# Improper Coloring of Sparse Graphs with a Given Girth, II: Constructions

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**Abstract:** A graph $G$ is $(j, k)$-*colorable* if $V(G)$ can be partitioned into two sets $V_j$ and $V_k$ so that the maximum degree of $G[V_j]$ is at most $j$ and of $G[V_k]$ is at most $k$. While the problem of verifying whether a graph is $(0, 0)$-colorable is easy, the similar problem with $(j, k)$ in place of $(0, 0)$ is NP-complete for all nonnegative $j$ and $k$ with $j + k \geq 1$. Let $F_{j,k}(g)$ denote

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the supremum of all $x$ such that for some constant $c_g$ every graph $G$ with girth $g$ and $|E(H)| \leq x|V(H)| + c_g$ for every $H \subseteq G$ is $(j, k)$-colorable. It was proved recently that $F_{0,1}(3) = 1.2$. In a companion paper, we find the exact value $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$. In this article, we show that increasing $g$ from 5 further on does not increase $F_{0,1}(g)$ much. Our constructions show that for every $g$, $F_{0,1}(g) \leq 1.25$. We also find exact values of $F_{j,k}(g)$ for all $g$ and all $k \geq 2j + 2$.  J. Graph Theory 81: 403–413, 2016

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## 1. INTRODUCTION

A *proper k-coloring* of a graph $G$ is a partition of $V(G)$ into $k$ independent sets $V_1, \dots, V_k$. A $(d_1, d_2, \dots, d_k)$*-coloring* of a graph $G$ is a partition of $V(G)$ into sets $V_1, V_2, \dots, V_k$ such that for every $1 \leq i \leq k$, the subgraph $G[V_i]$ of $G$ induced by $V_i$ has maximum degree at most $d_i$. If $d_1 = \cdots = d_k = 0$, then a $(d_1, d_2, \dots, d_k)$-coloring is simply a proper $k$-coloring. If at least one of the $d_i$ is positive, then a $(d_1, d_2, \dots, d_k)$-coloring is called *improper* or *defective*. Several papers on improper colorings of planar graphs with restrictions on girth and of sparse graphs have appeared.

In [10] and this article, we consider improper colorings with just two colors, the $(j, k)$-colorings. Even such colorings are not simple if $(j, k) \neq (0, 0)$. In particular, Esperet, Montassier, Ochem, and Pinlou [7] proved that the problem of verifying whether a given planar graph of girth 9 has a (0,1)-coloring is NP-complete. Since the problem is hard, it is natural to consider related extremal problems.

The maximum average degree, $\text{mad}(G)$, of a graph $G$ is the maximum of $\frac{2|E(H)|}{|V(H)|}$ over all subgraphs $H$ of $G$. It measures sparseness of $G$. Kurek and Ruciński [11] called graphs with low maximum average degree *globally sparse*. In particular,

$$\text{if } G \text{ is a planar graph of girth } g, \text{ then } \text{mad}(G) < \frac{2g}{g-2}. \tag{1}$$

We will use the following slight refinement of the notion of $\text{mad}(G)$. For $a, b \in \mathbf{R}$, a graph $G$ is $(a, b)$*-sparse* if $|E(H)| < a|V(H)| + b$ for all $H \subseteq G$. For example, every forest is $(1, 0)$-sparse, and every graph $G$ with $\text{mad}(G) < a$ is $(a/2, 0)$-sparse. We also say that $G$ is *almost* $(a, b)$*-sparse* if $|E(G)| = a|V(G)| + b$ and $|E(H)| < a|V(H)| + b$ for all $H \subsetneq G$. For example, every $k$-regular connected graph $G$ is almost $(k/2, 0)$-sparse. Note that every almost $(a, b)$-sparse graph is $(a, b')$-sparse for all $b' > b$. Almost $(a, b)$-sparse graphs could be considered as critical: they become $(a, b)$-sparse after deleting any edge.

Glebov and Zambalaeva [8] proved that every planar graph $G$ with girth at least 16 is $(0, 1)$-colorable. Then, Borodin and Ivanova [1] proved that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ is $(0, 1)$-colorable. By (1), this implies that every planar graph $G$ with girth at least 14 is $(0, 1)$-colorable. Borodin and Kostochka [2] proved that every graph $G$ with $\text{mad}(G) < \frac{12}{5}$ is $(0, 1)$-colorable, and this is sharp. This implies that every planar graph $G$ with girth at least 12 is $(0, 1)$-colorable. As mentioned above, Esperet et al. [7] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0, 1)$-coloring is NP-complete. Dorbec, Kaiser, Montassier, and Raspaud [5] mention that because of these results, the remaining open question is whether all planar graphs with girth 10 or 11 are $(0, 1)$-colorable. Our results in [10] yield the positive answer for planar graphs with girth 11.

In [10] and this article, instead of considering planar graphs with given girth, we consider graphs with given girth that are $(a, b)$-sparse for small $a$. A recent result by Borodin and Kostochka [3] can be stated in the language of $(a, b)$-sparse graphs as follows.

**Theorem 1.1** ([3]). *Let $k \geq 2j+2$ and $G$ be a graph. If $G$ is $(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1})$-sparse, then it is $(j, k)$-colorable. Moreover, the result is sharp in the sense that there are infinitely many almost $(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1})$-sparse graphs that are not $(j, k)$-colorable.*

Our first result gives triangle-free sharpness examples for Theorem 1.1.

**Theorem 1.2.** *Let $j \geq 0$ and $k \geq j+1$. Then there are infinitely many triangle-free almost $(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1})$-sparse graphs that are not $(j, k)$-colorable. Furthermore, for every $k \geq 1$, there are infinitely many almost $(2 - \frac{k+2}{2(k+1)}, \frac{1}{k+1})$-sparse graphs of girth 5 that are not $(0, k)$-colorable.*

When $k \geq 2j+2$, the graphs we construct in Theorem 1.2 are $(j, k)$-*critical* in the sense that each proper subgraph of every such graph is $(j, k)$-colorable by Theorem 1.1, but the graphs themselves are not.

Let $F_{j,k}(g)$ denote the supremum of all positive $a$ such that there is some (possibly negative) $b$ with the property that every $(a, b)$-sparse graph $G$ with girth $g$ is $(j, k)$-colorable. The above-mentioned result in [2] implies $F_{0,1}(3) = \frac{12}{5} = 1.2$. In [10], we prove the exact result that $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$ and also find the best possible value of $b$. In this article, we extend this result in two directions: to large girth and to $(j, k)$-colorings instead of (0, 1)-colorings.

Since $F_{0,0}(4)$ and $F_{0,1}(4)$ are already known, with Theorem 1.2 we have the values of $F_{0,k}(4)$ for all $k \geq 0$.

Our second result concerns graphs with large girth.

**Theorem 1.3.** *For all $k \geq j \geq 0$ and $g \geq 3$, $F_{j,k}(g) \leq 2 - \frac{(k+2)}{(j+2)(k+1)}$.*

So, we have $F_{0,1}(3) = 1.2$, $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9} = 1.222\ldots$, $F_{0,1}(g) \leq 1.25$ for all $g$, and if $k \geq 2j+2$ then $F_{j,k}(g) = 2 - \frac{(k+2)}{(j+2)(k+1)}$ for all $g$.

**Remark.** The case $j = k$ seems to be quite different. Apart from the trivial equality $F_{0,0}(g) = 1$, the only known to us exact result is $F_{1,1}(3) = \frac{7}{5}$ [4]. The value $\frac{7}{5}$ does not fit the formula in Theorem 1.1 and differs from the lower bound by Havet and Sereni in [9]. Even $F_{2,2}(3)$ is not known.

## 2. ON $(j, k)$-COLORING OF TRIANGLE-FREE GRAPHS

For a graph $G$ and $W \subseteq V(G)$, $0 \leq j \leq k$, let *the $(j, k)$-potential of $W$ in $G$* be defined as

$$\phi(W, G) = \phi_{j,k}(W, G) = \left(2 - \frac{k+2}{(j+2)(k+1)}\right)|W| - |E(G[W])|.$$

(We will drop the subscripts $j, k$ and $G$ if they are clear from the context.)

Note that for a graph $G$, the condition

$$\phi_{j,k}(W, G) > -\frac{1}{k+1} \text{ for all } W \subseteq V(G) \tag{2}$$

is equivalent to the statement that $G$ is $(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1})$-sparse.

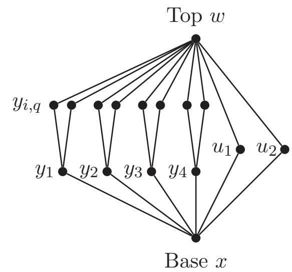


FIGURE 1. Graph $L(1, 3)$.

In this section, we prove Theorem 1.2, that is, we show that for all $k \geq j+1$, there are infinitely many triangle-free graphs $G$ with $\phi_{j,k}(W, G) \geq -\frac{1}{k+1}$ for all $W \subseteq V(G)$, but not $(j, k)$-colorable. We also show that for all $k \geq 2$, there are infinitely many graphs $G$ of girth 5 with $\phi_{0,k}(W, G) \geq -\frac{1}{k+1}$ for all $W \subseteq V(G)$, and not $(0, k)$-colorable. Together with Theorem 1.1, this means that for all $k \geq 2j+2$, $F_{j,k}(4) = F_{j,k}(3)$. Recall that this is not the case for $(j, k) = (0, 1)$ by our result in [10].

For $j \neq k$, we consider a $(j, k)$-coloring of a graph $G$ as a 2-coloring of $V(G)$ with color $j$ and color $k$ such that the vertices of color $j$ (respectively, $k$) induce a subgraph with maximum degree at most $j$ (respectively, $k$). We remark that this convention does not apply to the case $j = k$.

Let graph $L(j, k)$ be defined as follows. Let

$$V(L(j,k)) = \{x, w\} \cup \{u_1, \ldots, u_{j+1}\} \cup \bigcup_{i=1}^{k+1} \{y_{i,1}, \ldots, y_{i,j+1}, y_i\}.$$

Vertex $x$ is adjacent to all vertices in $\{u_1, \ldots, u_{j+1}\} \cup \{y_1, \ldots, y_{k+1}\}$, vertex $w$ is adjacent to all vertices in $\{u_1, \ldots, u_{j+1}\} \cup \bigcup_{i=1}^{k+1} \{y_{i,1}, \ldots, y_{i,j+1}\}$, for every $i \in [1, k+1]$, vertex $y_i$ is adjacent to all vertices in $\{y_{i,1}, \ldots, y_{i,j+1}\}$, and there are no other edges (see Fig. 1). We will call $x$ *the base* and $w$ *the top* of $L(j, k)$.

By construction, $L(j, k)$ is triangle-free and $L(0, k)$ has girth 5. We need the following simple property of $L(j, k)$.

**Claim 2.1.** *In every $(j, k)$-coloring $f$ of $L(j, k)$, $x$ has a neighbor of color $k$.*

***Proof.*** Suppose $f(y_1) = \cdots = f(y_{k+1}) = f(u_1) = \cdots = f(u_{j+1}) = j$. Then, for every $1 \leq i \leq k+1$ at least one of $y_{i,1}, \ldots, y_{i,j+1}$ must be colored with $k$. So, $w$ has at least $k+1$ neighbors of color $k$ and $j+1$ neighbors $u_1, \ldots, u_{j+1}$ of color $j$, a contradiction to the definition of $(j, k)$-coloring. ■

A $(j, k)$-*flag* in a graph $G$ is a pendant block isomorphic to $L(j, k)$ whose unique cut vertex is the base vertex $x$ in $L(j, k)$. Claim 2.1 immediately implies the following.

**Claim 2.2.** *In every $(j, k)$-coloring $f$ of a graph $G$, for any $x \in V(G)$,*

(a) *if $x$ is the base of $k+1$ distinct $(j, k)$-flags, then $f(x) = j$;*
(b) *if $x$ is the base of $k$ distinct $(j, k)$-flags and $f(x) = k$, then $x$ has no neighbors of color $k$ outside of these $k$ blocks.*

Another helpful property of $(j, k)$-flags is that they are sparse.

**Claim 2.3.** *Let graph $G$ consist of $q$ distinct $(j,k)$-flags, $W_1, W_2, \dots, W_q$, with a common base $x$, and for $i = 1, \dots, q$, let $w_i$ be the top of $W_i$.*

(a) *If $\emptyset \neq W \subseteq W_i$, then $\phi(W) \geq \phi(\{x\}) - \frac{1}{k+1}$, and equality holds only for $W = W_i$.*
(b) *If $\emptyset \neq W \subseteq V(G)$, then $\phi(W) \geq \phi(\{x\}) - \frac{q}{k+1}$ and equality holds only for $W = V(G)$.*

***Proof.*** To prove (a), choose among the nonempty subsets of $W_i$ a set $W$ of the smallest potential $\phi(W)$. Since deleting an isolated or pendant vertex from a set decreases the potential and the claim holds for a 1-element $W$, we may assume

$$\delta(G[W]) \geq 2. \tag{3}$$

If $\emptyset \neq W \subset W_i$ and $w_i \notin W$, then $W$ induces a forest, a contradiction to (3). So $w_i \in W$.

Since adding to a set $U$ of vertices, a vertex with at least two neighbors in $U$ decreases the potential by (3),

$$\begin{aligned}&\text{for all } 1 \leq h \leq j+1 \text{ and } 1 \leq h' \leq k+1,\ u_h \in W \text{if and only if } x \in W\\ &\text{and } y_{h',h} \in W \text{ if and only if } y_{h'} \in W.\end{aligned} \tag{4}$$

Suppose $x \notin W$. Then by (4), $W \cap \{u_1, \dots, u_{j+1}\} = \emptyset$. Also, if in this case $y_h \in W$ then by (4), all $y_{h,1}, \dots, y_{h,j+1}$ are in $W$ and

$$\begin{aligned}\phi(W) - \phi(W - \{y_{h,1}, \dots, y_{h,j+1}, y_h\}) &\geq \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(j+2) - (2j+2)\\ &= \frac{k}{k+1},\end{aligned}$$

a contradiction to the choice of $W$. Thus, $x \in W$. Then by (4), $\{u_1, \dots, u_{j+1}\} \subset W$. Also adding each $y_h$ together with $y_{h,1}, \dots, y_{h,j+1}$ decreases the potential by exactly $\frac{1}{k+1}$. So, the unique subset of $W_i$ with the minimum possible potential is $W_i$ itself and

$$\phi(W_i) - \phi(\{x\}) = \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(|W_i| - 1) - |E(G[W_i])|$$

$$= \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(j+2)(k+2) - ((2j+3)(k+2) - 1) = -\frac{1}{k+1},$$

as claimed. This proves $(a)$.

To prove (b), suppose that $W$ intersects exactly $r > 0$ of $W_1, \dots, W_q$. If $x \notin W$, then

$$\phi(W) = \sum_{i=1}^{q} \phi(W \cap W_i) > r\left(\phi(\{x\}) - \frac{1}{k+1}\right) \geq \phi(\{x\}) - \frac{r}{k+1}.$$

If $x \in W$, then $r = q$ and

$$\phi(W) = \sum_{i=1}^{q} \phi(W \cap W_i) - (q-1)\phi(\{x\}) \geq \phi(\{x\}) - \frac{q}{k+1}. \tag{5}$$

By (a), equality in (5) holds only when $W \cap W_i = W_i$ for all $i$, which means $W = V(G)$. ■

**Basic construction.** We construct a graph $H_0 = H_0(j,k)$ from a star $K_{1,j+1}$ with the center $x_0$ and leaves $x_1, \dots, x_{j+1}$ by adding $k+1$ $(j,k)$-flags to each of $x_0, x_1, \dots, x_{j+1}$.

(When we say "add $(j, k)$-flags to a vertex $x$," we mean that $x$ will be the base of the added flags.)

By construction, $H_0(j, k)$ is triangle-free and $H_0(0, k)$ has girth 5. If $H_0$ has a $(j, k)$-coloring $f$, then by Claim 2.2(a), $f(x_0) = \ldots = f(x_{j+1}) = j$, and vertex $x_0$ of color $j$ has $j + 1$ neighbors $x_1, \cdots, x_{j+1}$ of color $j$, a contradiction. Thus,

$$H_0 \text{ is not } (j, k) - \text{colorable.} \tag{6}$$

Now we want to prove that $H_0$ satisfies (2).

**Claim 2.4.** *If $W \subseteq V(H_0)$, then $\phi(W) \geq -\frac{1}{k+1}$, and equality holds only for $W = V(H_0)$.*

***Proof.*** Choose a largest $W \subset V(H_0)$ among the sets with minimum $\phi(W)$. As in the proof of Claim 2.3, $\delta(H_0[W]) \geq 2$. By Claim 2.3(a), if $L$ is any $(j, k)$-flag in $H_0$ and $W \cap L \neq \emptyset$, then $L \subseteq W$ otherwise $\phi(W \cup L) < \phi(W)$.

It follows that if we know which vertices in $X = \{x_0, \ldots, x_{j+1}\}$ are in $W$, then we know $W$. Similarly, if $x_0 \in W$ and $x_i \notin W$ for some $i$, then by Claim 2.3(b), adding to $W$ vertex $x_i$ and all the $k + 1$ $(j, k)$-flags containing $x_i$ we get a set $W'$ with

$$\phi(W') \leq \phi(W) + \phi(\{x_i\}) - \frac{k+1}{k+1} - 1 < \phi(W),$$

a contradiction to the minimality of $\phi(W)$. So, $W = V(H_0)$ is the unique set of minimum potential among the sets containing $x_0$.

If $x_0 \notin W$, then every component of $H_0[W]$ is a subgraph of a graph $G$ described in Claim 2.3 and so has a nonnegative potential. So, in this case $\phi(W) \geq 0$. ■

Thus, $H_0$ is the first in the series of examples proving Theorem 1.2.

In order to generalize $H_0$, we need one more notion. A vertex $v$ in a graph $G$ is a *remote $(j, k)$-base* if it is the base of $k + 1$ $(j, k)$-flags $W_1, \ldots, W_{k+1}$ in $G$ and has exactly one neighbor outside of $W_1 \cup \ldots \cup W_{k+1}$. This unique neighbor of $v$ will be called *the main neighbor* of $v$.

**Claim 2.5.** *Suppose a graph $H$ has no $(j, k)$-colorings, and $v \in V(H)$ is a remote $(j, k)$-base contained in $(j, k)$-flags $W_1, \ldots, W_{k+1}$ with the main neighbor $x$.*

(a) *For any $(j, k)$-coloring $f'$ of $H' = H - (W_1 - v)$ (if it exists), $f'(v) = k$ and $v$ has $k$ neighbors of color $k$ in $H'$.*
(b) *For any $(j, k)$-coloring $f''$ of $H'' = H - \bigcup_{i=1}^{k+1} W_i$ (if it exists), $f''(x) = j$ and $x$ has $j$ neighbors of color $j$ in $H''$.*

***Proof.*** If $H'$ has a $(j, k)$-coloring $f'$ with $f'(v) = j$, then $f'$ can be extended to $W_1$ by coloring all neighbors of $v$ in $W_1$ and the top vertex of $W_1$ with $k$ and the remaining vertices with $j$. But $H$ has no $(j, k)$-colorings. Thus, if a $(j, k)$-coloring $f'$ of $H'$ exists, then $f'(v) = k$, and by Claim 2.1 each of $W_2, \ldots, W_{k+1}$ contains a neighbor of $v$ of color $k$. This proves (a).

Similarly, if $H''$ has a $(j, k)$-coloring $f''$ with either $f''(x) = k$ or with $f''(x) = j$ and at most $j - 1$ neighbors of color $j$, then we can extend $f''$ to the whole $H$ by letting $f''(v) = j$, coloring all its neighbors in $W_1 \cup \ldots \cup W_{k+1}$ and the tops of $W_1, \ldots, W_{k+1}$ with $k$, and the remaining vertices in $W_1 \cup \ldots \cup W_{k+1}$ with $j$. ■

**General construction.** Recall that $H = H_0$ has the following properties:

(P1) $H$ is not $(j, k)$-colorable;
(P2) $H$ has no triangles and if $j = 0$, then $H$ has girth 5;
(P3) $\phi(W) \geq -\frac{1}{k+1}$ for each $W \subseteq V(H)$, and equality holds only for $W = V(H)$;
(P4) $H$ has at least two remote bases (if $j = 0$, then $x_0$ also is a remote base in $H_0(0, k)$).

We now show how to use a graph $H$ satisfying (P1)–(P4) to construct a larger graph satisfying (P1)–(P4). Take two copies, $H_1$ and $H_2$ of $H$. For $h = 1, 2$, choose in $H_h$ a remote base $v_h$ contained in $(j, k)$-flags $W_{h,1}, \ldots, W_{h,k+1}$ with the main neighbor $x_h$. Let $H' = H_1 - (W_{1,1} - v_1)$ and $H'' = H_2 - \bigcup_{i=1}^{k+1} W_{2,i}$. We get the new graph $\widetilde{H}$ by adding to $H' \cup H''$ a new vertex $z$ adjacent to $v_1$ in $V(H')$ and to $x_2$ in $V(H'')$.

Property (P2) for $\widetilde{H}$ directly follows from (P2) for $H_1$ and $H_2$. Since $H_1 \cup H_2$ had at least four remote bases and we destroyed only two of them when creating $H'$ and $H''$, (P4) holds for $\widetilde{H}$.

Suppose $\widetilde{H}$ has a $(j, k)$-coloring $f$. Then by Claim 2.5(a), $f(v_1) = k$ and $v_1$ has $k$ neighbors of color $k$ in $V(H')$. Thus, we need $f(z) = j$. But by Claim 2.5(b), $f(x_2) = j$ and $x_2$ has $j$ neighbors of color $j$ in $V(H'')$. This contradiction proves (P1) for $\widetilde{H}$.

To prove (P3), consider a set $W$ of minimum potential in $\widetilde{H}$. If $z \notin W$, then by (P3) for $H$, $\phi(W) = \phi(W \cap V(H')) + \phi(W \cap V(H'')) \geq 0 + 0 = 0$ since each of $W \cap V(H')$ and $W \cap V(H'')$ is proper subset of $V(H')$ and $V(H'')$, respectively. Suppose $z \in W$. Then, similarly to (3), $v_1, x_2 \in W$. Let $W' = W \cap V(H')$ and $W'' = W \cap V(H'')$. Since adding to $W''$ vertex $v_2$ together with all $k+1$ $(j, k)$-flags containing $v_2$ would decrease the potential of $W''$ by $\frac{k+2}{(j+2)(k+1)}$, we conclude that $\phi(W'') \geq \frac{k+2}{(j+2)(k+1)} - \frac{1}{k+1}$ with equality only when $W'' = V(H'')$. Similarly, $\phi(W') \geq 0$ with equality only when $W' = V(H')$. Thus,

$$\phi(W) \geq \phi(W') + \phi(W'') + \phi(\{z\}) - 2 \geq 0 + \frac{k+2}{(j+2)(k+1)} - \frac{1}{k+1}$$
$$+(2 - \frac{k+2}{(j+2)(k+1)}) - 2 \geq \frac{-1}{k+1},$$

with equality only when $W = V(\widetilde{H})$.

This construction yields Theorem 1.2.

## 3. ON $(j, k)$-COLORING OF GRAPHS WITH LARGE GIRTH

In this section, we prove Theorem 1.3. First, we inductively define the tree $T'_d(j, k)$ that will be a gadget to construct graphs we want. For $i = 0, 1, \ldots, k$, let $S_i$ be a copy of the star $K_{1,j+1}$ with the center $c_i$. We subdivide each of the $j+1$ edges of each star $S_i$ once and add edges $c_0c_i$ for $i = 1, 2, 3, \ldots, k$. The resulting tree is $T_1(j, k)$ and $c_0$ is called *the center of* $T_1(j, k)$. Note that $T_1(j, k)$ has $(k+1)(j+1)$ leaves. Assume we already have defined the tree $T_{d-1}(j, k)$ and it has $(k+1)^{d-1}(j+1)^{d-1}$ leaves. Let $T^0$ be a copy of $T_1(j, k)$ with the center $c_0$ and $T^1, \ldots, T^{(k+1)(j+1)}$ be disjoint copies of $T_{d-1}(j, k)$ with the centers $c_1, \ldots, c_{(k+1)(j+1)}$. Let $x_1, \ldots, x_{(k+1)(j+1)}$ be the leaves of $T^0$. The tree $T_d(j, k)$ with *the center* $c_0$ is obtained by gluing $c_i$ with $x_i$ for all $i = 1, \ldots, (k+1)(j+1)$. Finally, the tree $T'_d(j, k)$ is obtained from two disjoint copies of $T_d(j, k)$ by adding an edge connecting their centers. The example of $T'_1(2, 3)$ is in Figure 2.

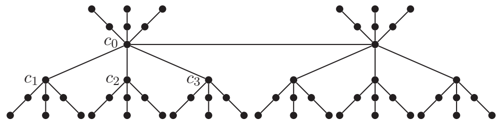


FIGURE 2. $T_1'(2, 3)$.

**Claim 3.1.** *For $d \geq 1$, let $f$ be a $(j, k)$-coloring of $T_d(j, k)$ with the center $c_0$ such that every neighbor of a leaf has color $j$. Then, $f(c_0) = k$ and $c_0$ has $k$ neighbors of color $k$.*

***Proof.*** We use induction on $d$.

Let $L$ be the set of all leaves of $T_1(j, k)$. If all the neighbors of $L$ are colored with the color $j$, then each of the remaining nonleaf vertices is adjacent to $j + 1$ vertices of color $j$, and thus has color $k$. These vertices form a star $K_{1,k}$ with the center $c_0$, which yields the claim for $d = 1$.

Assume the statement holds for $d - 1$. Let $T^0, T^1, \dots, T^{(k+1)(j+1)}$ be the trees from the definition of $T_d(j, k)$ and $c_0, c_1, \dots, c_{(k+1)(j+1)}$ be their centers. Let $f$ be a $(j, k)$-coloring of $T_d(j, k)$ such that every neighbor of a leaf has color $j$. By the induction assumption, for each $i = 1, \dots, (k + 1)(j + 1)$, $f(c_i) = k$, and $c_i$ has $k$ neighbors of color $k$ in $T^i$. It follows that the neighbor of $c_i$ in $T^0$ has color $j$. Again by the induction assumption, the conclusion holds for $c_0$. ■

**Claim 3.2.** *For $k \geq j$ and $d \geq 1$, in every $(j, k)$-coloring of $T_d'(j, k)$, some neighbor of a leaf has color $k$.*

***Proof.*** Tree $T_d'(j, k)$ contains two disjoint copies $T_1$ and $T_2$ of $T_d(j, k)$ with centers $c_1, c_2$ connected by edge $c_1c_2$. If $f$ is a $(j, k)$-coloring of $T_d'(j, k)$ such that every neighbor of a leaf has color $j$, then by the Claim 3.1, for $i = 1, 2$, the center $c_i$ of $T^i$ has color $k$ and has $k$ neighbors of color $k$ in $T^i$. Since $c_1$ and $c_2$ are adjacent, each of them has $k + 1$ neighbors of the color $k$, a contradiction. ■

**Claim 3.3.** *Let $k \geq j$. Let $L$ be the set of leaves in $T_d(j, k)$ and $B = V(T_d(j, k)) - L$. Then for every subgraph $T$ of $T_d(j, k)$,*

$$|E(T)| \leq \left(2 - \frac{(k+2)}{(j+2)(k+1)}\right) |B \cap V(T)|. \tag{7}$$

***Proof.*** First, suppose that $d = 1$. Recall that in this case, $B = C \cup D$, where $D$ is the set of vertices of degree 2 adjacent to $L$, $|D| = |L| = (j + 1)(k + 1)$, $C = \{c_1, \dots, c_{k+1}\}$ is the set of centers of the original stars, each $c_i$ is adjacent to $j + 1$ vertices in $D$, and in addition $c_1$ is adjacent to each vertex in $C - c_1$. Thus, there are three types of edges: Type 1—the edges connecting $D$ with $L$, Type 2—the edges connecting $D$ with $C$, and Type 3—the edges connecting $c_1$ with $C - c_1$. We will prove (7) using discharging. Let every $e \in E(T)$ have charge $ch(e) = 1$ so that $\sum_{e \in E(T)} ch(e) = |E(T)|$. Now each $e \in E(T)$ distributes its charge to its endvertices according to the following rules.

*Rule 1*: Each edge $d\ell$ of Type 1 gives all its charge to the end $d \in D$.

*Rule 2*: Each edge $c_id$ of Type 2 gives charge $1 - \frac{(k+2)}{(j+2)(k+1)}$ to the end $d \in D$ and charge $\frac{(k+2)}{(j+2)(k+1)}$ to the end $c_i \in C$.

*Rule 3*: Each edge $c_1c_i$ of Type 3 gives charge $\frac{k}{k+1}$ to $c_i \in C - c_1$ and charge $\frac{1}{k+1}$ to $c_1$.

By the rules, only vertices of $V(T) \cap B$ may receive a positive charge and total charge on them will be exactly $|E(T)|$. Thus, it is enough to prove that for every $v \in V(T) \cap B$,

$$ch(v) \le 2 - \frac{(k+2)}{(j+2)(k+1)}. \tag{8}$$

If $v \in D$, then it gets at most 1 by Rule 1 and at most $1 - \frac{(k+2)}{(j+2)(k+1)}$ by Rule 2, so (8) holds for $v$. If $v = c_i$ for some $2 \le i \le k+1$, then it gets at most $(j+1)\frac{(k+2)}{(j+2)(k+1)}$ by Rule 2 and at most $\frac{k}{k+1}$ by Rule 3, so

$$ch(v) \le (j+1)\frac{(k+2)}{(j+2)(k+1)} + \frac{k}{k+1} = 2 - \frac{(k+2)}{(j+2)(k+1)}.$$

Finally, if $v = c_1$, then it again gets at most $(j+1)\frac{(k+2)}{(j+2)(k+1)}$ by Rule 2 and at most $k\frac{1}{k+1}$ by Rule 3, so again (8) holds for $v$. This proves Case $d = 1$.

Suppose now that $d \ge 2$. Then, $T_d(j,k)$ is obtained from several copies of $T_1(j,k)$ by gluing leaves of some copies with the centers of some others. So, if we do the discharging from $E(T)$ to $V(T) \cap B$ in each copy of $T_1(j,k)$ forming $T_d(j,k)$ by the Rules 1–3 above, then again only vertices of $V(T) \cap B$ may receive a positive charge and the total charge on them will be exactly $|E(T)|$. Moreover, since by Rule 1 the leaves of each copy of $T_1(j,k)$ will get zero charge from this copy, as we have checked above, (8) will hold for every $v \in V(T) \cap B$. This proves the claim.

***Proof of Theorem 1.3.*** Our goal is to show that for any $\epsilon > 0$, $g \ge 3$ and $k \ge j \ge 0$,

$$\text{there is a } \left(2 - \frac{(k+2)}{(j+2)(k+1)} + \epsilon, 0\right) - \text{ sparse non} - (j,k)$$
$$- \text{colorable graph } G \text{ of girth } g. \tag{9}$$

Recall that $G$ is $(2 - \frac{(k+2)}{(j+2)(k+1)} + \epsilon, 0)$-sparse if and only if $\text{mad}(G) < 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$. We use induction on $j+k$. If $j = k = 0$, then any odd cycle of length at least $g$ is almost $(1,0)$-sparse and not $(0,0)$-colorable. Assume that $k \ge 1$ and (9) is proved for all pairs $(j',k')$ with $j'+k' < j+k$ and $j' \le k'$.

CASE 1: $j < k$. Then there is a graph $G_0$ with girth $g$, which is not $(j, k-1)$-colorable and with

$$\text{mad}(G_0) < 4 - \frac{2(k+1)}{(j+2)k} + 2\epsilon \le 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{10}$$

Let $V(G_0) = \{v_1, v_2, \ldots, v_n\}$. Fix an integer $d > \frac{1}{\epsilon}$. Let $M$ be the number of leaves in $T'_d(j,k)$. By an old result of Erdős and Hajnal [6], there exists a non-$n$-colorable $nM$-uniform hypergraph $H$ with girth $g$. We construct our graph $G$ using $H$ and many copies of $G_0$ and $T'_d(j,k)$ as follows:

(i) Partition each $e \in E(H)$ into $n$ subsets $e_1, \ldots, e_n$ of size $M$;
(ii) Replace each vertex $x$ in $H$ with a copy $G_0(x)$ of $G_0$;
(iii) For each $e \in H$ and $1 \le i \le n$, if $e_i = \{x_1, \ldots, x_M\}$, we take a copy $T(e,i)$ of $T'_d(j,k)$ with the set of leaves, say, $L(e,i) = \{\ell_1, \ldots, \ell_M\}$ and for $h = 1, \ldots, M$, glue $\ell_h$ with the vertex $v_i$ in the copy $G_0(x_h)$ of $G_0$. We will say that $T(e,1), \ldots, T(e,n)$ *belong to* $e$ and denote $B(e,i) = V(T(e,i)) - L(e,i)$.

Let us check that the obtained graph $G$ has the properties we need: (a) the girth of $G$ is at least $g$, (b) $G$ is not $(j,k)$-colorable, and (c) $\mathrm{mad}(G) < 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$.

For an edge $e \in E(H)$, let $G(e)$ denote the subgraph of $G$ formed by the copies $G_0(x)$ of $G_0$ for all $nM$ vertices $x \in e$ plus all the copies $T(e,i)$ of $T'_d(j,k)$ for $i = 1,\ldots,n$. If $G$ has a cycle $C$ of length less than $g$, then $C$ is not contained in a copy of $G_0$ since $G_0$ has girth $g$. Moreover, then $C$ is not contained in any $G(e)$, since all edges of $G(e)$ in $\bigcup_{i=1}^{n} T(e,i)$ are cut-edges in $G(e)$. Since $H$ is a linear hypergraph, $C$ yields a (hypergraph) cycle in $H$, and any such cycle has at least $g$ edges, a contradiction to the choice of $C$. This proves (a).

Suppose we have a $(j,k)$-coloring $f$ of $G$. Since $G_0$ is not $(j,k-1)$-colorable, each graph $G_0(x)$ has a vertex $v_i$ of color $k$ with $k$ neighbors in $G_0(x)$ of color $k$ in $f$. Let $i(x)$ be the minimum $i$ such that $G_0(x)$ has a vertex $v_i$ of color $k$ with $k$ neighbors in $G_0(x)$ of color $k$ in $f$. We define a coloring $\phi$ of $H$ as follows: for each $x \in V(H)$, let $\phi(x) = i(x)$. Then $\phi$ is an $n$-coloring of $H$, and $H$ has no proper $n$-colorings. Thus, there is a monochromatic $e \in E(H)$. Suppose $f(x) = i$ for each $x \in e$. By construction, all the leaves of the copy $T(e,i)$ of $T'_d(j,k)$ are in $e_i$; each of these leaves is of color $k$ and has $k$ neighbors of color $k$ in $\bigcup_{x\in e_i} G_0(x)$. Thus, none of these leaves has a neighbor of color $k$ in $T(e,i)$. This contradicts Claim 3.2. Thus, (b) holds.

In order to prove (c), consider some $W \subseteq V(G)$ with the largest $\frac{|E(G[W])|}{|W|}$. If this ratio is at most 1, then (c) holds; otherwise by the maximality of the average degree, $G[W]$ has no isolated vertices and no leaves. Let $W' = \bigcup_{x\in V(H)}(W \cap V(G_0(x)))$. Then, $W - W' = \bigcup_{e\in E(H)} \bigcup_{i=1}^{n} (W \cap B(e,i))$. Since each component of $G[W']$ is contained in some $G_0(x)$, by (10), the average degree of $G[W']$ is less than $4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$. We can obtain $W$ from $W'$ by a sequence of adding the sets $W \cap B(e,i)$, one by one. We will show that after every such step,

$$\text{the average degree of the obtained subgraph remains less than } 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{11}$$

Indeed, suppose it is the turn to add to a current set $W''$ the set $W \cap B(e,i)$. Let $c_1c'_1$ be the edge in $T(e,i)$ connecting the centers $c_1$ and $c'_1$ of the two disjoint copies of $T_d(j,k)$. If $\{c_1, c'_1\} \not\subset W$, then by Claim 3.3, adding $W \cap B(e,i)$ to $W''$ adds at most $(2 - \frac{(k+2)}{(j+2)(k+1)})|W \cap B(e,i)|$ edges, as claimed. So let $\{c_1, c'_1\} \subset W$. Since $G[W]$ has no leaves, $W$ contains the vertices of disjoint paths from $c_1$ and $c'_1$ to $L(e,i)$ and thus $|W \cap B(e,i)| \geq 6d$. Again by Claim 3.3, adding $W \cap B(e,i)$ to $W''$ adds at most $1 + (2 - \frac{(k+2)}{(j+2)(k+1)})|W \cap B(e,i)|$ edges. Since $d > 1/\epsilon$ and $|W \cap B(e,i)| \geq 6d$, the last expression is less than $(2 - \frac{(k+2)}{(j+2)(k+1)} + \epsilon)|W \cap B(e,i)|$, as claimed. This proves (c).

CASE 2: $0 < j = k$. Then there is a graph $G_0$ with girth $g$, which is not $(k-1,k)$-colorable and with

$$\mathrm{mad}(G_0) < 4 - \frac{2(k+2)}{(k+1)^2} + 2\epsilon \leq 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{12}$$

Now, we simply repeat the proof of Case 1 with the only twist that using $j = k$, we consider $G_0$ as not $(k,k-1)$-colorable instead of not $(k-1,k)$-colorable. ■

**Concluding remark.** Studying improper colorings with more colors, one can consider the function $F_{a_1,a_2,\ldots,a_t}(g)$ generalizing $F_{j,k}(g)$. Using similar techniques, we can prove the following extension of Theorem 1.3.

**Theorem 3.4.** *Let* $a_1 \le a_2 \le \cdots \le a_t$*,* $t \ge 2$ *and* $g \ge 3$*. Then,* $F_{a_1,a_2,\ldots,a_t}(g) \le t - \frac{(a_2+2)}{(a_1+2)(a_2+1)}$.

Since we do not know how sharp is this bound, we do not supply a proof of Theorem 3.4.

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