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On the Corrádi–Hajnal theorem and a question of Dirac [☆]

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A R T I C L E I N F O

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A B S T R A C T

In 1963, Corrádi and Hajnal proved that for all $k \geq 1$ and $n \geq 3k$, every graph G on n vertices with minimum degree $\delta(G) \geq 2k$ contains k disjoint cycles. The bound $\delta(G) \geq 2k$ is sharp. Here we characterize those graphs with $\delta(G) \geq 2k-1$ that contain k disjoint cycles. This answers the simple-graph case of Dirac's 1963 question on the characterization of $(2k-1)$-connected graphs with no k disjoint cycles.
Enomoto and Wang refined the Corrádi–Hajnal Theorem, proving the following Ore-type version: For all $k \geq 1$ and $n \geq 3k$, every graph G on n vertices contains k disjoint cycles, provided that $d(x) + d(y) \geq 4k - 1$ for all distinct nonadjacent vertices x, y. We refine this further for $k \geq 3$ and $n \geq 3k + 1$: If G is a graph on n vertices such that $d(x) + d(y) \geq 4k - 3$ for all distinct nonadjacent vertices x, y, then G has k vertex-disjoint cycles if and only if the independence number $\alpha(G) \leq n - 2k$ and G is not one of two small exceptions in the case $k = 3$. We also show how the case

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$k = 2$ follows from Lovász' characterization of multigraphs with no two disjoint cycles.

1. Introduction

For a graph $G = (V, E)$, let $|G| = |V|$, $\|G\| = |E|$, $\delta(G)$ be the minimum degree of G, and $\alpha(G)$ be the independence number of G. Let $\overline{G}$ denote the complement of G and for disjoint graphs G and H, let $G \vee H$ denote $G \cup H$ together with all edges from $V(G)$ to $V(H)$. The degree of a vertex v in a graph H is $d_H(v)$; when H is clear, we write $d(v)$.

In 1963, Corrádi and Hajnal proved a conjecture of Erdős by showing the following:

Theorem 1.1 *([5]). Let $k \in \mathbb{Z}^+$. Every graph G with* (i) $|G| \geq 3k$ *and* (ii) $\delta(G) \geq 2k$ *contains k disjoint cycles.*

Clearly, hypothesis (i) in the theorem is sharp. Hypothesis (ii) also is sharp. Indeed, if a graph G has k disjoint cycles, then $\alpha(G) \leq |G| - 2k$, since every cycle contains at least two vertices of $G - I$ for any independent set I. Thus $H := \overline{K_{k+1}} \vee K_{2k-1}$ satisfies (i) and has $\delta(H) = 2k - 1$, but does not have k disjoint cycles, because $\alpha(H) = k + 1 > |H| - 2k$. There are several works refining Theorem 1.1. Dirac and Erdős [7] showed that if a graph G has many more vertices of degree at least $2k$ than vertices of degree at most $2k - 2$, then G has k disjoint cycles. Dirac [6] asked:

Question 1.2. Which $(2k - 1)$-connected graphs do not have k disjoint cycles?

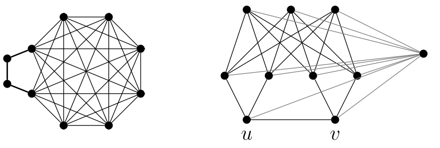
He also resolved his question for $k = 2$ by describing all 3-connected multigraphs on at least 4 vertices in which every two cycles intersect. It turns out that the only simple 3-connected graphs with this property are wheels. Lovász [20] fully described all multigraphs in which every two cycles intersect.

The following result in this paper yields a full answer to Dirac's question for simple graphs.

Theorem 1.3. *Let $k \geq 2$. Every graph G with* (i) $|G| \geq 3k$ *and* (ii) $\delta(G) \geq 2k-1$ *contains k disjoint cycles if and only if*

(H3) *$\alpha(G) \leq |G| - 2k$, and*
(H4) *if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$ and if $k = 2$ then G is not a wheel.*

Since for every independent set I in a graph G and every $v \in I$, $N(v) \subseteq V(G) - I$, if $\delta(G) \geq 2k - 1$ and $|I| \geq |G| - 2k + 1$, then $|I| = |G| - 2k + 1$ and $N(v) = V(G) - I$ for every $v \in I$. It follows that every graph G satisfying (ii) and not satisfying (H3) contains



(a) $\mathbf{Y_1}$ (b) $\mathbf{Y_2}$

Fig. 1. Graphs $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

$K_{2k-1,|G|-2k+1}$ and is contained in $K_{|G|} - E(K_{|G|-2k+1})$. The conditions of Theorem 1.3 can be tested in polynomial time.

Most likely, Dirac intended his question to refer to multigraphs; indeed, his result for $k = 2$ is for multigraphs. But the case of simple graphs is the most important in the question. In [18] we heavily use the results of this paper to obtain a characterization of $(2k-1)$-connected multigraphs that contain k disjoint cycles, answering Question 1.2 in full.

Studying Hamiltonian properties of graphs, Ore introduced the *minimum Ore-degree* σ_2: If G is a complete graph, then $\sigma_2(G) = \infty$, otherwise $\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$. Enomoto [8] and Wang [21] generalized the Corrádi–Hajnal Theorem in terms of σ_2:

Theorem 1.4 *([8,21]). Let $k \in \mathbb{Z}^+$. Every graph G with* (i) $|G| \geq 3k$ *and*

(E2) $\sigma_2(G) \geq 4k - 1$

contains k disjoint cycles.

Again $H := \overline{K_{k+1}} \vee K_{2k-1}$ shows that hypothesis (E2) of Theorem 1.4 is sharp. What happens if we relax (E2) to (H2): $\sigma_2(G) \geq 4k - 3$, but again add hypothesis (H3)? Here are two interesting examples.

Example 1.5. Let $k = 3$ and $\mathbf{Y}_1$ be the graph obtained by twice subdividing one of the edges wz of K_8, i.e., replacing wz by the path $wxyz$. Then $|\mathbf{Y}_1| = 10 = 3k + 1$, $\sigma_2(\mathbf{Y}_1) = 9 = 4k - 3$, and $\alpha(\mathbf{Y}_1) = 2 \leq |\mathbf{Y}_1| - 2k$. However, $\mathbf{Y}_1$ does not contain $k = 3$ disjoint cycles, since each cycle would need to contain three vertices of the original K_8 (see Fig. 1(a)).

Example 1.6. Let $k = 3$. Let Q be the graph obtained from $K_{4,4}$ by replacing a vertex v and its incident edges vw, vx, vy, vz by new vertices u, u' and edges $uu', uw, ux, u'y, u'z$; so $d(u) = 3 = d(u')$ and contracting uu' in Q yields $K_{4,4}$. Now set $\mathbf{Y}_2 := K_1 \vee Q$. Then $|\mathbf{Y}_2| = 10 = 3k + 1$, $\sigma_2(\mathbf{Y}_2) = 9 = 4k - 3$, and $\alpha(\mathbf{Y}_2) = 4 \leq |\mathbf{Y}_2| - 2k$. However, $\mathbf{Y}_2$ does not contain $k = 3$ disjoint cycles, since each 3-cycle contains the only vertex of K_1 (see Fig. 1(b)).

Our main result is:

Theorem 1.7. *Let $k \in \mathbb{Z}^+$ with $k \geq 3$. Every graph G with*

(H1) $|G| \geq 3k+1$,
(H2) $\sigma_2(G) \geq 4k-3$, *and*
(H3) $\alpha(G) \leq |G| - 2k$

contains k disjoint cycles, unless $k = 3$ and $G \in \{\mathbf{Y}_1, \mathbf{Y}_2\}$. Furthermore, there is a polynomial time algorithm that either produces k disjoint cycles or demonstrates that one of the hypotheses fails.

Theorem 1.7 is proved in Section 2. In Section 3 we discuss the case $k = 2$. In Section 4 we discuss connections to equitable colorings and derive Theorem 1.3 from Theorem 1.7 and known results.

Now we show examples demonstrating the sharpness of hypothesis (H2) that $\sigma(G) \geq 4k-3$, then discuss some unsolved problems, and then review our notation.

Example 1.8. Let $k \geq 3$, $Q = K_3$ and $G_k := \overline{K_{2k-2}} \vee (\overline{K_{2k-3}} + Q)$. Then $|G_k| = 4k-2 \geq 3k+1$, $\delta(G_k) = 2k-2$ and $\alpha(G_k) = |G_k| - 2k$. If G_k contained k disjoint cycles, then at least $4k - |G_k| = 2$ would be 3-cycles; this is impossible, since any 3-cycle in G_k contains an edge of Q. This construction can be extended. Let $k = r + t$, where $k + 3 \leq 2r \leq 2k$, $Q' = K_{2t}$, and put $H = G_r \vee Q'$. Then $|H| = 4r - 2 + 2t = 2k + 2r - 2 \geq 3k + 1$, $\delta(H) = 2r - 2 + 2t = 2k - 2$ and $\alpha(H) = 2r - 2 = |H| - 2k$. If H contained k disjoint cycles, then at least $4k - |H| = 2t + 2$ would be 3-cycles; this is impossible, since any 3-cycle in H contains an edge of Q or a vertex of Q'.

There are several special examples for small k. The constructions of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ can be extended to $k = 4$ at the cost of lowering σ_2 to $4k - 4$. Below is another small family of special examples. The blow-up of G by H is denoted by $G[H]$; that is, $V(G[H]) = V(G) \times V(H)$ and $(x, y)(x', y') \in E(G[H])$ if and only if $xx' \in E(G)$, or $x = x'$ and $yy' \in E(H)$.

Example 1.9. For $k = 4$, $G := C_5[\overline{K_3}]$ satisfies $|G| = 15 \geq 3k + 1$, $\delta(G) = 2k - 2$ and $\alpha(G) = 6 < |G| - 2k$. Since girth$(G) = 4$, we see that G has at most $\frac{|G|}{4} < k$ disjoint cycles. This example can be extended to $k = 5, 6$ as follows. Let $I = \overline{K_{2k-8}}$ and $H = G \vee I$. Then $|G| = 2k + 7 \geq 3k + 1$, $\delta = 2k - 2$ and $\alpha(G) = 6 < |G| - 2k = 7$. If H has k disjoint cycles then each of the at least $k - (2k - 8) = 8 - k$ cycles that do not meet I use 4 vertices of G, and the other cycles use at least 2 vertices of G. Then $15 = |G| \geq 2k + 2(8 - k) = 16$, a contradiction.

Unsolved problems 1. For every fixed k, we know only a finite number of extremal examples. It would be very interesting to describe all graphs G with $\sigma_2(G) = 4k - 4$

that do not have k disjoint cycles, but this most likely would need new techniques and approaches.

2. Recently, there were several results in the spirit of the Corrádi–Hajnal Theorem giving degree conditions on a graph G sufficient for the existence in G of k disjoint copies of such subgraphs as chorded cycles [1,3] and Θ-graphs [4]. It could be that our techniques can help in similar problems.

3. One also may try to sharpen the above-mentioned theorem of Dirac and Erdős [7].

Notation A *bud* is a vertex with degree 0 or 1. A vertex is *high* if it has degree at least $2k-1$, and *low* otherwise. For vertex subsets A, B of a graph $G = (V, E)$, let

$$\|A, B\| := \sum_{u \in A} |\{uv \in E(G) : v \in B\}|.$$

Note A and B need not be disjoint. For example, $\|V, V\| = 2\|G\| = 2|E|$. We will abuse this notation to a certain extent. If A is a subgraph of G, we write $\|A, B\|$ for $\|V(A), B\|$, and if $\mathcal{A}$ is a set of disjoint subgraphs, we write $\|\mathcal{A}, B\|$ for $\left\|\bigcup_{H\in\mathcal{A}} V(H), B\right\|$. Similarly, for $u \in V(G)$, we write $\|u, B\|$ for $\|\{u\}, B\|$. Formally, an edge $e = uv$ is the set $\{u, v\}$; we often write $\|e, A\|$ for $\|\{u, v\}, A\|$.

If T is a tree or a directed cycle and $u, v \in V(T)$ we write uTv for the unique subpath of T with endpoints u and v. We also extend this: if $w \notin T$, but has exactly one neighbor $u \in T$, we write wTv for $w(T+w+wu)v$. Finally, if w has exactly two neighbors $u, v \in T$, we may write wTw for the cycle $wuTvw$.

2. Proof of Theorem 1.7

Suppose $G = (V, E)$ is an edge-maximal counterexample to Theorem 1.7. That is, for some $k \geq 3$, (H1)–(H3) hold, and G does not contain k disjoint cycles, but adding any edge $e \in E(\overline{G})$ to G results in a graph with k disjoint cycles. The edge e will be in precisely one of these cycles, so G contains $k-1$ disjoint cycles, and at least three additional vertices. Choose a set $\mathcal{C}$ of disjoint cycles in G so that:

(O1) $|\mathcal{C}|$ is maximized;
(O2) subject to (O1), $\sum_{C\in\mathcal{C}} |C|$ is minimized;
(O3) subject to (O1) and (O2), the length of a longest path P in $R := G - \bigcup\mathcal{C}$ is maximized;
(O4) subject to (O1), (O2), and (O3), $\|R\|$ is maximized.

Call such a $\mathcal{C}$ an *optimal set*. We prove in Subsection 2.1 that R is a path, and in Subsection 2.2 that $|R| = 3$. We develop the structure of $\mathcal{C}$ in Subsection 2.3. Finally, in Subsection 2.4, these results are used to prove Theorem 1.7.

Our arguments will have the following form. We will make a series of claims about our optimal set $\mathcal{C}$, and then show that if any part of a claim fails, then we could have improved $\mathcal{C}$ by replacing a sequence $C_1, \ldots, C_t \in \mathcal{C}$ of at most three cycles by another sequence of cycles $C'_1, \ldots, C'_{t'}$. Naturally, this modification may also change R or P. We will express the contradiction by writing "$C'_1, \ldots, C'_t, R', P'$ beats $C_1, \ldots, C_t, R, P$," and may drop R' and R or P' and P if they are not involved in the optimality criteria.

This proof implies a polynomial time algorithm. We start by adding enough extra edges—at most $3k$—to obtain from G a graph with a set $\mathcal{C}$ of k disjoint cycles. Then we remove the extra edges in $\mathcal{C}$ one at a time. After removing an extra edge, we calculate a new collection $\mathcal{C}'$. This is accomplished by checking the series of claims, each in polynomial time. If a claim fails, we calculate a better collection (again in polynomial time) and restart the check, or discover an independent set of size greater than $|G| - 2k$. (Observe that if I is an independent set with $|I| \geq n-2k+1$, then there exists $x \in I$ with $I = V \smallsetminus N(x)$, so the existence of $|I|$ can be checked in polynomial time.) As there can be at most n^4 improvements, corresponding to adjusting the four parameters (O1)–(O4), this process ends in polynomial time.

We now make some simple observations. Recall that $|\mathcal{C}| = k - 1$ and R is acyclic. By (O2) and our initial remarks, $|R| \geq 3$. Let a_1 and a_2 be the endpoints of P. (Possibly, R is an independent set, and $a_1 = a_2$.)

Claim 2.1. *For all $w \in V(R)$ and $C \in \mathcal{C}$, if $\|w, C\| \geq 2$ then $3 \leq |C| \leq 6 - \|w, C\|$. In particular,* (a) *$\|w, C\| \leq 3$,* (b) *if $\|w, C\| = 3$ then $|C| = 3$, and* (c) *if $|C| = 4$ then the two neighbors of w in C are nonadjacent.*

Proof. Let $\overrightarrow{C}$ be a cyclic orientation of C. For distinct $u, v \in N(w) \cap C$, the cycles $wu\overrightarrow{C}vw$ and $wu\overleftarrow{C}vw$ have length at least $|C|$ by (O2). Thus $2\,\|C\| \leq \|wu\overrightarrow{C}vw\| + \|wu\overleftarrow{C}vw\| = \|C\| + 4$, so $|C| \leq 4$. Similarly, if $\|w, C\| \geq 3$ then $3\|C\| \leq \|C\| + 6$, and so $|C| = 3$. $\square$

The next claim is a simple corollary of condition (O2).

Claim 2.2. *If $xy \in E(R)$ and $C \in \mathcal{C}$ with $|C| \geq 4$ then $N(x) \cap N(y) \cap C = \emptyset$.*

2.1. R is a path

Suppose R is not a path. Let L be the set of buds in R; then $|L| \geq 3$.

Claim 2.3. *For all $C \in \mathcal{C}$, distinct $x, y, z \in V(C)$, $i \in [2]$, and $u \in V(R - P)$:*

(a) *$\{ux, uy, a_iz\} \nsubseteq E$;*
(b) *$\|\{u, a_i\}, C\| \leq 4$;*
(c) *$\{a_ix, a_iy, a_{3-i}z, zu\} \nsubseteq E$;*

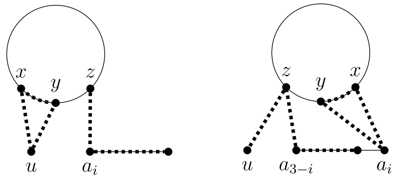


Fig. 2. Claim 2.3.

(d) *if* $\|\{a_1, a_2\}, C\| \geq 5$ *then* $\|u, C\| = 0$*;*
(e) $\|\{u, a_i\}, R\| \geq 1$*; in particular* $\|a_i, R\| = 1$ *and* $|P| \geq 2$*;*
(f) $4 - \|u, R\| \leq \|\{u, a_i\}, C\|$ *and* $\|\{u, a_i\}, D\| = 4$ *for at least* $|\mathcal{C}| - \|u, R\|$ *cycles* $D \in \mathcal{C}$.

Proof. (a) Else $ux(C - z)yu, Pa_iz$ beats C, P by (O3) (see Fig. 2(a)).

(b) Else $|C| = 3$ by Claim 2.1. Then there are distinct $p, q, r \in V(C)$ with $up, uq, a_ir \in E$, contradicting (a).

(c) Else $a_ix(C - z)ya_i, (P - a_i)a_{3-i}zu$ beats C, P by (O3) (see Fig. 2(b)).

(d) Suppose $\|\{a_1, a_2\}, C\| \geq 5$ and $p \in N(u) \cap C$. By Claim 2.1, $|C| = 3$. Pick $j \in [2]$ with $pa_j \in E$, preferring $\|a_j, C\| = 2$. Then $V(C) - p \subseteq N(a_{3-j})$, contradicting (c).

(e) Since a_i is an end of the maximal path P, we get $N(a_i) \cap R \subseteq P$; so $a_iu \notin E$. By (b)

$$4(k-1) \geq \|\{u, a_i\}, V \smallsetminus R\| \geq 4k - 3 - \|\{u, a_i\}, R\|. \tag{2.1}$$

Thus $\|\{u, a_i\}, R\| \geq 1$. Hence $G[R]$ has an edge, $|P| \geq 2$, and $\|a_i, P\| = \|a_i, R\| = 1$.

(f) By (2.1) and (e), $\|\{u, a_i\}, V \smallsetminus R\| \geq 4|\mathcal{C}| - \|u, R\|$. Using (b), this implies the second assertion, and $\|\{u, a_i\}, C\| + 4(|\mathcal{C}| - 1) \geq 4|\mathcal{C}| - \|u, R\|$ implies the first assertion. □

Claim 2.4. $|P| \geq 3$. *In particular,* $a_1a_2 \notin E(G)$.

Proof. Suppose $|P| \leq 2$. Then $\|u, R\| \leq 1$. As $|L| \geq 3$, there is a bud $c \in L \smallsetminus \{a_1, a_2\}$. By Claim 2.3(f), there exists $C = z_1 \dots z_t z_1 \in \mathcal{C}$ such that $\|\{c, a_1\}, C\| = 4$ and $\|\{c, a_2\}, C\| \geq 3$.

If $\|c, C\| = 3$ then the edge between a_1 and C contradicts Claim 2.3(a). If $\|c, C\| = 1$ then $\|\{a_1, a_2\}, C\| = 5$, contradicting Claim 2.3(d). Therefore, we assume $\|c, C\| = 2 = \|a_1, C\|$ and $\|a_2, C\| \geq 1$. By Claim 2.3(a), $N(a_1) \cup N(a_2) = N(c)$, so there exists $z_i \in N(a_1) \cap N(a_2)$ and $z_j \in N(c) - z_i$. Then $a_1a_2z_ia_1, cz_jz_{j\pm1}$ beats C, P by (O3). □

Claim 2.5. *Let* $c \in L - a_1 - a_2$*,* $C \in \mathcal{C}$*, and* $i \in [2]$.

(a) $\|a_1, C\| = 3$ *if and only if* $\|c, C\| = 0$*, and if and only if* $\|a_2, C\| = 3$.
(b) *There is at most one cycle* $D \in \mathcal{C}$ *with* $\|a_i, D\| = 3$.

(c) *For every* $C \in \mathcal{C}$, $\|a_i, C\| \geq 1$ *and* $\|c, C\| \leq 2$.
(d) *If* $\|\{a_i, c\}, C\| = 4$ *then* $\|a_i, C\| = 2 = \|c, C\|$.

Proof. (a) If $\|c, C\| = 0$ then by Claims 2.1 and 2.3(f), $\|a_i, C\| = 3$. If $\|a_i, C\| \geq 3$ then by Claim 2.3(b), $\|c, C\| \leq 1$. By Claim 2.3(f), $\|a_{3-i}, C\| \geq 2$, and by Claim 2.3(d), $\|c, C\| = 0$.

(b) As $c \in L$, $\|c, R\| \leq 1$. Thus Claim 2.3(f) implies $\|c, D\| = 0$ for at most one cycle $D \in \mathcal{C}$.

(c) Suppose $\|c, C\| = 3$. By Claim 2.3(a), $\|\{a_1, a_2\}, C\| = 0$. By Claims 2.4 and 2.3(d):

$$4k - 3 \leq \|\{a_1, a_2\}, R \cup C \cup (V - R - C)\| \leq 2 + 0 + 4(k-2) = 4k - 6,$$

a contradiction. Thus $\|c, C\| \leq 2$. Thus by Claim 2.3(f), $\|a_i, C\| \geq 1$.

(d) Now (d) follows from (a) and (c). □

Claim 2.6. *R has no isolated vertices.*

Proof. Suppose $c \in L$ is isolated. Fix $C \in \mathcal{C}$. By Claim 2.3(f), $\|\{c, a_1\}, C\| = 4$. By Claim 2.5(d), $\|a_1, C\| = 2 = \|c, C\|$; so $d(c) = 2(k-1)$. By Claim 2.3(a), $N(a_1) \cap C = N(c) \cap C$. Let $w \in V(C) \smallsetminus N(c)$. Then $d(w) \geq 4k - 3 - d(c) = 2k - 1 = 2|\mathcal{C}| + 1$. Therefore, either $\|w, R\| \geq 1$ or $|N(w) \cap D| = 3$ for some $D \in \mathcal{C}$. In the first case, $c(C - w)c$ beats C by (O4). In the second case, by Claim 2.5(c) there exists some $x \in N(a_1) \cap D$. Then $c(C - w)c, w(D - x)w$ beats C, D by (O3). □

Claim 2.7. *L is an independent set.*

Proof. Suppose $c_1c_2 \in E(L)$. By Claim 2.4, $c_1, c_2 \notin P$. By Claim 2.3(f) and using $k \geq 3$, there is $C \in \mathcal{C}$ with $\|\{a_1, c_1\}, C\| = 4$ and $\|\{a_1, c_2\}, C\|$, $\|\{a_2, c_1\}, C\| \geq 3$. By Claim 2.5(d), $\|a_1, C\| = 2 = \|c_1, C\|$; so $\|a_2, C\|$, $\|c_2, C\| \geq 1$. By Claim 2.3(a), $N(a_1) \cap C, N(a_2) \cap C \subseteq N(c_1) \cap C$. Then there are distinct $x, y \in N(c_1) \cap C$ with $xa_1, xa_2, ya_1 \in E$. If $xc_2 \in E$ then $c_1c_2xc_1$, ya_1Pa_2 beats C, P by (O3). Else $a_1Pa_2xa_1$, $c_1(C - x)c_2c_1$ beats C, P by (O1). □

Claim 2.8. *If $|L| \geq 3$ then for some $D \in \mathcal{C}$, $\|l, C\| = 2$ for every $C \in \mathcal{C} - D$ and every $l \in L$.*

Proof. Suppose some $D_1, D_2 \in \mathcal{C}$ and $l_1, l_2 \in L$ satisfy $D_1 \neq D_2$ and $\|l_1, D_1\| \neq 2 \neq \|l_2, D_2\|$.

CASE 1: $l_j \notin \{a_1, a_2\}$ for some $j \in [2]$. Say $j = 1$. For $i \in [2]$: $\|\{a_i, l_1\}, D_1\| \neq 4$ by Claim 2.5(d); $\|\{a_i, l_1\}, D_2\| = 4$ by Claim 2.3(f); $\|a_i, D_2\| = 2$ by Claim 2.5(d). Then $l_2 \notin \{a_1, a_2\}$. By Claim 2.7, $l_1l_2 \notin E(G)$. Claim 2.5(c) yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k-3) = 4k - 4.$$

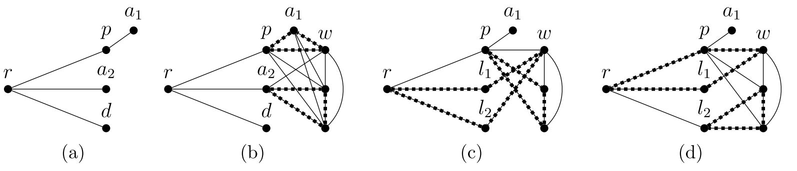


Fig. 3. Claim 2.10.

CASE 2: $\{l_1, l_2\} \subseteq \{a_1, a_2\}$. Let $c \in L - l_1 - l_2$. As above, $\|\{l_1, c\}, D_1\| \neq 4$, and so $\|c, D_2\| = 2 = \|l_1, D_2\|$. This implies $l_1 \neq l_2$. By Claim 2.5(a, c), $\|l_2, D_2\| = 1$. Thus $\|\{l_2, c\}, D_1\| = 4$; so $\|c, D_1\| = 2$, and $\|l_1, D_1\| = 1$. With Claim 2.4, this yields the contradiction:

$$\begin{aligned} 4k-3 &\leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \\ &\leq 2+3+3+4(k-3) = 4k-4. \qquad \square \end{aligned}$$

Claim 2.9. *R is a subdivided star (possibly a path).*

Proof. Suppose not. Then we claim R has distinct leaves $c_1, d_1, c_2, d_2 \in L$ such that c_1Rd_1 and c_2Rd_2 are disjoint paths. Indeed, if R is disconnected then each component has two distinct leaves by Claim 2.6. Else R is a tree. As R is not a subdivided star, it has distinct vertices s_1 and s_2 with degree at least three. Deleting the edges and interior vertices of s_1Rs_2 yields disjoint trees containing all leaves of R. Let T_i be the tree containing s_i, and pick $c_i, d_i \in T_i$.

By Claim 2.8, using $k \geq 3$, there is a cycle $C \in \mathcal{C}$ such that $\|l, C\| = 2$ for all $l \in L$. By Claim 2.3(a), $N(a_1) \cap C = N(l) \cap C = N(a_2) \cap C =: \{w_1, w_3\}$ for $l \in L - a_1 - a_2$. Then replacing C in $\mathcal{C}$ with $w_1c_1Rd_1w_1$ and $w_3c_2Rd_2w_3$ yields k disjoint cycles. $\square$

Claim 2.10. *R is a path or a star.*

Proof. By Claim 2.9, R is a subdivided star. If R is neither a path nor a star then there are vertices r, p, d with $\|r, R\| \geq 3$, $\|p, R\| = 2$, $d \in L - a_1 - a_2$ and (say) $pa_1 \in E$. Then a_2Rd is disjoint from pa_1 (see Fig. 3(a)). By Claim 2.5(c), $d(d) \leq 1 + 2(k-1) = 2k-1$. Then:

$$\|p, V - R\| \geq 4k - 3 - \|p, R\| - d(d) \geq 4k - 5 - (2k-1) = 2k - 4 \geq 2. \tag{2.2}$$

In each of the following cases, $R \cup C$ has two disjoint cycles, contradicting (O1).

CASE 1: $\|p, C\| = 3$ for some $C \in \mathcal{C}$. Then $|C| = 3$. By Claim 2.5(a), if $\|d, C\| = 0$ then $\|a_1, C\| = 3 = \|a_2, C\|$. Then for $w \in C$, wa_1pw and $a_2(C - w)a_2$ are disjoint cycles (see Fig. 3(b)). Else by Claim 2.5(c), $\|d, C\|$, $\|a_2, C\| \in \{1, 2\}$. By Claim 2.3(f), $\|\{d, a_2\}, C\| \geq 3$, so there are $l_1, l_2 \in \{a_2, d\}$ with $\|l_1, C\| \geq 1$ and $\|l_2, C\| = 2$; say $w \in N(l_1) \cap C$. If $l_2w \in E$ then wl_1Rl_2w and $p(C - w)p$ are disjoint cycles (see Fig. 3(c)); else l_1wpRl_1 and $l_2(C - w)l_2$ are disjoint cycles (see Fig. 3(d)).

CASE 2: There are distinct $C_1, C_2 \in \mathcal{C}$ with $\|p, C_1\|$, $\|p, C_2\| \geq 1$. By Claim 2.8, for some $i \in [2]$ and all $c \in L$, $\|c, C_i\| = 2$. Let $w \in N(p) \cap C_i$. If $wa_1 \in E$ then $D := wpa_1w$ is a cycle and $G[(C_i - w) \cup a_2Rd]$ contains cycle disjoint from D. Else, if $w \in N(a_2) \cup N(d)$, say $w \in N(c)$, then $a_1(C_i - w)a_1$ and $cwpRc$ are disjoint cycles. Else, by Claim 2.1 there exist vertices $u \in N(a_2) \cap N(d) \cap C_i$ and $v \in N(a_1) \cap C_i - u$. Then ua_2Rdu and $a_1v(C_i - u)wpa_1$ are disjoint cycles.

CASE 3: Otherwise. Then using (2.2), $\|p, V - R\| = 2 = \|p, C\|$ for some $C \in \mathcal{C}$. In this case, $k = 3$ and $d(p) = 4$. By (H2), $d(a_2), d(d) \geq 5$. Say $\mathcal{C} = \{C, D\}$. By Claim 2.3(b), $\|\{a_2, d\}, D\| \leq 4$. Thus,

$$\|\{a_2, d\}, C\| = \|\{a_2, d\}, (V - R - D)\| \geq 10 - 2 - 4 = 4.$$

By Claim 2.5(c, d), $\|a_2, C\| = \|d, C\| = 2$ and $\|a_1, C\| \geq 1$. Say $w \in N(a_1) \cap C$. If $wp \in E$ then $dRa_2(C - w)d$ contains a cycle disjoint from wa_1pw. Else, by Claim 2.3(a) there exists $x \in N(a_2) \cap N(d) \cap C$. If $x \neq w$ then xa_2Rdx and $wa_1p(C - x)w$ are disjoint cycles. Else $x = w$, and xa_2Rdx and $p(C - w)p$ are disjoint cycles. □

Lemma 2.11. *R is a path.*

Proof. Suppose R is not a path. Then it is a star with root r and at least three leaves, any of which can play the role of a_i or a leaf in $L - a_1 - a_2$. Thus Claim 2.5(c) implies $\|l, C\| \in \{1, 2\}$ for all $l \in L$ and $C \in \mathcal{C}$. By Claim 2.8 there is $D \in \mathcal{C}$ such that for all $l \in L$ and $C \in \mathcal{C} - D$, $\|l, C\| = 2$. By Claim 2.3(f) there is $l \in L$ such that for all $c \in L - l$, $\|c, D\| = 2$. Fix distinct leaves $l', l'' \in L - l$.

Let $Z = N(l') - R$ and $A = V \smallsetminus (Z \cup \{r\})$. By the first paragraph, every $C \in \mathcal{C}$ satisfies $|Z \cap C| = 2$, so $|A| = |G| - 2k + 1$. For a contradiction, we show that A is independent.

Note $A \cap R = L$, so by Claim 2.7, $A \cap R$ is independent. By Claim 2.3(a),

$$\text{for all } c \in L \text{ and for all } C \in \mathcal{C},\ N(c) \cap C \subseteq Z. \tag{2.3}$$

Therefore, $\|L, A\| = 0$. By Claim 2.1(c), for all $C \in \mathcal{C}$, $C \cap A$ is independent. Suppose, for a contradiction, A is not independent. Then there exist distinct $C_1, C_2 \in \mathcal{C}$, $v_1 \in A \cap C_1$, and $v_2 \in A \cap C_2$ with $v_1v_2 \in E$. Subject to this choose C_2 with $\|v_1, C_2\|$ maximum. Let $Z \cap C_1 = \{x_1, x_2\}$ and $Z \cap C_2 = \{y_1, y_2\}$.

CASE 1: $\|v_1, C_2\| \geq 2$. Choose $i \in [2]$ so that $\|v_1, C_2 - y_i\| \geq 2$. Then define $C_1^* := v_1(C_2 - y_i)v_1$, $C_2^* := l'x_1(C_1 - v_1)x_2l'$, and $P^* := y_il''rl$ (see Fig. 4(a)). By (2.3), P^* is a path and C_2^* is a cycle. Then C_1^*, C_2^*, P^* beats C_1, C_2, P by (O3).

CASE 2: $\|v_1, C_2\| \leq 1$. Then for all $C \in \mathcal{C}$, $\|v_1, C\| \leq 2$ and $\|v_1, C_2\| = 1$; so $\|v_1, \mathcal{C}\| = \|v_1, C_2 \cup (\mathcal{C} - C_2)\| \leq 1 + 2(k - 2) = 2k - 3$. By (2.3) $\|v_1, L\| = 0$ and $d(l) \leq 2k - 1$. By $(H2)$, $\|v_1, r\| = \|v_1, R\| = (4k - 3) - \|v_1, \mathcal{C}\| - d(l) \leq (4k - 3) - (2k - 3) - (2k - 1) = 1$, and $v_1r \in E$. Let $C_1^* := l'x_1(C_1 - v_1)x_2l'$, $C_2^* := l''y_1(C_2 - v_2)y_2l''$, and $P^* := v_2v_1rl$ (see Fig. 4(b)). Then C_1^*, C_2^*, P^* beats C_1, C_2, P by (O3). □

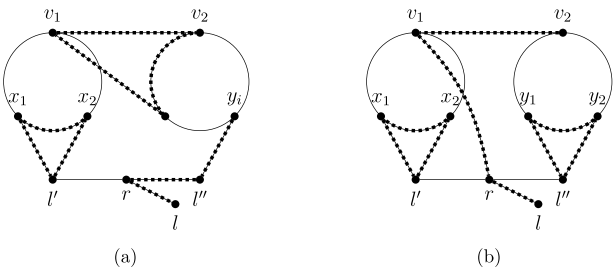


Fig. 4. Claim 2.10.

2.2. $|R| = 3$

By Lemma 2.11, R is a path, and by Claim 2.4, $|R| \geq 3$. Next we prove $|R| = 3$. First, we prove a claim that will also be useful in later sections.

Claim 2.12. *Let C be a cycle, $P = v_1v_2 \dots v_s$ be a path in R, and $1 < i < s$. At most one of the following two statements holds.*

(1) (a) $\|x, v_1Pv_{i-1}\| \geq 1$ *for all* $x \in C$ *or* (b) $\|x, v_1Pv_{i-1}\| \geq 2$ *for two* $x \in C$*;*
(2) (c) $\|y, v_iPv_s\| \geq 2$ *for some* $y \in C$ *or* (d) $N(v_i) \cap C \neq \emptyset$ *and* $\|v_{i+1}Pv_s, C\| \geq 2$.

Proof. Suppose (1) and (2) hold. If (c) holds then the disjoint graphs $G[v_iPv_s + y]$ and $G[v_1Pv_{i-1} \cup C - y]$ contain cycles. Else (d) holds, but (c) fails; say $z \in N(v_i) \cap C$ and $z \notin N(v_{i+1}Pv_s)$. If (a) holds then $G[v_1Pv_i + z]$ and $G[v_{i+1}Pv_s \cup C - z]$ contain cycles. If (b) holds then $G[v_1Pv_{i-1} + w]$ and $G[v_iPv_s \cup C - w]$ contain cycles, where $\|w, v_1Pv_{i-1}\| \geq 2$. □

Suppose, for a contradiction, $|R| \geq 4$. Say $R = a_1a_1'a_1'' \dots a_2''a_2'a_2$. It is possible that $a_1'' \in \{a_2'', a_2'\}$, etc. Set $e_i := a_ia_i' = \{a_i, a_i'\}$ and $F := e_1 \cup e_2$.

Claim 2.13. *If $C \in \mathcal{C}$, $h \in [2]$ and $\|e_h, C\| \geq \|e_{3-h}, C\|$ then $\|C, F\| \leq 7$; if $\|C, F\| = 7$ then*

$$|C| = 3,\ \|a_h, C\| = 2,\ \ \|a_h', C\| = 3,\ \ \|a_h''Ra_{3-h}, C\| = 2,\ \textit{and}\ N(a_h) \cap C = N(e_{3-h}) \cap C.$$

Proof. We will repeatedly use Claim 2.12 to obtain a contradiction to (O1) by showing that $G[C \cup R]$ contains two disjoint cycles. Suppose $\|C, F\| \geq 7$ and say $h = 1$. Then $\|e_1, C\| \geq 4$. There is $x \in e_1$ with $\|x, C\| \geq 2$. Thus $|C| \leq 4$ by Claim 2.1, and if $|C| = 4$ then no vertex in C has two adjacent neighbors in F. Then (1) holds with $v_1 = a_1$ and $v_i = a_2'$, even when $|C| = 4$.

If $\|e_1, C\| = 4$, as is the case when $|C| = 4$, then $\|e_2, C\| \geq 3$. If $|C| = 4$ there is a cycle $D := yza_2'a_2y$ for some $y, z \in C$. As (a) holds, $G[a_1Ra_2'' \cup C - y - z]$ contains another disjoint cycle. Thus, $|C| = 3$. As (c) must fail with $v_i = a_2'$, (a) and (c) hold for $v_i = a_1'$

and $v_1 = a_2$, a contradiction. Then $\|e_1, C\| \geq 5$. If $\|a_1, C\| = 3$ then (a) and (c) hold with $v_1 = a_1$ and $v_i = a_1'$. Now $\|a_1, C\| = 2$, $\|a_1', C\| = 3$ and $\|a_1'' Ra_2, C\| \geq 2$. If there is $b \in P - e_1$ and $c \in N(b) \cap V(C) \smallsetminus N(a_1)$ then $G[a_1' Ra_2 + c]$ and $G[a_1(C - c)a_1]$ both contain cycles. For every $b \in R - e_1$, $N(b) \cap C \subseteq N(a_1)$. Then if $\|a_1'' Ra_2, C\| \geq 3$, (c) holds for $v_1 = a_1$ and $v_1 = a_1''$, contradicting that (1) holds. Now $\|a_1'' Ra_2, C\| = \|e_1, C\| = 2$ and $N(a_1) = N(e_2)$. □

Lemma 2.14. $|R| = 3$ *and* $m := \max\{|C| : C \in \mathcal{C}\} = 4$.

Proof. Let $t = |\{C \in \mathcal{C} : \|F, C\| \leq 6\}|$ and $r = |\{C \in \mathcal{C} : |C| \geq 5\}|$. It suffices to show $r = 0$ and $|R| = 3$: then $m \leq 4$, and $|V(\mathcal{C})| = |G| - |R| \geq 3(k-1) + 1$ implies some $C \in \mathcal{C}$ has length 4. Choose R so that:

(P1) R has as few low vertices as possible, and subject to this,
(P2) R has a low end if possible.

Let $C \in \mathcal{C}$. By Claim 2.13, $\|F, C\| \leq 7$. By Claim 2.1, if $|C| \geq 5$ then $\|a, C\| \leq 1$ for all $a \in F$; so $\|F, C\| \leq 4$. Thus $r \leq t$. Hence

$$2(4k-3) \leq \|F, (V \smallsetminus R) \cup R\| \leq 7(k-1) - t - 2r + 6 \leq 7k - t - 2r - 1. \tag{2.4}$$

Therefore, $5 - k \geq t + 2r \geq 3r \geq 0$. Since $k \geq 3$, this yields $3r \leq t + 2r \leq 2$, so $r = 0$ and $t \leq 2$, with $t = 2$ only if $k = 3$.

CASE 1: $k - t \geq 3$. That is, there exist distinct cycles $C_1, C_2 \in \mathcal{C}$ with $\|F, C_i\| \geq 7$. In this case, $t \leq 1$: if $k = 3$ then $\mathcal{C} = \{C_1, C_2\}$ and $t = 0$; if $k > 3$ then $t < 2$. For both $i \in [2]$, Claim 2.13 yields $\|F, C_i\| = 7$, $|C_i| = 3$, and there is $x_i \in V(C_i)$ with $\|x_i, R\| = 1$ and $\|y, R\| = 3$ for both $y \in V(C_i - x_i)$. Moreover, there is a unique index $j = \beta(i) \in [2]$ with $\left\|a_j', C_i\right\| = 3$. For $j \in [2]$, put $I_j := \{i \in [2] : \beta(i) = j\}$; that is, $I_j = \{i \in [2] : \|a_j', C_i\| = 3\}$. Then $V(C_i) - x_i = N(a_{\beta(i)}) \cap C_i = N(e_{3-\beta(i)}) \cap C_i$. As $x_i a_{\beta(i)} \notin E$, one of $x_i, a_{\beta(i)}$ is high. As we can switch x_i and $a_{\beta(i)}$ (by replacing C_i with $a_{\beta(i)}(C_i - x_i)a_{\beta(i)}$ and R with $R - a_{\beta(i)} + x_i$), we may assume $a_{\beta(i)}$ is high.

Suppose $I_j \neq \emptyset$ for both $j \in [2]$; say $\|a_1', C_1\| = \|a_2', C_2\| = 3$. Then for all $B \in \mathcal{C}$ and $j \in [2]$, a_j is high, and either $\|a_j, B\| \leq 2$ or $\|F, B\| \leq 6$. Since $t \leq 1$, we get

$$\begin{aligned} 2k - 1 \leq d(a_j) = \|a_j, B \cup F\| + \|a_j, \mathcal{C} - B\| &\leq \|a_j, B\| + 1 + 2(k-2) + t \\ &\leq 2k - 2 + \|a_j, B\|. \end{aligned}$$

Thus $N(a_j) \cap B \neq \emptyset$ for all $B \in \mathcal{C}$. Let $y_j \in N(a_{3-j}) \cap C_j$. Then using Claim 2.13, $y_j \in N(a_j)$, and $a_1'(C_1 - y_1)a_1'$, $a_2'(C_2 - y_2)a_2'$, $a_1 y_1 a_2 y_2 a_1$ beats C_1, C_2 by (O1).

Otherwise, say $I_1 = \emptyset$. If $B \in \mathcal{C}$ with $\|F, B\| \leq 6$ then $\|e_1, B\| + 2\|a_2, B\| \leq \|F, B\| + \|a_2, B\| \leq 9$. Thus, using Claim 2.13,

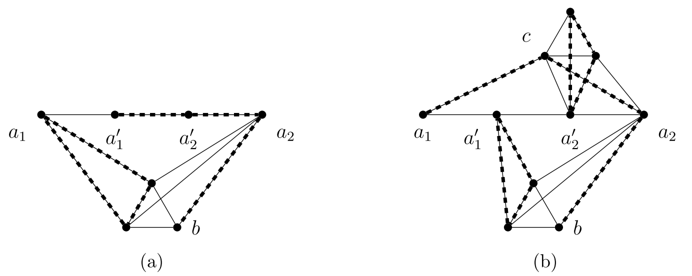


Fig. 5. Lemma 2.14, Case 1.

$$2(4k-3) \le d(a_1) + d(a_1') + 2d(a_2) = 5 + \|e_1, \mathcal{C}\| + 2\|a_2, \mathcal{C}\| \le 5 + 6(k-1-t) + 9t$$
$$\Rightarrow 2k \le 5 + 3t.$$

Since $k - t \ge 3$ (by the case), we see $3(k-t) + (5+3t) \ge 3(3) + 2k$ and so $k \ge 4$. Since $t \le 1$, in fact $k = 4$ and $t = 1$, and equality holds throughout: say B is the unique cycle in $\mathcal{C}$ with $\|F, B\| \le 6$. Then $\|a_2, B\| = \|e_1, B\| = 3$. Using Claim 2.13, $d(a_1) + d(a_1') = \|e_1, R\| + \|e_1, \mathcal{C} - B\| + \|e_1, B\| = 3 + 4 + 3 = 10$, and $d(a_1), d(a_2) \ge (4k-3) - d(a_2) = 13 - (1+4+3) = 5$, so $d(a_1) = d(a_2) = 5$. Note a_1 and a_2 share no neighbors: they share none in R because R is a path, they share none in $\mathcal{C} - B$ by Claim 2.13, and they share no neighbor $b \in B$ lest $a_1a_1'ba_1$ and $a_2(B-b)a_2$ beat B by (O1). Thus every vertex in $V - e_1$ is high.

Since $\|e_1, B\| = 3$, first suppose $\|a_1, B\| \ge 2$, say $B - b \subseteq N(a_1)$. Then $a_1(B-b)a_1$, $a_1'a_2'a_1b$ beat B, R by (P1) (see Fig. 5(a)). Now suppose $\|a_1', B\| \ge 2$, this time with $B - b \subseteq N(a_1')$. Since $d(a_1) = 5$ and $\|a_1, R \cup B\| \le 2$, there exists $c \in C \in \mathcal{C} - B$ with $a_1c \in E(G)$. Now $c \in N(a_2)$ by Claim 2.13, so $a_1'(B-b)a_1'$, $a_2'(C-c)a_2'$, and a_1ca_2b beat B, C, and R by (P1) (see Fig. 5(b)).

CASE 2: $k - t \le 2$. That is, $\|F, C\| \le 6$ for all but at most one $C \in \mathcal{C}$. Then, since $5 - k \ge t$, we get $k = 3$ and $\|F, V\| \le 19$. Say $\mathcal{C} = \{C, D\}$, so $\|F, C \cup D\| \ge 2(4k-3) - \|F, R\| = 2(4 \cdot 3 - 3) - 6 = 12$. By Claim 2.13, $\|F, C\|$, $\|F, D\| \le 7$. Then $\|F, C\|$, $\|F, D\| \ge 5$. If $|R| \ge 5$, then for the (at most two) low vertices in R, we can choose distinct vertices in R not adjacent to them. Then $\|R, V - R\| \ge 5|R| - 2 - \|R, R\| = 3|R|$. Thus we may assume $\|R, C\| \ge \lceil 3|R|/2 \rceil \ge |R| + 3 \ge 8$. Let $w' \in C$ be such that $q = \|w', R\| = \max\{\|w, R\| : w \in C\}$. Let $N(w') \cap R = \{v_{i_1}, \dots, v_{i_q}\}$ with $i_1 < \ldots < i_q$. Suppose $q \ge 4$. If $\|v_1Rv_{i_2}, C - w'\| \ge 2$ or $\|v_{i_2+1}Rv_s, C - w'\| \ge 2$, then $G[C \cup R]$ has two disjoint cycles. Otherwise, $\|R, C - w'\| \le 2$, contradicting $\|R, C\| \ge |R| + 3$. Similarly, if $q = 3$, then $\|v_1Rv_{i_2-1}, C - w'\| \le 1$ and $\|v_{i_2+1}Rv_s, C - w'\| \le 1$ yielding $\|v_{i_2}, C\| = \|R, C\| - \|(R - v_{i_2}), C - w'\| - \|R - v_{i_2}, w'\| \ge (|R| + 3) - 2 - (3-1) \ge 4$, a contradiction to Claim 2.1(a). Therefore, $q \le 2$, and hence $|R| + 3 \le \|R, C\| \le 2|C|$. It follows that $|R| = 5$, $|C| = 4$ and $\|w, R\| = 2$ for each $w \in C$. This in turn yields that $G[C \cup R]$ has no triangles and $\|v_i, C\| \le 2$ for each $i \in [5]$. By Claim 2.13, $\|F, C\| \le 6$, so $\|v_3, C\| = 2$. Thus we may assume that for some $w \in C$, $N(w) \cap R = \{v_1, v_3\}$. Then

$\|e_2, C\| = \|e_2, C - w\| \leq 1$, lest there exist a cycle disjoint from $wv_1v_2v_3w$ in $G[C \cup R]$. Therefore, $\|e_1, C\| \geq 8 - 1 - 2 = 5$, a contradiction to Claim 2.1(b). This yields $|R| \leq 4$.

Claim 2.15. *Either a_1 or a_2 is low.*

Proof. Suppose a_1 and a_2 are high. Then since $\|R, V\| \leq 19$, we may assume a_1' is low. Suppose there is $c \in C$ with $ca_2 \in E$ and $\|a_1, C - c\| \geq 2$. If $a_1'c \in E$, then $R \cup C$ contains two disjoint cycles; so $a_1'c \notin E$ and hence c is high. Thus either $a_1(C-c)a_1$ is shorter than C or the pair $a_1(C-c)a_1$, $ca_2a_2'a_1'$ beats C, R by (P2). Thus if $ca_2 \in E$ then $\|a_1, C - c\| \leq 1$. As a_2 is high, $\|a_2, C\| \geq 1$ and hence $\|a_1, C\| = \|a_1, C \smallsetminus N(a_2)\| + \|a_1, N(a_2)\| \leq 2$. Similarly, $\|a_1, D\| \leq 2$. Since a_1 is high, we see $\|a_1, C\| = \|a_1, D\| = 2$, and $d(a_1) = 5$. Hence

$$N(a_2) \cap C \subseteq N(a_1) \cap C \quad \text{and} \quad N(a_2) \cap D \subseteq N(a_1) \cap D. \tag{2.5}$$

As a_2 is high, $d(a_2) = 5$ and in (2.5) equalities hold. Also $d(a_1') = 4 \leq d(a_2')$.

If there are $c \in C$ and $i \in [2]$ with $ca_i, ca_i' \in E$ then by (O2), $|C| = 3$. Also $ca_i'a_ic$, $a_{3-i}'a_{3-i}(C - c)$ beats C, R by either (P1) or (P2). (Recall $N(a_1) \cap C = N(a_2) \cap C$ and neighbors of a_2 in C are high.) Then $N(a_i) \cap N(a_i') = \emptyset$. Thus the set $N(a_1) - R = N(a_2) - R$ contains no low vertices. Also, if $\|a_1', C\| \geq 1$ then $|C| = 3$: else C has the form $c_1c_2c_3c_4c_1$, where $a_1c_1, a_1c_3 \in E$, and so $a_1a_1'c_1c_2a_1$, $c_3c_4a_2a_2'$ beats C, R by either (P1) or (P2). Thus $|C| = 3$ and $a_1'c \in E$ for some $c \in V(C) - N(a_1)$. If $\|a_2', C\| \geq 1$, we have disjoint cycles $ca_1'a_2'c$, $a_1(C-c)a_1$ and D. Then $\|a_1', C\| = 0$, so $d(a_1') \leq 2 + |D \smallsetminus N(a_1)| \leq 4$. Now a_1' and a_2' are symmetric, and we have proved that $\|a_1', C\| + \|a_2', C\| \leq 1$. Similarly, $\|a_1', D\| + \|a_2', D\| \leq 1$, a contradiction to $d(a_1'), d(a_2') \geq 4$. □

By Claim 2.15, we can choose notation so that a_1 is low.

Claim 2.16. *If a_1' is low then each $v \in V \smallsetminus e_1$ is high.*

Proof. Suppose $v \in V - e_1$ is low. Since a_1 is low, all vertices in $R - e_1$ are high, so $v \in C$ for some $C \in \mathcal{C}$. Then $C' := ve_1v$ is a cycle and so by (O2), $|C| = 3$. Since a_2 is high, $\|a_2, C\| \geq 1$. As v is low, $va_2 \notin E$. Since a_1' is low, it is adjacent to the low vertex v, and $\|a_1', C - v\| \leq 1$. Then $C', a_2'a_2(C - v)$ beats C, R by (P1). □

Claim 2.17. *If $|C| = 3$ and $\|e_1, C\|$, $\|e_2, C\| \geq 3$, then either*

(a) $\|c, e_1\| = 1 = \|c, e_2\|$ *for all $c \in V(C)$ or*
(b) *a_1' is high and there is $c \in V(C)$ with $\|c, R\| = 4$ and $C - c$ has a low vertex.*

Proof. If (a) fails then $\|c, e_i\| = 2$ for some $i \in [2]$ and $c \in C$. If $\|e_{3-i}, C - c\| \geq 2$ then there is a cycle $C' \subseteq C \cup e_{3-i} - c$, and $R \cup C$ contains disjoint cycles ce_ic and C'. Else,

$$\|c, R\| = \|c, e_i\| + (\|C, e_{3-i}\| - \|C - c, e_{3-i}\|) \geq 2 + (3 - 1) = 4 = |R|.$$

If $C - c$ has no low vertices then ce_1c, $e_2(C - c)$ beats C, R by (P1). Then $C - c$ contains a low vertex c'. If a'_1 is low then $c'a'_1a_1c'$ and $ca_2a'_2c$ are disjoint cycles. Thus, (b) holds. $\square$

CASE 2.1: $|D| = 4$. By (O2), $G[R \cup D]$ does not contain a 3-cycle. Then $5 \leq d(a_2) \leq 3 + \|a_2, C\| \leq 6$. Thus $d(a_1), d(a'_1) \geq 3$.

Suppose $\|e_1, D\| \geq 3$. Pick $v \in N(a_1) \cap D$ with minimum degree, and $v' \in N(a'_1) \cap D$. Since $N(a_1) \cap D$ and $N(a'_1) \cap D$ are nonempty, disjoint and independent, we see $vv' \in E$. Say $D = vv'ww'v$. As $D = K_{2,2}$ and low vertices are adjacent, $D' := a_1a'_1v'va_1$ is a 4-cycle and v is the only possible low vertex in D. Note $a_1w \notin E$: else $a_1ww'va_1$, $v'a'_1a'_2a_2$ beats D, R by (P1). As $\|e_1, D\| \geq 3$, $a'_1w' \in E$. Also note $\|e_2, ww'\| = 0$: else $G[a_2, a'_2, w, w']$ contains a 4-path R', and D', R' beats D, R by (P1). Similarly, replacing D' by $D'' := a_1a'_1w'va_1$ yields $\|e_2, v'\| = 0$. Then $\|e_1 \cup e_2, D\| \leq 3 + 1 = 4$, a contradiction. Thus

$$\|e_1, D\| \leq 2 \quad \text{and so} \quad \|R, D\| \leq 6. \tag{2.6}$$

Suppose $d(a'_1) = 3$. Then $\|a'_1, D\| \leq 1$. Then there is $uv \in E(D)$ with $\|a'_1, uv\| = 0$. Thus $d(u), d(v), d(a_2) \geq 6$, and $\|a_2, C\| = 3$. Now $|C| = 3$, $|G| = 11$, and there is $w \in N(u) \cap N(v)$. If $w \in C$ put $C' = a_2(C - w)a_2$; else $C' = C$. In both cases, $|C'| = |C|$ and $|wuvw| = 3 < |D|$, so C', $wuvw$ beats C, D by (O2). Thus $d(a'_1) \geq 4$. If $d(a_1) = 3$ then $d(a_2), d(a'_2) \geq 9 - 3 = 6$, and $\|a_2, C\| \geq 3$. By (2.6),

$$\|R, C\| \geq 3 + 4 + 6 + 6 - \|R, R\| - \|R, D\| \geq 19 - 6 - 6 = 7,$$

contradicting Claim 2.13. Then $d(a_1) = 4 \leq d(a'_1)$ and by (2.6), $\|e_1, C\| \geq 3$. Thus (2.6) fails for C in place of D; so $|C| = 3$. As $\|a_2, C\| \geq 2$ and $\|a'_2, C\| \geq 1$, Claim 2.17 implies either (a) or (b) of Claim 2.17 holds. If (a) holds then (a) and (d) of Claim 2.12 both hold, and so $G[C \cup R]$ has two disjoint cycles. Else, Claim 2.17 gives a'_1 is high and there is $c \in \mathcal{C}$ with $\|c, R\| = 4$. As a'_1 is high, $\|R, C\| \geq 7$. Now $\|c, R\| = 4$ contradicts Lemma 2.13.

CASE 2.2: $|C| = |D| = 3$ and $\|R, V\| = 18$. Then $d(a_1) + d(a'_2) = 9 = d(a'_1) + d(a_2)$, a_1 and a'_1 are low, and by Claim 2.16 all other vertices are high. Moreover, $d(a'_1) \leq d(a_1)$, since

$$18 = \|R, V\| = d(a'_1) - d(a_1) + 2d(a_1) + d(a'_2) + d(a_2) \geq d(a'_1) - d(a_1) + 9 + 9.$$

Suppose $d(a'_1) = 2$. Then $d(v) \geq 7$ for all $v \in V - a_1a'_1a'_2$. In particular, $C \cup D \subseteq N(a_2)$. If $d(a_1) = 2$ then $d(a'_2) \geq 7$, and $G = \mathbf{Y_1}$. Else $\|a_1, C \cup D\| \geq 2$. If there is $c \in C$ with $V(C) - c \subseteq N(a_1)$, then $a_1(C - c)a_1$, $a'_1a'_2a_2c$ beats C, R by (P1). Else $d(a_1) = 3$, $d(a'_2) = 6$, and there are $c \in C$ and $d \in D$ with $c, d \in N(a_1)$. If $ca'_2 \in E$ then $C \cup R$ contains disjoint cycles $a_1ca'_2a'_1a_1$ and $a_2(C - c)a_2$, so assume not. Similarly, assume

$da'_2 \notin E$. Since $d(d) \geq 7$ and $a'_1, a'_2 \notin N(d)$, we see $cd \in E(G)$. Then there are three disjoint cycles $a'_2(C-c)a'_2$, $a_2(D-d)a_2$, and a_1cda_1. Thus $d(a'_1) \geq 3$.

Suppose $d(a'_1) = 3$. Say $a'_1v \in E$ for some $v \in D$. As $d(a_2) \geq 6$, $\|a_2, D\| \geq 2$. Then $e_2 + D - v$ contains a 4-path R'. Thus $a_1v \notin E$: else ve_1v, R' beats D, R by (P1). Also $\|a_1, D-v\| \leq 1$: else $a_1(D-v)a_1, va'_1a'_2a_2$ beats D, R by (P1). Then $\|a_1, D\| \leq 1$.

Suppose $\|a_1, C\| \geq 2$. Pick $c \in C$ with $C - c \subseteq N(a_1)$. Then

$$a_2c \notin E: \tag{2.7}$$

else $a_1(C-c)a_1$, $a'_1a'_2a_2c$ beats C, R by (P1). Then $\|a_2, C\| = 2$ and $\|a_2, D\| = 3$. Also $a_1c \notin E$: else picking a different c violates (2.7). As $a'_1c \notin E$, $\|c, D\| = 3$ and $a'_2c \in E(G)$. Then $a_1(C-c)a_1$, $a_2(D-v)a_2$ and $cva'_1a'_2c$ are disjoint cycles. Otherwise, $\|a_1, C\| \leq 1$ and $d(a_1) \leq 3$. Then $d(a_1) = 3$ since $d(a_1) \geq d(a'_1)$.

Now $d(a'_2) = 6$. Say $D = vbb'v$ and $a_1b \in E$. As $b'a'_1 \notin E$, $d(b') \geq 9 - 3 = 6$. Since $\|e_2, V\| = 12$, we see that a_2 and a'_2 have three common neighbors. If one is b' then $D' := a_1a'_1vba_1$, $b'e_2b'$, and C are disjoint cycles; else $\|b', C\| = 3$ and there is $c' \in C$ with $\|c', e_2\| = 2$. Then D', $c'e_2c'$ and $b'(C-c')b'$ are disjoint cycles. Thus, $d(a'_1) = 4$.

Since a_1 is low and $d(a_1) \geq d(a'_1)$, we see $d(a_1) = d(a'_1) = 4$ and $\|\{a_1, a'_1\}, C \cup D\| = 5$, so we may assume $\|e_1, C\| \geq 3$. If $\|e_2, C\| \geq 3$, then because a'_1 is low, Claim 2.17(a) holds. Now, $V(C) \subseteq N(e_1)$ and there is $x \in e_1 = xy$ with $\|x, C\| \geq 2$. First suppose $\|x, C\| = 3$. As x is low, $x = a_1$. Pick $c \in N(a_2) \cap C$, which exists because $\|a_2, C \cup D\| \geq 4$. Then $a_1(C-c)a_1$, $a'_1a'_2a_2c$ beats C, R by (P1). Now suppose $\|x, C\| = 2$. Let $c \in C \setminus N(x)$. Then $x(C-c)x$, yce_2 beats C, R by (P1).

CASE 2.3: $|C| = |D| = 3$ and $\|R, V\| = 19$. Say $\|C, R\| = 7$ and $\|D, R\| = 6$.

CASE 2.3.1: a'_1 is low. Then $\|a'_1, C \cup D\| \leq 4 - \|a'_1, R\| = 2$, so by Claim 2.13, $\|e_2, C\| = 5$ with $\|a_2, C\| = 2$. Then $5 \leq d(a_2) \leq 6$.

If $d(a_2) = 5$ then $d(a_1) = d(a'_1) = 4$ and $d(a'_2) = 6$. Then $\|a_2, D\| = 2$ and $\|a'_2, D\| = 1$. Say $D = b_1b_2b_3b_1$, where $a_2b_2, a_2b_3 \in E$. As a'_1 is low, (a) of Claim 2.17 holds. Then $\|b_1, a_1a'_1a'_2\| = 2$, and there is a cycle $D' \subseteq G[b_1a_1a'_1a'_2]$. Then $a_2(D-b_1)a_2$ and D' are disjoint.

If $d(a_2) = 6$ then $\|a_2, D\| = 3$. Let $c_1 \in C - N(a_2)$. By Claim 2.13, $\|c_1, R\| = 1$, so c_1 is high, and $\|c_1, D\| \geq 2$. If $\|a'_2, D\| \geq 1$, then (a) and (d) hold in Claim 2.12 for $v_1 = a_2$ and $v_i = a'_2$, so $G[D \cup c_1a'_2a_2]$ has two disjoint cycles, and $c_2e_1c_3c_2$ contains a third. Therefore, assume $\|a'_2, D\| = 0$, and so $d(a'_2) = 5$. Thus $d(a_1) = d(a'_1) = 4$. Again, $\|e_1, D\| = 3 = \|a_2, D\|$. Now there are $x \in e_1$ and $b \in V(D)$ with $D - b \subseteq N(x)$. As a'_1 is low and has two neighbors in R, if $\|x, D\| = 3$ then $x = a_1$. Anyway, using Claim 2.17, $G[R + b - x]$ contains a 4-path R', and $x(D-b)x$, R' beats D, R by (P1).

CASE 2.3.2: a'_1 is high. Since $19 = \|R, V\| \geq d(a_1) + d(a'_1) + 2(9 - d(a_1)) \geq 23 - d(a_1)$, we get $d(a_1) = 4$ and $d(a'_1) = d(a'_2) = d(a_2) = 5$. Choose notation so that $C = c_1c_2c_3c_1$, $D = b_1b_2b_3b_1$, and $\|c_1, R\| = 1$. By Claim 2.13, there is $i \in [2]$ with $\|a_i, C\| = 2$, $\|a'_i, C\| = 3$, and $a_ic_1 \notin E$. If $i = 1$ then every low vertex is in $N(a_1) - a'_1 \subseteq D \cup C'$, where $C' = a_1c_2c_3a_1$. Then C', $c_1a'_1a'_2a_2$ beats C, R by (P1). Thus let $i = 2$. Now $\|a_2, C\| = 2 = \|a_2, D\|$.

Say $a_2b_2, a_2b_3 \in E$. Also $\|a'_2, D\| = 0$ and $\|e_1, D\| = 4$. Then $\|b_j, e_1\| = 2$ for some $j \in [3]$. If $j = 1$ then $b_1e_1b_1$ and $a_2b_2b_3a_2$ are disjoint cycles. Else, say $j = 2$. By inspection, all low vertices are contained in $\{a_1, b_1, b_3\}$. If b_1 and b_3 are high then $b_2e_1b_2$, $b_1b_3e_2$ beats D, R by (P1). Else there is a 3-cycle $D' \subseteq G[D + a_1]$ that contains every low vertex of G. Pick D' with $b_1 \in D'$ if possible. If $b_2 \notin D'$ then D' and $b_2a'_1a'_2a_2b_2$ are disjoint cycles. If $b_3 \notin D'$ then D', $b_3a_2a'_2a'_1$ beats D, R by (P1). Else $b_1 \notin D'$, $a_1b_1 \notin E$, and b_1 is high. If $b_1a'_1 \in E$ then D', $b_1a'_1a'_2a_2$ beats D, R by (P1). Else, $\|b_1, C\| = 3$. Then D', $b_1c_1c_2b_1$, and $c_3e_2c_3$ are disjoint cycles. $\square$

2.3. Key lemma

Now $|R| = 3$; say $R = a_1a'a_2$. By Lemma 2.14 the maximum length of a cycle in $\mathcal{C}$ is 4. Fix $C = w_1 \dots w_4w_1 \in \mathcal{C}$.

Lemma 2.18. *If $D \in \mathcal{C}$ with $\|R, D\| \geq 7$ then $|D| = 3$, $\|R, D\| = 7$ and $G[R \cup D] = K_6 - E(K_3)$.*

Proof. Since $\|R, D\| \geq 7$, there exists $a \in R$ with $\|a, D\| \geq 3$. By Claim 2.1, $|D| = 3$. If $\|a_i, D\| = 3$ for any $i \in [2]$, then (a) and (c) in Claim 2.12 hold, violating (O1). Then $\|a_1, D\| = \|a_2, D\| = 2$ and $\|a', D\| = 3$. If $G[R \cup D] \neq K_6 - K_3$ then $N(a_1) \cap D \neq N(a_2) \cap D$. Then there is $w \in N(a_1) \cap D$ with $\|a_2, D - w\| = 2$. Then $wa_1a'w$ and $a_2(D - w)a_2$ are disjoint cycles. $\square$

Lemma 2.19. *Let $D \in \mathcal{C}$ with $D = z_1 \dots z_tz_1$. If $\|C, D\| \geq 8$ then $\|C, D\| = 8$ and*

$$W := G[C \cup D] \in \{K_{4,4},\quad K_1 \vee K_{3,3},\quad \overline{K}_3 \vee (K_1 + K_3)\}.$$

Proof. First suppose $|D| = 4$. Suppose

$$W \text{ contains two disjoint cycles } T \text{ and } C' \text{ with } |T| = 3. \tag{2.8}$$

Then $\mathcal{C}' := \mathcal{C} - C - D + T + C'$ is an optimal choice of $k - 1$ disjoint cycles, since $\mathcal{C}$ is optimal. By Lemma 2.14, $|C'| \leq 4$. Thus $\mathcal{C}'$ beats $\mathcal{C}$ by (O2).

CASE 1: $\Delta(W) = 6$. By symmetry, assume $d_W(w_4) = 6$. Then $\|\{z_i, z_{i+1}\}, C - w_4\| \geq 2$ for some $i \in \{1, 3\}$. Then (2.8) holds with $T = w_4z_{4-i}z_{5-i}w_4$.

CASE 2: $\Delta(W) = 5$. Say $z_1, z_2, z_3 \in N(w_1)$. Then $\|\{z_i, z_4\}, C - w_1\| \geq 2$ for some $i \in \{1, 3\}$. Then (2.8) holds with $T = w_1z_{4-i}z_2w_1$.

CASE 3: $\Delta(W) = 4$. Then W is regular. If W has a triangle then (2.8) holds. Else, say $w_1z_1, w_1z_3 \in E$. Then $z_1, z_3 \notin N(w_2) \cup N(w_4)$, so $z_2, z_4 \in N(w_2) \cup N(w_4)$, and $z_1, z_3 \in N(w_3)$.

Now, suppose $|D| = 3$.

CASE 1: $d_W(z_h) = 6$ for some $h \in [3]$. Say $h = 3$. If $w_i, w_{i+1} \in N(z_j)$ for some $i \in [4]$ and $j \in [2]$, then $z_3w_{i+2}w_{i+3}z_3$, $z_jw_iw_{i+1}z_j$ beats C, D by (O2). Else for all $j \in [2]$, $\|z_j, C\| = 2$, and the neighbors of z_j in C are nonadjacent. If $w_i \in N(z_1) \cap N(z_2) \cap C$, then $z_3w_{i+1}w_{i+2}z_3$, $z_1z_2w_iz_1$ are preferable to C, D by (O2). Whence $W = K_1 \vee K_{3,3}$.

CASE 2: $d_W(z_h) \leq 5$ for every $h \in [3]$. Say $d(z_1) = 5 = d(z_2)$, $d(z_3) = 4$, and $w_1, w_2, w_3 \in N(z_1)$. If $N(z_1) \cap C \neq N(z_2) \cap C$ then $W - z_3$ contains two disjoint cycles, preferable to C, D by (O2); if $w_i \in N(z_3)$ for some $i \in \{1, 3\}$ then $W - w_4$ contains two disjoint cycles. Then $N(z_3) = \{w_2, w_4\}$, and so $W = \overline{K}_3 \vee (K_1 + K_3)$, where $V(K_1) = \{w_4\}$, $w_2z_1z_2w_2 = K_3$, and $V(K_3) = \{w_1, w_3, z_3\}$. $\square$

Claim 2.20. *For $D \in \mathcal{C}$, if $\|\{w_1, w_3\}, D\| \geq 5$ then $\|C, D\| \leq 6$. If also $|D| = 3$ then $\|\{w_2, w_4\}, D\| = 0$.*

Proof. Assume not. Let $D = z_1 \ldots z_tz_1$. Then $\|\{w_1, w_3\}, D\| \geq 5$ and $\|C, D\| \geq 7$. Say $\|w_1, D\| \geq \|w_3, D\|$, $\{z_1, z_2, z_3\} \subseteq N(w_1)$, and $z_l \in N(w_3)$.

Suppose $\|w_1, D\| = 4$. Then $|D| = 4$. If $\|z_h, C\| \geq 3$ for some $h \in [4]$ then there is a cycle $B \subseteq G[w_2, w_3, w_4, z_h]$; so B, $w_1z_{h+1}z_{h+2}w_1$ beats C, D by (O2). Else there are $j \in \{l-1, l+1\}$ and $i \in \{2, 3, 4\}$ with $z_iw_j \in E$. Then $z_lz_j[w_iw_3]z_l$, $w_1(D - z_l - z_j)w_1$ beats C, D by (O2), where $[w_iw_3] = w_3$ if $i = 3$.

Else, $\|w_1, D\| = 3$. By assumption, there is $i \in \{2, 4\}$ with $\|w_i, D\| \geq 1$. If $|D| = 3$, applying Claim 2.12 with $P := w_1w_iw_3$ and cycle D yields two disjoint cycles in $(D \cup C) - w_{6-i}$, contradicting (O2). Therefore, suppose $|D| = 4$. Because $w_1z_1z_2w_1$ and $w_1z_2z_3w_1$ are triangles, there do not exist cycles in $G[\{w_i, w_3, z_3, z_4\}]$ or $G[\{w_i, w_3, z_1, z_4\}]$ by (O2). Then $\|\{w_i, w_3\}, \{z_3, z_4\}\|$, $\|\{w_i, w_3\}, \{z_1, z_4\}\| \leq 1$. Since $\|\{w_i, w_3\}, D\| \geq 3$, one has a neighbor in z_2. If both are adjacent to z_2, then $w_iw_3z_2w_i$, $w_1z_1z_4z_3w_1$ beat C, D by (O2). Then $\|\{w_i, w_3\}, z_2\| = 1 = \|\{w_i, w_3\}, z_1\| = \|\{w_i, w_3\}, z_3\|$. Let z_m be the neighbor of w_i. Then $w_iw_1z_mw_i$, $w_3(D - z_m)w_3$ beat C, D by (O2).

Suppose $|D| = 3$ and $\|\{w_1, w_3\}, D\| \geq 5$. If $\|\{w_2, w_4\}, D\| \geq 1$, then $C \cup D$ contains two triangles, and these are preferable to C, D by (O2). $\square$

For $v \in N(C)$, set type$(v) = i \in [2]$ if $N(v) \cap C \subseteq \{w_i, w_{i+2}\}$. Call v *light* if $\|v, C\| = 1$; else v is *heavy*. For $D = z_1 \ldots z_tz_1 \in \mathcal{C}$, put $H := H(D) := G[R \cup D]$.

Claim 2.21. *If $\|\{a_1, a_2\}, D\| \geq 5$ then there exists $i \in [2]$ such that*

(a) $\|C, H\| \leq 12$ *and* $\|\{w_i, w_{i+2}\}, H\| \leq 4$*;*
(b) $\|C, H\| = 12$*;*
(c) $N(w_i) \cap H = N(w_{i+2}) \cap H = \{a_1, a_2\}$ *and* $N(w_{3-i}) \cap H = N(w_{5-i}) \cap H = V(D) \cup \{a'\}$*.*

Proof. By Claim 2.1, $|D| = 3$. Choose notation so that $\|a_1, D\| = 3$ and $z_2, z_3 \in N(a_2)$.

(a) Using that $\{w_1, w_3\}$ and $\{w_2, w_4\}$ are independent and Lemma 2.19:

$$\|C, H\| = \|C, V - (V - H)\| \geq 2(4k - 3) - 8(k - 2) = 10. \tag{2.9}$$

Let $v \in V(H)$. As $K_4 \subseteq H$, $H - v$ contains a 3-cycle. If $C + v$ contains another 3-cycle then these 3-cycles beat C, D by (O2). Thus, type(v) is defined for all $v \in N(C) \cap H$, and $\|C, H\| \leq 12$. If only five vertices of H have neighbors in C then there is $i \in [2]$ such that at most two vertices in H have type i. Then $\|\{w_i, w_{i+2}\}, H\| \leq 4$. Else every vertex in H has a neighbor in C. By (2.9), H has at least four heavy vertices.

Let H' be the spanning subgraph of H with $xy \in E(H')$ iff $xy \in E(H)$ and $H - \{x, y\}$ contains a 3-cycle. If $xy \in E(H')$ then $N(x) \cap N(y) \cap C = \emptyset$ by (O2). Now, if x and y have the same type, then they are both light. By inspection, $H' \supseteq z_1 a_1 a' a_2 z_2 + a_2 z_3$.

Let type$(a_2) = i$. If a_2 is heavy then its neighbors a', z_2, z_3 have type $3 - i$. Either z_1, a_1 are both light or they have different types. Anyway, $\|\{w_i, w_{i+2}\}, H\| \leq 4$. Else a_2 is light. Then because there are at least four heavy vertices in H, at least one of z_1, a_1 is heavy and so they have different types. Also any type-i vertex in a', z_2, z_3 is light, but at most one vertex of a, z_2, z_3 is light because there are at most two light vertices in H. Then $\|\{w_i, w_{i+2}\}, H\| \leq 4$.

(b) By (a), there is i with $\|\{w_i, w_{i+2}\}, H\| \leq 4$; thus

$$\|\{w_i, w_{i+2}\}, V - H\| \geq (4k - 3) - 4 = 4(k - 2) + 1.$$

Now $\|\{w_i, w_{i+2}\}, D'\| \geq 5$ for some $D' \in \mathcal{C} - C - D$. By (a), Claim 2.20, and Lemma 2.19,

$$12 \geq \|C, H\| = \|C, V - D' - (V - H - D')\| \geq 2(4k - 3) - 6 - 8(k - 3) = 12.$$

(c) By (b), $\|C, H\| = 12$, so each vertex in H is heavy. Thus type(v) is the unique proper 2-coloring of H', and (c) follows. □

Lemma 2.22. *There exists $C^* \in \mathcal{C}$ such that $3 \leq \|\{a_1, a_2\}, C^*\| \leq 4$ and $\|\{a_1, a_2\}, D\| = 4$ for all $D \in \mathcal{C} - C^*$. If $\|\{a_1, a_2\}, C^*\| = 3$ then one of a_1, a_2 is low.*

Proof. Suppose $\|\{a_1, a_2\}, D\| \geq 5$ for some $D \in \mathcal{C}$; set $H := H(D)$. Using Claim 2.21, choose notation so that $\|\{w_1, w_3\}, H\| \leq 4$. Now

$$\|\{w_1, w_3\}, V - H\| \geq 4k - 3 - 4 = 4(k - 2) + 1.$$

Thus there is a cycle $B \in \mathcal{C} - D$ with $\|\{w_1, w_3\}, B\| \geq 5$; say $\|\{w_1, B\}\| = 3$. By Claim 2.20, $\|C, B\| \leq 6$. Note by Claim 2.21, if $|B| = 4$ then for an edge $z_1 z_2 \in N(w_1)$, $w_1 z_1 z_2 w_1$ and $w_2 w_3 a_2 a' w_2$ beat B, C by (O2). Then $|B| = 3$. Using Claim 2.21(b) and Lemma 2.19,

$$2(4k - 3) \leq \|C, V\| = \|C, H \cup B \cup (V - H - B)\| \leq 12 + 6 + 8(k - 3) = 2(4k - 3).$$

Thus, $\|C, D'\| = 8$ for all $D' \in \mathcal{C} - C - D$. By Lemma 2.19, $\|\{w_1, w_3\}, D'\| = \|\{w_2, w_4\}, D'\| = 4$. By Claim 2.21(c) and Claim 2.20,

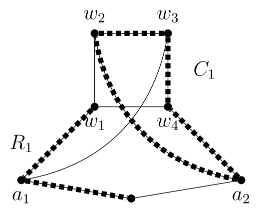


Fig. 6. Claim 2.24.

$$4k-3 \le \|\{w_2,w_4\}, H \cup B \cup (V-H-B)\| \le 8+1+4(k-3) = 4k-3,$$

and so $\|\{w_2,w_4\},B\| = 1$. Say $\|w_2,B\| = 1$. Since $|B| = 3$, by Claim 2.12, $G[B \cup C - w_4]$ has two disjoint cycles that are preferable to C, B by (O2). This contradiction implies $\|\{a_1,a_2\},D\| \le 4$ for all $D \in \mathcal{C}$. Since $\|\{a_1,a_2\},V\| \ge 4k-3$ and $\|\{a_1,a_2\},R\| = 2$, we get $\|\{a_1,a_2\},D\| \ge 3$, and equality holds for at most one $D \in \mathcal{C}$, and only if one of a_1 and a_2 is low. $\square$

2.4. Completion of the proof of Theorem 1.7

For an optimal $\mathcal{C}$, let $\mathcal{C}_i := \{D \in \mathcal{C} : |D| = i\}$ and $t_i := |\mathcal{C}_i|$. For $C \in \mathcal{C}_4$, let $Q_C := Q_C(\mathcal{C}) := G[R(\mathcal{C}) \cup C]$. A 3-path R' is *$\mathcal{D}$-useful* if $R' = R(\mathcal{C}')$ for an optimal set $\mathcal{C}'$ with $\mathcal{D} \subseteq \mathcal{C}'$; we write *$D$-useful* for $\{D\}$-useful.

Lemma 2.23. Let $\mathcal{C}$ be an optimal set and $C \in \mathcal{C}_4$. Then $Q = Q_C \in \{K_{3,4}, K_{3,4} - e\}$.

Proof. Since $\mathcal{C}$ is optimal, Q does not contain a 3-cycle. Then for all $v \in V(C)$, $N(v) \cap R$ is independent and $\|a_1,C\|, \|a_2,C\| \le 2$. By Lemma 2.22, $\|\{a_1,a_2\},C\| \ge 3$. Say $a_1w_1, a_1w_3 \in E$ and $\|a_2,C\| \ge 1$. Then type(a_1) and type(a_2) are defined.

Claim 2.24. *type*(a_1) = *type*(a_2).

Proof. Suppose not. Then $\|w_i,R\| \le 1$ for all $i \in [4]$. Say $a_2w_2 \in E$. If $w_ia_j \in E$ and $\|a_{3-j},C\| = 2$, let $R_i = w_ia_ja'$ and $C_i = a_{3-j}(C-w_i)a_{3-j}$ (see Fig. 6). Then R_i is $(\mathcal{C} - C + C_i)$-useful. Let $\lambda(X)$ be the number of low vertices in $X \subseteq V$. As Q does not contain a 3-cycle, $\lambda(R) + \lambda(C) \le 2$. We claim:

$$\forall D \in \mathcal{C} - C, \quad \|a',D\| \le 2. \tag{2.10}$$

Fix $D \in \mathcal{C} - C$, and suppose $\|a',D\| \ge 3$. By Claim 2.1, $|D| = 3$. Since

$$\begin{aligned}\|C,D\| &= \|C,\mathcal{C}\| - \|C,\mathcal{C}-D\| \\ &\ge 4(2k-1) - \lambda(C) - \|C,R\| - 8(k-2) \\ &= 12 - \|C,R\| - \lambda(C) \ge 6 + \lambda(R),\end{aligned} \tag{2.11}$$

we get that $\|w_i, D\| \geq 2$ for some $i \in [4]$. If R_i is defined, R_i is $\{C_i, D\}$-useful. By Lemma 2.22, $\|\{w_i, a'\}, D\| \leq 4$. As $\|w_i, D\| \geq 2$, $\|a', D\| \leq 2$, proving (2.10). Then R_i is not defined, so a_2 is low with $N(a_2) \cap C = \{w_2\}$ and $\|w_2, D\| \leq 1$. Then by (2.11), $\|C - w_2, D\| \geq 6$. Note $G[a' + D] = K_4$, so for any $z \in D$, $D - z + a'$ is a triangle, so by (O2) the neighbors of z in C are independent. Then $\|C - w_2, D\| = 6$ with $N(z) \cap C = \{w_1, w_3\}$ for every $z \in D$. Then $\|w_2, D\| = 1$, say $zw_2 \in E(G)$, and now $w_2 w_3 z w_2$, $w_1(D - z)w_1$ beat C, D by (O2).

If $\|a', C\| \geq 1$ then $a'w_4 \in E$ and $N(a_2) \cap C = \{w_2\}$. Now R_2 is C_2-useful, $\text{type}(a') \neq \text{type}(w_2)$ with respect to C_2, and the middle vertex a_2 of R_2 has no neighbors in C_2. Thus we may assume $\|a', C\| = 0$. Then a' is low:

$$d(a') = \|a', C \cup R\| + \|a', \mathcal{C} - C\| \leq 0 + 2 + 2(k-2) = 2k - 2. \tag{2.12}$$

Thus all vertices of C are high. Using Lemma 2.19, this yields:

$$4 \geq \|C, R\| = \|C, V - (V - R)\| \geq 4(2k-1) - 8(k-1) = 4. \tag{2.13}$$

As this calculation is tight, $d(w) = 2k - 1$ for every $w \in C$. Thus $d(a') \geq 2k - 2$, so (2.12) is tight. Hence $\|a', D\| = 2$ for all $D \in \mathcal{C} - C$.

Pick $D = z_1 \ldots z_t z_1 \in \mathcal{C} - C$ with $\|\{a_1, a_2\}, D\|$ maximum. By Lemma 2.22, $3 \leq \|\{a_1, a_2\}, D\| \leq 4$. Say $\|a_i, D\| \geq 2$. By (2.13), $\|C, D\| = 8$. By Lemma 2.19,

$$W := G[C \cup D] \in \{K_{4,4},\ \overline{K}_3 \vee (K_3 + K_1),\ K_1 \vee K_{3,3}\}.$$

CASE 1: $W = K_{4,4}$. Then $\|D, R\| \geq 5 > |D| = 4$, so $\|z, R\| \geq 2$ for some $z \in V(D)$. Let $w \in N(z) \cap C$. Either w and z have a common neighbor in $\{a_1, a_2\}$ or z has two consecutive neighbors in R. Regardless, $G[R + w + z]$ contains a 3-cycle D' and $G[W - w - z]$ contains a 4-cycle C'. Thus C', D' beats C, D by (O2).

CASE 2: $W = \overline{K}_3 \vee (K_3 + K_1)$. As $\|\{a', a_i\}, D\| \geq 4 > |D|$, there is $z \in V(D)$ with $D' := za'a_iz \subseteq G$. Also $W - z$ contains a 3-cycle C', so C', D' beats C, D by (O2).

CASE 3: $W = K_1 \vee K_{3,3}$. Some $v \in V(D)$ satisfies $\|v, W\| = 6$. There is no $w \in W - v$ such that w has two adjacent neighbors in R: else a and v would be contained in disjoint 3-cycles, contradicting the choice of C, D. Then $\|w, R\| \leq 1$ for all $w \in W - v$, because $\text{type}(a_1) \neq \text{type}(a_2)$. Similarly, no $z \in D - v$ has two adjacent neighbors in R. Thus

$$2 + 3 \leq \|a', D\| + \|\{a_1, a_2\}, D\| = \|R, D\| = \|R, D - v\| + \|R, v\| \leq 2 + 3,$$

so $\|\{a_1, a_2\}, D\| = 3$, $R \subseteq N(v)$, and $N(a_i) \cap K_{3,3}$ is independent. By Lemma 2.22 and the maximality of $\|\{a_1, a_2\}, D\| = 3$, $k = 3$. Thus $G = \mathbf{Y}_2$, a contradiction. □

Returning to the proof of Lemma 2.23, we have $\text{type}(a_1) = \text{type}(a_2)$. Using Lemma 2.22, choose notation so that $a_1w_1, a_1w_3, a_2w_1 \in E$. Then Q has bipartition $\{X, Y\}$ with $X := \{a', w_1, w_3\}$ and $Y := \{a_1, a_2, w_2, w_4\}$. The only possible nonedges between X and Y are $a'w_2$, $a'w_4$ and a_2w_3. Let $C' := w_1Rw_1$. Then $R' := w_2w_3w_4$ is C'-useful. By Lemma 2.22, $\|\{w_2, w_4\}, C'\| \geq 3$. It is already true that $w_2, w_4 \in N(w_1)$; so because Q has no C_3, (say) $a'w_2 \in E$. Now, let $C'' := a_1a'w_2w_3a_1$. Then $R'' := a_2w_1w_4$ is C''-useful; so $\|\{a_2, w_4\}, C''\| \geq 3$. Again, Q contains no C_3, so $a'w_4$ or a_2w_3 is an edge of G. Thus $Q \in \{K_{3,4}, K_{3,4} - e\}$. □

Proof of Theorem 1.7. Using Lemma 2.23, one of two cases holds:

(C1) For some optimal set $\mathcal{C}$ and $C' \in \mathcal{C}_4$, $Q_{C'} = K_{3,4} - x_0y_0$;
(C2) for all optimal sets $\mathcal{C}$ and $C \in \mathcal{C}_4$, $G[R \cup C] = K_{3,4}$.

Fix an optimal set $\mathcal{C}$ and $C' \in \mathcal{C}_4$, where $R = y_0x'y$ with $d(y_0) \leq d(y)$, such that in (C1), $Q_{C'} = K_{3,4} - x_0y_0$. By Lemmas 2.22 and 2.23, for all $C \in \mathcal{C}_4$, $1 \leq \|y_0, C\| \leq \|y, C\| \leq 2$ and $\|y_0, C\| = 1$ only in Case (C1) when $C = C'$. Put $H := R \cup \bigcup \mathcal{C}_4$, $S = S(\mathcal{C}) := N(y) \cap H$, and $T = T(\mathcal{C}) := V(H) \smallsetminus S$. As $\|y, R\| = 1$ and $\|y, C\| = 2$ for each $C \in \mathcal{C}_4$, $|S| = 1 + 2t_4 = |T| - 1$.

Claim 2.25. *H is a bipartite graph with parts S and T. In case (C1), $H = K_{2t_4+1,2t_4+2} - x_0y_0$; else $H = K_{2t_4+1,2t_4+2}$.*

Proof. By Lemma 2.23, $\|x', S\| = \|y, T\| = \|y_0, T\| = 0$.

By Lemmas 2.22 and 2.23, $\|y_0, S\| = |S| - 1$ in (C1) and $\|y_0, S\| = |S|$ otherwise. We claim that for every $t \in T - y_0$, $\|t, S\| = |S|$. This clearly holds for y, so take $t \in H - \{y, y_0\}$. Then $t \in C$ for some $C \in \mathcal{C}_4$. Let $\mathcal{R}^* := tx'y_0$ and $\mathcal{C}^* := y(C - t)y$. (Note R^* is a path and C^* is a cycle by Lemma 2.23 and the choice of y_0.) Since R^* is C^*-useful, by Lemmas 2.22 and 2.23, and by choice of y_0, $\|t, S\| = \|y, S\| = |S|$. Then in (C1), $H \supseteq K_{2t_4+1,2t_4+2} - x_0y_0$ and $x_0y_0 \notin E(H)$; else $H \supseteq K_{2t_4+1,2t_4+2}$.

Now we easily see that if any edge exists inside S or T, then $C_3 + (t_4 - 1)C_4 \subseteq H$, and these cycles beat $\mathcal{C}_4$ by (O2). □

By Claim 2.25 all pairs of vertices of T are the ends of a $\mathcal{C}_3$-useful path. Now we use Lemma 2.22 to show that they have essentially the same degree to each cycle in $\mathcal{C}_3$.

Claim 2.26. *If $v \in T$ and $D \in \mathcal{C}_3$ then $1 \leq \|v, D\| \leq 2$; if $\|v, D\| = 1$ then v is low and for all $C \in \mathcal{C}_3 - D$, $\|v, C\| = 2$.*

Proof. By Claim 2.25, $H + x_0y_0$ is a complete bipartite graph. Let $y_1, y_2 \in T - v$ and $u \in S - x_0$. Then $R' = y_1uv$, $R'' = y_2uv$, and $R''' = y_1uy_2$ are $\mathcal{C}_3$-useful. By Lemma 2.22,

$$3 \leq \|\{v, y_1\}, D\|, \|\{v, y_2\}, D\|, \|\{y_1, y_2\}, D\| \leq 4.$$

Say $\|y_1, D\| \leq 2 \leq \|y_2, D\|$. Thus

$$1 \le \|\{v, y_1\}, D\| - \|y_1, D\| = \|v, D\| = \|\{v, y_2\}, D\| - \|y_2, D\| \le 2.$$

Suppose $\|v, D\| = 1$. By Claim 2.25 and Lemma 2.22, for any $v' \in T - v$,

$$4k - 3 \le \|\{v, v'\}, H \cup (\mathcal{C}_3 - D) \cup D\| \le 2(2t_4 + 1) + 4(t_3 - 1) + 3 = 4k - 3.$$

Thus for all $C \in \mathcal{C}_3 - D_0$, $\|\{v, v'\}, C\| = 4$, and so $\|v, C\| = 2$. Hence v is low. □

Next we show that all vertices in T have essentially the same neighborhood in each $C \in \mathcal{C}_3$.

Claim 2.27. *Let $z \in D \in \mathcal{C}_3$ and $v, w \in T$ with w high.*

(1) *If $zv \in E$ and $zw \notin E$ then $T - w \subseteq N(z)$.*
(2) *$N(v) \cap D \subseteq N(w) \cap D$.*

Proof. (1) Since w is high, Claim 2.26 implies $\|w, D\| = 2$. Since $zw \notin E$, we see $D' := w(D - z)w$ is a 3-cycle. Let $u \in S - x_0$. Then $zvu = R(\mathcal{C}')$ for some optimal set $\mathcal{C}'$ with $\mathcal{C}_3 - D + D' \subseteq \mathcal{C}'$. By Claim 2.25, $T(\mathcal{C}') = S + z$ and $S(\mathcal{C}') = T - w$. If (C2) holds, then $T - w = S(\mathcal{C}') \subseteq N(z)$, as desired. Suppose (C1) holds, so there are $x_0 \in S$ and $y_0 \in T$ with $x_0 y_0 \notin E$. By Claims 2.25 and 2.26, $d(y_0) \le (|S| - 1) + 2t_3 = 2k - 2$, so y_0 is low. Since w is high, we see $y_0 \in T - w$. But now apply Claims 2.25 and 2.26 to $T(\mathcal{C}')$: $d(x_0) \le |S(\mathcal{C}')| - 1 + 2t_3 = 2k - 2$, and x_0 is low. As $x_0 y_0 \notin E$, this is a contradiction. Now $T - w = S(\mathcal{C}') \subseteq N(z)$.

(2) Suppose there exists $z \in N(v) \cap D \setminus N(w)$. By (1), $T - w \subseteq N(z)$. Let $w' \in T - w$ be high. By Claim 2.26, $\|w', D\| = 2$. Now there exists $z' \in N(w) \cap D \setminus N(w')$ and $z \neq z'$. By (1), $T - w' \subseteq N(z')$. As $|T| \ge 4$ and at least three of its vertices are high, there exists a high $w'' \in T - w - w'$. Since $w''z, w''z' \in E$, there exists $z'' \in N(w) \cap D \setminus N(w'')$ with $\{z, z', z''\} = V(D)$. By (1), $T - w'' \subseteq N(z'')$. Since $|T| \ge 4$, there exists $x \in T \setminus \{w, w', w''\}$. Now $\|x, D\| = 3$, contradicting Claim 2.26. □

Let $y_1, y_2 \in T - y_0$ and let $x \in S$ with $x = x_0$ if $x_0 y_0 \notin E$. By Claim 2.25, $y_1 x y_2$ is a path, and $G - \{y_1, y_2, x\}$ contains an optimal set $\mathcal{C}'$. Recall y_0 was chosen in T with minimum degree, so y_1 and y_2 are high and by Claim 2.26 $\|y_i, D\| = 2$ for each $i \in [2]$ and each $D \in \mathcal{C}_3$. Let $N = N(y_1) \cap \bigcup \mathcal{C}_3$ and $M = \bigcup \mathcal{C}_3 \setminus N$ (see Fig. 7). By Claim 2.25, T is independent. By Claim 2.27, for every $y \in T$, $N(y) \cap \bigcup \mathcal{C}_3 \subseteq N$, so $E(M, T) = \emptyset$. Since $y_2 \neq y_0$, also $N(y_2) \cap \bigcup \mathcal{C}_3 = N$.

Claim 2.28. *M is independent.*

Proof. First, we show

$$\|z, S\| > t_4 \text{ for all } z \in M. \tag{2.14}$$

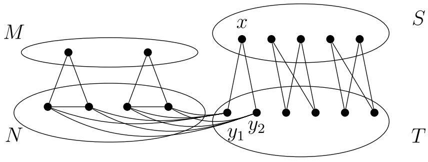


Fig. 7. Configuration of G, showing sets M, N, S, and T.

If not then there exists $z \in D \in \mathcal{C}_3$ with $\|z, S\| \leq t_4$. Since $\|M, T\| = \|T, T\| = 0$,

$$\begin{aligned}\|\{y_1, z\}, \mathcal{C}_3\| &\geq 4k - 3 - \|\{z, y_1\}, S\| \geq 4(t_4 + t_3 + 1) - 3 - (2t_4 + 1 + t_4) \\ &= t_4 + 4t_3 > 4t_3.\end{aligned}$$

Then there is $D' = z'z_1'z_2'z' \in \mathcal{C}_3$ with $\|\{z, y_1\}, D'\| \geq 5$ and $z' \in M$. As $\|y_1, D\| = 2$, $\|z, D'\| = 3$. Since $D^* := zz'z_2'z$ is a cycle, xy_2z_1' is D^*-useful. As $\|z_1', D^*\| = 3$, this contradicts Claim 2.26, proving (2.14).

Suppose $zz' \in E(M)$; say $z \in D \in \mathcal{C}_3$ and $z' \in D' \in \mathcal{C}_3$. By (2.14), there is $u \in N(z) \cap N(z') \cap S$. Then $zz'uz$, $y_1(D - z)y_1$ and $y_2(D' - z')y_2$ are disjoint cycles, contrary to (O1). □

By Claims 2.25 and 2.28, M and T are independent; as remarked above $E(M, T) = \emptyset$. Then $M \cup T$ is independent. This contradicts (H3), since

$$|G| - 2k + 1 = 3t_3 + 4t_4 + 3 - 2(t_3 + t_4 + 1) + 1 = t_3 + 2t_4 + 2 = |M \cup T| \leq \alpha(G).$$

The proof of Theorem 1.7 is now complete. □

3. The case $k = 2$

Lovász [20] observed that any (simple or multi-) graph can be transformed into a multigraph with minimum degree at least 3, without affecting the maximum number of disjoint cycles in the graph, by using a sequence of operations of the following three types: (i) deleting a bud; (ii) suppressing a vertex v of degree 2 that has two neighbors x and y, i.e., deleting v and adding a new (possibly parallel) edge between x and y; and (iii) increasing the multiplicity of a loop or edge with multiplicity 2. Here loops and two parallel edges are considered cycles, so forests have neither. Also K_s and $K_{s,t}$ denote simple graphs. Let W_s^* denote a wheel on s vertices whose spokes, but not outer cycle edges, may be multiple. The following theorem characterizes those multigraphs that do not have two disjoint cycles.

Theorem 3.1 *(Lovász [20]). Let G be a multigraph with $\delta(G) \geq 3$ and no two disjoint cycles. Then G is one of the following:* (1) K_5, (2) W_s^*, (3) $K_{3,|G|-3}$ *together with a multigraph on the vertices of the (first) 3-class, and* (4) *a forest F and a vertex x with possibly some loops at x and some edges linking x to F.*

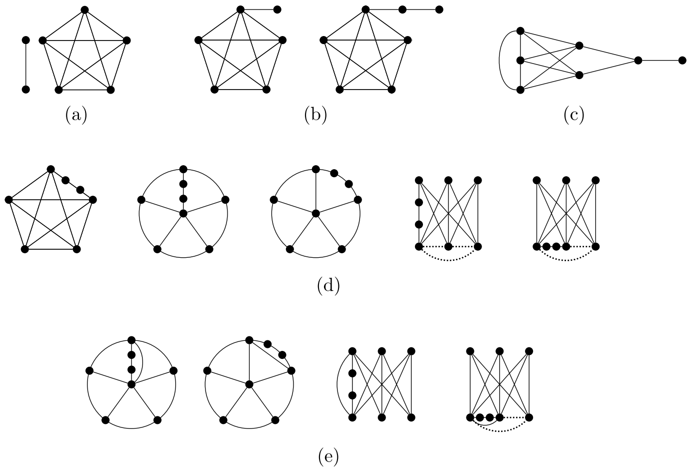


Fig. 8. Theorem 3.2.

Let $\mathcal{G}$ be the class of simple graphs G with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that do not have two disjoint cycles. Fix $G \in \mathcal{G}$. A vertex in G is low if its degree is at most 2. The low vertices form a clique Q of size at most 2—if $|Q| = 3$, then Q is a component-cycle, and $G - Q$ has another cycle. By Lovász's observation, G can be reduced to a graph H of type (1–4). Reversing this reduction, G can be obtained from H by adding buds and subdividing edges. Let $Q' := V(G) \smallsetminus V(H)$. It follows that $Q \subseteq Q'$. If $Q' \neq Q$, then Q consists of a single leaf in G with a neighbor of degree 3, so G is obtained from H by subdividing an edge and adding a leaf to the degree-2 vertex. If $Q' = Q$, then Q is a component of G, or $G = H + Q + e$ for some edge $e \in E(H, Q)$, or at least one vertex of Q subdivides an edge $e \in E(H)$. In the last case, when $|Q| = 2$, e is subdivided twice by Q. As G is simple, H has at most one multiple edge, and its multiplicity is at most 2.

In case (4), because $\delta(H) \geq 3$, either F has at least two buds, each linked to x by multiple edges, or F has one bud linked to x by an edge of multiplicity at least 3. This case cannot arise from G. Also, $\delta(H) = 3$, unless $H = K_5$, in which case $\delta(H) = 4$. Then Q is not an isolated vertex, lest deleting Q leave H with $\delta(H) \geq 5 > 4$; and if Q has a vertex of degree 1 then $H = K_5$. Else all vertices of Q have degree 2, and Q consists of the subdivision vertices of one edge of H. We have the following lemma.

Lemma 3.2. *Let G be a graph with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that does not have two disjoint cycles. Then G is one of the following (see Fig. 8):*

(a) *$K_5 + K_2$;*
(b) *K_5 with a pendant edge, possibly subdivided;*
(c) *K_5 with one edge subdivided and then a leaf added adjacent to the degree-2 vertex;*
(d) *a graph H of type (1–3) with no multiple edge, and possibly one edge subdivided once or twice, and if $|H| = 6 - i$ with $i \geq 1$ then some edge is subdivided at least i times;*

(e) *a graph H of type (2) or (3) with one edge of multiplicity two, and one of its parallel parts is subdivided once or twice—twice if $|H| = 4$.*

4. Connections to equitable coloring

A proper vertex coloring of a graph G is *equitable* if any two color classes differ in size by at most one. In 1970 Hajnal and Szemerédi proved:

Theorem 4.1 *([9]). Every graph G with $\Delta(G) + 1 \leq k$ has an equitable k-coloring.*

For a shorter proof of Theorem 4.1, see [17]; for an $O(k|G|^2)$-time algorithm see [16].

Motivated by Brooks' Theorem, it is natural to ask which graphs G with $\Delta(G) = k$ have equitable k-colorings. Certainly such graphs are k-colorable. Also, if k is odd then $K_{k,k}$ has no equitable k-coloring. Chen, Lih, and Wu [2] conjectured (in a different form) that these are the only obstructions to an equitable version of Brooks' Theorem:

Conjecture 4.2 *([2]). If G is a graph with $\chi(G), \Delta(G) \leq k$ and no equitable k-coloring then k is odd and $K_{k,k} \subseteq G$.*

In [2], Chen, Lih, and Wu proved Conjecture 4.2 holds for $k = 3$. By a simple trick, it suffices to prove the conjecture for graphs G with $|G| = ks$. Combining the results of the two papers [13] and [14], we have:

Theorem 4.3. *Suppose G is a graph with $|G| = ks$. If $\chi(G), \Delta(G) \leq k$ and G has no equitable k-coloring, then k is odd and $K_{k,k} \subseteq G$ or both $k \geq 5$ [13] and $s \geq 5$ [14].*

A graph G is k-*equitable* if $|G| = ks$, $\chi(G) \leq k$ and every proper k-coloring of G has s vertices in each color class. The following strengthening of Conjecture 4.2, if true, provides a characterization of graphs G with $\chi(G), \Delta(G) \leq k$ that have an equitable k-coloring.

Conjecture 4.4 *([12]). Every graph G with $\chi(G), \Delta(G) \leq k$ has no equitable k-coloring if and only if k is odd and $G = H + K_{k,k}$ for some k-equitable graph H.*

The next theorem collects results from [12]. Together with Theorem 4.3 it yields Corollary 4.6.

Theorem 4.5 *([12]). Conjecture 4.2 is equivalent to Conjecture 4.4. Indeed, for any k_0 and n_0, Conjecture 4.2 holds for $k \leq k_0$ and $|G| \leq n_0$ if and only if Conjecture 4.4 holds for $k \leq k_0$ and $|G| \leq n_0$.*

Corollary 4.6. *A graph G with $|G| = 3k$ and $\chi(G), \Delta(G) \leq k$ has no equitable k-coloring if and only if k is odd and $G = K_{k,k} + K_k$.*

We are now ready to complete our answer to Dirac's question for simple graphs.

Proof of Theorem 1.3. Assume $k \geq 2$ and $\delta(G) \geq 2k-1$. It is apparent that if any of (i), (H3), or (H4) in Theorem 1.3 fail, then G does not have k disjoint cycles. Now suppose G satisfies (i), (H3), and (H4). If $k = 2$ then $|G| \geq 6$ and $\delta(G) \geq 3$. Thus G has no subdivided edge, and only (d) of Lemma 3.2 is possible. By (i), $G \neq K_5$; by (H4), G is not a wheel; and by (H3), G is not type (3) of Theorem 3.1. Then G has 2 disjoint cycles. Finally, suppose $k \geq 3$. Since G satisfies (ii), we see $G \notin \{\mathbf{Y}_1, \mathbf{Y}_2\}$ and G satisfies (H2). If $|G| \geq 3k+1$, then G has k disjoint cycles by Theorem 1.7. Otherwise, $|G| = 3k$ and G has k disjoint cycles if and only if its vertices can be partitioned into disjoint K_3's. This is equivalent to $\overline{G}$ having an equitable k-coloring. By (ii), $\Delta(\overline{G}) \leq k$, and by (H3), $\omega(\overline{G}) \leq k$. Then by Brooks' Theorem, $\chi(\overline{G}) \leq k$. By (H4) and Corollary 4.6, $\overline{G}$ has an equitable k-coloring. □

Next we turn to Ore-type results on equitable coloring. To complement Theorem 1.7, we need a theorem that characterizes when a graph G with $|G| = 3k$ that satisfies (H2) and (H3) has k disjoint cycles, or equivalently, when its complement $\overline{G}$ has an equitable coloring. The complementary version of $\sigma_2(G)$ is the *maximum Ore-degree* $\theta(H) := \max_{xy \in E(H)}(d(x) + d(y))$. Then $\theta(\overline{G}) = 2|G| - \sigma_2(G) - 2$, and if $|G| = 3k$ and $\sigma_2(G) \geq 4k-3$ then $\theta(\overline{G}) \leq 2k+1$. Also, if G satisfies (H3) then $\omega(\overline{G}) \leq k$. This would correspond to an Ore–Brooks-type theorem on equitable coloring.

Several papers, including [10,11,19], address equitable colorings of graphs G with $\theta(G)$ bounded from above. For instance, the following is a natural Ore-type version of Theorem 4.1.

Theorem 4.7 *([10]). Every graph G with $\theta(G) \leq 2k-1$ has an equitable k-coloring.*

Even for proper (not necessarily equitable) coloring, an Ore–Brooks-type theorem requires forbidding some extra subgraphs when θ is 3 or 4. It was observed in [11] that for $k = 3, 4$ there are graphs for which $\theta(G) \leq 2k+1$ and $\omega(G) \leq k$, but $\chi(G) \geq k+1$. The following theorem was proved for $k \geq 6$ in [11] and then for $k \geq 5$ in [19].

Theorem 4.8. *Let $k \geq 5$. If $\omega(G) \leq k$ and $\theta(G) \leq 2k+1$, then $\chi(G) \leq k$.*

In the subsequent paper [15] we prove an analog of Theorem 1.7 for $3k$-vertex graphs.

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