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THE STRUCTURE OF LARGE INTERSECTING FAMILIES

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ABSTRACT. A collection of sets is *intersecting* if every two members have nonempty intersection. We describe the structure of intersecting families of r-sets of an n-set whose size is quite a bit smaller than the maximum $\binom{n-1}{r-1}$ given by the Erdős-Ko-Rado Theorem. In particular, this extends the Hilton-Milner theorem on nontrivial intersecting families and answers a recent question of Han and Kohayakawa for large n. In the case $r = 3$ we describe the structure of all intersecting families with more than 10 edges. We also prove a stability result for the Erdős matching problem. Our short proofs are simple applications of the Delta-system method introduced and extensively used by Frankl since 1977.

1. INTRODUCTION

An r-uniform hypergraph H, or simply an *r-graph*, is a family of r-element subsets of a finite set. We associate an r-graph H with its edge set and call its vertex set $V(H)$. Say that H is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in F$. A *matching* in H is a collection of pairwise disjoint sets from H. A *vertex cover* (henceforth cover) of H is a set of vertices intersecting every edge of H. Write $\nu(H)$ for the size of a maximum matching and $\tau(H)$ for the size of a minimum cover of H. Say that H is *trivial* or a *star* if $\tau(H) = 1$, otherwise call H *nontrivial*.

A fundamental problem in the extremal theory of finite sets is to determine the maximum size of an n-vertex r-graph H with $\nu(H) \leq s$. The case $s = 1$ is when H is intersecting, and in this case the Erdős-Ko-Rado Theorem [3] states that the maximum is $\binom{n-1}{r-1}$ for $n \geq 2r$ and if $n > 2r$, then equality holds only if $\tau(H) = 1$. More generally, Erdős [2] proved the following.

Theorem 1 (Erdős [2])**.** *For $r \geq 2$, $s \geq 1$ and n sufficiently large, every n-vertex r-graph H with $\nu(H) \leq s$, satisfies*

$$|H| \leq em(n, r, s) := \binom{n}{r} - \binom{n-s}{r} \sim s\binom{n}{r-1}, \tag{1}$$

and if equality in (1) *holds, then H is the r-graph $EM(n, r, s)$ described below.*

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Construction 1. *Let $EM(n,r,s)$ be the n-vertex r-graph that has s special vertices $x_1,\dots,x_s$ and the edge set consists of all the r-sets intersecting $\{x_1,\dots,x_s\}$. In particular, $EM(n,r,1)$ is a full star.*

There has been a lot of recent activity on Theorem 1 for small n (see, e.g., [10,11,16,17]).

Hilton and Milner [15] proved a strong stability result for the Erdős-Ko-Rado Theorem:

Theorem 2 (Hilton-Milner [15], Proposition $\mathcal{T}$)**.** *Suppose that $2 \le r \le n/2$ and $|H|$ is an n-vertex intersecting r-graph with $\tau(H) \ge 2$. Then*

$$|H| \le hm(n,r) := \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 \sim r\binom{n}{r-2}. \tag{2}$$

Moreover, if $4 \le r < n/2$ and (2) holds with equality, then H is the r-graph $HM(n,r)$ described below.

Construction 2. *For $n \ge 2r$, let $HM(n,r)$ be the following r-graph on n vertices: Choose an r-set $X = \{x_1,\dots,x_r\}$ and a special vertex $x \notin X$, and let $HM(n,r)$ consist of the set X and all r-sets containing x and a vertex of X.*

Observe that $HM(n,r)$ is intersecting, $\tau(HM(n,r)) = 2$, and $|HM(n,r)| = hm(n,r)$. Bollobás, Daykin and Erdős [1] extended Theorem 2 to r-graphs with matching number s in the way Theorem 1 extends the Erdős-Ko-Rado Theorem.

Theorem 3 (Bollobás-Daykin-Erdős [1], Theorem 1)**.** *Suppose $r \ge 2$, $s \ge 1$ and $n > 2r^3 s$. If H is an n-vertex r-graph with $\nu(H) \le s$ and $|H| > em(n,r,s-1) + hm(n-s+1,r)$, then $H \subseteq EM(n,r,s)$.*

The bound of Theorem 3 is also sharp: take a copy of $HM(n-s+1,r)$, add an extra set S of $s-1$ vertices and all edges intersecting with S. Han and Kohayakawa [14] refined Theorem 2 using the following construction.

Construction 3. *For $r \ge 3$, the n-vertex r-graph $HM'(n,r)$ has $r+2$ distinct special vertices $x, x_1,\dots,x_{r-1}, y_1, y_2$ and all edges e such that*
1) $\{x, x_i\} \subset e$ for any $i \in [r-1]$, or
2) $\{x, y_1, y_2\} \subset e$, or
3) $e = \{x_1,\dots,x_{r-1},y_1\}$, or $e = \{x_1,\dots,x_{r-1},y_2\}$.

Note that $HM'(n,r)$ is intersecting, $\tau(HM'(n,r)) = 2$, and $HM'(n,r) \not\subseteq HM(n,r)$. Let $hm'(n,r) = |HM'(n,r)|$ so that

$$hm'(n,r) = \binom{n-1}{r-1} - \binom{n-r}{r-1} + \binom{n-r-2}{r-3} + 2 \sim (r-1)\binom{n}{r-2}.$$

The result of [14] for $r \ge 5$ is:

Theorem 4 (Han-Kohayakawa [14])**.** *Ler $r \ge 5$ and $n > 2r$. If H is an n-vertex intersecting r-graph, $\tau(H) \ge 2$ and $|H| \ge hm'(n,r)$, then $H \subseteq HM(n,r)$ or $H = HM'(n,r)$.*

They also resolved the cases $r = 4$ and $r = 3$, where the statements are similar but somewhat more involved.

For large n, Frankl [8] gave an exact upper bound on the size of intersecting n-vertex r-graphs H with $\tau(H) \ge 3$. He introduced the following family. We write $A + a$ to mean $A \cup \{a\}$.

Construction 4 ([8])**.** *The vertex set $[n]$ of the n-vertex r-graph $FP(n,r)$ contains a special subset $X = \{x\} \cup Y \cup Z$ with $|X| = 2r$ such that $|Y| = r$, $|Z| = r-1$, where a subset $Y_0 = \{y_1, y_2\}$ of Y is specified. The edge set of $FP(n,r)$ consists of all r-subsets of $[n]$ containing a member of the family*

$$G = \{A \subset X : |A| = 3, x \in A, A \cap Y \neq \emptyset, A \cap Z \neq \emptyset\} \cup \{Y, Y_0 + x, Z + y_1, Z + y_2\}.$$

By construction, $FP(n,r)$ is an intersecting r-graph with $\tau(FP(n,r)) = 3$. Frankl proved the following.

Theorem 5 (Frankl [8])**.** *Let $r \geq 3$ and n be sufficiently large. Then every intersecting n-vertex r-graph H with $\tau(H) \geq 3$ satisfies $|H| \leq |FP(n,r)|$. Moreover, if $r \geq 4$, then equality is attained only if $H = FP(n,r)$.*

He used the following folklore result.

Proposition 6. *Every intersecting 3-graph H with $\tau(H) \geq 3$ satisfies $|H| \leq 10$.*

Note that Erdős and Lovász [4] proved the more general result that for every $r \geq 2$ each intersecting r-graph H with $\tau(H) = r$ has at most r^r edges. But their proof gives the bound 25 for $r = 3$, while Proposition 6 gives 10.

In this short paper, we determine for large n, the structure of H in the situations described above when $|H|$ is somewhat smaller than the bounds in Theorems 4 and 2. In particular, our Theorem 7 below answers for large n the question of Han and Kohayakawa [14] at the end of their paper. We also use Theorem 5 to describe large dense hypergraphs H with $\nu(H) \leq s$ and $\tau(H) = 2$. Related results can be found in [8, 9].

2. Results

First we characterize the nontrivial intersecting r-graphs that have a bit fewer edges than $hm'(n,r)$. We need to describe three constructions before we can state our result.

Construction 5. *For $r \geq 3$ and $t = n - r$, the n-vertex r-graph $HM(n,r,t)$ has r distinct special vertices $x, x_1, \ldots, x_{r-1}$ and all edges e such that*

1) $\{x, x_i\} \subset e$ for any $i \in [r-1]$, or

2) $\{x_1, \ldots, x_{r-1}\} \subset e$.

Similarly, for $r \geq 3$ and $1 \leq t \leq r-1$, the n-vertex r-graph $HM(n,r,t)$ has $r+t$ distinct special vertices $x, x_1, \ldots, x_{r-1}, y_1, y_2, \ldots, y_t$ and all edges e such that

1) $\{x, x_i\} \subset e$ for any $i \in [r-1]$, or

2) $e = \{x_1, \ldots, x_{r-1}, y_j\}$ for all $1 \leq j \leq t$, and

3) $\{x, y_1, \ldots, y_t\} \subseteq e$.

Let $hm(n,r,t) = |HM(n,r,t)|$. Note that $HM(n,r,1) = HM(n,r)$, and $HM(n,r,2) = HM'(n,r)$. For n large, we have the inequalities

$$hm(n,r) = hm(n,r,1) > \cdots > hm(n,r,r-1) = hm(n,r,r) < hm(n,r,n-r).$$

Note that $HM(n,r,t)$ is intersecting, $\tau(HM(n,r,t)) = 2$, and $HM(n,r,t) \not\subseteq HM(n,r,t-1)$. Also, for fixed $r \geq 4$ and $2 \leq t \leq n-r$,

$$hm(n,r,t) \sim (r-1)\binom{n}{r-2}.$$

Construction 6. *The n-vertex r-graph $HM(n,r,0)$ has 3 special vertices x, x_1, x_2 and all edges that contain at least two of these 3 vertices.*

By definition,

$$|HM(n,r,0)| = 3\binom{n-3}{r-2} + \binom{n-3}{r-3}. \tag{3}$$

Construction 7. *The n-vertex r-graph $HM''(n,r)$ has $r+3$ special vertices x, $x_1, \ldots, x_{r-2}$ and y_1, y_1', y_2, y_2' and all edges e such that*
1) $\{x, x_i\} \subset e$ for some $i \in [r-2]$, or
2) $\{x, y_1, y_2\} \subseteq e$, or $\{x, y_1, y_2'\} \subseteq e$ or $\{x, y_1', y_2\} \subseteq e$ or $\{x, y_1', y_2'\} \subseteq e$, or
3) $e = \{x_1, \ldots, x_{r-2}, y_1, y_1'\}$, or $e = \{x_1, \ldots, x_{r-2}, y_2, y_2'\}$.

Note that $HM''(n,r)$ is intersecting, $\tau(HM''(n,r)) = 2$, and $HM''(n,r) \not\subseteq HM(n,r,t)$ for any t. Let $hm''(n,r) = |HM''(n,r)|$ so that for $r \geq 5$,

$$\begin{aligned} hm''(n,r) &= \binom{n-1}{r-1} - \binom{n-r+1}{r-1} + 4\binom{n-r-3}{r-3} \\ &\quad + 4\binom{n-r-3}{r-4} + \binom{n-r-3}{r-5} + 2 \\ &\sim (r-2)\binom{n}{r-2}. \end{aligned} \tag{4}$$

Theorem 7. *Fix $r \geq 4$. Let n be sufficiently large. If H is an n-vertex intersecting r-graph with $\tau(H) \geq 2$ and $|H| > hm''(n,r)$, then $H \subseteq HM(n,r,t)$ for some $t \in \{1, \ldots, r-1, n-r\}$ or $r = 4$ and $H \subseteq HM(n,4,0)$. The bound on H is sharp due to $HM''(n,r)$.*

When $r = 3$ we are able to obtain stronger results than Theorem 7, and describe the structure of *almost all* intersecting 3-graphs. We will use the following construction.

Construction 8. *Let $n \geq 6$.*
- *For $i = 0, 1, 2$, let*

$$H_i(n) = HM(n,3,i) \qquad \text{and} \qquad H(n) = EM(n,3,1).$$

- *The n-vertex 3-graph $H_3(n)$ has special vertices v_1, v_2, y_1, y_2, y_3 and its edges are the $n-2$ edges containing $\{v_1, v_2\}$ and the 6 edges each of which contains one of v_1, v_2 and two of y_1, y_2, y_3.*
- *Each of the n-vertex 3-graphs $H_4(n)$ and $H_5(n)$ has 6 special vertices $v_1, v_2, z_{1,1}$ $z_{1,1}', z_{2,1} z_{2,1}'$ and contains all edges containing $\{v_1, v_2\}$. Apart from these, $H_4(n)$ contains edges*

$$v_1 z_{1,1} z_{1,1}', v_1 z_{2,1} z_{2,1}', v_2 z_{1,1} z_{2,1}, v_2 z_{1,1} z_{2,1}', v_2 z_{1,1}' z_{2,1}, v_2 z_{1,1}' z_{2,1}'$$

and $H_5(n)$ contains edges

$$v_1 z_{1,1} z_{1,1}', v_1 z_{2,1} z_{2,1}', v_1 z_{1,1} z_{2,1}', v_2 z_{1,1} z_{2,1}', v_2 z_{1,1} z_{2,1}, v_2 z_{1,1}' z_{2,1}'.$$

Theorem 8. *Let H be an intersecting 3-graph and $n = |V(H)| \geq 6$. If $\tau(H) \leq 2$, then H is contained in one of $H(n), H_0(n), \ldots, H_5(n)$. This yields that*
(a) if $|H| \geq 11$, then H is contained in one of $H(n), H_0(n), \ldots, H_5(n)$;
(b) if $|H| > n+4$, then H is contained in $H(n), H_0(n), H_1(n)$ or $H_2(n)$.

The restriction $|H| \geq 11$ cannot be weakened because of K_5^3 and $|H| > n+4$ cannot be weakened because $|H_3(n)| = |H_4(n)| = |H_5(n)| = n+4$.

To prove an analog of Theorem 8 for r-graphs, we need an extension of Construction 8:

Construction 9. *Let $n \geq r+1$. For $i = 0, \dots, 5$, let the r-graph $H_i^r(n)$ have the vertex set of the 3-graph $H_i(n)$ and the edge set of $H_i^r(n)$ consist of all r-tuples containing an edge of $H_i(n)$.*

By definition, $H_0^r(n) = HM(n, r, 0)$. Each $H_i^r(n)$ is intersecting, since each $H_i(n)$ is intersecting. Using Theorem 5, we extend Theorem 8 as follows:

Theorem 9. *Let $r \geq 4$ be fixed and n be sufficiently large. Then there is $C > 0$ such that for every intersecting n-vertex r-graph H with $|H| > |FP(n, r)| = O(n^{r-3})$, one can delete from H at most Cn^{r-4} edges so that the resulting r-graph H' is contained in one of $H_0^r(n), \dots, H_5^r(n), EM(n, r, 1)$.*

The results above naturally extend to r-graphs H with $\nu(H) \leq s$. For example, Theorem 7 extends to the following result which implies Theorem 3 for large n.

Theorem 10. *Fix $r \geq 4$ and $s \geq 1$. Let n be sufficiently large. If H is an n-vertex r-graph with $\nu(H) \leq s$ and $|H| > em(n, r, s-1) + hm''(n-s+1, r)$, then $V(H)$ contains a subset $Z = \{z_1, \dots, z_{s-1}\}$ such that either $\tau(H - Z) = 1$ or $H - Z \subseteq HM(n-s+1, r, t)$ for some $t \in \{1, \dots, r-1, n-s+1-r\}$ or $r = 4$ and $H - Z \subseteq HM(n-s+1, 4, 0)$. The bound on $|H|$ is sharp.*

Theorems 4 and 9 can be extended in a similar way. We leave this to the reader.

3. Proof of Theorem 7

The main tool used in the proof is the Delta-system method developed by Frankl (see, e.g. [6,8]). Recall that a *k-sunflower* S is a collection of distinct sets $S_1, \dots, S_k$ such that for every $1 \leq i < j \leq k$, we have $S_i \cap S_j = \bigcap_{\ell=1}^k S_\ell$. The common intersection of the S_i is the *core* of S. We will use the following fundamental result of Erdős and Rado [5].

Lemma 11 (Erdős-Rado Sunflower Lemma [5])**.** *For every $k, r \geq 2$ there exists $f(k, r) < k^r r!$ such that the following holds: every r-graph H with no k-sunflower satisfies $|H| < f(k, r)$.*

Proof of Theorem 7. Let $r \geq 4$ and H be an n-vertex intersecting r-graph with $\tau(H) \geq 2$ and $|H| > hm''(n, r)$. Define $B^*(H)$ to be the set of $T \subset V(H)$ such that

(i) $0 < |T| < r$, and

(ii) T is the core of an $(r+1)^{|T|}$-sunflower in H.

Define

$$B'(H) = \{T \in B^*(H) : \nexists U \in B^*(H), U \subsetneq T\}$$

to be the set of all inclusion minimal elements in $B^*(H)$. Next, let

$$B''(H) = \{e \in H : \nexists T \subsetneq e, T \in B^*(H)\}$$

be the set of edges in H that contains no member of $B^*(H)$. Finally, set

$$B(H) = B'(H) \cup B''(H).$$

Let B_i be the family of the sets in $B(H)$ of size i. Note that $B_1 = \emptyset$ for otherwise we have an $(r+1)$-sunflower with core of size one and since H is intersecting, this forces H to be trivial. Similarly, if $2 \le i \le r-1$ and $T, T' \in B_i$, then $T \cap T' \ne \emptyset$, since otherwise H would have disjoint edges $A \supset T$ and $A' \supset T'$. Thus for each $2 \le i \le r-1$, B_i is an intersecting family. The following crucial claim proved by Frankl can be found in Lemma 1 in [6, 8]. □

Claim. B_i contains no $(r+1)^{i-1}$-sunflower.

Proof of Claim. Suppose for contradiction that $S_1, \dots, S_{(r+1)^{i-1}}$ is an $(r+1)^{i-1}$-sunflower in B_i with core K. By definition of B_i, there is an $(r+1)^i$-sunflower $\mathcal{S}_1 = S_{1,1}, \dots, S_{1,(r+1)^i}$ in H with core S_1. Since $|S_2 \cup \cdots \cup S_{(r+1)^{i-1}}| < (r+1)(r+1)^{i-1} = (r+1)^i$, and $\mathcal{S}_1$ is an $(r+1)^i$-sunflower, there is a $k = k(1)$ such that

$$(S_{1,k(1)} - S_1) \cap (S_2 \cup S_3 \cup \cdots \cup S_{(r+1)^{i-1}}) = \emptyset.$$

Next, we use the same argument to define $S_{2,k(2)}$ such that $S_{2,k(2)} - S_2$ is disjoint from $S_{1,k(1)} \cup S_3 \cup \cdots \cup S_{(r+1)^{i-1}}$ and then $S_{3,k(3)}$ such that $S_{3,k(3)} - S_3$ is disjoint from $S_{1,k(1)} \cup S_{2,k(2)} \cup S_3 \cup \cdots \cup S_{(r+1)^{i-1}}$ and so on. Continuing in this way we finally obtain edges $S_{j,k(j)}$ of H for all $1 \le j \le (r+1)^{i-1}$ that form an $(r+1)^{i-1}$-sunflower with core K. This implies that $K \ne \emptyset$ as H is intersecting. Since $|K| \le i-1$, there exists a nonempty $K' \subseteq K$ such that $K' \in B(H)$. But $K' \subsetneq S_j$ for all j, so this contradicts the fact that $S_j \in B(H)$. □

Applying the Claim and Lemma 11 yields $|B_i| < f((r+1)^{i-1}, i)$ for all $i > 1$. Every edge of H contains an element of $B(H)$ so we can count edges of H by the sets in $B(H)$. So for $q = |B_2|$ we have

$$hm''(n,r) < |H| \le \sum_{B \in B_2} \binom{n-2}{r-2} + \sum_{i=3}^{r} \sum_{B \in B_i} \binom{n-i}{r-i} < q\binom{n-2}{r-2} + (r-2) f((r+1)^{r-1}, r)\binom{n}{r-3}.$$

Since $hm''(n,r) \sim (r-2)\binom{n}{r-2}$, this gives $q \ge r-2$. On the other hand, B_2 is intersecting and thus the pairs in B_2 form either the star $K_{1,q}$ or a K_3.

Case 1. B_2 is a K_3. Then to keep H intersecting, $H \subseteq HM(n,r,0)$. If $r \ge 5$, then by (3) and (4), $|HM(n,r,0)| < hm''(n,r) < |H|$, a contradiction. Thus $r = 4$ and $H \subseteq HM(n,4,0)$, as claimed.

Since Case 1 is proved, we may assume that B_2 is a star with center x and the set of leaves $X = \{x_1, \dots, x_q\}$.

Case 2 $(q \ge r-1)$. If $q \ge r$, then $q = r$ and since H is nontrivial, $H \subseteq HM(n,r)$ and we are done. We may therefore assume that $q = r-1$. Since $\tau(H) \ge 2$, there exists e such that $x \notin e \in H$, and since H is intersecting we may assume that $e = e_1 = X \cup \{y_1\}$. We may also assume that all edges of H that omit x are of the form $e_i = X \cup \{y_i\}$, where $1 \le i \le t$. If $t = 1$, then $H \subseteq HM(n,r)$ and we are done, so assume that $t \ge 2$. Any edge of H containing x that omits X must contain all $\{y_1, \dots, y_t\}$. Consequently, $H \subseteq HM(n,r,t)$ for some $t \in \{1, \dots, r-1, n-r\}$.

Case 3 $(q = r-2)$. Let F_0 be the set of edges in H that contain x and intersect X, F_1 be the set of edges of H disjoint from X and F_2 be the set of edges disjoint from

x. Then $H = F_0 \cup F_1 \cup F_2$, all edges in F_1 contain x and all edges in F_2 contain X. Since $|F_0| \le \binom{n-1}{r-1} - \binom{n-r+1}{r-1}$, by (4),

$$(5)\ \ |F_1 \cup F_2| > 4\binom{n-r-3}{r-3} + 4\binom{n-r-3}{r-4} + \binom{n-r-3}{r-5} + 2 > 4\binom{n-r-2}{r-3}.$$

Let G be the graph of pairs ab such that $x \notin \{a,b\}$ and $X \cup \{a,b\} \in F_2$. Then $|G| = |F_2|$ and $V(G) \subseteq V(H) - X - \{x\}$.

Case 3.1 ($\tau(G) = 1$). Then $G = K_{1,s}$ for some $1 \le s \le n-r$. Let the partite sets of G be x_{r-1} and Y. Then every edge in F_1 must contain either x_{r-1} or Y. Thus $H \subseteq HM(n,r,t)$ for some $t \in \{1, \dots, r-1, n-r\}$, as claimed.

Case 3.2 ($\tau(G) \ge 2$ and $\nu(G) = 1$). Then $G = K_3$ and every edge in F_1 must contain at least two vertices of G. Then $|F_1| < 3\binom{n-r-1}{r-3} \sim 3\binom{n}{r-3}$ and thus $|F_1 \cup F_2| = |F_1| + 3 \sim 3\binom{n}{r-3}$, contradicting (5).

Case 3.3 ($\nu(G) \ge 3$). Let f_1, f_2, f_3 be disjoint edges in G. Then each edge in F_1 has at least 4 vertices in $f_1 \cup f_2 \cup f_3 \cup \{x\}$ and thus $|F_1| = O(n^{r-4})$. If $F_1 = \emptyset$, then $H \subseteq HM(n,r,n-r)$, as claimed. Suppose there is $e_0 \in F_1$. Then each $f \in G$ meets $e_0 - x$ and thus $|G| = |F_2| \le (r-1)(n-2r+2) + \binom{r-1}{2}$. Thus if $r \ge 5$, then $|F_1 \cup F_2| \le O(n^{r-4}) + O(n) = o(n^{r-3})$, contradicting (5). Moreover, if $r = 4$, then $|F_2| \le 3(n-6)+3$ and $|F_1 \cup F_2| \le O(n^{r-4}) + 3n < 4\binom{n-6}{1}$, again contradicting (5).

Case 3.4 ($\nu(G) = 2$). Say that a vertex v is *big* if $d_G(v) \ge 2r$. Let $v_1, \dots, v_s$ be all the big vertices in G. Since $\nu(G) = 2$, $s \le 2$. Since H is intersecting,

(6) Every edge in F_1 contains all big vertices.

Suppose first, $s = 2$. Then to have $\nu(G) = 2$, all edges in F_2 are incident with v_1 or v_2; thus $|F_2| < 2n$. On the other hand, in this case by (6), $|F_1| \le \binom{n-r-1}{r-3}$. Together, this contradicts (5).

Suppose now, $s = 1$. Then to have $\nu(G) = 2$, we need $|F_2| \le d_G(v_1) + 2r \le n + 2r$. On the other hand, since $\nu(G) = 2$, G has an edge $v'v''$ disjoint from v_1. It follows that each edge in F_1 meets $v'v''$. By this and (5), $|F_1| \le 2\binom{n-r}{r-3}$ and thus $|F_1 \cup F_2| \le n + 2r + 2\binom{n-r}{r-3}$, contradicting (5).

Finally, suppose $s = 0$. Let edges y_1y_1' and y_2y_2' form a matching in G. If G has no other edges, then H is contained in $HM''(n,r)$. So there is a third edge in G. Still, since $\nu(G) = 2$, each edge of G is incident with $\{y_1, y_1', y_2, y_2'\}$ which by $s = 0$ yields $|F_2| = |G| < 8r$. If an edge in G is y_1y_3, then each each edge in F_1 contains $\{y_1, y_2\}$ or $\{y_1, y_2'\}$ or $\{y_1', y_2, y_3\}$ or $\{y_1', y_2', y_3\}$; thus $|F_1| \le 2\binom{n-r}{r-3} + 2\binom{n-r}{r-4} \sim 2\binom{n-r}{r-3}$. This together with $|F_2| \le 8r$ contradicts (5). If this third edge is y_1y_2, then we get a similar contradiction. □

4. On 3-graphs

Lemma 12. *Let $n \ge 6$ and H be an intersecting 3-graph. If H has a vertex x such that $H - x$ has at most two edges, then H is contained in one of $H(n), H_0(n), H_1(n), H_2(n), H_4(n)$.*

Proof. If $H - x$ has no edges, then $H \subseteq H(n)$, and if $H - x$ has one edge, then $H \subseteq H_1(n)$. Suppose $H - x$ has two edges, e_1 and e_2. If $|e_1 \cap e_2| = 2$, then we may assume $e_1 = \{x_1, x_2, y_1\}$ and $e_2 = \{x_1, x_2, y_2\}$. In this case, each edge in $H - e_1 - e_2$ contains x and either intersects $\{x_1, x_2\}$ or coincides with $\{x, y_1, y_2\}$. This means $H \subseteq H_2(n)$.

If $|e_1 \cap e_2| = 1$, then we may assume $e_1 = \{y, v_1, w_1\}$ and $e_2 = \{y, v_2, w_2\}$. In this case, each edge in $H - e_1 - e_2$ contains x and either contains y or intersects each of $\{v_1, w_1\}$ and $\{v_2, w_2\}$. This means $H \subseteq H_4(n)$. □

Proof of of Theorem 8. Let $n \geq 6$ and H be an n-vertex intersecting 3-graph with $\tau(H) \leq 2$ not contained in any of $H(n), H_0(n), \ldots, H_5(n)$. Write H_i for $H_i(n)$. If $\tau(H) = 1$, then $H \subseteq H(n)$. So, suppose a set $\{v_1, v_2\}$ covers all edges of H, but H is not a star. Let $E_0 = \{e \in H : \{v_1, v_2\} \subset e\}$, and for $i = 1, 2$, let $E_i = \{e \in H : v_{3-i} \notin e\}$. By Lemma 12, $|E_1|, |E_2| \geq 3$. For $i = 1, 2$, let F_i be the subgraph of the link graph of v_i formed by the edges in E_i. If $\tau(F_i) \geq 3$, then any edge $e \in E_{3-i}$ does not cover some edge $f \in F_i$ and thus is disjoint from $f + v_1 \in H$, a contradiction. Thus $\tau(F_1) \leq 2$ and $\tau(F_2) \leq 2$.

Case 1 ($\tau(F_1) = 1$). Suppose x_1 is a dominating vertex in F_1. Since $|F_1| = |E_1| \geq 3$, x_1 is **the** dominating vertex in F_i and we may assume that $x_1x_2, x_1x_3, x_1x_4 \in F_1$. But to cover these 3 edges, each edge in F_2 must contain x_1. Thus $H \subseteq H_0(n)$, as claimed.

Case 2 ($\tau(F_1) = \tau(F_2) = 2$). If say F_1 contains a triangle $T = y_1y_2y_3$, then F_2 cannot contain an edge not in T and thus $F_2 = T$ and by symmetry $F_1 = T$. Thus H is contained in H_4.

So the remaining case is that each of F_i contains a matching $M_i = \{z_{1,i}z'_{1,i}, z_{2,i}z'_{2,i}\}$. Since each edge of F_1 intersects each edge of F_2, we may assume $z_{1,2} = z_{1,1}, z'_{1,2} = z_{2,1}, z_{2,2} = z'_{1,1}, z'_{2,2} = z'_{2,1}$. The only other edges that may have F_2 are $f_1 = z_{1,1}z'_{2,1}$ and $f_2 = z'_{1,1}z_{2,1}$. Since $|F_2| \geq 3$, we may assume $f_1 \in F_2$. Then the only third edge that F_1 may contain is also f_1. It follows that H is contained in H_5. This proves the main part of the theorem.

To prove part (a), assume H is an intersecting n-vertex 3-graph with $|H| \geq 11$. Since $|K_5^3| = 10 < |H|$, $n \geq 6$. By Proposition 6, $\tau(H) \leq 2$. So part (a) is implied by the main claim of the theorem. Part (b) follows from the fact that each of H_3, H_4, H_5 has $n + 4$ edges. □

5. Proof of Theorem 9

Let H be as in the statement. By Theorem 5, $\tau(H) \leq 2$. So, suppose a set $\{v_1, v_2\}$ covers all edges of H. Let $E_0 = \{e \in H : \{v_1, v_2\} \subset e\}$, and for $i = 1, 2$, let $E_i = \{e \in H : v_{3-i} \notin e\}$.

For $E_1 \cup E_2$, construct the family $B(H) = B_1 \cup B_2 \cup \ldots B_r$ as in the previous proofs. Recall that by the minimality of the sets in B_i,

$$X \not\subseteq Y \text{ for all distinct } X, Y \in B(H), \tag{7}$$

and since H is intersecting,

$$B(H) \text{ is intersecting.} \tag{8}$$

If $B_1 \neq \emptyset$, say $\{v_0\} \in B_1$, then by (7) and (8), and $B(H) = \{\{v_0\}\}$. This means either $H \subseteq H(n,r)$ (when $v_0 \in \{v_1, v_2\}$), or $H \subseteq H_0^r(n)$ (when $v_0 \notin \{v_1, v_2\}$), and the theorem holds. So, let $B_1 = \emptyset$.

Let H' be obtained from H by deleting all edges not containing a member of $B' = B_2 \cup B_3$. Then $|H - H'| \leq Cn^{r-4}$. Since $\{v_1, v_2\}$ dominates H,

$$\text{each } D \in B' \text{ must contain either } v_1 \text{ or } v_2. \tag{9}$$

For $i = 1, 2$, let B_i' be the set of the members of B' containing v_i.

Define the auxiliary 3-graph H'' with vertex set $V(H)$ as follows. The edges of H'' are all members of B_3 and each triple f that contains a member of B_2 and is contained in an $e \in H'$.

By (8), H'' is intersecting. By (9), $\tau(H'') \leq 2$. If $\tau(H'') = 1$, then H' is a star. Suppose $\tau(H'') = 2$. By Theorem 8, H'' is contained in one of $H(n), H_0(n), \ldots, H_5(n)$. But then H' is contained in one of $H_0^r(n), \ldots, H_5^r(n), EM(n, r, 1)$, as claimed. □

6. Proof of Theorem 10

Recall that $r \geq 4, s \geq 1, n$ is sufficiently large and H is an n-vertex r-graph with $\nu(H) \leq s$ and $|H| > em(n, r, s-1) + hm''(n - s + 1, r)$. We are to show that $V(H)$ contains a subset $Z = \{z_1, \ldots, z_{s-1}\}$ such that either $\tau(H - Z) = 1$ or $H - Z \subseteq HM(n - s + 1, r, t)$ for some $t \in \{1, \ldots, r-1, n-s+1-r\}$ or $r = 4$ and $H - Z \subseteq HM(n - s + 1, 4, 0)$.

Define $B(H)$ and B_i as in the previous proofs with the slight change that $T \in B(H)$ lies in an $(rs)^{|T|+1}$-sunflower (instead of an $(r+1)^{|T|}$-sunflower). Then the following claim holds (with an identical proof).

Claim. B_i contains no $(rs)^i$-sunflower.

Using the Claim and Lemma 11 we obtain $|B_i| < f((rs)^i, i)$ for all $1 \leq i \leq r$. As before, setting $h = |B_1|$ we have

$$|H| \leq \sum_{B \in B_1} \binom{n-1}{r-1} + \sum_{i=2}^{r} \sum_{B \in B_i} \binom{n-i}{r-i} < h\binom{n-1}{r-1} + (r-1)f((rs)^r, r)\binom{n}{r-2}.$$

Since $|H| > em(n, r, s-1) + hm''(n-s+1, r) \sim s\binom{n}{r-1}$ and n is large, this immediately gives $h \geq s - 1$. Consider distinct vertices $z_1, \ldots, z_{s-1} \in B_1$ and the set of edges $F \subset H$ omitting $z_1, \ldots, z_{s-1}$. If F is not intersecting, then let e, e' be two disjoint edges in F. There exists a matching $e_1, \ldots, e_{s-1}$ in H with $z_i \in e_i$ and $(e \cup e') \cap e_i = \emptyset$ for all $1 \leq i \leq s-1$. Note that we can produce the e_i one by one since each z_i forms the core of an $(rs)^2$-sunflower in H due to the definition of B_1. We obtain the matching $e, e', e_1, \ldots, e_{s-1}$ contradicting $\nu(H) \leq s$. Consequently, we may assume that F is intersecting. Because $|H| > em(n, r, s-1) + hm''(n-s+1, r)$ we have $|F| > hm''(n-s+1, r)$. Now we apply Theorem 7 to F to conclude that Theorem 10 holds. □

7. Concluding remarks

Say that a hypergraph H is *t-irreducible*, if $\nu(H) = t$ and $\nu(H - x) = t$ for every $x \in V(H)$. Frankl [10] presented a family of n-vertex t-irreducible r-graphs

$PF(n,r,t)$ such that

$$pf(n,r,t) = |PF(n,r,t)| \sim r\binom{t-1}{2}\binom{n}{r-2}.$$

He also proved

Theorem 13 ([10])**.** *Let $r \geq 4$, $t \geq 1$, and let n be sufficiently large. Then every n-vertex t-irreducible r-graph H has at most $pf(n,r,t)$ edges with equality only if $H = PF(n,r,t)$.*

Using this result, one can prove the following.

Lemma 14. *For every $r \geq 3$, $s \geq t \geq 2$, if n is large, and H is an n-vertex r-graph with $\nu(H) = s$ and*

$$|H| > em(n,r,s-t) + pf(n-s+t,r,t),$$

then there exists $X \subseteq V(H)$ with $|X| = s-t+1$ such that $\nu(H-X) = t-1$. The bound on $|H|$ is sharp.

This in turn implies the following claim.

Theorem 15. *For every $r \geq 3$ and $s \geq 2$ there exists $c > 0$ such that the following holds. If n is large, and H is an n-vertex r-graph with $\nu(H) = s$ and*

$$|H| > em(n,r,s-2) + pf(n-s+2,r,2),$$

then either

1) there exists $H' \subset H$ with $|H'| < cn^{r-3}$ and $\tau(H-H') \leq s$ or

2) there exist an $X \subset V(H)$ with $|X| = s-1$ and $u,v,w \in V(H-X)$ such that every edge of $H-X$ contains at least two elements of $\{u,v,w\}$.

We leave the details of the proofs to the reader.

Most of the proofs in this paper are rather simple applications of the early version of the Delta-system method. There has been renewed interest in stability versions for problems in extremal set theory, so the general message of this work is that the Delta-system method can quickly give some structural information about problems in extremal set theory, a fact that was already shown in several papers by Frankl and Füredi in the 1980s. For more advanced recent applications of the Delta-system method, see the papers of Füredi [12] and Füredi-Jiang [13].

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