Strengthening Theorems of Dirac and Erdős on Disjoint Cycles

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Abstract: Let $k \geq 3$ be an integer, $H_k(G)$ be the set of vertices of degree at least $2k$ in a graph G, and $L_k(G)$ be the set of vertices of degree at most $2k-2$ in G. In 1963, Dirac and Erdős proved that G contains k (vertex) disjoint cycles whenever $|H_k(G)| - |L_k(G)| \geq k^2 + 2k - 4$. The main result of this article is that for $k \geq 2$, every graph G with $|V(G)| \geq 3k$ containing at most t disjoint triangles and with $|H_k(G)| - |L_k(G)| \geq 2k + t$ contains k

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disjoint cycles. This yields that if $k \geq 2$ and $|H_k(G)| - |L_k(G)| \geq 3k$, then G contains k disjoint cycles. This generalizes the Corrádi–Hajnal Theorem, which states that every graph G with $H_k(G) = V(G)$ and $|H_k(G)| \geq 3k$ contains k disjoint cycles. J. Graph Theory 85: 788–802, 2017

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1. INTRODUCTION

For a graph G, let $|G| = |V(G)|$, $\|G\| = |E(G)|$, and $\delta(G)$ be the minimum degree of a vertex in G. The complement of G is denoted by $\overline{G}$. The join $G \vee G'$ of two graphs is $G \cup G' \cup \{xx' : x \in V(G) \text{ and } x' \in V(G')\}$. Let SK_m denote the graph obtained by subdividing one edge of the complete m-vertex graph K_m. For a positive integer k, define $H_k(G)$ to be the subset of vertices with degree at least $2k$ and $L_k(G)$ to be the subset of vertices of degree at most $2k-2$ in G. When we say that two cycles *are disjoint*, we mean that they are vertex-disjoint.

Resolving a conjecture of Erdős, Corrádi, and Hajnal [2] proved the following theorem.

Theorem 1.1. [2] *Let G be a graph and k a positive integer. If* $|G| \geq 3k$ *and* $\delta(G) \geq 2k$*, then G contains k disjoint cycles.*

Since each cycle has at least three vertices, the condition $|G| \geq 3k$ is necessary. The condition $\delta(G) \geq 2k$ is also sharp, as witnessed by the graph $G_{n,k} = \overline{K}_{n-2k+1} \vee K_{2k-1}$ for $n \geq 3k$. Indeed, any cycle in $G_{n,k}$ must contain at least two vertices from the K_{2k-1} and therefore $G_{n,k}$ contains at most $k-1$ disjoint cycles.

Theorem 1.1 prompted a series of refinements and extensions for both undirected graphs (see, e.g., [5–12, 14]) and directed graphs (see, e.g., [3, 4, 13, 15]). In particular, Dirac and Erdős [5] proved in 1963 the following theorem.

Theorem 1.2. [5] *Let* $k \geq 3$ *be an integer and G be a graph with* $|H_k(G)| - |L_k(G)| \geq k^2 + 2k - 4$*. Then G contains k disjoint cycles.*

The bound $k^2 + 2k - 4$ is not best possible. Dirac and Erdős provided the following examples of a graph G without k disjoint cycles such that $|H_k(G)| - |L_k(G)| = 2k - 1$. Let $n \geq 3k$ be odd. Let $V(G) = X \cup Y \cup Z$ with $|X| = 2k - 1$ and $|Y| = |Z| = \frac{n-2k+1}{2}$. Let the set of edges of G consist of a perfect matching connecting Y with Z, all edges between X and Y, and all edges inside X. Then, $H_k(G) = X \cup Y$ and $L_k(G) = Z$, but G has no k disjoint cycles, since every cycle must contain at least two vertices of X.

Dirac and Erdős also proved that for a planar graph G weaker restrictions on the difference $|H_k(G)| - |L_k(G)|$ provide that G contains k cycles.

Theorem 1.3. [5] *Let* $k \geq 3$ *be an integer and G be a planar graph such that* $|H_k(G)| - |L_k(G)| \geq 5k - 7$*. Then G contains k disjoint cycles.*

The main result of this article is the following theorem.

Theorem 1.4. *Let $k \geq 2$ be an integer and G be a graph such that $|G| \geq 3k$. Let t be the maximum number of disjoint triangles contained in G. If*

$$|H_k(G)| - |L_k(G)| \geq 2k + t,$$

then G contains k disjoint cycles.

Theorem 1.4 is sharp, as witnessed by the graph SK_{3k-1}. Let u be the newly created vertex (of degree 2) and observe that $L_k(SK_{3k-1}) = \{u\}$, $|H_k(SK_{3k-1})| = 3k - 1$, and G contains $k - 1$ disjoint triangles. Since $|SK_{3k-1}| = 3k$, any set of k disjoint cycles must partition $V(SK_{3k-1})$ into triangles, but the vertex u is not contained in any triangle. As above, $3k$ vertices are necessary for the existence of k cycles. However, if the bound on $|H_k(G)| - |L_k(G)|$ in Theorem 1.4 is slightly strengthened, then the condition $|G| \geq 3k$ holds automatically.

Corollary 1.5. *Let $k \geq 2$ be an integer and G be a graph. Let t be the maximum number of disjoint triangles contained in G. If*

$$|H_k(G)| - |L_k(G)| \geq 2k + t + 1,$$

then G contains k disjoint cycles.

Corollary 1.5 is sharp since K_{3k-1} contains only $k - 1$ disjoint cycles and $|H_k(K_{3k-1})| - |L_k(K_{3k-1})| = 3k - 1$. Corollary 1.5 requires a short proof that is given in Section 5. A straightforward consequence of Corollary 1.5 is the following stronger version of Theorem 1.2.

Corollary 1.6. *Let $k \geq 2$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq 3k$. Then G contains k disjoint cycles.*

Observe that the special case $H_k(G) = V(G)$ of Corollary 1.6 is equivalent to Theorem 1.1 for $k \geq 2$. Using the techniques of Theorem 1.4, we will also prove the following stronger version of Theorem 1.3.

Theorem 1.7. *Let $k \geq 2$ be an integer and G be a planar graph. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

The condition that G be planar is necessary. Indeed, consider the nonplanar graph SK_5. If u is the newly created vertex, then $H_2(SK_5) = V(SK_5) - u$ and $L_2(SK_5) = \{u\}$, but SK_5 does not have two disjoint cycles. The bound $2k$ in Theorem 1.7 is sharp (see, e.g., $K_5 - e$ for $k = 2$), however only for small values of k. Since the average degree of every planar graph is less than 6, for $k \geq 5$ much weaker restrictions provide existence of k disjoint cycles in planar graphs.

Finally, we prove that the bound $2k$ is also sufficient if the graph G contains no two disjoint triangles.

Theorem 1.8. *Let $k \geq 3$ be an integer and G be a graph such that G does not contain two disjoint triangles. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

Our proofs are based on the approach and ideas of Dirac and Erdős [5]. We also heavily use an extension of Theorem 1.1 from [10] (Theorem 2.1).

The remainder of this article is organized as follows. The next section outlines the notation that we will use throughout the article, and introduces some tools to be used in the proof. In Section 3 we will prove several lemmas for the base case, and in Section 4 we prove the main result. In Sections 5–7, we use Theorem 1.4 to prove Corollary 1.5, Theorem 1.7, and Theorem 1.8, respectively.

2. NOTATION AND TOOLS

For disjoint sets $U, U' \subseteq V(G)$, we write $\|U, U'\|$ for the number of edges from U to U'. If $U = \{u\}$, then we will write $\|u, U'\|$ instead of $\|\{u\}, U'\|$. For $x \in V(G)$, $N_G(x)$ is the set of vertices adjacent to x in G and $d_G(x) = |N_G(x)|$. When the choice of G is clear, we simplify the notation to $N(x)$ and $d(x)$, respectively. For an edge $xy \in E(G)$, G/xy denotes the graph obtained from G by contracting xy, and v_{xy} denotes the vertex resulting from contracting xy. By $\alpha(G)$ we denote the *independence number* of G.

We say that $x, y, z \in V(G)$ *form a triangle* $T = xyz$ in G if $G[\{x, y, z\}]$ is a triangle. We say $v \in T$, if $v \in \{x, y, z\}$. A set $\mathcal{T}$ of triangles is a set of subgraphs of G such that each subgraph is a triangle and all the triangles are disjoint. For a set $\mathcal{S}$ of graphs, let $V(\mathcal{S}) = \bigcup\{V(S) : S \in \mathcal{S}\}$. For a graph G, we write t_G for the maximum number of disjoint triangles contained in G.

$$\begin{aligned}&\textit{When the graph } G \textit{ is clear from context, we will use } t \textit{ instead of } t_G.\\&\textit{Similarly, when the integer } k \textit{ is also clear, we will use } H \textit{ and } L \textit{ for } H_k(G)\\&\textit{and } L_k(G), \textit{ respectively. The sizes of } H \textit{ and } L \textit{ will be denoted by } h \textit{ and } \ell,\\&\textit{respectively.}\end{aligned} \qquad (1)$$

As shown in [10], if a graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k - 1$ does not contain a large independent set, then with two exceptions, G contains k disjoint cycles:

Theorem 2.1. [10] *Let $k \geq 2$. Let G be a graph with $|G| \geq 3k$ and $\delta(G) \geq 2k - 1$ such that G does not contain k disjoint cycles. Then,*

(1) $\alpha(G) \geq |G| - 2k + 1$, *or*
(2) *k is odd and $G = 2K_k \vee \overline{K_k}$, or*
(3) *$k = 2$ and G is a wheel.*

We will use the following corollary of Theorem 2.1 throughout the article.

Corollary 2.2. *Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 3k$. If $|H| \geq 2k$ and $\delta(G) \geq 2k - 1$ (i.e., $L = \emptyset$), then G contains k disjoint cycles.*

Proof. First, if $G = 2K_k \vee \overline{K_k}$, then $|H| = k$, a contradiction. Next, if $\alpha(G) \geq |G| - 2k + 1$, then let U be an independent set of size $|G| - 2k + 1$. For each $u \in U$, $d(u) \leq 2k - 1$, so $H \subseteq V(G) \setminus U$ and $|H_k(G)| \leq 2k - 1$. Finally, if $k = 2$, then G is not a wheel, since the wheel has only one vertex of degree at least 4. Therefore, by Theorem 2.1, G contains k disjoint cycles. ∎

Call a graph G minimal if among graphs with a certain property, $|G|$ is minimal, and subject to this, $\|G\|$ is minimal. Dirac and Erdős [5] observed the following.

Property 2.3. *Let $k \geq 2$ be an integer and $f : \mathbb{N} \to \mathbb{Z}$ a function. Suppose G is minimal among the graphs without k disjoint cycles satisfying $|H| - |L| \geq f(k)$. Then,*

(1) $\delta(G) \geq 2$, *and*
(2) *if $uv \in E(G)$, then $d(u) \in \{2k-1, 2k\}$ or $d(v) \in \{2k-1, 2k\}$.*

Indeed, if such a graph G contained a vertex v with $d(v) \leq 1$, then $G - v$ is a smaller counterexample. Similarly, if *(2)* does not hold, then $G - uv$ is a smaller counterexample.

3. GRAPHS WITH TWO DISJOINT CYCLES

In this section we prove several lemmas that will serve as the base case $k = 2$ for our various proofs. In notation, we will use convention (1) with $k = 2$.

Lemma 3.1. *Every triangle-free graph G with $h \geq \ell + 4$ contains two disjoint cycles.*

Proof. Let G be a minimal counterexample. As G is triangle-free, and $h \geq 4$, $|G| \geq 8$. By Property 2.3, $\delta(G) \geq 2$, and by Corollary 2.2, $\delta(G) = 2$. Say $d(x) = 2$ and $N(x) = \{y, z\}$. By Property 2.3, $d(y), d(z) \in \{3, 4\}$. Set $G' = G/xy$. Since G is triangle-free, $d_{G'}(v) = d_G(v)$ for all $v \in V(G) \setminus \{x, y\}$. As $x \in L$, this implies $|H_2(G')| \geq |L_2(G')| + 4$. Since G is minimal, G' has a triangle, say $v_{xy}zw$. Then $C := yxzw$ is a 4-cycle in G. Let $W = V(G) \setminus C$.

As $x \in L$, $|C \cap H| - |C \cap L| \leq 2$. So, since $h - \ell \geq 4$, $|H \cap W| - |L \cap W| \geq 2$. Thus

$$\sum_{u \in W} d(u) \geq 3|W| + |H \cap W| - |L \cap W| \geq 3|W| + 2. \tag{2}$$

Each $v \in W$ has no two adjacent neighbors as G is triangle-free, and is not adjacent to x as $N(x) \subset C$. Thus if $\|v, C\| \geq 2$ then $N(v) \cap C = \{y, z\}$. As $d(y) \leq 4$, there are at most two such v. So $\|W, C\| \leq |W| + 2$. Hence by (2),

$$2\|G[W]\| = \sum_{u \in W} d(u) - \|W, C\| \geq (3|W| + 2) - (|W| + 2) = 2|W|.$$

Therefore $\|G[W]\| \geq |W|$, and so $G[W]$ contains a cycle (disjoint from C). ■

The *2-core* of a graph G is the union of all $G' \subseteq G$ with $\delta(G') \geq 2$. It can be obtained from G by iterative deletion of vertices of degree at most 1.

Lemma 3.2. *Suppose the 2-core of G contains at least six vertices, and it is not isomorphic to SK_5. If $h \geq \ell + 4$, then G contains two disjoint cycles.*

Proof. Let G be a minimal counterexample. If there exists a vertex of degree at most 1, then removing it yields a smaller counterexample. So G is its own 2-core and $\delta(G) \geq 2$. Also $|G| \geq 6$ and by Corollary 2.2, $L \neq \emptyset$. Thus $h \geq 5$, and $|G| \geq 7$, since G is not isomorphic to SK_5. Pick $x \in L$. Let $N(x) = \{y, z\}$.

Suppose $yz \notin E(G)$. Set $G' = G/xy$. Then $|G'| = |G| - 1 \geq 6$. Since $d(x) = 2$, all $v \in V(G')$ satisfy $d_{G'}(v) = d_G(v)$. So G' is its own 2-core, $|H_2(G')| - |L_2(G')| = h - \ell + 1 \geq 5$, and G' is not isomorphic to SK_5. As $|G'| < |G|$, by the minimality of G, G' has two disjoint cycles. But then so does G.

Otherwise $yz \in E(G)$. Now xyz is a triangle in G, so $G' := G - \{x, y, z\}$ is acyclic, and $\|G'\| < |G'|$. Since $h \geq \ell + 4$ and $x \in L$, we have $|H \cap V(G')| - |L \cap V(G')| \geq 3$. So $\sum_{v \in V(G')} d_G(v) \geq 3|G'| + 3$. As $N(x) = \{y, z\}$,

$$\|V(G'), \{y, z\}\| = \|V(G'), V(G) \setminus V(G')\| \geq 3|G'| + 3 - 2(|G'| - 1) \geq |G'| + 5.$$

Thus $d(y), d(z) \geq 6$. Let $G^* = G - x$. Then $|G^*| \geq 6$, $d_{G^*}(y), d_{G^*}(z) \geq 5$, and $d_{G^*}(v) = d_G(v)$ for all $v \in V(G^*) \setminus \{y, z\}$. So $|H_2(G^*)| - |L_2(G^*)| \geq 5$ and G^* coincides with its 2-core. As $|G^*| < |G|$, by the minimality of G, G^* has two disjoint cycles. But then so does G. ■

Lemma 3.3. *Every graph G containing a triangle $X = x_1x_2x_3$ has two disjoint cycles provided (a) $|H \setminus X| - |L \setminus X| \geq 2$ and (b) $\|v, X\| \leq 2$ for all $v \in V(G) \setminus X$.*

Proof. Let G be a minimal counterexample to the lemma. Then $G - X$ is acyclic. Let $Y = V(G) \setminus X$. By (a), there is $u \in H \setminus X$, and by (b), $\|u, Y\| \geq 2$. This yields $|G| \geq 6$. First, we show:

$$\text{If } v \in Y \text{ and } \|v, Y\| \leq 1, \text{ then } \|v, Y\| = 1 \text{ and } \|v, X\| = 2. \tag{3}$$

Indeed, by (b), $\|v, X\| \leq 2$. So if $\|v, Y\| = 0$ or $\|v, X\| \leq 1$ and $\|v, Y\| \leq 1$, then $v \in L \setminus X$. Thus $G - v \subset G$ satisfies (a) and (b). Then by the minimality of G, $G - v$ has two disjoint cycles, and hence so does G.

By (a), there are $z, z' \in H \setminus X$. If they are in the same component of $G - X$, then let Q be the interior of the unique z, z'-path in $G - X$ and put $G' = G - X - Q - zz'$; otherwise put $G' = G - X$. Pick maximum paths $P = y_1 \ldots z \ldots y_2$ and $P' = y'_1 \ldots z' \ldots y'_2$ in G'. Perhaps $z = y_1$ or $z' = y'_1$, but $z, z' \in H$ implies $|P|, |P'| \geq 2$. For $i \in \{1, 2\}$, if $y_i \neq z$ and $N(y_i) \cap Q \neq \emptyset$, then $G[P \cup Q]$ contains a cycle, a contradiction. Then,

$$\text{either } d_{G-X}(y_i) = d_{G'}(y_i) \text{ or } y_i = z. \tag{4}$$

So, if $y_i \neq z$, then by (3), $\|y_i, X\| = 2$. Otherwise $y_i = y_1 = z$, and $\|z, X\| \geq d_G(z) - d_{G-X}(z) \geq d_G(z) - d_{G'}(z) - 1 \geq 2$. So in any case, $\|y_i, X\| = 2$ and a similar argument shows $\|y'_i, X\| = 2$. Now y_1 and y_2 have a common neighbor, say x_1 in X, and $G[P + x_1]$ contains a cycle C_1. If y'_1 and y'_2 have a common neighbor $x_i \in X - x_1$, then $G[P' + x_i]$ contains a cycle disjoint from C_1. Otherwise, one of y'_1 and y'_2 is adjacent to x_2 and the other to x_3. Then $G[P' \cup \{x_2, x_3\}]$ contains a cycle disjoint from C_1. ■

4. PROOF OF THEOREM 1.4

Recall that we use convention (1). Let k be the smallest integer such that there exists a graph G without k disjoint cycles satisfying $|H| - |L| \geq 2k + t$ and $|G| \geq 3k$. By Lemmas 3.1 and 3.2, $k \geq 3$. Choose such G to be minimal.

Lemma 4.1. $|G| \geq 3k + 1$.

Proof. Suppose that $|G| = 3k$. Create the graph $G' \supseteq G$ by adding edges to G until, for each $x \in L$, $N_{G'}(x) = V(G') - x$. Then $\delta(G') \geq 2k - 1$, so by Corollary 2.2, G' contains k disjoint cycles. As $|G'| = 3k$, these cycles are triangles, and at most ℓ of them contain edges from $E(G') \setminus E(G)$. Thus $t \geq k - \ell$ and so $h \geq \ell + 2k + t \geq 3k = |G|$. Hence $H = V(G)$ and by Theorem 1.1, G contains k disjoint cycles. ■

Lemma 4.2. *Each $x \in L$ is in a triangle in G.*

Proof. Suppose x is not in a triangle. By Property 2.3, $d(x) \geq 2$. Let $y \in N(x)$ and set $G' = G/xy$. Then $d_{G'}(v_{xy}) \geq d(y)$ and $d_{G'}(z) = d_G(z)$ for all $z \in V(G') - v_{xy}$. Since any triangle in G' not containing v_{xy} is also a triangle in G, $t' := t_{G'} \leq t + 1$. Thus $H \subseteq H_k(G')$ and $L_k(G') + x \subseteq L$. So,

$$|H_k(G')| - |L_k(G')| \geq h - (\ell - 1) \geq (\ell + 2k + t) - \ell + 1 = 1 + 2k + t \geq 2k + t'.$$

By Lemma 4.1, $|G'| \geq 3k$. As G is minimal, G' has k disjoint cycles and so does G. ■

By Corollary 2.2, $L \neq \emptyset$. Fix an $x \in L$. Let $\mathcal{T}$ be a set of disjoint triangles in G such that (a) $x \in V(\mathcal{T})$, and (b) subject to (a), $|\mathcal{T}|$ is maximum. By Lemma 4.2, $|\mathcal{T}| \geq 1$. Let $T_0 = T_0(\mathcal{T})$ be the triangle in $\mathcal{T}$ containing x; say $T_0 = xyz$.

Define an auxiliary digraph $\mathcal{D} = \mathcal{D}(\mathcal{T})$ with $V(\mathcal{D}) = \mathcal{T}$ and $\overrightarrow{TU} \in E(\mathcal{D})$ if and only if $T, U \in \mathcal{T}$ and $\|v, U\| = 3$ for some $v \in T$. If $v \in T$ and $\|v, U\| = 3$, we say the vertex v *witnesses* the edge $\overrightarrow{TU}$. We say a vertex T is *reachable from a vertex S* if there exists a directed ST-path in $\mathcal{D}(\mathcal{T})$. Let $\mathcal{R} = \mathcal{R}(\mathcal{T}) \subseteq \mathcal{T}$ be the set of triangles from which T_0 is reachable in $\mathcal{D}(\mathcal{T})$. Let $r = |\mathcal{R}|$. Since $T_0 \in \mathcal{R}$, $r \geq 1$. Finally, define $B = B(\mathcal{T}) = \{v \in V(G) \setminus V(\mathcal{T}) : \|v, T_0\| = 3\}$. By the definitions of $\mathcal{R}$ and B, if $\|v, T_0\| = 3$ for a vertex v, then $v \in V(\mathcal{R}) \cup B$.

Lemma 4.3. *If $|B| \leq 1$, then $\|v, T\| = 3$ for some vertex $v \notin V(\mathcal{R}) \cup B$ and triangle $T \in \mathcal{R}$.*

Proof. Suppose $|B| \leq 1$ and $\|v, T\| \leq 2$ for every $v \notin V(\mathcal{R}) \cup B$ and $T \in \mathcal{R}$.

Case 1: $r \leq k - 2$. Let $G' = G - V(\mathcal{R})$ and observe $t_{G'} \leq t - r$. We will find $k' := k - r$ disjoint cycles in G'. For each $v \notin V(\mathcal{R}) \cup B$, $\|v, V(\mathcal{R})\| \leq 2r$, so $d_{G'}(v) \geq d_G(v) - 2r$. Thus $H \setminus (V(\mathcal{R}) \cup B) \subseteq H_{k'}(G')$ and $L_{k'}(G') \subseteq (L \setminus V(\mathcal{R})) \cup B$. As $x \in L \cap V(\mathcal{R})$ and $|B| \leq 1$,

$$|H_{k'}(G')| \geq h - (3r - 1) - |B| \geq h - 3r \quad \text{and} \quad |L_{k'}(G')| \leq (\ell - 1) + |B| \leq \ell.$$

Combining these inequalities yields

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 3r) - \ell \geq 2(k - r) + (t - r) \geq 2k' + t_{G'}.$$

As $k' \geq 2$ and $|G'| = |G| - 3r \geq 3k'$, G' contains k' disjoint cycles by the minimality of G, and thus G has k disjoint cycles.

Case 2: $r = k - 1$. Let $\mathcal{R}^- = \mathcal{R} - T_0$ and consider $G' = G - V(\mathcal{R}^-)$. For each $v \notin V(\mathcal{R}) \cup B$, since $\|v, V(\mathcal{R}^-)\| \leq 2(r - 1)$, $d_{G'}(v) \geq d_G(v) - 2k + 4$. This implies that $H \setminus (V(\mathcal{R}) \cup B) \subseteq H_2(G') \setminus T_0$ and, since each vertex in B is adjacent to three vertices in $T_0 \subseteq G'$, $L_2(G') \setminus T_0 \subseteq L \setminus V(\mathcal{R})$. Therefore, since $x \in L \cap V(\mathcal{R})$ and $|B| \leq 1$,

$$|H_2(G') \setminus T_0| \geq h - (3r - 1) - |B| \geq h - 3k + 3 \quad \text{and} \quad |L_2(G') \setminus T_0| \leq \ell - 1.$$

Since $t = k - 1$, these inequalities give

$$|H_2(G') \setminus T_0| - |L_2(G') \setminus T_0| \geq (h - 3k + 3) - \ell + 1 \geq (\ell + 2k + (k - 1)) - 3k + 3 - \ell + 1 = 3.$$

If $\|u, T_0\| = 3$, then u is the unique vertex in B; in this case let e be an edge from u to T_0 and let $G'' = G' - e$. Otherwise, let $G'' = G'$. Since in both cases, $d_{G''}(v) = d_{G'}(v)$ for $v \in V(G') \setminus (T_0 + u)$, $|H_2(G'') \setminus T_0| - |L_2(G'') \setminus T_0| \geq 2$. By Lemma 3.3, G'' contains two disjoint cycles, and so G contains k disjoint cycles, a contradiction. ■

Lemma 4.4. *If $v \notin V(\mathcal{R}) \cup B$ and $\|v, T\| = 3$ for some $T \in \mathcal{R}$, then there are a vertex $v' \in V(\mathcal{T})$ and a set $\mathcal{T}'$ of disjoint triangles such that $xyz \in \mathcal{T}'$, $|\mathcal{T}'| = |\mathcal{T}|$, $B(\mathcal{T}') = B + v'$, and $V(\mathcal{T}') = V(\mathcal{T}) + v - v'$.*

Proof. Let $T = T_j, T_{j-1}, \dots, T_0$ be a $T \to T_0$ path in $\mathcal{D}(\mathcal{T})$ and, for each $i \in [j]$, let v_i witness the edge $\overrightarrow{T_i T}_{i-1}$. Define the triangle T_j' to be $T_j - v_j + v$ and the triangle T_i' to be $T_i - v_i + v_{i+1}$ for all $i \in [j-1]$. Then, $\mathcal{T}' = (\mathcal{T} \setminus \{T_1, \dots, T_j\}) \cup \{T_1', \dots, T_j'\}$ is a set of $|\mathcal{T}|$ disjoint triangles in G, $v' := v_1 \notin V(\mathcal{T}') \cup B$, and $\|v', T_0\| = 3$. Thus $B + v' = B(\mathcal{T}')$. ■

Now choose $\mathcal{T}$ subject to (a) and (b) so that B is maximum.

Lemma 4.5. *$|B| = 2$. Moreover, $\|v, T_0 \cup B\| \leq 2$ for all $v \notin V(\mathcal{T}) \cup B$.*

Proof. As B is maximum, Lemmas 4.3 and 4.4 imply $|B| \geq 2$. Fix a vertex $u_1 \in B$ and let T_0' be the triangle xyu_1. Observe $\mathcal{T}' = \mathcal{T} - T_0 + T_0'$ is a set of $|\mathcal{T}|$ disjoint triangles in G. Let $\mathcal{R}' = \mathcal{R}(\mathcal{T}')$, $r' = |\mathcal{R}'|$, $B' = B'(\mathcal{T}')$ and note $z \in B'$. If $|B'| \geq 2$, let $\mathcal{T}'' = \mathcal{T}'$. Otherwise by Lemma 4.3, there are $v \notin V(\mathcal{R}') \cup B'$ and $T \in \mathcal{R}'$ with $\|v, T\| = 3$. By Lemma 4.4, there are $z' \in V(\mathcal{T}')$ and a set $\mathcal{T}''$ of triangles satisfying $T_0' \in \mathcal{T}''$, $|\mathcal{T}''| = |\mathcal{T}'|$, and $B(\mathcal{T}'') = \{z, z'\}$.

If $|B| \geq 3$, then pick $u_2 \in B \setminus \{u_1, v\}$. As $V(\mathcal{T}'') \setminus V(\mathcal{T}) \subseteq \{u_1, v\}$, $u_2 \notin V(\mathcal{T}'')$. Thus $\mathcal{T}'' - T_0' \cup \{xu_1z', yu_2z\}$ is a set of $|\mathcal{T}| + 1$ disjoint triangles containing x, contradicting (b). So $|B| = 2$.

Lastly, if $v \notin V(\mathcal{T}) \cup B$ and $\|v, T_0 \cup B\| \geq 3$, then v has neighbors $w \in T_0$ and $u \in B$. Thus vuw and $(T_0 - w) \cup (B - u)$ are disjoint triangles in $T_0 \cup (B + v)$, contradicting (b). ■

Let $T^* := G[T_0 \cup B]$. Define a second auxiliary digraph $\mathcal{D}^*(\mathcal{T})$ to have vertex set $\mathcal{T} - T_0 + T^*$ and $\overrightarrow{TU} \in E(\mathcal{D}^*(\mathcal{T}))$ if and only if $\|v, U\| \geq 3$ for some $v \in T$. Again, we say the vertex v *witnesses* the edge $\overrightarrow{TU}$. Define the set of graphs $\mathcal{R}^*$ to be T^* together with the set of triangles from which T^* is reachable in $\mathcal{D}^*(\mathcal{T})$. Let $r^* = |\mathcal{R}^*|$.

Lemma 4.6. *If $v \in V(G) \setminus V(\mathcal{R}^*)$, then $\|v, T\| \leq 2$ for each $T \in \mathcal{R}^*$.*

Proof. Suppose $v \in V(G) \setminus V(\mathcal{R}^*)$, $T \in \mathcal{R}^*$, and $\|v, T\| \geq 3$. Let $T = T_j$, $T_{j-1}, \dots, T_1, T^*$ be a $T \to T^*$ path in $\mathcal{D}^*(\mathcal{T})$. By Lemma 4.5, v is adjacent to at most two vertices in T^*, so $j \geq 1$.

Let v_1 witness the edge $\overrightarrow{T_1 T^*}$ and, for $i \in \{2, \dots, j\}$, let v_i witness the edge $\overrightarrow{T_i T}_{i-1}$. As in the proof of Lemma 4.4, define the triangle T_j' to be $T_j - v_j + v$ and the triangle T_i' to be $T_i - v_i + v_{i+1}$ for all $i \in [j-1]$. If $\|v_1, T_0\| = 3$, then $\mathcal{T}' = \mathcal{T} \setminus \{T_1, \dots, T_j\} \cup \{T_1', \dots, T_j'\}$ is a set of $|\mathcal{T}|$ triangles in G, but $B + v_1 = B(\mathcal{T}')$, contradicting the maximality of B. Otherwise, there exist a vertex $w \in N(v_1) \cap T_0$, a vertex $u \in N(v_1) \cap B$, and a triangle $T_0' = (T_0 - w) \cup (B - u)$. Then $\mathcal{T}' = \mathcal{T} \setminus \{T_0, T_1, \dots, T_j\} \cup \{v_1wu, T_0', \dots, T_j'\}$ is a set of $|\mathcal{T}| + 1$ disjoint triangles in G, contradicting the maximality of $\mathcal{T}$. ■

Proof of Theorem 1.4. Let $G' = G - V(\mathcal{R}^*)$. Set $k' = k - r^*$ and $t' = t - r^*$. Then $k' \geq 1$. By Lemma 4.5, B has the form $\{w_1, w_2\}$, and by Lemma 4.6, every $v \in V(G')$ satisfies

$$d_{G'}(v) \geq d_G(v) - 2r^*. \tag{5}$$

Thus $H \setminus V(\mathcal{R}^*) \subseteq H_{k'}(G')$ and $L_{k'}(G') \subseteq (L \setminus V(\mathcal{R}^*))$. As $x \in L \cap T_0$, this implies

$$|H \cap H_{k'}(G')| \geq h - (3r^* - 1) - |B| \geq h - 3r^* - 1 \quad \text{and}$$
$$|L_{k'}(G')| \leq |L \cap V(G')| \leq \ell - 1.$$

Combining these inequalities yields

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 3r^* - 1) - (\ell - 1) \geq 2(k - r^*) + (t - r^*) = 2k' + t' \tag{6}$$

and

$$|H \cap H_{k'}(G')| - |L \cap V(G')| \geq (h - 3r^* - 1) - (\ell - 1) \geq 2(k - r^*) + (t - r^*) = 2k' + t'. \tag{7}$$

Case 1: $|G'| = 3k' - 1$. As $H'_k(G') \neq \emptyset$, $\Delta(G') \geq 2k'$ and $2k' + 1 \leq |G'| = 3k' - 1$. So $k' \geq 2$. Let $G^+ = G' \vee K_1$, where $V(K_1) = \{u\}$. Then $|G^+| = 3k'$ and $t_{G^+} \leq t' + 1$. So

$$|H_{k'}(G^+)| - |L_{k'}(G^+)| \geq |H_{k'}(G') + u| - |L_{k'}(G')| \geq 2k' + t' + 1 \geq 2k' + t_{G^+}.$$

As $|G|$ is minimal, G^+ has a set $\mathcal{S}'$ of k' disjoint triangles. Since $|G^+| = 3k'$, we may assume $T' = uu_1u'_1 \in \mathcal{S}'$. Let $T'' = xyw_1$ and $\mathcal{S} = (\mathcal{S}' - T') \cup (\mathcal{R} - T_0 + T'')$. Thus $t = k - 1$, $h \geq 2k + t + \ell = 3k$ and $\ell = 1$. So $H = V(G) - x$. Let $u_2u'_2 := zw_2$, $U = \{u_1, u'_1, u_2, u'_2\}$, and note that $u_1u'_1, u_2u'_2 \in E(G - V(\mathcal{S}))$.

Since $U \subseteq H$, and $G[U]$ is acyclic, $\|U, V(G) \setminus U\| \geq 8k - 6 > 8(k - 1)$. Thus $\|U, T\| \geq 9$ for some $T = q_1q_2q_3 \in \mathcal{S}$. Say $\|q_1, U\| \leq \|q_2, U\| \leq \|q_3, U\|$. Then $q_2u_iu'_i$ is a triangle for some $i \in [2]$. Now $\|\{q_1, q_3\}, \{u_{3-i}, u'_{3-i}\}\| \geq 2$, so $\{q_1, q_3, u_{3-i}, u'_{3-i}\}$ contains a cycle. Thus G has k disjoint cycles, a contradiction.

Case 2: $|G'| \geq 3k'$. If $k' \geq 2$ then (5), (6), and the minimality of G imply G' contains k' cycles and so G contains k cycles. So assume $k' = 1$ and G' is acyclic.

By (7), $|H \cap H_{k'}(G')| - |L \cap V(G')| \geq 2$. Thus, there is a component G_0 of G' with

$$|H \cap H_{k'}(G_0)| - |L \cap V(G_0)| \geq 1. \tag{8}$$

By (5), $|G_0| \geq 3$. Let $W_0 = V(G_0)$ and $G'_0 = G[T^* \cup W_0]$. By Lemma 4.5 and the fact that G_0 has no isolated vertices,

$$d_{G'_0}(v) \geq \begin{cases} 4, & \text{if } v \in H \cap W_0; \\ 1, & \text{if } v \in L \cap W_0; \\ 3, & \text{if } v \in W_0 \setminus (L \cup H). \end{cases}$$

By this and (8),

$$\begin{aligned} \|W_0, T^*\| &= \sum_{v \in W_0} d_{G'_0}(v) - 2\|G_0\| \geq 2.5|W_0| + 1.5(|H \cap W_0| - |L \cap W_0|) - 2(|W_0| - 1) \\ &\geq 0.5|W_0| + 1.5 + 2 \geq 5. \end{aligned}$$

It follows that there are $w \in T_0$ and $u \in B$ such that $\|\{w, u\}, W_0\| \geq 2$. Then $G[W_0 \cup \{w, u\}]$ contains a cycle, and $(T_0 - w) \cup (B - u)$ induces a triangle. This gives k disjoint cycles. ■

5. PROOF OF COROLLARY 1.5

Suppose an integer $k \geq 2$ and a graph G satisfy $h - \ell \geq 2k + t + 1$, and G has no k disjoint cycles. By Lemmas 3.1 and 3.2, $k \geq 3$. Let $|G| = 3k' + r$, where $k' = \lfloor |G|/3 \rfloor$

and $0 \le r \le 2$. By Theorem 1.4, $3k-1 \ge |G| \ge h \ge 2k+1 \ge 7$, so $k-1 \ge k' \ge 2$. Pick $R \subset V(G)$ so that $G' := G - R$ has t disjoint triangles. Let $r = |R|$. Then $t_{G'} = t$, and $d_{G'}(v) \ge d_G(v) - 2$ for each $v \in V(G')$. Thus

$$|H_{k'}(G')| - |L_{k'}(G')| \ge |H \setminus R| - \ell \ge 2k + t + 1 - r \ge 2k' + t_{G'} + 1.$$

By Theorem 1.4, G' has k' disjoint triangles, so $t_{G'} = k'$ and $|H_{k'}(G')| \ge 3k' + 1 > |G'|$, a contradiction.

6. PROOF OF THEOREM 1.7

The proof will be by contradiction. Consider the smallest k such that there exists a counterexample G, and choose such G to be minimal. If $k = 2$, then $h \ge 4$, so $G = K_5$ or $|G| \ge 6$. As G is planar, G contains neither K_5 nor SK_5. Thus by Lemma 3.2, G has two disjoint cycles. Hence $k \ge 3$.

We first show that $L \ne \emptyset$. Since G is planar, $\|G\| \le 3|G| - 6$ and the average degree is less than 6. If $k \ge 4$, then $L \ne \emptyset$ follows immediately. If $k = 3$ and $\delta(G) = 5$, then since $h \ge 2k = 6$, $\|G\| \ge \frac{1}{2}(36 + 5(|G| - 6))$. This implies $|G| \ge 18 = 6k$, and by Corollary 2.2, $L \ne \emptyset$.

Let $x \in L$. We claim that

$$\text{for every } y \in N(x), \text{ the edge } xy \text{ is contained in a triangle.} \tag{9}$$

Indeed, if xy is not in a triangle, then consider the graph $G^* = G/xy$. The degree of every vertex other than x or y remains unchanged and the degree of v_{xy} is at least the degree of y. Therefore, $|H_k(G^*)| \ge |L_k(G^*)| + 2k$ and by the minimality of G, G^* contains k disjoint cycles. Expanding the edge xy yields k-disjoint cycles in G. This proves (9).

Fix a plane drawing of G. Every triangle T separates the plane into the exterior region $R_1(T)$ and interior region $R_2(T)$. Among all triangles containing x, choose T' so that $R_2(T')$ contains the fewest vertices. Let $T' = xyz$, $R_1 = R_1(T')$, and $R_2 = R_2(T')$. By (9), R_2 contains no neighbors of x.

Suppose G has two vertices v_1 and v_2 adjacent to all three vertices of T'. By the choice of T' and R_2, both v_1 and v_2 are in R_1. The planar drawing induced by $T' \cup \{v_1, v_2\}$ contains no edges in the interior of R_2. Adding a vertex v in R_2 adjacent to all three vertices of T' gives a planar embedding of $K_{3,3}$, a contradiction. So G has at most one vertex v_1 adjacent to all three vertices of T'.

Let $G' = G - T'$, $k' = k - 1$. Then for each $u \in V(G) - v_1$, $d_{G'}(u) \ge d_G(u) - 2$ and $d_{G'}(v_1) = d_G(v_1) - 3$. It follows that $|H \cap \{v_1\}| + |L_{k'}(G') \cap \{v_1\}| \le 1$. Hence

$$|H_{k'}(G')| - |L_{k'}(G')| \ge (h - 2 - |H \cap \{v_1\}|) - (\ell - 1 + |L_{k'}(G') \cap \{v_1\}|) \ge 2k - 2 = 2k'.$$

By the minimality of G, G' contains $k-1$ disjoint cycles, and so G contains k disjoint cycles.

7. PROOF OF THEOREM 1.8

Following Dirac and Erdős [5], let $V_{\ge s}(G)$ (respectively, $V_{\le s}(G)$) denote the set of vertices of G of degree at least s (respectively, at most s). In these terms, $H = H_k(G) = V_{\ge 2k}(G)$ and $L = L_k(G) = V_{\le 2k-2}(G)$. The following lemma may be of interest on its own.

Lemma 7.1. *Let G be a triangle-free graph with $V(G) \neq \emptyset$. If*

$$|V_{\geq 2k+1}(G)| - |V_{\leq 2k-1}(G)| \geq 2k-2, \tag{10}$$

then G has k disjoint cycles.

Proof. Suppose the lemma does not hold and consider the smallest k such that there exists a counterexample. Among all such counterexamples, choose the graph G to be minimal. First consider $k = 1$. Since $|V_{\geq 3}(G)| \geq |V_{\leq 1}(G)|$, G contains a component with average degree at least 2. Therefore, G contains a cycle and the claim holds. Now, let $k \geq 2$.

By (10), the sum of degrees of the vertices in $V_{\geq 2k}(G)$ is greater than the sum of degrees of the vertices in $V_{\leq 2k-1}(G)$. Thus there are vertices $u, v \in V_{\geq 2k}(G)$ such that $uv \in E(G)$. Since G is triangle-free, $N(v) \cap N(u) = \emptyset$ and so $|G| \geq 4k$. Since G has no k disjoint cycles, by Theorem 1.1, G has a vertex $x \in V_{\leq 2k-1}(G)$.

As in Property 2.3, if $d(x) \leq 1$, then $G - x$ is a smaller counterexample, so $d(x) \geq 2$. Let $y \in N(x)$. Since G is triangle-free, contracting the edge xy does not change the degree of any vertex distinct from x, y. By the minimality of G, G/xy contains either k disjoint cycles or a triangle. If G/xy contains k disjoint cycles, then G does as well. Otherwise, let $v_{xy}zw$ be a triangle in G/xy. Then by symmetry we may assume $xyzw$ is a 4-cycle in G. Every vertex in $G - \{w, x, y, z\}$ is adjacent to at most two vertices in $\{w, x, y, z\}$.

Let $k' = k - 1$ and $G' = G - \{w, x, y, z\}$. Then, for each $v \in V(G')$, $d_{G'}(v) \geq d_G(v) - 2$, so $|V_{\geq 2k'+1}(G')| \geq |V_{\geq 2k+1}(G)| - 3$ and $|V_{\leq 2k'-1}(G')| \leq |V_{\leq 2k-1}(G)| - 1$. Therefore,

$$|V_{\geq 2k'+1}(G')| - |V_{\leq 2k'-1}(G')| \geq |V_{\geq 2k+1}(G)| - 3 - (|V_{\leq 2k-1}(G)| - 1) \geq 2k - 2 - 2 = 2k' - 2.$$

By the minimality of G, G' contains k' disjoint cycles. Hence G contains k disjoint cycles. ■

Suppose that Theorem 1.8 is false and let k be the smallest integer such that there exists a counterexample. Among all counterexamples, choose G to be minimal.

Lemma 7.2. $|G| \geq 4k - 1$ *and* $L \neq \emptyset$.

Proof. Suppose $|G| \leq 4k - 2$. For all $u \in H$, $|N(u) \cap H| \geq 2$ and if also $w \in H$ then $|N(w) \cap N(u)| \geq 2$. It suffices to show that G has two disjoint triangles. As $h \geq 2k \geq 6$, if $G[H]$ is a complete graph, then we are done, so assume there are $x, y \in H$ with $xy \notin E(G)$.

Choose $w \in N(x) \cap H$, $z \in N(y) \cap H - w$, and $v \in N(w) \cap N(x) - z$. If $N(y) \cap N(z) \neq \{v, w\}$, then there are two triangles in G; else put $Q = \{v, w, y, z\}$ and $P = N(x) \setminus Q$. Now $|P| \geq 2k - 3 \geq k$. If there is $u \in P$ with $d(u) \geq 2k - 1$, then there is $t \in N(x) \cap N(u)$. Thus txu is a triangle, and $Q - t$ contains another triangle. So $V(P) \subseteq L$ and $|L| \geq k$. Therefore, $|G| \geq h + \ell \geq 2\ell + 2k \geq 4k$. ■

Lemma 7.3. *If $x \in L$, then x is not contained in a triangle.*

Proof. Let $x \in L$ and suppose T_0 is a triangle in G containing x. Let $B = B(T_0) = \{v \in V(G) : \|v, T_0\| = 3\}$ and fix $T_0 = xyz$ so that $|B|$ is minimized. Let $k' = k - 1$, $G' = G - T_0$. For each $v \in V(G') \setminus B$, $d_{G'}(v) \geq d_G(v) - 2$, so $|H_{k'}(G')| \geq |H \setminus (B \cup T_0)|$ and $|L_{k'}(G')| \leq |L \setminus (B \cup T_0)| \leq \ell - 1$.

If $|B| \leq 1$, then $|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 3) - (\ell - 1) \geq 2k - 2 = 2k'$. Since G' is triangle-free, by Theorem 1.4, G' contains $k - 1$ disjoint cycles. Then G contains k disjoint cycles. Similarly, if $|B| = 2$ and $B \cup T_0$ contains at most three vertices in H, then G' contains k disjoint cycles. So we may assume that $|B| \geq 2$ and $B \cup T_0$ contains at least four vertices in H. We complete the proof in three cases.

Case 1: *B is an independent set.* Let $u_1, u_2 \in B$ and $T_1 = xyu_1$. If $v \notin B \cup T_0$ and $\|v, T_1\| = 3$, then xu_1v and yzu_2 are two disjoint triangles in G. Let $k' = k - 1$, $G'' = G - T_1$. For each $v \in V(G'') - z$, $d_{G''}(v) \geq d_G(v) - 2$ and $d_{G''}(z) = d_G(z) - 3$. So possibly $z \in H \setminus H_{k'}(G'')$ or $z \in L_{k'}(G'') \setminus L$, but not both, that is, $|\{z\} \cap H| + |\{z\} \cap L''| \leq 1$. Therefore,

$$\begin{aligned} |H_{k'}(G'')| - |L_{k'}(G'')| &\geq (h - 2 - |\{z\} \cap H|) - (\ell - 1 + |\{z\} \cap L_{k'}(G'')| \\ &\geq (h - \ell) - 1 - (|\{z\} \cap H| + |\{z\} \cap L_{k'}(G'')|) \\ &\geq 2k - 2 = 2k'. \end{aligned} \tag{11}$$

By Theorem 1.4, G'' contains $k - 1$ disjoint cycles. Then G contains k disjoint cycles.

Case 2: $|B| \geq 3$. Let $u_1, u_2, u_3 \in B$ and, by Case 1 assume $u_1u_2 \in E(G)$. Then xu_1u_2 and yzu_3 are two triangles in G, a contradiction.

Case 3: $|B| = 2$. Let $u_1, u_2 \in B$ and, by Case 1, assume $u_1u_2 \in E(G)$. In particular $B \cup T_0 \cong K_5$ and every vertex in $B \cup T_0$ apart from x is in H. If $v \notin B \cup T_0$ is adjacent to two vertices in $B \cup T_0$, then G contains two disjoint triangles, a contradiction. Let $k' = k - 1$ and $G' = G - (B \cup T_0)$. For each $v \in V(G')$, $d_{G'}(v) \geq d_G(v) - 1$. In particular, $|V_{2k'+1}(G')| \geq h - 4$ and $|V_{2k'-1}(G')| \leq \ell - 1$. Therefore,

$$|V_{2k'+1}(G')| - |V_{2k'-1}(G')| \geq (h - 4) - (\ell - 1) \geq 2k - 3 = 2k' - 1. \tag{12}$$

The graph G' is triangle-free, so by Lemma 7.1, G' contains $k - 1$ disjoint cycles. Then G contains k disjoint cycles. ■

Lemma 7.4. *If $x, z \in L$, then $|N_G(x) \cap N_G(y)| \leq 1$.*

Proof. Suppose $w, y \in N_G(x) \cap N_G(z)$. Then $X = wxyz$ is a copy of C_4 in G. If $v \notin X$ is adjacent to at least three vertices in X, then either x or z is contained in a triangle, contradicting Lemma 7.3. Let $G' = G - X$. For each $v \in V(G')$, $d_{G'}(v) \geq d_G(v) - 2$. Therefore,

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 2) - (\ell - 2) \geq 2k = 2k' + 2. \tag{13}$$

Since G' contains at most one triangle, by Theorem 1.4, G' contains $k - 1$ disjoint cycles. Then G contains k disjoint cycles. ■

Let $L = \{x_1, \ldots, x_\ell\}$ and, for each i, let $y_i \in N_G(x_i)$. Starting with the graph $G = G_0$, we construct a sequence of graphs by defining $G_i = G_{i-1} / x_iy_i$. For simplicity, if we contract the edge x_iy_i, we label the contracted vertex in G_i as y_i. We terminate this process if G_i contains k cycles or when $i = \min\{\ell, k - 1\}$. Suppose, after terminating the process, we have defined graphs $G_0, \ldots, G_r$ for some nonnegative integer r.

Lemma 7.5. *For the graphs $G_0, \ldots, G_r$ and $i \in \{0, \ldots, r\}$, all of the following hold:*

(1) $|G_i| = |G_0| - i \geq 3k$;
(2) *if $i < r$, then G_i contains $i + 1$ disjoint triangles;*
(3) *L_i is an independent set;*
(4) *if $x \in L_k(G_i)$, then $N_{G_i}(x)$ is an independent set;*
(5) *if $x, x' \in L_k(G_i)$, then $|N_{G_i}(x) \cap N_{G_i}(x')| \leq 1$;*
(6) *$L_k(G_i) = L_0 - \{x_1, \ldots, x_i\}$ and $H_k(G_i) \supseteq H_k(G_0)$;*
(7) *if $i \geq 1$ and G_i contains k disjoint cycles, then G_{i-1} does as well.*

Proof. For all i, (1) holds by Lemma 7.2 and (7) holds since a contraction cannot increase the number of disjoint cycles.

The proof of (2)–(6) will be by induction on i. By assumption, G_0 contains at most one triangle. If G is triangle-free, then by Theorem 1.4, G_0 contains k disjoint cycles, so (2) holds for $i = 0$. Since G_0 is a minimum counterexample, (3) holds for $i = 0$ by Property 2.3. Further, (4) and (5) hold for $i = 0$ by Lemma 7.3 and Lemma 7.4, respectively. And (6) is trivial for $i = 0$.

Suppose that $r \geq 1$ and consider $i \in \{1, \dots, r\}$. Assume that (2)–(6) hold for all $j < i$. Recall that $G_i = G_{i-1}/x_iy_i$. By (4) for $i-1$, $d_{G_i}(y_i) \geq d_{G_{i-1}}(y_i)$ and no other vertex v is adjacent to both x_i and y_i, so $d_{G_i}(v) = d_{G_{i-1}}(v)$. Thus, (6) holds.

To see that (3) holds, observe if $uv \notin E(G_{i-1})$, then $uv \in E(G_i)$ only if $u, v \in N_{G_{i-1}}(x_i)$. Since L_{i-1} is an independent set and $L_k(G_i) \supseteq L_k(G_{i-1})$ by (6), $L_k(G_i)$ is also an independent set.

If $x \in L_k(G_i)$, then by (6), $x \in L_k(G_{i-1})$ also and $x \neq x_i$. Let $y, y' \in N_{G_i}(x)$ and note that since (4) holds for G_{i-1}, $yy' \notin E(G_{i-1})$. Edges are only added to G_i between pairs of vertices in $N_{G_{i-1}}(x_i)$. Since (5) holds for $i-1$, $|N_{G_{i-1}}(x_i) \cap N_{G_{i-1}}(x)| \leq 1$, so y and y' cannot both be in $N_{G_{i-1}}(x_i) \cap N_{G_{i-1}}(x)$. Thus, $yy' \notin E(G)$ and (4) holds for i.

If $x, x' \in L_k(G_i)$, then by (6), $x, x' \in L_k(G_{i-1})$ and $|N_{G_{i-1}}(x) \cap N_{G_{i-1}}(x')| \leq 1$. Since $L_k(G_{i-1})$ is an independent set, $N_{G_i}(x) = N_{G_{i-1}}(x)$ and $N_{G_i}(x') = N_{G_{i-1}}(x')$, so $|N_{G_i}(x) \cap N_{G_i}(x')| \leq 1$ and (5) holds.

Finally, by (2), G_{i-1} contains exactly i disjoint triangles. Contracting an edge introduces increases the number of disjoint triangles by at most 1, so G_i contains at most $i+1$ disjoint triangles. By (6),

$$|H_k(G_i)| - |L_k(G_i)| \geq h - (\ell - i) \geq 2k + i. \tag{14}$$

Since $|G_i| \geq 3k$, if G_i contains i disjoint triangles, by Theorem 1.4, G_i contains k disjoint cycles and $i = r$. Therefore, if $i < r$ then G contains exactly $i+1$ disjoint triangles and (2) holds. ■

We are now ready to complete the proof. If $r < \min\{\ell, k-1\}$, then we stopped the process because G_r contains k disjoint cycles. If $r = k-1 = \min\{\ell, k-1\}$, then G_{k-2} contains $k-1$ disjoint triangles and G_{k-1} contains at least this many disjoint triangles. If G_{k-1} contains only $k-1$ disjoint triangles, then by Lemma 7.5 (6),

$$|H_k(G_{k-1})| - |L_k(G_{k-1})| \geq h - (\ell - (k-1)) \geq 2k + (k-1) = 3k - 1. \tag{15}$$

Lemma 7.2 implies that G_{k-1} contains $3k$ vertices and by Theorem 1.4, $G_r = G_{k-1}$ contains k disjoint cycles. Finally if $r = \ell = \min\{\ell, k-1\}$, then $L_r = \emptyset$ and $|H_k(G_r)| \geq 2k$. Corollary 2.2 implies $G_r = G_\ell$ contains k disjoint cycles. Therefore, in any case G_r contains k disjoint cycles and by Lemma 7.5 (7), G contains k disjoint cycles as well.

8. CONCLUDING REMARKS

Remark 1. As mentioned earlier, there are graphs G with $|G| \geq 3k$ and $|H_k(G)| - |L_k(G)| \geq 2k$ that have no k disjoint cycles, but all examples that we know have rather few vertices. The largest such graph G that we can construct has $4k$ vertices and is obtained as follows.

Let F be a copy of K_{3k-1}. Choose $W \subset V(F)$ with $|W| = k$ and delete all edges between the vertices in W. Then add $k+1$ new vertices $x_0, x_1, \ldots, x_k$, and make x_0 adjacent to $x_1, \ldots, x_k$ and all vertices in W. In other words, let $(K_{2k-1} \cup K_1) \vee \overline{K}_k$ be the 2-core of G, and complete the construction by adding k leaves adjacent to x_0, where $V(K_1) = \{x_0\}$.

Now $L_k(G) = \{x_1, \ldots, x_k\}$, and $H_k(G) = V(G) \setminus L_k(G)$. This graph has no k disjoint cycles, since its 2-core has $3k$ vertices, and x_0 does not belong to any triangle.

Is it true that every graph G with $|G| \geq 4k+1$ and $|H_k(G)| - |L_k(G)| \geq 2k$ has k disjoint cycles?

Remark 2. Lemma 7.1 suggests that considering $|V_{\geq 2k+1}(G)| - |V_{\leq 2k-1}(G)|$ instead of $|H_k(G)| - |L_k(G)|$ may result in different bounds providing the existence of k disjoint cycles. It could be that the claim of Lemma 7.1 holds not only for triangle-free graphs. That is, it could be that *for any nonempty graph G with $|V_{\geq 2k+1}(G)| - |V_{\leq 2k-1}(G)| \geq 2k-2$, G contains k disjoint cycles.* This is trivially true for $k = 1$.

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