Tight Descriptions of 3-Paths in Normal Plane Maps

Dedicated to Andre Raspaud on the occasion of his 70th birthday.

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Abstract: We prove that every normal plane map (NPM) has a path on three vertices (3-path) whose degree sequence is bounded from above by one of the following triplets: $(3, 3, \infty)$, (3,15,3), (3,10,4), (3,8,5), (4,7,4), (5,5,7), (6,5,6), (3,4,11), (4,4,9), and (6,4,7). This description is tight in the sense that no its parameter can be improved and no term dropped. We also pose a problem of describing all tight descriptions of 3-paths in NPMs and make a modest contribution to it by showing that there exist precisely

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three one-term descriptions: $(5, \infty, 6)$, $(5, 10, \infty)$, and $(10, 5, \infty)$. J. Graph Theory 85: 115–132, 2017

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1. INTRODUCTION

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least 3. Let δ be the minimum vertex degree, and w_k be the minimum degree sum of a path on k vertices in an NPM or a graph. The degree of a vertex or face x, that is the number of edges incident with x (loops and cut-edges are counted twice), is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc. An edge uv is an *(i, j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type (i, j, k)* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$. A path uvw is an *off-(i, j, k)-path* if $d(u) \geq i$, $d(v) \geq j$, and $d(w) \geq k$.

Already in 1904, Wernicke [23] proved that every NPM M_5 with $\delta(M_5) = 5$ contains a 5-vertex adjacent to a 6^--vertex, and Franklin [13] strengthened this to the existence of at least two 6^--neighbors, which implies that M_5 satisfies $w_3 \leq 17$. Franklin's bound 17 is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

It follows from Lebesgue's results in [21] that each NPM has an edge $e = uw$ of *weight* $w(e) = d(u) + d(w)$ at most 14 (more specifically, a (3, 11)-, or (4, 7)-, or (5, 6)-edge, where bounds 7 and 6 are sharp). For 3-connected plane graphs, Kotzig [20] proved a precise result: $w_2 \leq 13$.

Note that $\delta(K_{2,t}) = 2$ and $w_2(K_{2,t}) = t + 2$, so w_2 is unbounded if $\delta \leq 2$. In 1972, Erdős (see [14]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all planar graphs with $\delta \geq 3$. Barnette (see [14]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [3]. More precisely, Borodin [4–6] proved that every NPM contains a (3, 10)-, or (4, 7)-, or (5, 6)-edge (as easy corollaries of some stronger structural facts with applications to coloring). More results on the structure of edges and some related results on minor faces and stars centered at minor vertices in NPMs can be found in [8-10, 17].

Now we turn to discussing the structure of 3-paths in NPMs.

Theorem 1.1 (Ando et al. [2]). *Every 3-polytope satisfies $w_3 \leq 21$, which is sharp.*

The sharpness of the bound $w_3 \leq 21$ in Theorem 1.1 is witnessed by the Jendrol' construction [16] (see Figure 2).

Jendrol' [15] proves that each 3-polytope has a 3-path uvw such that $\max\{d(u), d(v), d(w)\} \leq 15$ (the bound is precise). Jendrol' [16] further shows that such a path must belong to one of 10 types, in which $d(u) + d(v) + d(w)$ varies from 23 to 16.

Theorem 1.2 (Jendrol' [16]). *Every 3-polytope has a 3-path of one of the following types: (10,3,10), (7,4,7), (6,5,6), (3,4,15), (3,6,11), (3,8,5), (3,10,3), (4,4,11), (4,5,7), or (4,7,5).*

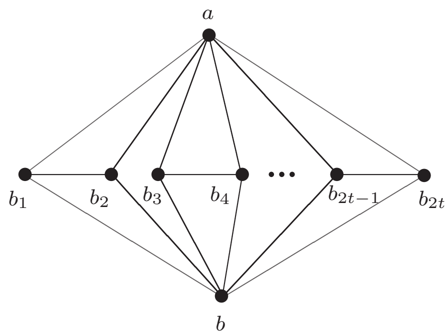

FIGURE 1. The graph $K^*_{2,2t}$ with only off-$(3, 3, \infty)$-paths and off-$(3, \infty, 3)$-paths.

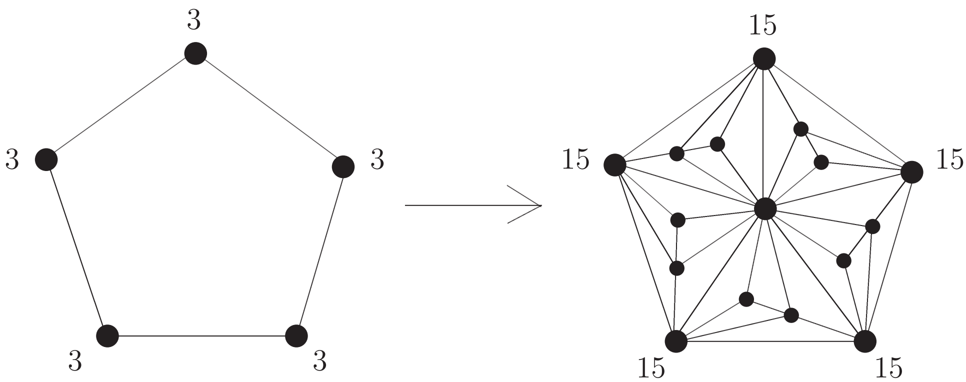

FIGURE 2. The Jendrol' construction [15] with only off-(3, 15, 3)-paths confirming the tightness of (Ta).

Note that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to famous Steinitz's theorem [22]. The requirement of 3-connectedness is essential for the finiteness of w_3, as shown by the construction $K^*_{2,2t}$ in Figure 1.

Borodin [7] showed that only the presence in an NPM of a $(K_4 - e)$-like configuration $K^*_{2,4}$, described as "two adjacent 3-vertices lying in two common 3-faces," is responsible for the unboundedness of w_3 in NPMs. The following refinement of Theorem 1.1 holds.

Theorem 1.3 (Borodin [7]). *Every NPM without a $K^*_{2,4}$ has*

(i) either $w_3 \leq 18$ or a vertex of degree ≤ 15 adjacent to two 3-vertices, and
(ii) either $w_3 \leq 17$ or $w_2 \leq 7$.

As mentioned above, the bounds $w_3 \leq 21$ and $w_3 \leq 17$ are tight. It had been open whether the bound $w_3 \leq 18$ in Theorem 1.3 is sharp or not; its sharpness was recently confirmed in Borodin et al. [11] (see Figure 7).

In particular, Ando et al.'s [2] precise bound $w_3 \leq 21$ is valid for all NPMs in which no two 3-vertices are adjacent.

Corollary 1.4 ([7]). *Every NPM with $w_2 > 6$ has $w_3 \leq 21$.*

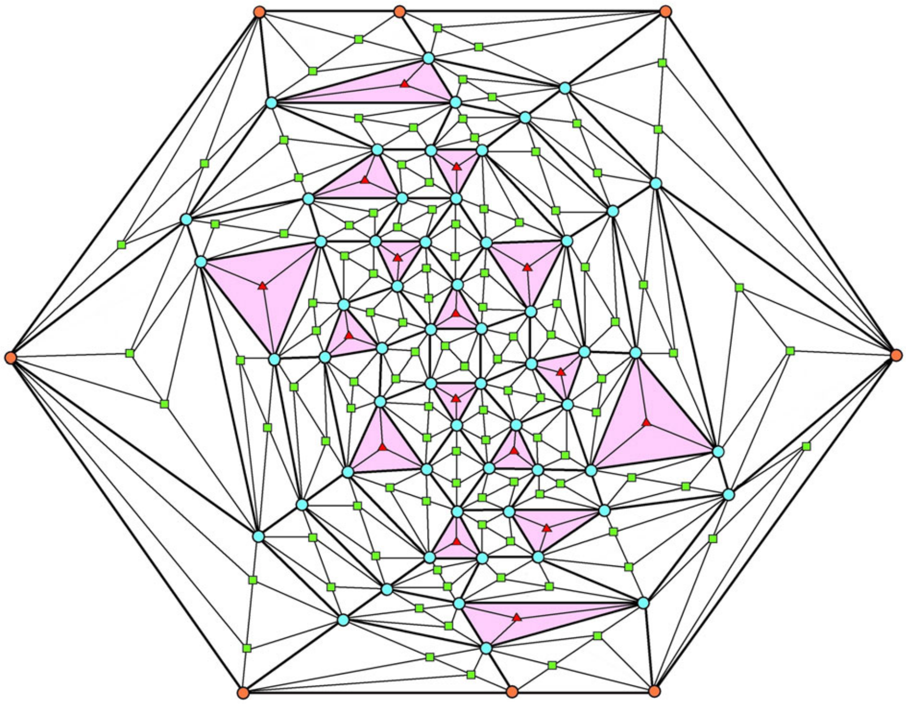

FIGURE 3. A half of the off-(3, 10, 4)-triangulation proving the tightness of (Tb) (Borodin et al. [11]).

Theorem 1.3 immediately implies that Franklin's precise bound $w_3 \leq 17$ is valid for all NPMs with $\delta \geq 4$.

Corollary 1.5 ([7]). *Every NPM without* 3*-vertices has* $w_3 \leq 17$.

The upper bound in the following statement is also immediate.

Corollary 1.6 ([7]). *Every* 3*-polytope with* $\delta \geq 4$ *has a path uvw such that* $max\{d(u), d(v), d(w)\} \leq 9$.

The bound 9 in Corollary 1.6 is sharp, as follows from Figure 5 (Th) .

A description of 3-paths is *tight* if no its parameter can be strengthened and no term dropped.

It is not known whether Theorem 1.2 is tight. Recently, Borodin et al. [11] gave the first provably tight description of 3-paths in arbitrary NPMs known to us.

Theorem 1.7 (Borodin et al. [11]). *Every NPM without* $K^*_{2,4}$*s has a* 3*-path of one of the following types:* (3, 4, 11)*,* (3, 7, 5)*,* (3, 10, 4)*,* (3, 15, 3)*,* (4, 4, 9)*,* (6, 4, 8)*,* (7, 4, 7)*,* (6, 5, 6)*, which description is tight.*

We note that Theorem 1.7 extends Franklin's Theorem and Theorem 1.1, but does not imply Theorem 1.3.

One of the purposes of this article is to prove the following tight refinement of Theorem 1.3.

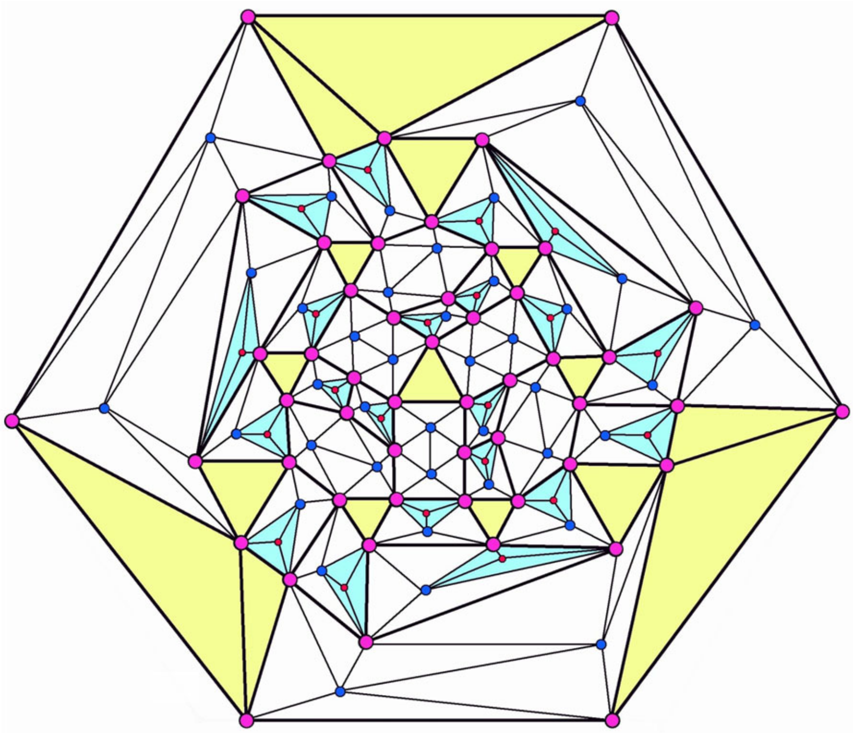

FIGURE 4. A half of the triangulation with only off-(3, 8, 5)-paths proving the tightness of (Tc).

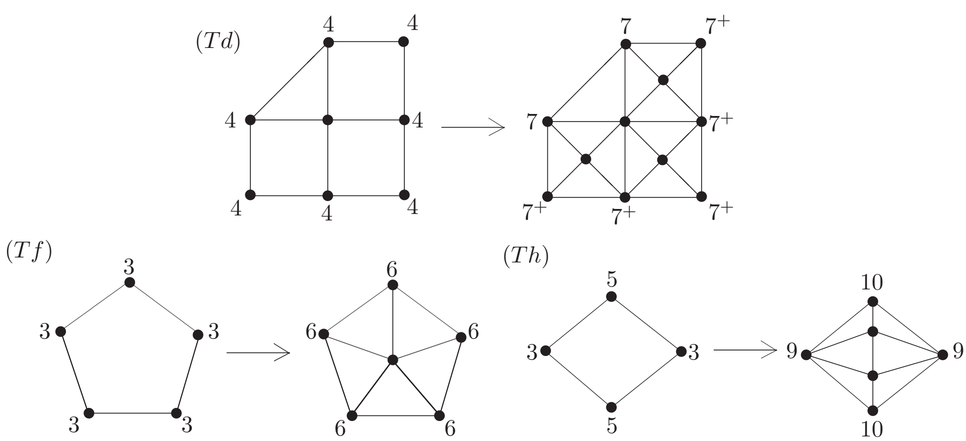

FIGURE 5. Constructions pertaining to (Td), (Tf), and (Th).

Theorem 1.8. *Every NPM without two adjacent* 3*-vertices lying in two common* 3*-faces has a* 3*-path of one of the following types:*

(Ta) (3,15,3),
(Tb) (3,10,4),

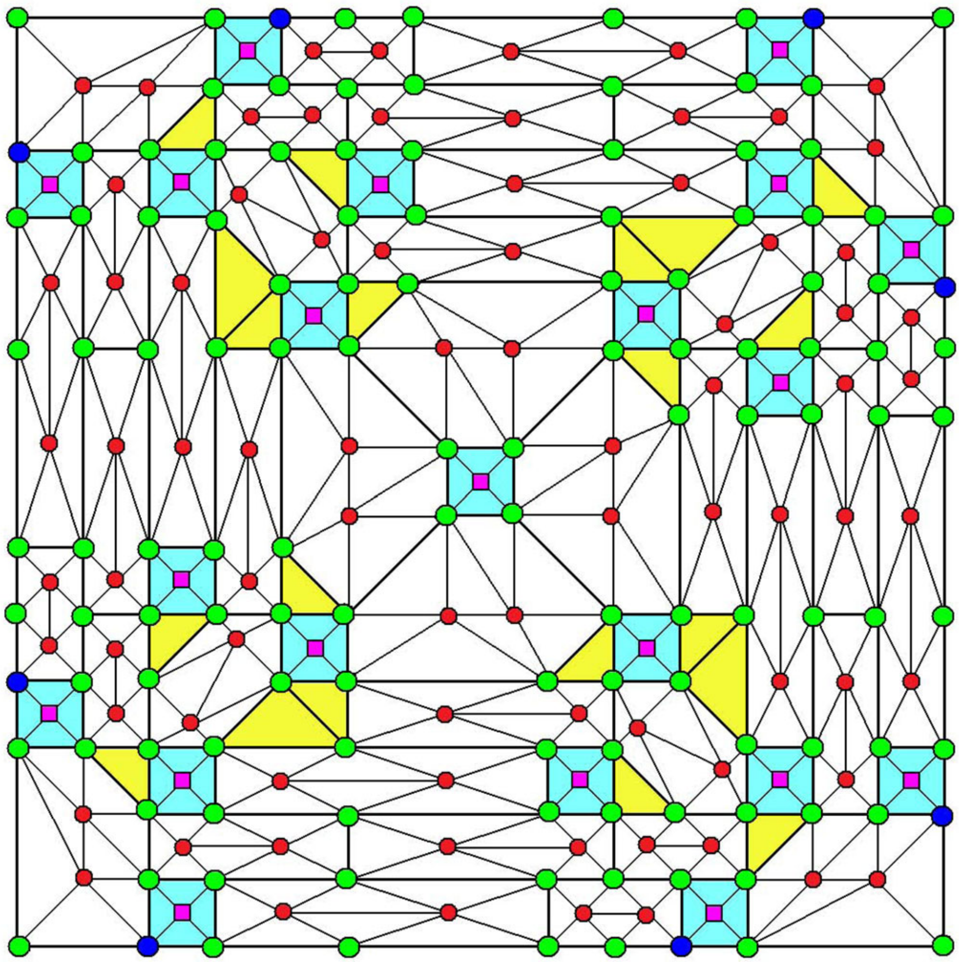

FIGURE 6. A half of the triangulation with only off-(5, 5, 7)-paths proving the tightness of (Te).

(Tc) (3,8,5),
(Td) (4,7,4),
(Te) (5,5,7),
(Tf) (6,5,6),
(Tg) (3,4,11),
(Th) (4,4,9),
(Ti) (6,4,7),

which description is tight, as confirmed by certain triangular 3-polytopes.

Corollary 1.9. *Every planar graph has a* 2^-*-vertex, or a* $(3, 3, \infty)$*-path, or a* 3*-path of one of the types* (Ta)–(Ti) *described in Theorem* 1.8.

It is not hard to see that Theorem 1.8 refines Theorem 1.1, Theorem 1.3, and Corollaries 1.4–1.6.

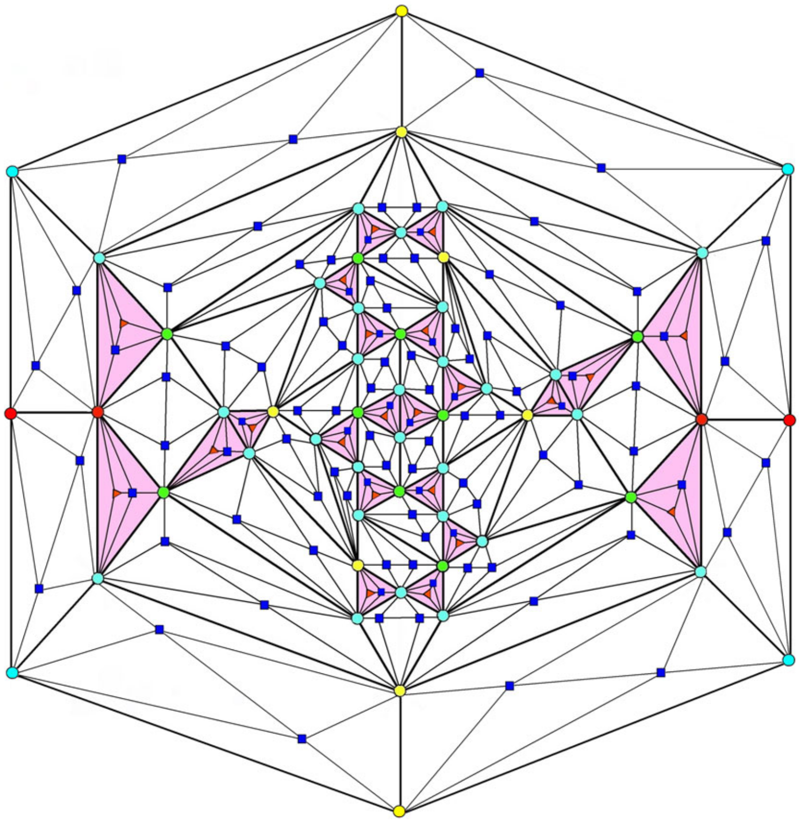

FIGURE 7. A bit more than a half of a triangulation with only off-(3, 4, 11)-paths among those mentioned in (Ta)–(Ti), which implies the sharpness of (Tg) (Borodin et al. [11]).

Having the two different tight descriptions of 3-paths in Theorems 1.7 and 1.8, it seems natural to pose the following challenging problem.

Problem 1.10. *Describe all tight descriptions of* 3*-paths in NPMs.*

Perhaps Problem 1.10 is not totally hopeless; at least we already have solutions for two special cases. The first concerns NPMs of *girth* (the length of minimal cycle) at least 4.

Theorem 1.11 (Borodin–Ivanova [10]). *There exist precisely seven tight descriptions of* 3*-paths in triangle-free NPMs:*

(*i*) {(5, 3, 6), (4, 3, 7)},
(*ii*) {(3, 5, 3), (3, 4, 4)},
(*iii*) {(5, 3, 6), (3, 4, 3)},
(*iv*) {(3, 5, 3), (4, 3, 4)},
(*v*) (5, 3, 7),
(*vi*) (3, 5, 4),
(*vii*) (5, 4, 6).

The second purpose of our article is to make another modest contribution to Problem 1.10 by describing all one-term descriptions in arbitrary NPMs.

Proposition 1.12. *There exist precisely three one-term tight descriptions of* 3*-paths in NPMs:*

(*i*) $(10, 5, \infty)$,
(*ii*) $(5, 10, \infty)$*, and*
(*iii*) $(5, \infty, 6)$.

We readily see that the one-term tight descriptions (v), (vi), and (vii) in Theorem 1.11 follow from the two-term tight descriptions (i), (ii), and (iii), respectively. This prompts the following general notion: a tight description is *irreducible* if it is not a consequence of another tight description. (Intuitively, the more terms has a tight description, and the smaller its parameters are, the more valuable the description is.)

Formally, suppose we have proved a description $D = \{(x_1, y_1, z_1), \ldots, (x_k, y_k, z_k)\}$ with $1 \leq k < \infty$ of 3-paths in a class **G** of NPMs. By $S(D)$ we denote the set of triplets (x, y, z) such that there is an i, $1 \leq i \leq k$, for which (x, y, z) belongs to the type (x_i, y_i, z_i), so that $x \leq x_i$, $y \leq y_i$, and $z \leq z_i$. A description D is *irreducible* if there is no description D' such that $S(D') \subset S(D)$.

Given **G**, the set of tight descriptions Ds for **G** forms a partially ordered set $P(\mathbf{G})$ by inclusion on the sets $S(D)$s of their triplets. The irreducible tight descriptions form the set of minimal elements in $P(\mathbf{G})$.

It seems to us that the most informative part of Problem 1.10 consists in describing all irreducible tight descriptions, so we pose also the following problem as a possible step toward Problem 1.10.

Problem 1.13. *Describe all irreducible tight descriptions of* 3*-paths in NPMs.*

In particular, the description in Theorem 1.8 is irreducible.

Claim 1.14. *The tight description of* 3*-paths given in Theorem 1.8 is irreducible.*

Problem 1.10 is meaningful also in a broader class of plane graphs with $\delta \geq 2$. It is an old fact rediscovered by many authors that each such graph with girth $g \geq 5k + 1$ and $k \geq 1$ has a path of k vertices of degree 2. For a proof, it suffices to contract all 2-vertices in a counterexample, take a 5^--face f^* in the graph with $\delta \geq 3$ obtained, and look at the preimage of f^* before the contraction. In particular, every plane graph with $\delta \geq 2$ and $g \geq 16$ has a (2, 2, 2)-path.

The behavior of 3-paths with low degree sum in sparse planar graphs with $\delta = 2$ was recently studied by Jendrol' and Maceková [18].

Theorem 1.15 (Jendrol' and Maceková [18])**.** *Every planar graph with* $\delta = 2$ *and girth* $g \geq 5$ *has a* 3*-path of one of the following types:*

(*i*) $(2, \infty, 2)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, *or* $(3, 3, 3)$ *if* $g = 5$,
(*ii*) $(2, 2, \infty)$, $(2, 3, 5)$, $(2, 4, 3)$, *or* $(2, 5, 2)$ *if* $g = 6$,
(*iii*) $(2, 2, 6)$, $(2, 3, 3)$, *or* $(2, 4, 2)$ *if* $g = 7$,
(*iv*) $(2, 2, 5)$ *or* $(2, 3, 3)$ *if* $8 \leq g \leq 9$,
(*v*) $(2, 2, 3)$ *or* $(2, 3, 2)$ *if* $10 \leq g \leq 15$*, and*
(*vi*) $(2, 2, 2)$ *if* $g \geq 16$.

Also, Jendrol' and Maceková [18] conjectured that every planar graph with $\delta = 2$ and girth $g = 5$ has either a $(2, \infty, 2)$-path or $w_3 \leq 9$. This conjecture was disproved by Aksenov et al. [1].

It is not known for which g within $5 \leq g \leq 15$ Theorem 1.2 is tight, but for $g = 4$ Jendrol' et al. [19] obtained a precise result, which extends Theorem 1.11(iv).

Theorem 1.16 (Jendrol' et al. [19]). *Every planar graph with $\delta = 2$ and girth $g \geq 4$ has a 3-path of one of the following types:* $(2, \infty, 2)$, $(2, 7, 3)$, $(3, 5, 3)$, $(4, 2, 5)$, *or* $(4, 3, 4)$, *which description is tight.*

We prove our main Theorem 1.8 in Sections 2 and 3; the proofs of Proposition 1.12 and Claim 1.14 are much shorter and are given in Sections 4 and 5, respectively.

2. THE TIGHTNESS OF THEOREM 1.8

In Figure 1, we see the graph $K^*_{2,2t}$, in which every path of three vertices contains a vertex of degree $2t$, where t can be arbitrarily large. This confirms the necessity of the assumption of Theorem 1.8.

The bounds in Theorem 1.8 are all sharp, as the following examples show. For each of the alternatives (Ta)–(Ti) in Theorem 1.8, we present a triangulation (see Figures 2–8) that contains only one type of 3-paths allowed in (Ta)–(Ti) and such that all upper bounds on the degrees of the corresponding vertices are attained.

(Ta). Figure 2 (Jendrol' [15]) shows how to transform the dodecahedron to a triangulation with vertices of degrees 3, 4, and 15 only and such that every 3-path goes through a 15-vertex. This construction confirms the tightness of (Ta).

(Tb). In Figure 3, we see a half of a plane triangulation. Its internal vertices have degrees 3, 4, or 10 only. No 3-vertex is adjacent to a 4^--vertex. The boundary vertices have the clockwise degree-sequence 97569756. We rotate the view by $\frac{\pi}{2}$ and glue the two halves along the boundary (equator).
Note that after gluing every equatorial vertex has degree at least 11 and is not adjacent to 3-vertices. As a result, we have only off-$(3, 10, 4)$-paths among the 3-paths mentioned in (Ta)–(Ti). Thus (Tb) is sharp.

(Tc). Figure 4 represents a semitriangulation to be glued along the equator with its copy to obtain a triangulation with no 3-paths other than off-$(3, 8, 5)$-paths. Indeed, there are no two 4^--vertices at distance at most 2, and there is no 3-path consisting of 7^--vertices. This proves the tightness of (Tc).

(Td). Figure 5(Td) shows how to transform the dual of the $(3, 4, 4, 4)$-Archimedean solid to a triangulation in which there are only vertices of degrees 4 and 7, where no two 4-vertices are adjacent to each other.

(Te). In Figure 6 the equator consists of four segments 877877778, so there is no problem to glue two halves of the desired triangulation.
There are no 3-vertices, so we should not bother about (Ta)–(Tc), (Tg). Each 4-vertex is surrounded by 7^+-vertices, and each 5-vertex has four 7^+-neighbors, so this triangulation avoids also (Tf), (Th), and (Ti).

(Tf). Figure 5(Tf) shows how to transform the dodecahedron to a triangulation with only off-$(6, 5, 6)$-paths. This is because there are no 4^--vertices and each 5-vertex is surrounded by 6-vertices, as desired.

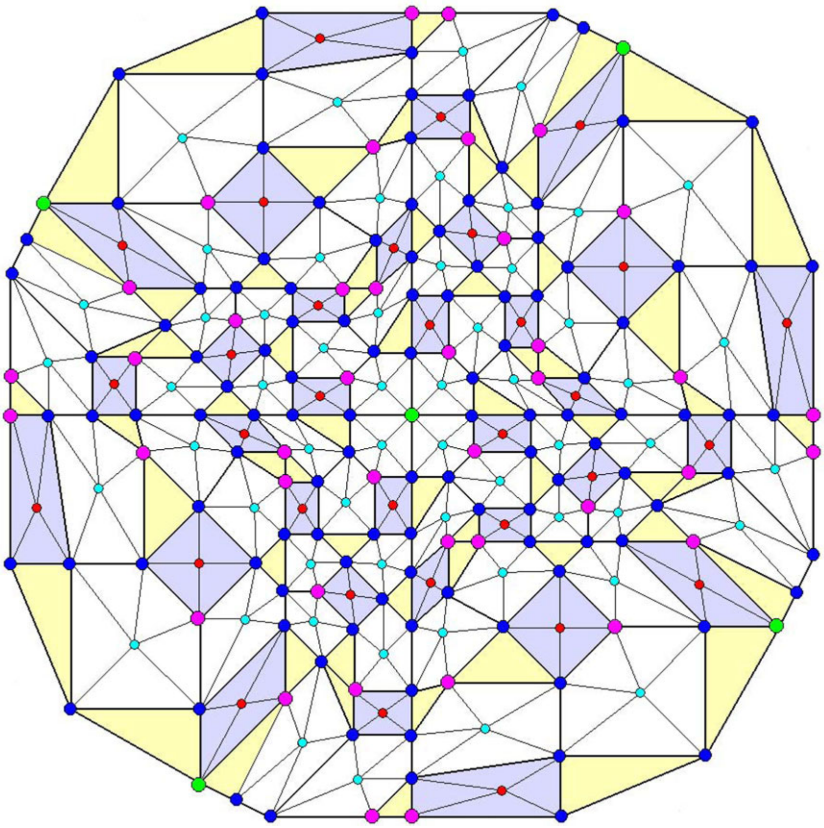

FIGURE 8. A half of the triangulation with only off-(6,4,7)-paths proving the tightness of (Ti).

(Tg). The picture in Figure 7 can be treated as a semispherical map extended by the tropical belt. Namely, the outside cycle and the concentric cycle next to it are analogues of tropics, with the invisible equator going in the middle between them. In particular, the equator separates the paired 4-vertices.

It is easy to see that the triangulation obtained by gluing two such semispherical maps along equator contains only vertices of degrees 3, 4, and at least 11. Since no vertex is adjacent to two 3-vertices, there are only paths of type $(3,4,11)$ among those appearing in (Ta)–(Ti) of Theorem 1.8. Moreover, all these paths of type $(3,4,11)$ are in fact off-$(3,4,11)$-paths. This proves that the bounds 4 and 11 in (Tg) are sharp.

(Th). Figure 5(Th) shows how to transform the $(3,5,3,5)$-Archimedean solid to a triangulation in which there are no 3-vertices, and each 4-vertex is surrounded by three 9^+-vertices, as desired.

(Ti). In Figure 8 we see a half of triangulation, with the equator consisting of four segments 87766778. There are precisely two ways to glue two halves; one of them is wrong, while the other is suitable.

There are no 3-vertices, which implies that obeying (Ta)–(Tc) and (Tg) is out of question. Since there are no adjacent 5^--vertices, it follows that our triangulation avoids (Te) and (Th). As for (Tf), we see that each 5^--vertex has at most one 6-neighbor. So, there are off-(6,4,7)-paths only.

3. PROVING THE MAIN STATEMENT OF THEOREM 1.8

Suppose that M' is a counterexample to Theorem 1.8.

A Constructing a Triangular Counterexample to Theorem 1.8

Let M be a counterexample on $V(M')$ with the most edges. From now on, by $d(v)$ of $v \in V(M)$ we mean the degree of v in M. We abbreviate the clause "since M does not contain a path xyz such that $d(x) \leq i$, $d(y) \leq j$ and $d(z) \leq k$" to "by non-(i, j, k)!."

(A) *M is a triangulation.*
Suppose there is a 4^+-face $f = abc \ldots$ Let b be a vertex of the minimum degree among all vertices incident with f and, moreover, $d(a) \leq d(c)$. It suffices to prove that $M + ac$ is also a counterexample to Theorem 1.8.
First observe that $M + ac$ cannot contain the $(K_4 - e)$-like configuration $K^*_{2,4}$ since $\delta(M) \geq 3$ and M does not contain the $K^*_{2,4}$ by assumption. Second, suppose $M + ac$ has a forbidden path zac or acz. This cannot happen if $z \neq b$ since a similar forbidden path should exist already in M due to the fact that $d(b) \leq d(a) \leq d(c)$ (namely, we could replace c or a in such a path, respectively, by b), a contradiction. Thus, we can assume that $z = b$.

Case 1. $d(b) = 3$. If $d(a) = 3$, then $d(c) \geq 12$ by non-$(3, 4, 11)$!, so the new paths bac and acb in $M + ac$ are either off-$(3, 13, 4)$-paths or off-$(3, 4, 13)$-paths, that is they are not forbidden. If $d(a) = 4$, then $d(c) \geq 10$ by non-$(4, 4, 9)$!, so the new paths in $M + ac$ are either off-$(3, 11, 5)$-paths or off-$(3, 5, 11)$-paths, so they are again allowed. Similarly, for $d(a) \in \{5, 6\}$ we have $d(c) \geq 7$ by non-$(6, 5, 6)$!, which means that the new paths are admissible because they are either off-$(3, 6, 8)$-paths or off-$(3, 8, 6)$-paths. The same is true if $7 \leq d(a) \leq d(c)$.

Case 2. $d(b) = 4$. If $d(a) = 4$, then $d(c) \geq 10$ by non-$(4, 4, 9)$!, and new off-$(4, 11, 5)$-paths or off-$(4, 5, 11)$-paths are admissible again. If $5 \leq d(a) \leq 6$, then $d(c) \geq 8$ by non-$(6, 4, 7)$!, so we can create only off-$(4, 6, 9)$-paths or off-$(4, 9, 6)$-paths. If $d(a) \geq 7$ then $d(c) \geq 7$ by $d(a) \leq d(c)$, so we can create only off-$(8, 4, 8)$-paths or off-$(4, 8, 8)$-paths.

Case 3. $d(b) \geq 5$. Here, adding the edge ac results in new off-$(5, 6, 8)$-paths or off-$(5, 8, 6)$-paths, which are again not forbidden, as desired.

The next property follows immediately from (A) and the assumptions of our theorem:

(B) *No* 3*-vertex of M is adjacent to a* 3*-vertex.*

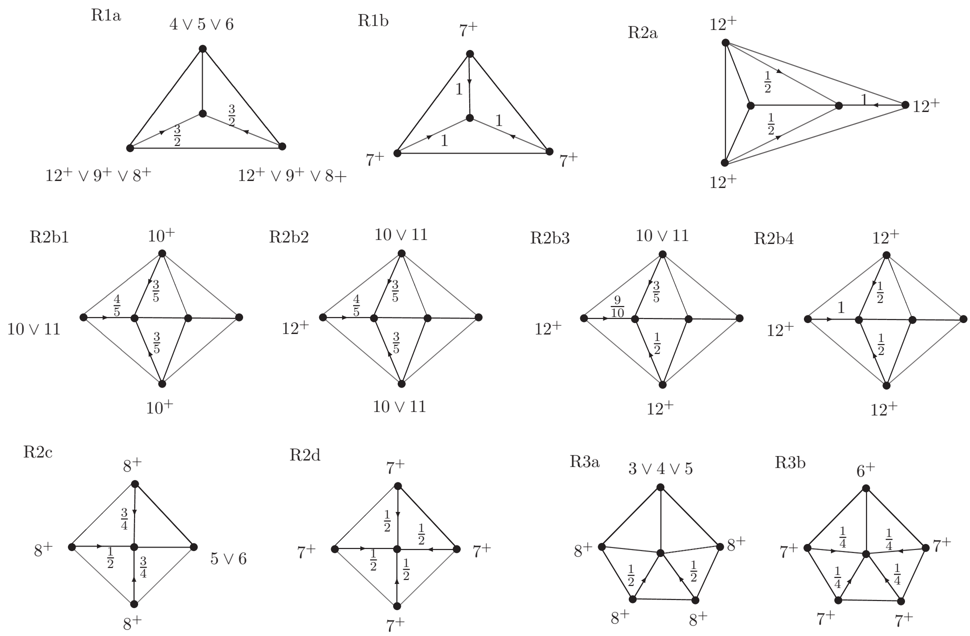

FIGURE 9. Rules of discharging.

B Discharging

Euler's formula $|V| - |E| + |F| = 2$ for M may be written as

$$\sum_{v \in V} (d(v) - 6) = -12. \tag{1}$$

Every vertex v contributes the *initial charge* $\mu(v) = d(v) - 6$ to (1), so only the charges of 5^--vertices are negative. Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(v)$ is nonnegative for all $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

Throughout the article, we denote the vertices adjacent to a vertex v in a cyclic order by $v_1, \dots, v_{d(v)}$. The rules of discharging are as follows (see Figure 9):

R1. *Suppose* $d(v) = 3$.
(a) If $4 \le d(v_1) \le 6$, then each of v_2, v_3 gives $\frac{3}{2}$ to v.
(b) If $d(v_i) \ge 7$, $1 \le i \le 3$, then each v_i gives 1 to v.
Note that $d(v_2) \ge 12$ and $d(v_3) \ge 12$ if $d(v_1) = 4$ due to non-(3, 4, 11)!, $d(v_2) \ge 9$ and $d(v_3) \ge 9$ if $d(v_1) = 5$ due to non-(3, 8, 5)!, and $d(v_2) \ge 8$ and $d(v_3) \ge 8$ if $d(v_1) = 6$ due to non-(6, 4, 7)!. Now $\mu'(v)$ is completely determined for $d(v) = 3$, and in all cases $\mu'(v) = \mu(v) + 3 = 0$.

R2. *Suppose* $d(v) = 4$.
(a) If $d(v_1) = 3$, then v_3 gives 1 to v, and each of v_2, v_4 gives $\frac{1}{2}$.
(b) If $d(v_1) = 4$, then:

(b1) if $d(v_3) \leq 11$, then v_3 gives $\frac{4}{5}$ to v, and each of v_2, v_4 gives $\frac{3}{5}$;

(b2) if $d(v_3) \geq 12$, $d(v_2) \leq 11$, and $d(v_4) \leq 11$, then v_3 gives $\frac{4}{5}$ to v, and each of v_2, v_4 gives $\frac{3}{5}$;

(b3) if $d(v_3) \geq 12$, $d(v_2) \geq 12$, and $d(v_4) \leq 11$, then v_3 gives $\frac{9}{10}$ to v, v_2 gives $\frac{1}{2}$, and v_4 gives $\frac{3}{5}$;

(b4) if $d(v_3) \geq 12$, $d(v_2) \geq 12$, and $d(v_4) \geq 12$, then v_3 gives 1 to v, and each of v_2, v_4 gives $\frac{1}{2}$.

(c) If $5 \leq d(v_1) \leq 6$, then each of v_2, v_4 gives $\frac{3}{4}$ to v, while v_3 gives $\frac{1}{2}$.

(d) Otherwise, v receives $\frac{1}{2}$ from each of $v_1, \ldots, v_4$.

Note that $d(v_i) \geq 12$ whenever $2 \leq i \leq 4$ if $d(v_1) = 3$ due to non-$(3, 4, 11)!$, and $d(v_i) \geq 10$ whenever $2 \leq i \leq 4$ if $d(v_1) = 4$ due to non-$(4, 4, 9)!$. Furthermore, if $5 \leq d(v_1) \leq 6$, then $d(v_i) \geq 8$ whenever $2 \leq i \leq 4$ due to non-$(6, 4, 7)!$. Otherwise $d(v_i) \geq 7$ whenever $1 \leq i \leq 4$. So each 4-vertex v has $\mu'(v) \geq \mu(v) + 2 = 0$.

R3. *Suppose* $d(v) = 5$.

(a) If $d(v_1) \leq 5$, then each of v_3, v_4 gives $\frac{1}{2}$ to v;

(b) otherwise, v receives $\frac{1}{4}$ from each 7^+-neighbor.

If $d(v_1) \leq 5$ in R3, then $d(v_i) \geq 8$ whenever $2 \leq i \leq 5$ due to non-$(5, 5, 7)!$. Otherwise we have $d(v_1) \geq 6$ and $d(v_i) \geq 7$ whenever $2 \leq i \leq 5$ due to non-$(6, 5, 6)!$. This implies that $\mu'(v) \geq \mu(v) + 1 = 0$.

C Checking $\mu'(v) \geq 0$ for $d(v) \geq 6$

CASE 1. $d(v) = 6$. Note that v does not participate in R1–R3, so we have $\mu'(v) = \mu(v) = 0$.

CASE 2. $d(v) = 7$. If v has a 3-neighbor, then v makes only one donation to its neighbors due to non-$(3, 8, 5)!$; namely, v gives 1 to its 3-neighbor by R1b.
Suppose v has no 3-neighbors. Then v does not make donation along two consecutive edges due to non-$(5, 5, 7)!$. Thus v makes at most three donations to its neighbors by R2d and R3b. By non-$(4, 7, 4)!$, there is at most one donation of $\frac{1}{2}$. This means that $\mu'(v) \geq 7 - 6 - \frac{1}{2} - 2 \times \frac{1}{4} = 0$.

CASE 3. $d(v) = 8$. If v has a 3-neighbor, then v is not adjacent to 5^--vertices due to $(3, 8, 5)!$. So v makes just one donation of $\frac{3}{2}$ by R1a, which implies that $\mu'(v) \geq 8 - 6 - \frac{3}{2} > 0$.
Suppose v is not adjacent to 3-vertices. Note that v makes at most two donations of $\frac{3}{4}$ by R2c. If v does two, then $\mu'(v) \geq 8 - 6 - 2 \times \frac{3}{4} - \frac{1}{2} = 0$ due to R2 and R3. If v makes precisely one donation of $\frac{3}{4}$, then v can make at most two donations of $\frac{1}{2}$ or $\frac{1}{4}$ by R2 and R3, so $\mu'(v) \geq 2 - \frac{3}{4} - 2 \times \frac{1}{2} > 0$. Finally, if v never sends $\frac{3}{4}$, then $\mu'(v) \geq 2 - 4 \times \frac{1}{2} > 0$.

CASE 4. $d(v) = 9$. Again, v has at most one 3-neighbor and cannot have both a 3-neighbor and a 4-neighbor by non-$(3, 10, 4)!$. Note that v cannot give charge to two consecutive neighbors by R1–R3. Indeed, suppose our v gives charge to v_1 and v_2. Inspecting our rules we can assume by symmetry that this could happen only if $d(v_1) \leq 5$ and $d(v_2) = 5$. However, we see from R3a that v actually does not give anything to v_2, a contradiction.

If $d(v_1) = 3$, then v spends at most $\frac{3}{2}$ on v_1 by R1 and at most $3 \times \frac{1}{2}$ on the other 5-neighbors by R3, which implies $\mu'(v) \geq 9 - 6 - 3 = 0$. If v has no 3-neighbors, then v makes at most four donations by R2 and R3, each of which of at most $\frac{3}{4}$, so $\mu'(v) \geq 9 - 6 - 4 \times \frac{3}{4} = 0$.

CASE 5. $d(v) = 10$. If v has a 3-neighbor, then other 4^--neighbors are still impossible due to non-$(3, 10, 4)!$, which means that we can again apply the argument in Case 3 combined with R1a. Since now v still has at most five 5-neighbors, $\mu'(v) \geq 10 - 6 - \frac{3}{2} - 5 \times \frac{1}{2} = 0$.

We need some definitions concerning any 10^+-vertex v. A *single* 5^-*-neighbor* of v is a 5^--vertex v_2 such that $d(v_1) \geq 6$ and $d(v_3) \geq 6$. By a *pair of* 5^-*-twins* we mean 5^--vertices v_2, v_3, which are surrounded by 6^+-vertices v_1, v_4 in the neighborhood of v due to non-$(5, 5, 5)!$.

Now we return to our 10-vertex v and assume that it has no 3-neighbors. It follows by R2, R3 that v gives at most $\frac{4}{5}$ to each single neighbor, and at most $2 \times \frac{3}{5}$ to each pair of 5^--twins.

To estimate the total expenditure of v, we use the following $\frac{2}{5}$*-averaging* of transfers from v to its neighbors w.r.t. the level of $\frac{2}{5}$. The donation to each single is shared evenly between two nearby faces, while that to a pair of twins is shared among the three nearest faces. As a result, each face takes away at most $\frac{2}{5}$ from v. Thus $\mu'(v) \geq 10 - 6 - 10 \times \frac{2}{5} = 0$.

CASE 6. $d(v) = 11$. Still, at most one 3-neighbor at v is possible. In contrast to Case 5, now v can have a 3-neighbor and a 4-neighbor simultaneously, but still no 4-neighbor of v is adjacent to a 3-vertex by non-$(3, 4, 11)!$.

If v has a 3-neighbor, then we cannot apply the $\frac{2}{5}$-argument to the donation from v literally. However, if $d(v_2) = 3$ and $d(v_3) = 5$, then $d(v_1) \geq 8$ and $d(v_4) \geq 8$ by non-$(5, 5, 7)!$. So we have $\mu'(v) \geq 11 - 6 - \frac{3}{2} - 8 \times \frac{2}{5} > 0$. Now if v has no 3-neighbors, then our $\frac{2}{5}$-averaging yields $\mu'(v) \geq 5 - 11 \times \frac{2}{5} > 0$. This completes the proof for Case 6.

To estimate the total donation of v whenever $d(v) \geq 12$, we look at the singles and twins in more detail. Each single, say v_2, takes at most 1 from v by R1b, R2, and R3, and we can split its donation by at most $\frac{1}{2}$ between faces incident with edge vv_2.

We distinguish two kinds of twins. Assuming that $d(v_1) \geq 8$, $d(v_2) \leq 5$, $d(v_3) \leq 5$, and $d(v_4) \geq 8$, the first, T_1, has $d(v_2) = 3$ and $d(v_3) = 4$, and takes $\frac{3}{2} + \frac{1}{2}$ from v by R1a and R2a. Any other twin T_2 is of the second kind. Note that T_2 takes at most $\frac{3}{2}$ from v, so we can split its donation by at most $\frac{1}{2}$ among the three faces incident with edges vv_2 and vv_3.

After this averaging, each face f incident with v takes $\frac{2}{3}$ from v if $f = vv_2v_3$ such that v_2, v_3 is a pair of twins of the kind T_1 defined in the previous paragraph, and at most $\frac{1}{2}$ otherwise.

CASE 7. $12 \leq d(v) \leq 15$. Note that v has at most one T_1 by non-$(3, 15, 3)!$. If v does have a T_1, then $\mu'(v) \geq d(v) - 6 - (\frac{3}{2} + \frac{1}{2}) - (d(v) - 3) \times \frac{1}{2} = \frac{d(v)-13}{2}$. So we are done unless $d(v) = 12$.

Here, we must argue more carefully. If v has a face f not incident with a 5^--vertex, then f takes nothing from v after averaging, which implies that $\mu'(v) \geq 12 - 6 - 2 - 8 \times \frac{1}{2} = 0$. Now it follows by parity that v has a T_2

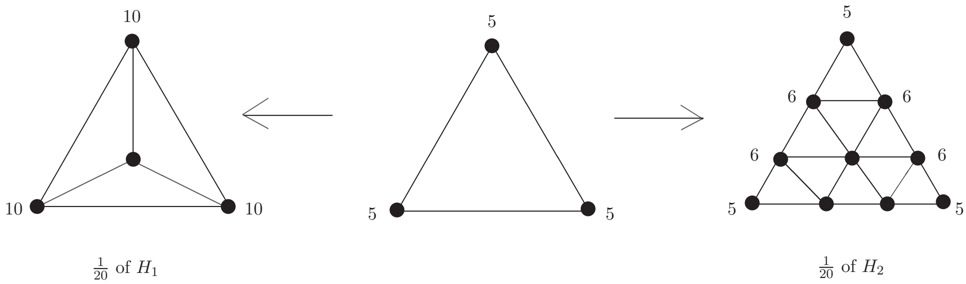

FIGURE 10. Two derivatives of the icosahedron.

in addition to our T_1. Note that v cannot have a T_2 with a 3-vertex by non-(3, 15, 3)!. This implies that T_2 takes at most 1 from v rather than $\frac{3}{2}$, so that $\mu'(v) \geq 6 - 2 - 1 - 6 \times \frac{1}{2} = 0$.

Finally, suppose v has no T_1; then $\mu'(v) \geq d(v) - d(v) \times \frac{1}{2} = \frac{d(v)-12}{2} \geq 0$.

CASE 8. $d(v) \geq 16$. Now R1a and R2a can be applied to v more than once. As a result, every face takes at most $(\frac{3}{2} + \frac{1}{2}) : 3 = \frac{2}{3}$ from v on the average. For $d(v) \geq 18$, this implies that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3} \geq 0$.

Suppose $16 \leq d(v) \leq 17$. Note that every face that is not incident with a 5^--vertex takes away nothing after averaging, so $\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} = \frac{d(v)-16}{3} \geq 0$. So we are done unless every face belongs to a single or twins. Due to parity, a 17-vertex has a single, which implies $\mu'(v) \geq 17 - 6 - 15 \times \frac{2}{3} - 1 = 0$. Since a 16-vertex necessarily has at least two singles, we have $\mu'(v) \geq 16 - 6 - 12 \times \frac{2}{3} - 2 \times 1 = 0$.

Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$, which contradicts (1):

$$0 \leq \sum_{v \in V} \mu'(v) = \sum_{v \in V} \mu(v) = -12.$$

This completes the proof of Theorem 1.8.

4. PROVING PROPOSITION 1.12

A Proving the Tightness of Proposition 1.12

For the description $(5, \infty, 6)$ in Proposition 1.12, we have to prove that each of its strengthenings $(4, \infty, 6)$, $(5, N, 6)$ with any finite integer N, and $(5, \infty, 5)$ is wrong. The first and third are violated by the graph H_2 in Figure 10, which has neither 4^--vertices nor two 5^--vertices at distance 2 from each other. The second description is invalid due to the Jendrol' graph in Figure 1, which has no 3-path of bounded weight.

Checking the tightness of $(10, 5, \infty)$ and $(5, 10, \infty)$ is also easy; it is based on the graphs in Figures 1 and 10. (It suffices to note in addition that H_1 has no (9, 9)-edges.)

B Proving the Main Statement of Proposition 1.12

By the main statement of Proposition 1.12(i–iii) we mean that every triangle-free NPM has a $(10, 5, \infty)$-path, $(5, 10, \infty)$-path, and $(5, \infty, 6)$-path, respectively. The first two facts follow from the existence of an edge of one of the types (3, 10), (4, 7), and (5, 6), as proved in Borodin [4–6].

So suppose that M' is a counterexample to the main statements in Proposition 1.12(iii). Let M be a counterexample on $V(M')$ with the most edges; then the following holds.

(*) *Each face $f = \ldots wxyz$ in M with $d(f) \geq 4$ satisfies $d(f) = 4$ and, up to symmetry, $d(w) \geq 7$, $d(x) \leq 5$, $d(y) \leq 5$, and $d(z) \geq 7$.*

Indeed, note that f cannot have two nonconsecutive 6^+-vertices a and b, for otherwise the edge ab in $M + ab$ is a $(7^+, 7^+)$-edge, so ab cannot be a part of a $(5, \infty, 6)$-path. Hence each vertex in the boundary $\partial(f)$ of f is adjacent to a 5^--vertex in $\partial(f)$. Let $d(x) \leq 5$; this means by the symmetry between w and y that $d(y) \leq 5$. Since $\partial(f)$ has no three consecutive 5^--vertices, it follows that $d(w) \geq 6$ and $d(z) \geq 6$. This implies that w and z are consecutive in $\partial(f)$, so $d(f) = 4$. Moreover, $d(w) \geq 7$ and $d(z) \geq 7$ by non-$(5, \infty, 6)!$, as desired.

It follows from (*) combined with non-$(5, \infty, 6)!$ that no 5^--vertex in M is adjacent to a 6-vertex.

Euler's formula $|V| - |E| + |F| = 2$ for M may be written as

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12. \tag{2}$$

This time each 4-face contributes the *initial charge* $\mu(v) = 2d(f) - 6 = 2$ to (2). Again, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is nonnegative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (2), equal to -12.

We now use the following rules of discharging.

R1. *Each 4-face gives 1 to each incident5^--vertex.*
R2. *Each 7^+-vertex v gives 1 to each adjacent 5^--vertex.*

We now check that $\mu'(x) \geq 0$ for all $x \in V \cup F$.

Suppose $f \in F$. If $d(f) = 3$, then $\mu'(f) = \mu(f) = 2 \times 3 - 6 = 0$. If $d(f) = 4$, then $\mu'(f) = 2 \times 4 - 6 - 2 \times 1 = 0$ by R1 due to (*).

Now suppose $v \in V$.

Case 1. $3 \leq d(v) \leq 5$. If v is incident with a 5^--vertex v_2, then (*) implies that v receives 1 from each of incident 4-faces v_1vv_2x and v_2vv_3y by R1 and 1 from each of the 7^+-vertices v_1, v_3 by R2, which results in $\mu'(v) \geq 3 - 6 + 4 \times 1 > 0$. Otherwise, v is surrounded by 7^+-vertices, so $\mu'(v) \geq d(v) - 6 + d(v) \times 1 \geq 0$ by R2, as desired.

Case 2. $d(v) = 6$. Since v cannot give charge away, we have $\mu'(v) = \mu(v) = 0$.

Case 3. $d(v) \geq 7$. Note that v cannot make more than one donation by R2 due to non-$(5, \infty, 6)!$, whence $\mu(v) \geq d(v) - 6 - 1 \geq 0$.

C Proving that There are No One-Term Tight Descriptions Other Than in Proposition 1.12

Suppose we have a theorem saying "Every NPM has an (X, Y, Z)-path, which description is tight." We can assume that $X \leq Z$.

CASE 1. $Y = \infty$. It follows from the graph H_2 in Figure 10 that $X \geq 5$ and $Z \geq 6$. Since the description $(5, \infty, 6)$ is tight by Proposition 1.12(iii), we have $X = 5$ and $Z = 6$.

CASE 2. $Y \neq \infty$. It follows from $K^*_{2,2t}$ that $Z = \infty$, and from the graph H_2 that $X \geq 5$ and $Y \geq 5$. Furthermore, the graph H_1 in Figure 10 has no (9, 9)-edges. This implies that in fact $X \geq 10$ or $Y \geq 10$. Since both descriptions $(5, 10, \infty)$ and $(10, 5, \infty)$ are tight, we have obtained that $XYZ = 5(10)\infty$ or $XYZ = (10)5\infty$, as desired.

5. PROVING CLAIM 1.14

Suppose $D' = \{(x_1, y_1, z_1), \ldots, (x_k, y_k, z_k)\}$ with $1 \leq k < \infty$ is a description of 3-paths in $K^*_{2,4}$-free NPMs and that D' is stronger than $D = \{(3, 15, 3), \ldots, (6, 4, 7)\}$ in Theorem 1.8. We have to prove that $S(D') \subset S(D)$. Our proof is based on the properties of the graphs depicted in Figures 2–8. Each of the 10 graphs is "responsible" for just one of the types in Theorem 1.8.

For example, the graph in Figure 2 has a 3-path corresponding to the triplet (3,15,3), but no other 3-path in this graph obeys any of the types included in Theorem 1.8. This means that $(3, 15, 3) \in S(D')$.

As for the graph in Figure 7, it has only vertices of degrees 3, 4, and 11^+. This graph contains a 3-path with the triplet (3,4,11), but no other 3-path in this graph obeys any of the types included in Theorem 1.8. Furthermore, this construction contains neither a $(3, 3, 11^-)$-path, nor $(3, 4^-, 10^-)$-path. Indeed, each 3-path goes through an 11^+-vertex, but no 11^+-vertex has more than one 3-neighbor. This implies that $(3, 4, 11) \in S(D')$.

Checking the other eight graphs in Figures 2–8, we see that for any type (X, Y, Z) mentioned in Theorem 1.8(b–f, h, i) the triplet (X, Y, Z) appears in $S(D')$. Thus $S(D') \supseteq S(D)$, a contradiction.

This confirms the desired irreducibility of the tight description in Theorem 1.8.

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