Harmonious Coloring of Trees with Large Maximum Degree

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Abstract

A harmonious coloring of G is a proper vertex coloring of G such that every pair of colors appears on at most one pair of adjacent vertices. The harmonious chromatic number of G, $h(G)$, is the minimum number of colors needed for a harmonious coloring of G. We show that if T is a forest of order n with maximum degree $\Delta(T) \geq \frac{n+2}{3}$, then

$$h(T) = \begin{cases} \Delta(T) + 2, & \text{if } T \text{ has non-adjacent vertices of degree } \Delta(T); \\ \Delta(T) + 1, & \text{otherwise.} \end{cases}$$

Moreover, the proof yields a polynomial-time algorithm for an optimal harmonious coloring of such a forest.

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1   Introduction.

Let G be a simple graph. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of G, respectively. A vertex of degree 1 in G is called a *leaf*. A *harmonious coloring* of G is a proper vertex coloring of G such that every pair of colors appears on at most one pair of adjacent vertices. The *harmonious chromatic number* of G, $h(G)$, is the minimum number of colors needed for any

harmonious coloring of G. The first paper [3] on harmonious coloring appeared in 1982. However, the proper definition of this notion is due to Hopcroft and Krishnamoorthy [4]. Harmonious coloring of a graph is essentially an edge-injective homomorphism from a graph G to a complete graph and the harmonious chromatic number of G is the minimum order of a complete graph that admits such homomorphism from G. Paths and cycles are among the first graphs whose harmonious chromatic numbers have been established [3]. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard. Moreover, Edwards and McDiarmid [2] showed that the problem remains hard even restricted to the class of trees. Since the problem is hard in the class of all trees, it makes sense to identify subclasses in which the problem is easier.

Since vertices at distance at most two in a graph G must have distinct colors in any harmonious coloring of G, $h(G) \geq \Delta(G) + 1$ for every graph G. In [1] it was shown that if T is a tree of order n and $\Delta(T) \geq \frac{n}{2}$, then $h(T) = \Delta(T) + 1$. Moreover, the proof yields a polynomial-time algorithm for an optimal harmonious coloring of such a tree. We strengthen this result by finding a wider class of trees T for which $h(T) = \Delta(T) + 1$.

Theorem 1. *Let*
$$\Delta \geq \frac{n+2}{3}. \tag{1}$$
If T is a forest of order n with $\Delta(T) = \Delta$, then

$$h(T) = \begin{cases} \Delta + 2, & \text{if } T \text{ has non-adjacent vertices of degree } \Delta; \\ \Delta + 1, & \text{otherwise.} \end{cases}$$

Moreover, there is a polynomial-time algorithm for an optimal harmonious coloring of such a forest.

In the next section we prove the lower bounds in the theorem and show that the bound $\Delta \geq \frac{n+2}{3}$ is sharp. In the last two sections we prove the upper bounds in the theorem.

Our notation is standard. In particular, for a graph G, $v \in V(G)$ and $W \subseteq V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v in G, $d_G(v) = |N_G(v)|$, $N_G[v] = \{v\} \cup N_G(v)$, and $G[W]$ is the subgraph of G induced by W.

2 Lower bounds

Since in each harmonious coloring f of a graph G, the colors of all neighbors of a vertex v are different and distinct from $f(v)$,

$$h(G) \geq 1 + \Delta(G) \qquad \text{for every graph } G. \tag{2}$$

Claim 1 *Let $k \geq 1$. If a graph G contains two non-adjacent vertices, say u_1 and u_2, of degree k, then $h(G) \geq k + 2$.*
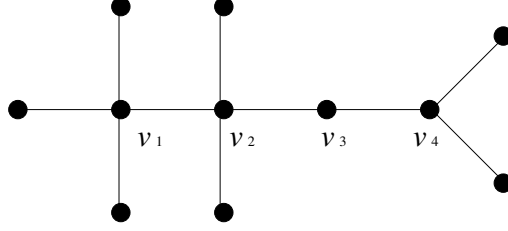
Figure 1: Tree T_4.

Proof. Suppose that G has a harmonious $(k+1)$-coloring f with colors in $A + \{\alpha_1, \ldots, \alpha_{k+1}\}$. Then by (2), for each $j = 1, \ldots, k+1$, the set $f^{-1}(\alpha_j)$ has a vertex in $N_G[u_1]$ and a vertex in $N_G[u_2]$. If $f(u_1) \neq f(u_2)$, then the pair $\{f(u_1), f(u_2)\}$ appears on two pairs of adjacent vertices: one pair in $N_G[u_1]$ and one pair in $N_G[u_2]$. And if $f(u_1) = f(u_2)$, then for each $\alpha \in A - f(u_1)$, the pair $\{f(u_1), \alpha\}$ appears on two pairs of adjacent vertices. So, f is not harmonious. Thus $h(G) \geq k+2$.

Now for every $\Delta \geq 3$ we present a tree T_Δ such that (i) $|V(T)| = 3\Delta - 1$, (ii) $\Delta(T) = \Delta$, (iii) T has no non-adjacent vertices of degree Δ, and (iv) $h(T) \geq \Delta + 2$. These examples show that the restriction (1) in Theorem 1 cannot be weakened.

Let T_Δ be obtained from a 4-vertex path (v_1, v_2, v_3, v_4) by adding $\Delta - 1$ leaves adjacent to v_1, $\Delta - 2$ leaves adjacent to v_2, and $\Delta - 2$ leaves adjacent to v_4. Tree T_4 is depicted in Fig. 1.

By construction, T_Δ is a tree with maximum degree Δ and $|V(T_\Delta)| = 4 + (\Delta - 1) + (\Delta - 2) + (\Delta - 2) = 3\Delta - 1$. So, (i) and (ii) hold; and (iii) is also evident. We establish (iv) by proving the following.

Claim 2 $h(T_\Delta) = \Delta + 2$.

Proof. Suppose f is a harmonious coloring of T_Δ with $\Delta + 1$ colors. We may assume that $f(v_1) = 1, f(v_2) = 2$, and $f(v_3) = 3$. Also we may assume that $f(N(v_1) - v_2) = \{3, 4, \ldots, \Delta + 1\}$. Then $f(N(v_2) - v_1 - v_3) = \{4, \ldots, \Delta + 1\}$. Since $d_{T_\Delta}(v_1) = d_{T_\Delta}(v_2) = \Delta$, no other vertices can be colored 1 or 2 in a harmonious $(\Delta + 1)$-coloring. Thus, we may assume $f(v_4) = 4$. Then $f(N(v_4) - v_3) \subset \{5, 6, \ldots, \Delta + 1\}$ should hold. But $N(v_4) - v_3$ has $\Delta - 2$ vertices and only $\Delta - 3$ colors are available. Therefore we cannot complete the coloring with $\Delta + 1$ colors. Thus $h(T_\Delta) > \Delta + 1$.

3   When $h(T) = \Delta + 1$

In this section, we present polynomial-time coloring procedures yielding that
(*****) *if (1) holds and T is an n-vertex forest with $\Delta(T) = \Delta$ such that T has no non-adjacent vertices of degree Δ, then $h(T) = \Delta + 1$.*

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First, observe that the statement holds for $\Delta \leq 2$: By (1), $n \leq 3\Delta - 2$. So, if $\Delta \leq 1$, then $n \leq 1$, and so $h(T) \leq 1 \leq 1 + \Delta$. If $\Delta = 2$, then $n \leq 4$ and hence T is a subgraph of the 4-vertex path P_4 whose harmonious chromatic number is $3 = \Delta + 1$. So, everywhere below

$$\Delta \geq 3. \tag{3}$$

Second, let us check that it is enough to prove (*) for trees. Indeed, if T is a disconnected n-vertex forest satisfying (1) and (3) with no non-adjacent vertices of degree Δ, then by adding an edge connecting two leaves or isolated vertices from different components of T, we again get a forest with these properties and fewer components. Thus in this section we will assume that T is a tree.

Let $v \in V(T)$ be a vertex of degree Δ. We will construct a harmonious coloring $f : V(T) \to V(K_{\Delta+1})$ step by step. The vertices of $H := K_{\Delta+1}$ will by denoted by Greek letters so that we do not mix them with the vertices of T. We start from mapping v and the Δ neighbors of v in T into the all different $\Delta + 1$ vertices of H. Let $f(w)$ denote the color of w. If $f(w)$ is not defined yet, then w is an *uncolored vertex*, otherwise it is a *colored vertex*.

We consider several cases.

Case 0: T consists of two stars with a path joining them. This case is straightforward.

Case 1: v is the only vertex of degree Δ in T, and T has no vertices of degree $\Delta - 1$. Suppose that we have already defined $f(w)$ for some vertices $w \in V(T)$ (in particular, $f(w)$ is defined for $w \in N_T[v]$). For $\alpha \in V(H)$, let

$$d(\alpha) := \sum_{x \in f^{-1}(\alpha)} d_T(x).$$

Also, we will say that vertices α and β of H are *T-adjacent*, if there are $x \in f^{-1}(\alpha)$ and $y \in f^{-1}(\beta)$ such that $xy \in E(T)$. Our procedure will color one vertex at each step. It works as follows:

(a) Choose a vertex $w \in V(T)$ such that $f(w)$ is defined and w has a neighbor u for which f is not defined and u is not a leaf. If there are no such vertex, then choose u which is a leaf.

(b) If there is $\gamma \in V(H) - f(w)$ such that (i) γ is not T-adjacent to $f(w)$ and (ii) $d(\gamma) + d_T(u) \leq \Delta$, then we let $f(u)$ be any γ satisfying (i)–(ii) and go to (a) of the next step.

(c) If no $\gamma \in V(H) - f(w)$ satisfies (i)–(ii), then we stop.

We need to prove that we do not stop until we embed all T. Note that after the initial coloring of $N_T[v]$, for every $\alpha \in V(H)$ we have $|f^{-1}(\alpha)| = 1$, and hence $d(\alpha) \leq \Delta$.

Suppose that we stopped in some step, before $f(x)$ was defined for every $x \in V(T)$. This means that at the moment of stopping, either every $\gamma \in V(H) - f(w)$ is T-adjacent to $f(w)$ or

$$d(\gamma) + d_T(u) \geq \Delta + 1 \quad \text{for every } \gamma \in V(H) - f(w) \text{ not } T\text{-adjacent to } f(w). \tag{4}$$

If the former holds, then since $f(u)$ is not defined yet, at the moment of defining $f(w)$ we had already had $d(f(w)) + d_T(w) \geq \Delta + 1$ and should have stopped then. Thus some $\gamma \in V(H) - f(w)$

is not T-adjacent to $f(w)$, and (4) holds. We may assume that $f(w) = \gamma_0$. Let $\gamma_1, \ldots, \gamma_r$ be the vertices of $H - \gamma_0$ not T-adjacent to γ_0, and $\gamma_{r+1}, \ldots, \gamma_\Delta$ be the vertices of H that are T-adjacent to γ_0. And let $\gamma_\Delta = f(v)$. By the above, $r \geq 1$.

By the choice of u, $u \neq v$. So in our case $d(u) \leq \Delta - 2$. Thus by (4),

$$d(\gamma_i) \geq \Delta + 1 - d_T(u) \geq 3 \quad \text{for every } 1 \leq i \leq r. \tag{5}$$

Since $f(v)$ is T-adjacent to every other vertex in H, according to our rules, $f(x) \neq f(v)$ for every $x \neq v$.

For every $W \subseteq V(T)$,

$$n - 1 = |E(T)| \geq \sum_{w \in W} d_T(w) - |E(T[W])|. \tag{6}$$

We may assume that $d(\gamma_1) \leq d(\gamma_2) \leq \ldots \leq d(\gamma_r)$. Let

$$W := \{v, u\} \cup f^{-1}(\{\gamma_0, \gamma_1, \ldots, \gamma_r\}).$$

Case 1.1: $r = 1$. Then, since γ_0 is T-adjacent to $\Delta - 1$ vertices in H and $uw \in E(T)$, $d(\gamma_0) \geq \Delta$. By (4), $d_T(u) + d(\gamma_1) \geq \Delta + 1$. So since γ_1 is not T-adjacent to γ_0, the graph $T[W]$ has exactly 3 edges, uw, vx_0 and vx_1, where x_i is a neighbor of v with $f(x_i) = \gamma_i$, for $i = 0, 1$. Thus using this W in (6), we have

$$|E(T)| \geq d_T(v) + d(\gamma_0) + d_T(u) + d(\gamma_1) - 3 \geq \Delta + \Delta + (\Delta + 1) - 3 = 3\Delta - 2.$$

So, $3\Delta - 2 \leq n - 1$, i.e., $\Delta \leq \frac{n+1}{3}$, a contradiction.

Case 1.2: $r = 2$. Similarly to Case 1.1, $d(\gamma_0) \geq \Delta - 1$ and $d_T(u) + d(\gamma_1) \geq \Delta + 1$. Now γ_1 and γ_2 are not T-adjacent to γ_0. So, graph $T[W]$ has at most 5 edges. Thus, using this W in (6), we have (using also (5) to estimate $d(\gamma_2)$)

$$|E(T)| \geq d_T(v) + d(\gamma_0) + d_T(u) + d(\gamma_1) + d(\gamma_2) - 5 \geq \Delta + (\Delta - 1) + (\Delta + 1) + 3 - 5 = 3\Delta - 2,$$

a contradiction as in Case 1.1.

Case 1.3: $r \geq 3$. Now $d(\gamma_0) \geq \Delta - r + 1$ and $\gamma_1, \ldots, \gamma_r$ are not T-adjacent to γ_0. So,

$$|E(T[W])| \leq (1 + r) + 1 + \sum_{i=1}^{r} \frac{d(\gamma_i) - 1}{2} \tag{7}$$

(here $r + 1$ counts the edges incident with v, 1 stands for the edge uw and $\sum_{i=1}^{r} \frac{d(\gamma_i)-1}{2}$ estimates from above the number of edges both ends of which are in $\{\gamma_1, \ldots, \gamma_r\}$). So by (6),

$$n - 1 = |E(T)| \geq d_T(v) + d_T(u) + \sum_{i=0}^{r} d(\gamma_i) - r - 2 - \sum_{i=1}^{r} \frac{d(\gamma_i) - 1}{2}$$

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$$\geq \Delta + (\Delta - r + 1) + (d_T(u) + d(\gamma_1)) - d(\gamma_1) - r - 2 + \sum_{i=1}^{r} \frac{d(\gamma_i) + 1}{2}$$

$$\geq (2\Delta - r + 1) + (\Delta + 1) - d(\gamma_1) - r - 2 + r\frac{d(\gamma_1) + 1}{2} = 3\Delta - 2r + (r - 2)\frac{d(\gamma_1)}{2} + \frac{r}{2}$$

$$= 3\Delta - 3 + (d(\gamma_1) - 3)\frac{r - 2}{2} \geq 3\Delta - 3 + \frac{d(\gamma_1) - 3}{2}.$$

Thus if $d(\gamma_1) \geq 4$ or if (7) is a strict inequality, or if $d(\gamma_0) > \Delta - r + 1$, then we have $n - 1 > 3\Delta - 3$, which yields $\Delta \leq \frac{n+1}{3}$, a contradiction. So, by (5) we may suppose that $d(\gamma_1) = 3$, $d(\gamma_0) = \Delta - r + 1$, and (7) holds with equality. In particular, by (5), $d_T(u) = \Delta - 2$. Since $r \geq 3$, we have $\Delta \geq 1 + r \geq 4$, and so u is not a leaf. Since we do not have Case 0, there is a leaf l not adjacent to u and not adjacent to v. Thus, according to our rule (a), l is not colored yet. Since $d(\gamma_0) = \Delta - r + 1$, and (7) holds with equality, l is adjacent neither to any vertex in $f^{-1}(\gamma_0)$ nor to any vertex in $f^{-1}(\{\gamma_1, \ldots, \gamma_r\})$. Hence the right-hand side of (6) does not count the edge incident with l. So, we have $n - 2 \geq 3\Delta - 3$, a contradiction to $\Delta > \frac{n+1}{3}$.

Therefore we do not stop until we color all the vertices in T.

Let $(u_1, \ldots, u_n)$ be an ordering of the vertices of T such that $u_1 = v$ and $d_T(u_1) \geq d_T(u_2) \geq \ldots \geq d_T(u_n)$. In these terms, Case 1 was the case $d_T(u_2) \leq \Delta - 2$. Let t be chosen so that $d_T(u_t) \geq 2$ and $d_T(u_{t+1}) = 1$.

Case 2: $d_T(u_3) \geq \Delta - 1$. Let $W' := \{u_1, \ldots, u_t\}$. Since T is connected, $T[W']$ is also connected. Then

$$\sum_{w \in W'} d_T(w) \geq \Delta + (\Delta - 1) + (\Delta - 1) + 2(t - 3) \quad \text{and} \quad |E(T[W'])| = t - 1.$$

So by (6),

$$n - 1 = |E(T)| \geq 3\Delta - 2 + 2(t - 3) - (t - 1) \geq (n + 2) + t - 7 = n + t - 5.$$

It follows that $t \leq 4$, and that if $t = 4$, then $d_T(u_2) = \Delta - 1$ and $d_T(u_4) = 2$.

Case 2.1: $t = 3$. The only 3-vertex tree is the 3-vertex path. So, $T[W']$ is the path (w_1, w_2, w_3). By (*), we may assume that $w_3 = u_3$. For $i = 1, 2, 3$, we let $f(w_i) = \gamma_{i-1}$. We place the leaves adjacent to w_1 into any $d_T(w_1) - 1$ vertices in $V(H) - \gamma_0 - \gamma_1$, the leaves adjacent to w_2 into any $d_T(w_2) - 2$ vertices in $V(H) - \gamma_0 - \gamma_1 - \gamma_2$, and the $\Delta - 2$ leaves adjacent to w_3 into the vertices in $V(H) - \gamma_0 - \gamma_1 - \gamma_2$.

Case 2.2: $t = 4$. As it was mentioned, in this case $d_T(u_2) = d_T(u_3) = \Delta - 1$ and $d_T(u_4) = 2$. Since $T[W']$ is connected and $d_T(u_4) = 2$, we may assume that u_3 is adjacent either to u_1 or to u_2. For $i = 1, 2, 3, 4$, we let $f(u_i) = \gamma_{i-1}$. Then for $j = 1, 2, 3, 4$, we place the leaves adjacent to u_j into the vertices of $H - \gamma_0 - \ldots - \gamma_{j-1}$ not occupied by the neighbors of u_j in W'. We can do it for $j = 1, 2$, since $d_T(u_j) = 1 + \Delta - j$ for these j. And u_3 was chosen so that $\gamma_2 = f(u_3)$ is T-adjacent to $\{\gamma_0, \gamma_1\}$. Finally, since $T[W']$ is connected, u_4 has at most one adjacent leaf. So if $\Delta \geq 4$ or u_4 has no adjacent leaves, then we are done. Thus we need only to handle the situation when $\Delta = 3$

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and each vertex of degree 2 in T has an adjacent leaf. Then $T[W'] = K_{1,3}$. For $i = 2, 3, 4$, let w_i be the leaf adjacent to u_i. In this case, we again let $f(u_i) = \gamma_{i-1}$ for $i = 1, 2, 3, 4$ and then let $f(w_2) = \gamma_2$, $f(w_3) = \gamma_3$, and $f(w_4) = \gamma_1$.

Case 3: $d_T(u_2) = \Delta$. Under Condition (*), $vu_2 \in E(G)$, and by Case 2, $d_T(u_3) \leq \Delta - 2$. Since $vu_2 \in E(G)$, $f(u_2)$ was defined at the first step. Then we can apply the procedure of Case 1, and the argument goes through since $d_T(u_3) \leq \Delta - 2$.

The only case, we have not yet considered is:

Case 4: $d_T(u_2) = \Delta - 1$ and $d_T(u_3) \leq \Delta - 2$. Let $P = (v_1, \ldots, v_q)$ be the path in T connecting $v_1 = v$ with $v_q = u_2$. Suppose v has exactly p adjacent non-leaves in T. We claim that

$$q + p \leq \Delta + 2, \tag{8}$$

since otherwise

$$n \geq q + (p - 1) + (d_T(v) - 1) + (d_T(u_2) - 1) \geq (\Delta + 3 - 1) + (\Delta - 1) + (\Delta - 1 - 1) = 3\Delta - 1,$$

a contradiction to (1).

By (8), we can place all the vertices of P and all remaining non-leaf neighbors of v into distinct vertices of H. After that, we place the leaves adjacent to v into distinct vertices of H not containing v or its neighbors. Then we again apply the procedure of Case 1 and the argument goes through since $d_T(u_3) \leq \Delta - 2$.

4 Finishing the proof

The only situation not covered in the previous section is that T has non-adjacent vertices v and z of degree Δ. We add a vertex w to T and make w adjacent to v to get a tree T' with maximum degree $\Delta + 1$. Then we may apply Case 1 to T' and color T' with $\Delta + 2$ colors. This harmonious coloring of T' gives a harmonious coloring of T with $\Delta + 2$ colors. This completes the proof of Theorem 1.

Remark. Since we colored vertices one by one with no recolorings, and the choice of every next vertex took polynomial time, the total time taken by our algorithm is polynomial.

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