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Note

A new tool for proving Vizing's Theorem

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ABSTRACT

The known proofs of the famous theorem of Vizing on edge coloring of (multi)graphs are not long but sophisticated. The main goal of this note is to present an auxiliary (multi)digraph that simplifies and facilitates proofs of it. The secondary goal is to use the approach for proofs of Vizing's Adjacency Lemma and the Andersen–Goldberg Theorem.

1. Introduction

Let $[k] := \{1, \dots, k\}$. We consider only loopless multigraphs and follow the notation and definitions in [9]. An *edge k-coloring* of a multigraph G is a mapping $\phi : E(G) \to [k]$ such that $\phi(e) \neq \phi(e')$ for any two edges e, e' sharing a vertex or two. In other words, for every $1 \leq i \leq k$, the *color class* $\phi^{-1}(i)$ is a matching. The *edge chromatic number*, $\chi'(G)$, of G is the smallest positive integer k such that G has an edge k-coloring.

For each pair $\{u, v\}$ of vertices of a multigraph G, let $M(u, v)$ denote the set of edges of G connecting u with v and let $\mu(u, v) = |M(u, v)|$. Let $\mu(G) = \max\{\mu(u, v) : u, v \in V(G)\}$. The classical result is:

Theorem 1 (*Vizing [6,8]*). *Let $D \geq 2$ and G be a multigraph with $\Delta(G) \leq D$. Let $\mu = \mu(G)$. Let $x, y \in V(G)$ and $e_0 \in M(x, y)$ be such that $G - e_0$ has an edge-$(D + \mu)$-coloring ϕ. Then G also has an edge-$(D + \mu)$-coloring.*

An immediate corollary of Theorem 1 is

Theorem 2 (*Vizing [6,8]*). *For every multigraph G, $\chi'(G) \leq \Delta(G) + \mu(G)$. In particular, $\chi'(G) \leq \Delta(G) + 1$ for every simple graph G.*

There are many different proofs of Theorems 1 and 2 (see, e.g. [5,9]). The main goal of this note is to present an auxiliary multidigraph that simplifies and facilitates proofs of it. We also use the same machinery for short proofs of two other well-known results on edge coloring. One of them belongs to Andersen [1] and Goldberg [3,4].

Theorem 3 (*Andersen [1], Goldberg [3,4]*). *Let G be a multigraph and S be the set of all paths (x, y, z) of length 2 in G. Then*

$$\chi'(G) \leq q(G) := \max\left\{\Delta(G), \max_{(x,y,z)\in S} \frac{d(x) + \mu(x, y) + d(z) + \mu(z, y)}{2}\right\}. \quad (1)$$

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For the other result, we need a definition: A (simple) graph G is *critical*, if $\chi'(G) = \Delta(G) + 1$ and $\chi'(G - e) < \chi'(G)$ for every $e \in E(G)$. *Vizing's Adjacency Lemma* stated below is a useful tool for studying edge colorings of graphs.

Theorem 4 (*Vizing [7]*)**.** *If G is a critical graph with maximum degree $D \geq 2$ and $xy \in E(G)$, then y has at least* $\max\{2, D - d(x) + 1\}$ *neighbors of degree D.*

In the next three sections we present proofs of Theorems 1, 3 and 4, respectively.

2. Proof of Theorem 1

The proof follows the structure of the original proof [6] and the proof in [5], but the use of the auxiliary digraph H' makes it simpler.

Let $q = D + \mu$ and ϕ be an edge-q-coloring of $G - e_0$, where $e_0 \in M(x, y)$. For every $v \in V(G)$, let $O(v) = O_\phi(v)$ denote the set of colors in $[q]$ not used to color the edges incident to v. Then

$$|O(v)| \geq q - d(v) \quad \text{for each } v \in V(G) \quad \text{and} \quad |O(u)| \geq q - d_{G-e_0}(u) = q - d(u) + 1, \quad \text{for } u \in \{x, y\}. \tag{2}$$

Construct the auxiliary multidigraph H as follows: $V(H) = N_G(y)$ and the number of edges from u to v equals the number of edges $e \in M(v, y)$ with $\phi(e) \in O(u)$. Let X be the set of vertices reachable in H from x and H' be the subdigraph $H[X]$ of H induced by X. By definition, $x \in X$. Since the outneighbors of a reachable from x vertex also are reachable from x,

$$N_H^+(v) \subseteq X \quad \text{for every } v \in X. \tag{3}$$

Suppose G has no edge-q-coloring. Then $O(x) \cap O(y) = \emptyset$. Let $\alpha \in O(y)$. We prove a sequence of three simple claims.

Claim 1. *$\alpha \notin O(v)$ for every $v \in X$.*

Proof. Suppose $\alpha \in O(v)$ and $(x_0, x_1, \ldots, x_s)$ where $x_0 = x$ and $x_s = v$ is an x, v-path in H'. By the definition of edges in H', for each $i \in \{1, \ldots, s\}$, there is an edge $e_i \in M(y, x_i)$ with $\phi(e_i) \in O(x_{i-1})$. Then we can recolor e_s with α and for every $i = 1, \ldots, s$, (re)color e_{i-1} with $\phi(e_i)$. This yields a coloring of G. □

Claim 1 yields that for every $v \in X$ and $\beta \in O(v)$, there is some $w \in N(y)$ and $e \in M(y, w)$ with $\phi(e) = \beta$. Then by the definition of H, $vw \in E(H')$. So by (2), (3),

$$d^+_{H'}(v) \geq q - d(v) \quad \text{for every } v \in X \quad \text{and} \quad d^+_{H'}(x) \geq q - d(x) + 1. \tag{4}$$

A $[\beta, \gamma]$-*path* in G is a path whose edges are alternately colored with β and γ. A $[\beta, \gamma](a, b)$-*path* is a $[\beta, \gamma]$-path from a to b in G.

Claim 2. *If $v \in X$ and $\beta \in O(v)$, then G contains an $[\alpha, \beta](v, y)$-path.*

Proof. If the claim is not true, choose a vertex $v \in X$ at minimum distance from x in H' for which there is $\beta \in O(v)$ such that the $[\alpha, \beta]$-path P starting at v does not end at y. Let z denote the other end of P. Let $(x_0, x_1, \ldots, x_s)$ where $x_0 = x$ and $x_s = v$ be a shortest x, v-path in H'. For each $i \in \{1, \ldots, s\}$, let $e_i \in M(y, x_i)$ be such that $\phi(e_i) \in O(x_{i-1})$. If $z \in X$, then by Claim 1 the last edge of P has color α, and $\beta \in O(z)$. So by the minimality of the distance of v from x, $z \notin \{x_0, \ldots, x_s\}$. Then we can switch the colors α and β on the edges of P, recolor e_s with α and for every $i = 1, \ldots, s$ (as in the proof of Claim 1), (re)color e_{i-1} with $\phi(e_i)$. □

Claim 3. *For all distinct $v, w \in X$, $O(v) \cap O(w) = \emptyset$.*

Proof. If $v, w \in X$ and $\beta \in O(v) \cap O(w)$, then by Claim 2, the $[\beta, \alpha]$-path starting at y must end at both v and w, an impossibility. □

Claim 3 yields that

$$d^-_{H'}(v) \leq \mu(v, y) \leq \mu \quad \text{for every } v \in X, \quad \text{and} \quad d^-_{H'}(x) \leq \mu(x, y) - 1. \tag{5}$$

Since $\sum_{v \in X} d^-_{H'}(v) = |E(H')| = \sum_{v \in X} d^+_{H'}(v)$ and $q = D + \mu$, by (4) and (5),

$$0 = \sum_{v \in X} \left(d^+_{H'}(v) - d^-_{H'}(v) \right) \geq 2 + \sum_{v \in X} (D + \mu - d(v) - \mu(v, y)) \geq 2, \quad \text{a contradiction.} \quad \square \tag{6}$$

Remark. The proof for simple graphs would be even simpler logically, since in this case H has no multiple edges.

3. Proof of Theorem 3

Let G be a counter-example to Theorem 3 with the fewest edges. By minimality, it is connected. Clearly, $|V(G)| \geq 3$. For $y \in V(G)$ and $v \in N(y)$, let $f_y(v) = d(v) + \mu(y, v)$. In terms of f, we can redefine $q(G)$ (see (1)) as follows:

$$q(G) := \max \left\{ \Delta(G), \max_{y \in V(G); x,z \in N(y), x \neq z} \frac{f_y(x) + f_y(z)}{2} \right\}. \tag{7}$$

Choose any $y \in V(G)$ with $d(y) = \Delta(G)$. Since G is connected and $|V(G)| \geq 3$, $|N(y)| \geq 2$. Let x be a vertex in $N(y)$ with the maximum $f_y(x)$ and $z \in N(y)$ have the largest $f_y(z)$ in $N(y) - x$.

Since $q(G)$ may be not an integer, let $q := \lfloor q(G) \rfloor$. By definition, q may be less than $f_y(x)$ but

$$\text{(a) } q \geq \frac{f_y(x) + f_y(z) - 1}{2} \quad \text{and} \quad \text{(b) } q \geq f_y(v) \quad \text{for every } v \in N(y) - x. \tag{8}$$

Let $e_0 \in M(y, x)$. Since $q(G)$ does not increase when we delete an edge, by the minimality of G, multigraph $G - e_0$ has an edge-q-coloring ϕ. Define $O(v)$, H, X and H' exactly as in the previous section. Then inequalities (2)–(5) and Claims 1–3 hold with proofs repeated word by word. The only extra observation is that since $q \geq \Delta(G)$, by (2) we have $O(y) \neq \emptyset$ and $O(x) \neq \emptyset$. It follows that there is $\alpha \in O(y)$ and $d^+_{H'}(x) \geq 1$; thus $|X| \geq 2$.

By (4) and (5), similarly to (6),

$$0 = \sum_{v \in X} \left(d^+_{H'}(v) - d^-_{H'}(v) \right) \geq 2 + \sum_{v \in X} (q - d(v) - \mu(v, y)) = 2 + \sum_{v \in X} (q - f_y(v)). \tag{9}$$

Since $|X| \geq 2$, we can choose $z' \in X - x$. Using (9), then (8)(b), and then (8)(a), we have

$$0 \geq 2 + (q - f_y(x)) + (q - f_y(z')) + \sum_{v \in X - x - z'} (q - f_v(v)) \geq 2 + (q - f_y(x)) + (q - f_y(z')) \geq 1,$$

since $f_y(z') \leq f_y(z)$. This is a contradiction. □

4. Proof of Theorem 4

Let G be a critical graph with maximum degree $D \geq 2$ and $xy \in E(G)$. Define $q := D$. By criticality, graph $G' = G - xy$ has an edge-q-coloring ϕ. Define $O(v)$, H, X and H' exactly as in Section 1. Again inequalities (2)–(5) and Claims 1–3 hold with proofs repeated word by word. Let $X' = \{v \in X : d(v) = D\}$. Since $q = D$ and $\mu = 1$, similarly to (9),

$$0 = \sum_{v \in X} \left(d^+_{H'}(v) - d^-_{H'}(v) \right) \geq 2 + \sum_{v \in X} (q - d(v) - \mu) = 2 + \sum_{v \in X} (D - 1 - d(v)). \tag{10}$$

A term $D - 1 - d(v)$ in the last sum is negative (and equals -1) iff $v \in X'$. Thus (10) yields $|X'| \geq 2$. Moreover, if $d(x) < D$ then by (10), $0 \geq 2 + (D - 1 - d(x)) - |X'|$, i.e. $|X'| \geq D - d(x) + 1$, as required. □

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Note added in proof: After the note was accepted, the author learned about a very different and interesting way to prove Theorems 1–4 in [2].

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