leq t+2$, i.e. $v_j \in B$. Then each $v \in B - v_j$ has t neighbors in A. Thus each $v \in B \setminus \{v_j, v_{t+1}, v_{t+2}\}$ has no neighbors in B, and each of v_{t+1}, v_{t+2} has at most one neighbor in B. If each of v_{t+1}, v_{t+2} is adjacent to v_j, then G' has a hamiltonian cycle using edges $v_{t+1}v_j$ and v_jv_{t+2}. Otherwise $G'[B]$ has at most one edge, as claimed.

Case 2: $j \geq t+3$, i.e. $v_j \in A$. Together with (23), this yields that G' contains $K_{t-1,t+3}$ with partite sets $A \setminus \{v_j\}$ and $B \cup \{v_j\}$. In particular, all pairs of vertices in $A \setminus \{v_j\}$ are adjacent. So, G' is obtained from $K_{2t+2} - E(K_{t+3})$ by adding at least $e(G') - \binom{2t+2}{2} + \binom{t+3}{2} \geq 7$ edges. If $G'[B \cup \{v_j\}]$ contains a linear forest with four edges, then G' has a hamiltonian cycle. So suppose

$$G'[B \cup \{v_j\}] \textit{ contains no linear forests with four edges.} \tag{24}$$

Case 2.1: $G'[B \cup \{v_j\}]$ contains a cycle C. By (24), $|C| \leq 4$ and if $|C| = 4$, then each other edge in $G'[B \cup \{v_j\}]$ has both ends in $V(C)$. Thus $G'[B \cup \{v_j\}]$ has at most 6 edges, a contradiction. So suppose $C = (x,y,z)$. If no other edge is incident with $V(C)$, then the set of the remaining at least four edges in $G'[B \cup \{v_j\}]$ contains a linear forest with two edges, a contradiction to (24). Thus we may assume that $G'[B \cup \{v_j\}]$ has an edge xu where $u \notin \{y,z\}$. Then by (24) and the fact that $G'[B \cup \{v_j\}]$ contains no 4-cycles, none of u, y, z is incident with other edges. On the other hand, if $G'[B \cup \{v_j\}]$ has an edge not incident with $V(C)$, this would contradict (24). Hence $G'[B \cup \{v_j\} \setminus \{x\}]$ has only the edge yz, as claimed.

Case 2.2: $G'[B \cup \{v_j\}]$ is a forest. By (24), there is $x \in B \cup \{v_j\}$ of degree at least 3 in $G'[B \cup \{v_j\}]$. If there is another vertex y of degree at least 3 in $G'[B \cup \{v_j\}]$, then we can choose two edges incident with x and two edges incident with y that together form a linear forest with four edges. So $G'[B \cup \{v_j\} \setminus \{x\}]$ is a linear forest, call it F, and thus has at most 3 edges. Each edge of F has at most one end adjacent to x and the degree of x in $G'[B \cup \{v_j\}]$ is at least four. So if F has exactly $m \in \{2, 3\}$ edges, then we can choose $4 - m$ edges incident with x so that together with F they form a linear forest. And if F has at most one edge, then the lemma holds. □

Proof for $k = 2t+1$. The proof is almost identical to the case $k = 2t+2$. By Lemma 4.3, $r(G_k) = t$. Let G' be the k-closure of G_k and $d'_1 \leq d'_2 \leq \ldots \leq d'_k$ be the vertex degrees in G'. As in (21), we have

$$d(u) + d(v) \leq k - 1 = 2t \qquad \textit{for every non-edge } uv \textit{ in } G'. \tag{25}$$

As in the proof in the case $k = 2t + 2$, $r(G') = t$. Let $V(G') = \{v_1, \ldots, v_k\}$ where $d_{G'}(v_i) = d'_i$ for all i. Instead of (22), we get the stronger claim

$$d'_{t+2} \geq t \ \textit{ and } \ d'_t \leq d'_{t+1} = t. \tag{26}$$

Let $B = \{v_1, \ldots, v_{t+1}\}$ and $A = V(G') \setminus B$. If $d'_{t+3} \leq t + 1$, then

$$\sum_{i=1}^{2t+1} d'_i \leq t|B| + (t+1)2 + (2t)(t-2) = 3t^2 - t + 2 \leq h(k, k, t-1),$$

a contradiction. Thus,

$$d'_{t+3} \geq t + 2 \ \text{ so by (25) and (26)}, \ G'[A] = K_t. \tag{27}$$

If there are distinct $v_{i_1}, v_{i_2} \in B$ and distinct $v_{j_1}, v_{j_2} \in A$ such that $v_{i_1}v_{j_1}$ and $v_{i_2}v_{j_2}$ are non-edges in G', then by (25) and (26),

$$\sum_{i=1}^{k} d'_i \leq (2t)2 + t(|B| - 2) + (2t)(|A| - 2) = 4t + t^2 - t + 2t^2 - 4t = 3t^2 - t$$
$$\leq h(k, k, t-1),$$

a contradiction. So, some v_j is incident with all non-edges of G' connecting A with B.

Case 1: $j \leq t + 1$, i.e. $v_j \in B$. Then each $v \in B - v_j$ has t neighbors in A. Thus by (26), each $v \in B - v_j$ has no neighbors in B, hence B is independent, as claimed.

Case 2: $j \geq t+2$, i.e. $v_j \in A$. Together with (27), this yields that $G' - v_j$ contains $K_{t-1,t+2}$ with partite sets $A \setminus \{v_j\}$ and $B \cup \{v_j\}$. In particular, each vertex in $A \setminus \{v_j\}$ is all-adjacent. So, G' is obtained from $K_k - E(K_{t+2})$ by adding at least four edges. If $G'[B \cup \{v_j\}]$ contains a linear forest with three edges, then G' has a hamiltonian cycle. Every graph with at least four edges not containing a linear forest with three edges is a star plus isolated vertices. And if $G'[B \cup \{v_j\}]$ is a star plus isolated vertices, then $G' \subseteq H_{k,k,t}$. □

4.2.2. The case $m > k$

Lemma 4.5. *Let $m > k \geq 9$.*

(1) *If $k \neq 10$, then $G_m \subseteq H_{m,k,t}$.*
(2) *If $k = 10$ then $G_m \subseteq H_{m,k,t}$ or $G_m \supseteq F_4$.*

Proof for $k = 2t+2$. G_m is an m-vertex 2-connected graph with $c(G_m) \leq 2t+1$ satisfying $e(G) \geq h(n,k,t-1)+1$. Since (R2) is not applicable,

$$T_{G_m}(xy) \geq t-1 \text{ for every non-separating edge } xy. \tag{28}$$

By Lemmas 3.2 and 3.1, (28) implies

$$\delta(G_m) \geq t \text{ and for each } v \in V(G_m) \text{ with } d(v) = t,\ G_m[N(v)] = K_{t+1}. \tag{29}$$

Let $C = (v_1, \dots, v_q)$ be a longest cycle in G_m. Since $\delta(G_m) \geq t$, Dirac's Theorem (Theorem 2.3) yields $q \geq 2t$. Obviously, $q \leq 2t+1$.

By (28) and Lemma 3.4, each edge of C is in at least $t-1$ triangles. By the maximality of C, the third vertex of each such triangle is in $V(C)$. So

$$\text{the minimum degree of } G_m[V(C)] \text{ is at least } t. \tag{30}$$

We now prove that

$$G_m[V(C)] \text{ is 3-connected.} \tag{31}$$

Indeed, assume (31) fails and $G_m[V(C)]$ has a separating set S of size 2. By symmetry, we may assume that $S = \{v_1, v_j\}$ and that $j \leq \lfloor q/2 \rfloor + 1 \leq t+1$. Then by (30), $j = t+1$ and $G_m[\{v_1, \dots, v_{t+1}\}] = K_{t+1}$. In particular,

$$v_1 v_{t+1} \in E(G_m). \tag{32}$$

Let $H_1 = G_m[\{v_1, \dots, v_{t+1}\}]$ and $H_2 = G_m[\{v_{t+1}, \dots, v_q, v_1\}]$. Similarly to H_1, graph H_2 is either K_{t+1} (when $q = 2t$) or is obtained from K_{t+2} by deleting some matching (when $q = 2t+1$).

Concerning almost complete graphs we need the following statement which is an easy consequence of Theorem 2.8 (or one can prove it directly).

For $p \geq 6$ and for any matching $M \subseteq K_p$, every two edges of $K_p - M$ are in a common hamiltonian cycle of $K_p - M$. (33)

Since G_m is 2-connected, each component F of $G_m - V(C)$ has at least two neighbors, say $y(F)$ and $y'(F)$, in C. If at least one of them, say $y'(F)$, is not in $S = \{v_1, v_{t+1}\}$, then we can construct a cycle longer than C as follows.

If $y(F) \in V(H_1) \setminus \{v_1, v_{t+1}\}$ and $y'(F) \in V(H_2) \setminus \{v_1, v_{t+1}\}$, then $H_1 - v_{t+1}$ has a hamiltonian $v_1, y(F)$-path P_1 (recall that $H_1 - v_{t+1}$ is a complete graph), and H_2 has a hamiltonian $v_1, y'(F)$-path P_2, by (33) and since $k \geq 4$. So $P_1 \cup P_2$ and a $y(F), y'(F)$-path through F form a longer than C cycle in G_m.

If both, $y(F)$ and $y'(F)$ are in the same H_j, then we let H'_j be the graph obtained from H_j by adding the edge $y(F)y'(F)$. Recall that by (32), $v_1 v_{t+1} \in E(H_j)$. If we have a hamiltonian cycle C' in H'_j containing $y(F)y'(F)$ and $v_1 v_{t+1}$, then let P be the v_1, v_{t+1}-path obtained from C' by deleting edge $v_1 v_{t+1}$ and replacing edge $y(F)y'(F)$ with a $y(F), y'(F)$-path P' through F, and then replace in C the v_1, v_{t+1}-path through $V(H_j)$ with the longer path P. There is such a C' if $|V(H_j)| \geq 6$ by (33), and also if $|V(H_j)| = 5$ because in the latter case $|V(H_j)| = t + 1$ with $t = 4$ and it is a complete graph.

Thus every component F of $G_m - V(C)$ is adjacent only to S, and S is a separating set in G_m. In particular, $H_1 - S = K_{t-1}$ and $H_2 - S$ are components of $G_m - S$. So, if $m \geq 3t + 1$, then rule (R3) is applicable, contradicting the definition of G_m. Hence $2t + 2 \leq m \leq 3t$. On the other hand, by (29), every component of $G_m - S$ has at least $t - 1$ vertices, and so $m - q \geq t - 1$. Therefore, $3t - 1 \leq m \leq 3t$.

If $m = 3t - 1$, then $q = 2t$, $H_2 = K_{t+1}$ and $H_3 := G_m - (V(C) - S) = K_{t+1}$. Hence

$$e(G_m) - h(m, k, t-1) - 1 = 3\binom{t+1}{2} - 2 - h(3t-1, k, t-1) - 1$$
$$= \frac{3t^2 + 3t - 4}{2} - \frac{5t^2 - 7t + 16}{2} = -t^2 + 5t - 10 < 0.$$

Similarly, if $m = 3t$, then the component sizes of $G_m - S$ are $t, t-1, t-1$. Thus in this case

$$e(G_m) - h(m, k, t-1) - 1 \leq t^2 + t + \binom{t+2}{2} - 2 - h(3t, k, t-1) - 1$$
$$= \frac{3t^2 + 5t}{2} - 1 - \frac{5t^2 - 5t + 14}{2} = -t^2 + 5t - 8 < 0.$$

These contradictions prove (31).

So by (31) and Theorem 2.7 for $n = q$, $s = 2t$ and $H = G_m[V(C)]$, one of three cases below holds:

Case 1: $\overline{K_t} + \overline{K_{q-t}} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_{q-t}}$. Let B be the independent set of size $q - t$ in $G_m[V(C)]$ and $A = V(C) \setminus B$. In this case, since $G_m[V(C)]$ has hamiltonian cycle C and an independent set B of size $q - t$, we need $q = 2t$.

Suppose that $G_m - V(C)$ has a component D with at least two vertices. By Menger's Theorem, there are two fully disjoint paths, say P_1 and P_2, connecting some two distinct vertices, say u and v, of D with two distinct vertices, say x and y, of C. Since $G_m[V(C)]$ contains $K_{t,t}$, it has an x, y-path with at least $2t - 1$ vertices. This path together with P_1, P_2 and a u, v-path in D form a cycle of length at least $2t + 1$, a contradiction to the maximality of C. Thus each component of $G_m - V(C)$ is a single vertex and is adjacent either only to vertices in A or only to vertices in B. Moreover, by (29), each such vertex has degree exactly t, and thus its neighborhood is a complete graph. Since B is independent, each $v \in V(G_m) - C$ is adjacent only to vertices in A. Thus $G_m = K_m - E(K_{m-t}) = H_{m,k-1,t} \subseteq H_{m,k,t}$.

Case 2: $\overline{K_3} + \ell K_{t-2} \subseteq G_m[V(C)] \subseteq K_3 + \ell K_{t-2}$, where $\ell = 2(q-3)/(2t-4)$. Again, since $G_m[V(C)]$ has hamiltonian cycle C and a separating set of size 3 (call this set A), $\ell \leq 3$. If $\ell \leq 2$, then $q \leq 3 + 2(t-2) < 2t$, a contradiction. Thus, $\ell = 3$ and $q = 3 + 3(t-2) = 3t - 3$. Since $2t \leq q \leq 2t + 1$, we get $t \in \{3, 4\}$. Since $t \geq 4$ by assumption, we obtain that $t = 4$ and $F_4 \subseteq G_m$.

Case 3: *For every two distinct $x, y \in V(C)$, the graph $G_m[V(C)]$ contains an x, y-path with at least $2t$ vertices.* Let $W = V(G_m) - V(C)$. Repeating the argument of the second paragraph of Case 1, we obtain that in our case

$$\textit{each component of } G_m[W] \textit{ is a singleton and so } N(w) \subseteq V(C) \textit{ for each } w \in W. \quad (34)$$

Since no $w \in W$ is adjacent to two consecutive vertices of C (by the maximality of C) and $q \leq 2t + 1$, by (29),

$$d_{G_m}(w) = t \textit{ for every } w \in W. \quad (35)$$

Fix some $w_1 \in W$. Then we may relabel the vertices of C so that $N_{G_m}(w_1) = \{v_1, v_3, v_5, \ldots, v_{2t-1}\}$. By (29), this also yields $G_m[\{v_1, v_3, \ldots, v_{2t-1}\}] = K_t$ and thus $d_{G_m}(v_i) \geq t + 1$ for all $i \in \{1, 3, \ldots, 2t - 1\}$. In particular,

$$d_{G_m}(v) \geq t + 1 \textit{ for every } v \in N_{G_m}(w_1). \quad (36)$$

Then for every $j \in \{2, 4, \ldots, 2t - 2\}$ (and for $j = 2t$ in the case $q = 2t$) we can replace v_j with w_1 in C and obtain another longest cycle. By (35) and (34), this yields $d_{G_m}(v_j) = t$ and

$$\begin{aligned} &N_{G_m}(v_j) \subseteq V(C) \textit{ for all } j \in \{2, 4, \ldots, 2t - 2\} \\ &\textit{(and for } j = 2t \textit{ in the case } q = 2t\textit{).} \end{aligned} \quad (37)$$

Case 3.1: $q = 2t$. Switching the roles of w_1 with v_j together with (36) yields

$$N_{G_m}(v_j) = \{v_1, v_3, v_5, \dots, v_{2t-1}\} \text{ for all } j = 2, 4, \dots, 2t. \tag{38}$$

By (35) and (38), $N_{G_m}(w) = \{v_1, v_3, v_5, \dots, v_{2t-1}\}$ for all $w \in V(G_m) - \{v_1, v_3, v_5, \dots, v_{2t-1}\}$. This means $G_m \subseteq H_{m,2t+2,t}$, as claimed.

Case 3.2: $q = 2t + 1$. Since $m \geq 2t + 3$, there is $w_2 \in W - w_1$. By (37), vertex w_2 is not adjacent to v_j for $j \in \{2, 4, \dots, 2t - 2\}$. Suppose that w_2 is adjacent to v_{2t} or v_{2t+1}, say $w_2v_{2t} \in E(G_m)$. Then by the maximality of C, $w_2v_{2t+1}, w_2v_{2t-1} \notin E(G_m)$. So the only possible t-element set of neighbors of w_2 is $\{v_1, v_3, \dots, v_{2t-3}, v_{2t}\}$. But then G_m has the $(2t + 2)$-cycle $(w_2, v_3, v_4, v_5, \dots, v_{2t-1}, w_1, v_1, v_{2t+1}, v_{2t}, w_2)$, a contradiction. Thus

$$N_{G_m}(w) = \{v_1, v_3, v_5, \dots, v_{2t-1}\} \text{ for all } w \in W. \tag{39}$$

Since we can replace in C any v_j for $j \in \{2, 4, \dots, 2t-2\}$ with w_1, (39) yields $N_{G_m}(v_j) = \{v_1, v_3, v_5, \dots, v_{2t-1}\}$ for all $j = 2, 4, \dots, 2t-2$. It follows that $\{v_1, v_3, v_5, \dots, v_{2t-1}\}$ covers all edges in G_m apart from edge $v_{2t}v_{2t+1}$. This means $G_m \subseteq H_{m,2t+2,t}$, as claimed. □

Proof for $k = 2t + 1$. Similarly to the proof for $k = 2t + 2$, we have (28) and (29). Let $C = (v_1, \dots, v_q)$ be a longest cycle in G_m. Since $\delta(G_m) \geq t$, by Theorem 2.3, $q \geq 2t$; so $c(G_m) < k$ yields $q = 2t$. Then repeating the argument for $k = 2t + 2$, we obtain (30) and finally (31). So by Theorem 2.7 for $n = s = 2t$ and $H = G_m[V(C)]$, one of three cases below holds:

Case 1: $\overline{K_t} + \overline{K_t} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_t}$. As in the proof for $k = 2t + 2$, we derive $G_m = K_m - E(K_{m-t}) = H_{m,k,t}$.

Case 2: $\overline{K_3} + \ell K_{t-2} \subseteq G_m[V(C)] \subseteq K_3 + \ell K_{t-2}$, where $\ell = 2(2t - 3)/(2t - 4)$. Again, since $G_m[V(C)]$ has hamiltonian cycle C and a separating set of size three (call this set A), $\ell \leq 3$. Since $t \geq 4$, $\ell \neq 3$. If $\ell \leq 2$, then $q \leq 3 + 2(t - 2) < 2t$, a contradiction.

Case 3: For every two distinct $x, y \in V(C)$, graph $G_m[V(C)]$ contains a hamiltonian x, y-path. Then for any component H of $G_m - V(C)$, let x and y be neighbors of H in $V(C)$. By the case, $G_m[V(C)]$ contains a $2t$-vertex path, say P. Then P together with an x, y-path through H forms a cycle with at least k vertices, a contradiction. But since $m > k$, such a component H does exist. □

4.3. Subgraphs of G_m

In this section, we define classes of graphs which we shall show are subgraphs of G_m, and these subgraphs will have the important property that they have many long paths and are preserved by the reverse of the contraction process in the Basic Procedure.

For a graph F and a nonnegative integer s, we denote by $\mathcal{K}^{-s}(F)$ the family of graphs obtained from F by deleting at most s edges.

Let $F_0 = F_0(t)$ denote the complete bipartite graph $K_{t,t+1}$ with partite sets A and B where $|A| = t$ and $|B| = t + 1$. Let $\mathcal{F}_0 := \mathcal{K}^{-t+3}(F_0)$, i.e., the family of subgraphs of $K_{t,t+1}$ with at least $t(t + 1) - t + 3$ edges.

Let $F_1 = F_1(t)$ denote the complete bipartite graph $K_{t,t+2}$ with partite sets A and B where $|A| = t$ and $|B| = t + 2$. Let $\mathcal{F}_1 := \mathcal{K}^{-t+4}(F_1)$, i.e., the family of subgraphs of $K_{t,t+2}$ with at least $t(t + 2) - t + 4$ edges.

Let $\mathcal{F}_2$ denote the family of graphs obtained from a graph in $\mathcal{K}^{-t+4}(F_1)$ by subdividing an edge a_1b_1 with a new vertex c_1, where $a_1 \in A$ and $b_1 \in B$. Note that any member $H \in \mathcal{F}_2$ has at least $|A||B| - (t - 3)$ edges between A and B and the pair a_1b_1 is not an edge.

Let $F_3 = F_3(t, t')$ denote the complete bipartite graph $K_{t,t'}$ with partite sets A and B where $|A| = t$ and $|B| = t'$. Take a graph from $\mathcal{K}^{-t+4}(F_3)$, select two non-empty subsets $A_1, A_2 \subseteq A$ with $|A_1 \cup A_2| \geq 3$ such that $A_1 \cap A_2 = \emptyset$ if $\min\{|A_1|, |A_2|\} = 1$, add two vertices c_1 and c_2, join them to each other and add the edges from c_i to the elements of A_i ($i = 1, 2$). The class of obtained graphs is denoted by $\mathcal{F}(A, B, A_1, A_2)$. The family $\mathcal{F}_3$ consists of these graphs when $|A| = |B| = t$, $|A_1| = |A_2| = 2$ and $A_1 \cap A_2 = \emptyset$. In particular, for $t = 4$ the family $\mathcal{F}_3$ consists of exactly one graph, call it $F_3(4)$.

Recall that F_4 is a 9-vertex graph with vertex set $A \cup B$, $A = \{a_1, a_2, a_3\}$, $B := \{b_1, b_2, \dots, b_6\}$ and edges of the complete bipartite graph $K(A, B)$ and three extra edges b_1b_2, b_3b_4, and b_5b_6. Define F'_4 as the (only) member of $\mathcal{F}(A, B, A_1, A_2)$ where $|A| = |B| = t = 4$, $A_1 = A_2$, and $|A_i| = 3$. Let $\mathcal{F}_4 := \{F_4, F'_4\}$, which is defined only for $t = 4$.

In this subsection we will prove two useful properties of graphs in $\mathcal{F}_0 \cup \dots \cup \mathcal{F}_4$: First we show that G_m contains one of them (Proposition 4.6) and then show that such graphs have long paths with given end-vertices (Lemma 4.8).

Proposition 4.6. *Let $k \geq 9$. If k is odd, then G_m contains a member of $\mathcal{F}_0$, and if k is even then G_m contains a member of $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$.*

Proof. By Proposition 4.2, $G_m \subseteq H_{m,k,t}$ or $m > k = 10$ and $F_4 \subseteq G_m$. In the latter case, the proof is complete. So assume $G_m \subseteq H_{m,k,t}$ and A, B, C are as in the definition of $H_{m,k,t}$. First suppose k is even and $C = \{c_1, c_2\}$. If $m = k$ then by (2),

$$e(H_{m,k,t}) - e(G_m) \leq h(m, k, t) - h(m, k, t - 1) - 1 = t - 4,$$

i.e. $G_m \in \mathcal{K}^{-t+4}(H_{m,k,t})$. Since $F_1(t) \subseteq H_{m,k,t}$, G_m contains a subgraph in $\mathcal{F}_1$. If $m > k$ then by (R2) and Lemma 3.2, we have $\delta(G_m) \geq t$. So, each $v \in B$ is adjacent to every $u \in A$ and each of c_1, c_2 has at least $t-1$ neighbors in A. Since $|B \cup \{c_1\}| \geq m-t-1 \geq t+2$, G_m contains a member of $\mathcal{K}^{-1}(F_1(t))$. Thus G_m contains a member of $\mathcal{F}_1$ unless $t = 4$, $m = 2t+3$ and c_1 has a nonneighbor $x \in A$. But then $c_1c_2 \in E(G_m)$, and so G_m contains either $F_3(4)$ or F'_4.

Similarly, if k is odd and $m = k$, then by (2), $G_m \in \mathcal{K}^{-t+3}(H_{m,k,t})$. Thus, since $H_{m,k,t} \supseteq F_0(t)$, G_m contains a subgraph in $\mathcal{F}_0$. If k is odd and $m > k$ then by (R2) we have $\delta(G_m) \geq t$. So, each $v \in V(G_m) - A$ is adjacent to every $u \in A$. Hence G_m contains $K_{t,m-t}$. $\square$

In order to prove Lemma 4.8, we will use Corollary 2.10 and the following implication of it.

Lemma 4.7. *Let $t \geq 4$ and $H \in \mathcal{F}(A, B, A_1, A_2)$ with $|B| \geq t-1$, $|A| = t$. Let P be a path $a_1c_1c_2a_2$ and L be a subtree of H with $|E(L)| \leq 2$ such that $P \cup L$ form a linear forest. Then*

$$H \text{ has a cycle } C \text{ of length } 2t+1 \text{ containing } P \cup L. \tag{40}$$

Proof. Choose some $B' \subseteq B$ with $|B'| = t-1$ such that $B \cap V(L) \subseteq B'$. Let Q be the bipartite graph whose t-element partite sets are A and $B' \cup \{c\}$ where c is a new vertex, and the edge set consists of $H[A \cup B']$ and all edges joining c to A. By the conditions of the lemma, the set $E' := E(L) \cup \{a_1c, ca_2\}$ forms a linear forest in Q. Since Q misses at most $t-4$ edges connecting A with $B' \cup \{c\}$, by Corollary 2.10 with $s = t$ and $i = 2$, Q has a hamiltonian cycle C' containing E'. Then the $(2t+1)$-cycle C in H obtained from C' by replacing path a_1ca_2 with P satisfies (40). $\square$

Lemma 4.8. *Let $H \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4$ and $x, y \in V(H)$.*

(*a*) *H contains an x,y-path of length at least $2t-2$;*
(*b*) *if H does not contain an x,y-path of length at least $2t-1$, then*
  (*b*0) *$H \in \mathcal{F}_0$ and $\{x, y\} \subseteq A$, or*
  (*b*1) *$H \in \mathcal{F}_1$ and $\{x, y\} \subseteq A$, or*
  (*b*2) *$H = F_4 \in \mathcal{F}_4$ and $\{x, y\} \subseteq A$;*
(*c*) *if H does not contain an x,y-path of length at least $2t$, then*
  (*c*0) *$H \in \mathcal{F}_0$, or*
  (*c*1) *$H \in \mathcal{F}_1$ and at least one of x, y is in A, or*
  (*c*2) *$H \in \mathcal{F}_2$ and either $\{x, y\} \subseteq A$ or $\{x, y\} = \{a_1, b_1\}$, or*
  (*c*3) *$H \in \mathcal{F}_3$ and $\{x, y\} \subseteq A$, or*
  (*c*4) *$H \in \mathcal{F}_4$ and $\{x, y\} \subseteq A$.*

Proof. The statements concerning $H \in \mathcal{F}_0 \cup \mathcal{F}_1$ are the easiest. Using Corollary 2.10 (or just using induction on t) it is easy to prove a bit more. Suppose that $H \in \mathcal{K}_{t,t+1}^{-(t-2)}(A, B)$, $t \geq 2$. Then every pair $x, y \in A \cup B$ is joined by a path of maximum possible length. This means that every pair of vertices $b_1, b_2 \in B$ is joined by a path of length $2t$, every pair $a \in A$, $b \in B$ is joined by a path of length $2t-1$, and every pair $a_1, a_2 \in A$ is joined by a path of length $2t-2$. For example, the proof for $H \in \mathcal{F}_0$, $a \in A$ and $b \in B$ is as follows. Consider H' obtained from H by adding edge ab if $ab \notin E(H)$ and deleting any $b' \in B - b$. Then by Corollary 2.10, H' has a hamiltonian cycle containing ab, which yields an a,b-path in H of length $2t-1$.

The cycle $(b_1b_2a_1b_3b_4a_2b_5b_6a_3b_1)$ and path $b_1b_2a_1b_3a_2b_4a_3b_5b_6$ in F_4 prove (b2) and the part of (c4) related to F_4.

Suppose now that $H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F_4'\}$; even in a more general setting suppose that $H \in \mathcal{F}(A, B, A_1, A_2)$ with $|B| = |A| = t$, $|A_1 \cup A_2| \geq 3$, $|A_2| \geq |A_1| \geq 1$ (and in case of $|A_1| = 1$ one has $A_1 \cap A_2 = \emptyset$). We prove the statements in reverse order, first (c2) and (c3), then (b), finally (a). When we comment below "Case BC" or "Case AA", this means that we consider paths from B to C or from A to A, respectively.

By Lemma 4.7, we already knew that c_1c_2 is contained in a cycle of length $2t+1$ so these two vertices are joined by a path of length $2t$ (Case CC). If $b \in B$, and $a_i \in A_i$, then the almost complete bipartite subgraph $H[A \cup B]$ contains a b, a_i-path of length $2t-1$, so b and c_{3-i} is joined in H by a path of length $2t+1$ (Case BC). Concerning $b_1, b_2 \in B$ we can define H^+ by adding an extra vertex a_{t+1} to A and joining it to each vertex of B. Applying Lemma 4.7 to H^+ (with $t+1$ in place of t) we get that it has a cycle C_{2t+3} through $b_1a_{t+1}b_2$. This cycle gives a b_1, b_2-path of length $2t+1$ in H (Case BB). In case of $x \in A$, $y \in A$ the high edge density of H implies that x and y have a common neighbor $b \in B$. One can find a path $P = a_1c_1c_2a_2$ such that P and xby form a linear forest. Then Lemma 4.7 yields a cycle C_{2t+1} through all these edges. Leaving out b one gets an x, y-path of length $2t-1$ in H (Case AA). In case of $x \in A$, $y \in B$ maybe we have to add the edge xy to obtain a cycle C_{2t+1} through it by Lemma 4.7. This yields an x, y-path of length $2t$ (Case AB). Finally, if $x \in A$, $y = c_i$ one uses a path c_i, c_{3-i}, x' and an x, x'-path of length $2t-2$ in $A \cup B$ to get an x, y-path of length $2t$, if this can be done. If such an $x' \neq x$ does not exist, then $x = a_1 \in A_1$, $|A_1| = 1$, and $y = c_2$. This is the case described in (c2) (Case AC). □

4.4. Reversing contraction

The aim of this section is to prove Lemma 4.9 below on preserving certain subgraphs during the reverse of the Basic Procedure.

Lemma 4.9 *(Main lemma on contraction). Let $k \geq 9$ and suppose F and F' are 2-connected graphs such that $F = F'/xy$ and $c(F') < k$.*

If k is even and F contains a subgraph $H \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4$, then F' has a subgraph $H' \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4$.

If k is odd and F contains a subgraph $H \in \mathcal{F}_0$, then F' has a subgraph $H' \in \mathcal{F}_0$.

Proof for *k* even. Case 1. $H \in \mathcal{F}_1$. Let $u = x * y$. If $u \notin V(H)$ then $H \subseteq F'$ and we are done. In case of $u \in A$ consider the sets $X := N_{F'}(x) \cap B$ and $Y := N_{F'}(y) \cap B$. If $X = X \cup Y$ then F' restricted to $(A \setminus \{u\}) \cup \{x\} \cup B$ contains a copy of H. If $X = X \cup Y \setminus \{y'\}$ for $y' \in V(H')$, then F' restricted to $(A \setminus \{u\}) \cup \{x\} \cup B \cup \{y\}$ contains a copy of a graph from $\mathcal{F}_2$ (with $a_1 := x$, $b_1 := y'$, and $c_1 := y$). We proceed in the same way if $Y = X \cup Y$ or if $|Y| = |X \cup Y| - 1$. In the remaining case $|X \setminus Y| \geq 2$ and $|Y \setminus X| \geq 2$, so one can choose five distinct elements b_0, x_1, x_2, y_1, y_2 from B such that

$\{x_1, x_2\} \subseteq X \setminus Y$ and $\{y_1, y_2\} \subseteq Y \setminus X$. Then the bipartite subgraph Q_0 of F' generated by the sets $A \setminus \{u\} \cup \{x, y\}$ and $B \setminus \{b_0\}$ contains the linear forest L consisting of the paths $x_1 x x_2$ and $y_1 y y_2$. If we define the graph Q by adding to Q_0 all edges joining x and y to $B \setminus \{b_0\}$, then Q has at least $(t+1)^2 - (t-4)$ edges. So by Corollary 2.10 for $s = t+1$ and $i = 2$, Q has a hamiltonian cycle C_{2t+2} containing all edges of L, and this cycle also appears in F', contradicting $c(F') < k$.

In case of $u \in B$ consider the sets $X := N_{F'}(x) \cap A$ and $Y := N_{F'}(y) \cap A$. If $|X \setminus Y| \leq 1$ or $|Y \setminus X| \leq 1$, then we proceed as above and find a subgraph H' of F either isomorphic to H or belonging to $\mathcal{F}_2$. If $|X \setminus Y| \geq 2$ and $|Y \setminus X| \geq 2$, then we have four distinct elements x_1, x_2, y_1, y_2 in A such that $\{x_1, x_2\} \subseteq X \setminus Y$ and $\{y_1, y_2\} \subseteq Y \setminus X$. Then F' contains a member of $\mathcal{F}_3$ with $(c_1, c_2) = (x, y)$, $A_1 := \{x_1, x_2\}$, and $A_2 := \{y_1, y_2\}$.

Case 2. $H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F'_4\}$. The proof in this case follows from two claims. We say that the graph H has the Property (W_ℓ) if the following holds.

(W_ℓ) *For all $z \in V(H)$ there exists $w \in N(z)$ such that for all $w' \in N(z) \setminus \{w\}$, the graph H has a cycle C_ℓ containing the path wzw'.*

Claim 1. *Suppose that the graph F contains a subgraph H satisfying Property (W_ℓ), and $c(F') \leq \ell$. Then F' has a subgraph H' isomorphic to H.*

Let $z = x * y$ and $V = V(F) - z = V(F') - x - y$. If $V(H) \subseteq V$, then there is nothing to prove.

Suppose that $z \in V(H) \subseteq V(F)$ and define $X := N_{F'}(x) \cap N_H(z)$ and $Y := N_{F'}(y) \cap N_H(z)$. Then $X \cup Y = N_H(z)$. Let $w \in N(z)$ be the vertex from the definition of the Property (W_ℓ). Since $N_H(z) = X \cup Y$, we may assume by symmetry that $w \in X$.

We claim that $Y - w = \emptyset$. Indeed, suppose there is $w' \in Y - w$. By Property (W_ℓ), H has a cycle C_ℓ containing the path wzw'. Then the path $C_\ell - z$ in F' together with the edges $w'y$, yx and xw forms a cycle of length $\ell + 1$, contradicting $c(F') \leq \ell$.

This implies that $N_{F'}(x)$ contains $N_H(z)$. So F' contains a copy of H with the vertex set $(V(H) \setminus \{z\}) \cup \{x\}$. □

Claim 2. *If $H \in \mathcal{F}_2 \cup \mathcal{F}_3$ or $H = F'_4$, then H satisfies Property (W_{2t+1}).*

We prove a bit more: every $H \in \mathcal{F}(A, B, A_1, A_2)$ with $|B| \geq t-1$, $|A| = t$ satisfies (W_{2t+1}). Indeed, for $z = c_i$ we can choose a $w := c_{3-i}$. For $z \in B$ we can choose a $w \in A$ arbitrarily. For $z \in A$ we can choose $w \in N(z) \subseteq B$ arbitrarily, except if $z \in A_i$ and $|A_i| = 1$. In this latter case we can use $w := c_i$. In each of these cases, given $L := wzw'$ one can find a path $P := a_1 c_1 c_2 a_2$ such that $P \cup L$ is a linear forest. Then Lemma 4.7 yields that H has a cycle C_{2t+1} through wzw'.

Since each $H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F'_4\}$ belongs to such $\mathcal{F}(A, B, A_1, A_2)$, this completes the proof of Claim 2. □

Case 3. $H = F_4$. Let $u = x * y$. By symmetry, we can consider only two cases: $u = a_1$ and $u = b_1$. First, suppose $u = a_1$ and $xb_1 \in E(F')$. Then since $c(F') \leq 9$, y is not adjacent to any of b_3, b_4, b_5, b_6. Thus x is adjacent to all of them, and if $yb_2 \in E(F')$, then the cycle $(yb_2b_1a_2b_3b_4a_3b_5b_6xy)$ contradicts $c(F') \leq 9$. So $xb_2 \in E(F')$ and the subgraph of F' with vertex set $V(H) \setminus \{u\} \cup \{x\}$ contains F_4.

Similarly, suppose $u = b_1$ and $xb_2 \in E(F')$. Then to avoid a 10-cycle in F', y has no neighbors in A and thus x is adjacent to all of A. So, again the subgraph of F' with vertex set $V(H) \setminus \{u\} \cup \{x\}$ contains F_4. □

Proof for *k* odd. First we prove the following statement (41) which is true for every $t \geq 2$. Let $H \in \mathcal{K}^{-t+2}(K(A, B))$ with $|A| = t$, $|B| = t + 1$. Let P be a path of length two in H. Then

$$H \text{ has a cycle } C \text{ of length } 2t \text{ containing } P. \tag{41}$$

If every vertex of $B \setminus P$ is joined to all vertices of A, then one can find a C_{2t} through P directly. Otherwise, there is a vertex $v \in B \setminus P$ of degree at most $t - 1$, so $H \setminus \{v\}$ is a subgraph of $K_{t,t}$ with at least $t^2 - t + 3$ edges. Then the statement follows from Corollary 2.10 for $s = t$ and $i = 1$.

Now suppose that $H \in \mathcal{F}_0$, $H \subseteq F$, $F = F'/xy$, and H, F, F' satisfy the constraints of Lemma 4.9. Then (41) implies that H satisfies property (W_{2t}). Thus by Claim 1, F' has a subgraph H' isomorphic to H. □

4.5. Completing the proof of Theorem 4.1

Proof for *k* even. Proposition 4.6 and Lemma 4.9 imply that there is a subgraph H of $G = G_n$ such that $H \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4$. Let $G' = G - V(H)$ and $S_1, \ldots, S_s$ be the components of G'. Each of S_i has at least two neighbors, say x_i and y_i in $V(H)$. Let ℓ_i denote the length of a longest x_i, y_i-path in $G[V(S_i) \cup \{x_i, y_i\}]$. Since $c(G) < k$, by Lemma 4.8(a) and (b),

$$\text{for all } i, \quad \ell_i \leq 3 \quad \text{and if } H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F_4'\}, \text{ then} \quad \ell_i \leq 2. \tag{42}$$

Case 1: $H \in \mathcal{F}_3 \cup \{F_4'\}$. By (42), $\ell_i \leq 2$ for all i and all choices of x_i and y_i. Since G is 2-connected, this yields that each S_i is a singleton, say v_i. Moreover, Lemma 4.8(c3) and (c4) imply $N(v_i) \subseteq A$ for all i. So G is contained in a graph in $\mathcal{G}_1(n, k)$, and the only edge outside A is c_1c_2.

Case 2: $H \in \mathcal{F}_2$. Again, by (42), $\ell_i \leq 2$ for all i and all choices of x_i and y_i. So again this yields that each S_i is a singleton, say v_i. But now Lemma 4.8(c2) implies that for all i, either $N(v_i) \subseteq A$ or $N(v_i) = \{a_1, b_1\}$. Thus G is contained in a graph in $\mathcal{G}_2(n, k)$, where the only possible star component of $G - A$ with at least three vertices is a star with center b_1 and c_1 a leaf.

Case 3: $H \in \mathcal{F}_1$. Suppose first that some x_i is in B. Then by Lemma 4.8(c3), $y_i \in A$ and by Lemma 4.8(b), $\ell_i = 2$. So, denoting the common neighbor of x_i and y_i in S_i by c_1, we get Case 2. Thus it is enough to consider below only the situation when

$$N(S_i) \cap V(H) \subseteq A \textit{ for every } i. \tag{43}$$

We consider three cases.

Case 3.1: For some $i \neq j$, $\ell_i \geq 3$ and $\ell_j \geq 3$, say $\ell_1 \geq 3$ and $\ell_2 \geq 3$. Then by (42), $\ell_1 = \ell_2 = 3$. For $i = 1, 2$, let (x_i, v_i, v_i', y_i) denote an x_i, y_i-path of length three in $G[V(S_i) \cup \{x_i, y_i\}]$. Also, by (43), $x_1, y_1, x_2, y_2 \in A$. Suppose first that $\{x_1, y_1\} \neq \{x_2, y_2\}$. We proceed as in the beginning of the proof of Lemma 4.9. Choose a $(t-2)$-element subset $B' \subseteq B$ and add two new vertices b_1' and b_2' and join them to all vertices of A. Then the obtained bipartite graph H' has at least $t^2 - t + 4$ edges so there is a hamiltonian cycle C' containing the linear forest $x_1 b_1' y_1 \cup x_2 b_2' y_2$ by Corollary 2.10. This C' corresponds to a cycle of length k in G, a contradiction.

It follows that every component S_i with $\ell_i \geq 3$ has exactly two neighbors in $V(H)$ and these two neighbors, say x_1, y_1, are the same for all such components; furthermore $x_1, y_1 \in A$. Furthermore, in order to have $\ell_i \leq 3$, all leaves of S_i have the same neighbor in A. Thus G is contained in a graph in $\mathcal{G}_3(n, k)$.

Case 3.2: There exists exactly one i with $\ell_i \geq 3$, say $\ell_1 \geq 3$. Then by (42), $\ell_1 = 3$. Let (x_1, v_1, v_1', y_1) be an x_1, y_1-path of length 3 in $G[V(S_i) \cup \{x_1, y_1\}]$. By (43), every other component S_i is a singleton, say v_i with $N(v_i) \subseteq A$. As in Case 3.2, in order to have $\ell_1 \leq 3$, S_1 should be a star, and if $S_1 \neq K_2, K_1$, then all leaves of S_1 are adjacent to the same vertex in A. Thus G is contained in a graph in $\mathcal{G}_1(n, k) \cup \mathcal{G}_2(n, k)$.

Case 3.3: $\ell_i \leq 2$ for all i. Here G is contained in a graph in $\mathcal{G}_1(n, k)$. Then each S_i is a singleton with all neighbors in A. It follows that $G - A$ is an independent set.

Case 4: $H = F_4$. By Lemma 4.8(c4), (43) holds. Together with (42), this yields that every component S of $G - A$ is a star and if $|S| \geq 3$, then all leaves of S have the same neighbor in A. It follows that $G \in \mathcal{G}_4(n, k)$. $\square$

Proof for k odd. By Proposition 4.6 and Lemma 4.9, G_n contains some $H \in \mathcal{F}_0$. Let $G' = G_n - H$ and $S_1, \ldots, S_s$ be the components of G'. Each of S_i has at least two neighbors, say x_i and y_i in $V(H)$. Let ℓ_i denote the length of a longest x_i, y_i-path in $G_n[V(S_i) \cup \{x_i, y_i\}]$. Since $c(G_n) \leq 2t$, by Lemma 4.8,

$$\textit{for all } i, \quad \ell_i \leq 2 \quad \text{and } \{x_i, y_i\} \subseteq A. \tag{44}$$

Then each S_i is a singleton with all neighbors in A. It follows that $G - A$ is an independent set. This completes the proof of Theorem 4.1 for k odd. $\square$

5. Proof of Theorem 1.4 for $k \leq 8$

Recall that Theorem 4.1 describes for $k \geq 9$ and $n \geq 3k/2$ the n-vertex 2-connected graphs with no cycle of length at least k and more than $h(n, k, t-1)$ edges. In this section, we will do the same for $4 \leq k \leq 8$ and $n \geq k$. We will use for this the classes $\mathcal{G}_i(n, k)$ defined in Section 4 and the notion of a J_3-*bridge*. For $A \subseteq V(G)$ and $S \subseteq V(G) \setminus A$, S forms a J_3-*bridge of* A *with endpoints* a_1, a_2 if $a_1, a_2 \in A$, $A' := \{a_1, a_2\}$ is a cutset of G, $G[S \cup A'] \cup \{a_1a_2\}$ is a 2-connected graph, $G[S]$ is connected, and the length of the longest a_1, a_2-path in $G[S \cup A']$ is three.

Furthermore, since the description (but not the proof) for $k = 8$ is more sophisticated, we will need four more special graph classes for $k = 8$: Each of the graph classes $\mathcal{G}_i(n, 8)$ $(5 \leq i \leq 8)$ contains 2-connected n-vertex graphs G with $c(G) < 8$ and having a special vertex set $A = \{a_1, a_2, \ldots, a_s\}$ with $G[A]$ being a complete graph and such that $G \setminus A$ consists of J_3-bridges and isolated vertices having exactly two neighbors in A.

If $G \in \mathcal{G}_5(n, 8)$, then $s = 3$ and a_1 is adjacent to each component in $G \setminus A$. So the edge a_2a_3 is contained in a unique triangle, namely $a_1a_2a_3$.

If $G \in \mathcal{G}_6(n, 8) \cup \mathcal{G}_7(n, 8)$, then $s = 4$ and the endpoints of all J_3-bridges are $\{a_1, a_2\}$ while one of the neighbors of some isolated vertex c of $G \setminus A$ is a_1 in case of $\mathcal{G}_6(n, 8)$ and $N(c) = \{a_3, a_4\}$ for all c in case of $\mathcal{G}_7(n, 8)$.

If $G \in \mathcal{G}_8(n, 8)$, then $s = 5$ and $N(S) = \{a_1, a_2\}$ for each component S of $G - A$.

Theorem 5.1. *Let $4 \leq k \leq 8$ and $n \geq k$. Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then either $7 \leq k \leq 8$ and $e(G) \leq h(n, k, t-1)$ or G is a subgraph of a graph in $\mathcal{G}(n, k)$, where*

(1) $\mathcal{G}(n, 4) = \emptyset$,
(2) $\mathcal{G}(n, 5) := \mathcal{G}_1(n, 5)$,
(3) $\mathcal{G}(n, 6) := \mathcal{G}_1(n, 6) \cup \mathcal{G}_2(n, 6)$,
(4) $\mathcal{G}(n, 7) := \{H_{n,7,3}\} \cup \mathcal{G}_1(n, 6) \cup \mathcal{G}_2(n, 6) \cup \mathcal{G}_3(n, 6)$,
(5) $\mathcal{G}(n, 8) := \bigcup_{1 \leq i \leq 8, i \neq 4} \mathcal{G}_i(n, 8)$.

The proof scheme is that we consider a graph G satisfying the conditions of the theorem and take a longest cycle C with vertex set, say $X := \{x_0, x_1, x_2, \ldots, x_r\}$. Moreover, we will assume that C has the maximum sum of the degrees of its vertices among the longest cycles in G. Analyzing possibilities, we will derive that $G \in \mathcal{G}(n, k)$.

A *bridge* of C is the vertex set of a component of $G - X$.

We start from a sequence of simple claims on the structure of bridges and the edges between X and the bridges. For brevity we denote by $d_C(i, j)$ the distance on C between x_j and x_i, i.e. $\min\{|j - i|, r + 1 - |j - i|\}$. For a bridge S and neighbors x, x' of S on C, an (x, x', S)-*path* is an x, x'-path whose all internal vertices are in S.

The maximality of $|C|$ implies our first claim:

Claim 5.2. *For every bridge S and any $x_i, x_j \in N(S) \cap X$, the length of any (x_i, x_j, S)-path is at most $d_C(i,j)$. In particular, if S contains distinct c_1, c_2 such that $x_ic_1, x_jc_2 \in E(G)$, then $d_C(i,j) \geq 3$.*

If $|S| \geq 2$, then by the 2-connectedness of G, there are two vertex-disjoint S, X-paths. Thus if $G[S]$ contains a cycle, then for some $x_i, x_j \in N(S) \cap X$, the length of the longest (x_i, x_j, S)-path is at least 4. Hence, since $|C| \leq k - 1 \leq 7$, by Claim 5.2, we get the next claim:

Claim 5.3. *For every bridge S of X and any distinct $x_i, x_j \in N(S) \cap X$, the length of any (x_i, x_j, S)-path is at most 3. In particular, $G[S]$ is acyclic (a tree).*

Suppose that for some bridge S, and two leaves c_1, c_2 of the tree $G[S]$, there is a c_1, c_2-path P in $G[S]$ of length at least 3. Then by Claim 5.3, each of c_1 and c_2 has exactly one neighbor in X, and this is the same vertex, say x_i. Again by the 2-connectedness of G, there is $x_j \in X \cap N(S) \setminus \{x_i\}$. Then there is an (x_j, x_i, S)-path of length at least 4 through either c_1 or c_2, which contradicts Claim 5.3. Thus we get:

Claim 5.4. *For every bridge S of X, $G[S]$ is a star. Moreover, if $|S| \geq 3$, then all leaves of $G[S]$ have degree 2 in G and the same neighbor, $x(S)$, in X.*

Suppose $|S| \geq 2$ and $|N(S) \cap X| \geq 3$, say $\{x, x', x''\} \subseteq N(S) \cap X$. Let c_1 be a leaf of $G[S]$. If $|S| \geq 3$, then by Claim 5.3 it has a unique neighbor in X, say x. It follows that there are an (x, x', S)-path and an (x, x'', S)-path of length at least 3. Also there is an (x', x'', S)-path of length at least 2. Then by Claim 5.2, the distance on C from x to x' and to x'' is at least 3 and between x' and x'' is at least 2. Thus $|X| \geq 3+3+2 = 8$, a contradiction. Similarly, if $S = \{c_1, c_2\}$, then by symmetry we may assume that $x \in N(c_1) \cap X$ and $\{x', x''\} \subseteq N(c_2) \cap X$. In this case again by Claim 5.2, $|X| \geq 3 + 3 + 2 = 8$, a contradiction. Thus summarizing this with the previous claims, we have proved the following.

Claim 5.5. *For every bridge S of X with $|S| \geq 2$, $|N(S) \cap X| = 2$. Moreover, if $|S| \geq 3$, then $G[S]$ is a star and all leaves of $G[S]$ have degree 2 in G and the same neighbor, $x(S)$, in X. In other words, each bridge S with $|S| \geq 2$ is a J_3-bridge of X.*

From Claims 5.2 and 5.5 we deduce:

Claim 5.6. *For every J_3-bridge S of X with endpoints x_i and x_j, $d_C(i,j) \geq 3$.*

If there are $i_1 < i_2 < i_3 < i_4 \leq r$ and bridges S_1 and S_2 such that G contains an (x_{i_1}, x_{i_3}, S_1)-path P_1 and an (x_{i_2}, x_{i_4}, S_2)-path P_2, then we can construct two new cycles C_1 and C_2 such that each of them contains the edges of P_1 and P_2 and each edge of C belongs to exactly one of C_1 and C_2. Then the total length of C_1 and C_2 is at least

$|E(C)| + 2(|E(P_1)| + |E(P_2)|) \geq (k-1) + 8 \geq 2k-1$. Thus at least one of them is longer than C, a contradiction. Thus we have:

Claim 5.7. *There are no $i_1 < i_2 < i_3 < i_4 \leq r$ and bridges S_1 and S_2 of X such that G contains an (x_{i_1}, x_{i_3}, S_1)-path and an (x_{i_2}, x_{i_4}, S_2)-path. In particular, since $k-1 \leq 7$, any two J_3-bridges share an endpoint.*

We now can prove Theorem 5.1. Indeed, by Claim 5.2, $|X| \geq 4$. This proves $\mathcal{G}(n,4) = \emptyset$, i.e., Part 1 of the theorem.

We will consider 3 cases according to the value of $|X|$. As mentioned above, $|X| \geq 4$.

Case 1: $4 \leq |X| \leq 5$. Then by Claims 5.5 and 5.6, each bridge is a singleton. Furthermore, by Claim 5.2 each such singleton has exactly two (necessarily nonconsecutive) neighbors in X. If $|X| = 4$, Claim 5.7 yields that this pair of neighbors is the same for all bridges, say it is $\{x_0, x_2\}$. Then G is contained in $H_{n,5,2}$ with $A = \{x_0, x_2\}$, as claimed. This proves Part 2.

Let $|X| = 5$. If also each bridge has the same pair of neighbors in X, say $\{x_0, x_2\}$, then since $n \geq |X| + 1 = 6$, x_1 is not adjacent to $\{x_3, x_4\}$ to avoid a 6-cycle. Thus in this case, G is contained in $H_{n,6,2}$ with $A = \{x_0, x_2\}$, and so $e(G) \leq h(n, 6, 2)$. Otherwise by Claim 5.7, there are exactly two distinct pairs of neighbors of the bridges, and they share a vertex. Suppose these pairs are $\{x_0, x_2\}$ and $\{x_0, x_3\}$ and for $j \in \{2, 3\}$, Y_j is the set of vertices adjacent to x_0 and x_j. Then to avoid a 6-cycle, edges x_1x_4, x_1x_3 and x_2x_4 are not present in G. Then $G \in \mathcal{G}_2(n, 6)$ with $A = \{x_0, x_2\}$, $B = Y_2 \cup \{x_3\}$ and $J = Y_3 \cup \{x_4\}$. Since $H_{n,6,2}$ contains $H_{n,5,2}$, this together with the previous paragraph proves Part 3 of the theorem.

Case 2: $|X| = 6$. By Claims 5.5–5.7, it is enough to consider the following three subcases.

Case 2.1: X has a bridge S with $|N(S) \cap X| \geq 3$. By Claim 5.5, S is a single vertex, say z, and by Claim 5.2, z has exactly 3 (nonconsecutive) neighbors on C, say x_0, x_2 and x_4. In view of the cycle $x_0zx_2x_3x_4x_5$ and the maximality of the degree sum of C, $d(x_1) \geq d(z) \geq 3$. By Claim 5.7, x_1 has no neighbors outside of C. In order to avoid a 7-cycle in G, $x_1x_3, x_1x_5 \notin E(G)$. So $x_1x_4 \in E(G)$. Similarly, $x_2x_5, x_0x_3 \in E(G)$, so G contains $K_{3,4}$ with parts $A = \{x_0, x_2, x_4\}$ and $B = \{x_1, x_3, x_5, z\}$. Moreover, B is independent. Let C be the vertex set of any component of $G - A - B$. If C has a neighbor in B or is not a singleton, then $G[A \cup B \cup C]$ has a cycle of length at least 7. Thus each component of $G - A - B$ is a singleton and has no neighbors in B. This means A meets all edges and so G is a subgraph of $H_{n,7,3}$.

Case 2.2: X has a J_3-bridge S. Then by Claim 5.2 and symmetry, we may assume $N(S) = \{x_0, x_3\}$. In this case, G has 3 internally disjoint x_0, x_3-paths of length 3. Thus to have $c(G) \leq 6$, $\{x_0, x_3\}$ separates internal vertices of distinct paths. It follows that $G - \{x_0, x_3\}$ is a collection of J_3-bridges of $\{x_0, x_3\}$ and isolated vertices each having only x_0 and x_3 as endpoints. Thus G is a subgraph of a graph in $\mathcal{G}_3(n, 6)$.

Case 2.3: $V \setminus X$ is independent and each $z \in V \setminus X$ has degree 2. By Theorem 1.3, for each $z \in V \setminus X$, graph $G[X \cup \{z\}]$ has at most $h(7,7,2) = 14$ edges, which yields $e(G) \le 2n = h(n,7,2)$. This proves Part 4 of Theorem 5.1.

Case 3: $|X| = 7$. By Claims 5.5–5.7, it is enough to consider the following four subcases.

Case 3.1: X has a bridge S with $|N(S) \cap X| \ge 3$. As in Case 2.1, S is a single vertex, say z, and we may assume $N(S) \cap X = \{x_0, x_2, x_4\}$. Again, similarly to Case 2.1, in view of the 7-cycle $x_0 z x_2 x_3 x_4 x_5 x_6$, we obtain that $d(x_1) \ge d(z) \ge 3$, and that (to avoid a long cycle in G) the third neighbor of x_1 is x_4. Similarly, $x_0 x_3 \in E(G)$. Thus, G has a subgraph consisting of $K_{3,3}$ with parts $A := \{x_0, x_2, x_4\}$ and $B := \{x_1, x_3, z\}$ and an attached 3-path $x_4 x_5 x_6 x_0$. Moreover, $d(x_1) = d(x_3) = d(z) = 3$ and these are isolated vertices in $G \setminus A$. Let Y be the vertex set of the component of $G - A$ containing $\{x_5, x_6\}$. If there is another component Y' of $G - A$ with $|Y'| \ge 2$, then to avoid a ≥ 8-cycle, G must be a subgraph of a graph in $\mathcal{G}_3(n,8)$. If all the bridges of A apart from A are singletons, then G is a subgraph of a graph in either $\mathcal{G}_1(n,8)$ (if $|Y| = 2$) or $\mathcal{G}_2(n,8)$ (if $|Y| \ge 3$).

Case 3.2: G has J_3-bridges S_1 and S_2 of X with $N(S_1) \ne N(S_2)$. By Claims 5.7 and 5.6, we may assume $N(S_1) = \{x_0, x_3\}$ and $N(S_2) = \{x_0, x_4\}$. By the 2-connectivity of G, we may assume that there is an (x_0, x_3, S_1)-path $x_0 y_1 y_2 x_3$ and an (x_4, x_0, S_2)-path $x_4 y_5 y_6 x_0$. Let $A = \{x_0, x_3, x_4\}$. Then the edges $y_1 y_2$, $y_5 y_6$, $x_1 x_2$, $x_5 x_6$ belong to distinct components of $G \setminus A$. Thus to avoid long cycles in G, no bridge of A is adjacent to both, x_3 and x_4 and none of the bridges S of A contains an (x_0, x_3, S)-path or an (x_0, x_4, S)-path of length at least 4. It follows that G is a subgraph of a graph in $\mathcal{G}_5(n,8)$.

Case 3.3: G has a J_3-bridge S of X, and every other J_3-bridge of X (if exists) has the same neighbors as S in X. We may assume that $N(S) \cap X = \{x_0, x_4\}$ and G contains an (x_0, x_4, S)-path $x_0 y_6 y_5 x_4$. Then the edges $y_5 y_6$, $x_1 x_2$, $x_5 x_6$ belong to three distinct components of $G \setminus \{x_0, x_4\}$. Let Y be the component of $G \setminus \{x_0, x_4\}$ containing $\{x_1, x_2, x_3\}$. By the case, all other components are either isolated vertices or J_3-bridges of $\{x_0, x_4\}$. Also, every vertex $y \in (Y \setminus \{x_1, x_2, x_3\})$ has only neighbors in X (i.e., $N(y) \subset \{x_0, x_1, \ldots, x_4\}$).

If $|Y| = 3$ we obtain that G is a subgraph of a member of $\mathcal{G}_8(n,8)$ with $A = \{x_0, x_1, x_2, x_3, x_4\}$. Suppose $|Y| \ge 4$. If there is $y \in Y \setminus \{x_1, x_2\}$ with $N_G(y) = \{x_0, x_3\}$, then to avoid an 8- or 9-cycle, $x_1 x_4 \notin E(G)$ and no $y' \in Y \setminus \{x_2, x_3\}$ has $N_G(y') = \{x_1, x_4\}$. So, either $\{x_0, x_3\}$ is a cut set in G or $x_2 x_4 \in E(G)$. In the former case, G is a subgraph of a graph in $\mathcal{G}_5(n,8)$ with $A = \{x_0, x_3, x_4\}$ and $a_1 = x_0$. In the latter case, in order to avoid an (x_0, x_4, Y)-path of length ≥ 5, graph $G[\{x_1, x_2, x_3, x_4, y\}]$ has only the 5 edges we already know and no vertex $y' \in Y - X - y$ has $N(y') \subseteq \{x_1, x_2, x_3, x_4, y\}$. This means G is a subgraph of a graph in $\mathcal{G}_6(n,8)$ with $A = \{x_0, x_4, x_2, x_3\}$, where $a_1 = x_0$ and $a_2 = x_4$. The case of $y \in Y \setminus \{x_1, x_2\}$ with $N_G(y) = \{x_1, x_4\}$ is symmetrical. If there is $y \in Y \setminus \{x_1\}$ with $N(y) = \{x_0, x_2\}$, then in order to avoid an (x_0, x_4, Y)-path of length ≥ 5, $x_1 x_3 \notin E(G)$ and every $y' \in Y - X$ is adjacent to x_2.

This means G is a subgraph of a graph in $\mathcal{G}_2(n,8) \cup \mathcal{G}_3(n,8)$ with $A = \{x_2, x_4, x_0\}$. The last possibility is that $N(y) = \{x_1, x_3\}$ for every $y \in Y - X$. Since $|Y| \geq 4$, this yields $x_2x_0, x_2x_4 \notin E(G)$. Thus G is a subgraph of a member of $\mathcal{G}_7(n,8)$ with $\{a_1, a_2\} := \{x_0, x_4\}$ and $\{a_3, a_4\} := \{x_1, x_3\}$.

Case 3.4: $G \setminus X$ consists of isolated vertices only, each having degree 2 in G. By Theorem 1.3, for each $z \in V \setminus X$, graph $G[X \cup \{z\}]$ has at most $h(8,8,2) = 19$ edges, which yields $e(G) \leq 2n + 3 = h(n,8,2)$. □

Theorem 5.1 yields the following analog of Theorem 4.1(1) for a smaller range of $e(G)$.

Corollary 5.8. *Suppose that G is a 2-connected, n-vertex graph with $c(G) < 7$, $n \geq 8$. If $e(G) \geq \lfloor (5n-6)/2 \rfloor$ then G is a subgraph of $H_{n,7,3}$, and this bound is best possible.* □

6. Concluding remarks

It could be that for $k \geq 11$, Theorem 1.4 holds already for $n \geq 5k/4$. Note that by Theorem 1.3, it does not hold for $n < 5k/4$. It may also be possible, albeit complicated, to describe the structure of 2-connected n-vertex graphs with no cycles of length at least $k = 2t+1$ and at least $h(n,k,t-2)$ edges. We leave these as avenues for further research.

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