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ON A LOWER BOUND FOR THE ISOPERIMETRIC NUMBER OF CUBIC GRAPHS

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ABSTRACT

It is proved that almost every cubic simple graph G on n vertices has the isoperimetric number not less than 1/4.95.

1. INTRODUCTION

Graphs under consideration are finite and may have loops and multiple edges. Graphs without loops and multiple edges will be called *simple*.
Let G be a graph. If $V_1 \subseteq V(G)$ and $V_2 = V(G)\backslash V_1$, then $E_G(V_1, V_2)$ denotes the set of edges of G having one end in V_1 and the other end in V_2. A *cut* (V_1, V_2) of G is the partition $V(G) = V_1 \cup V_2$ with $0 < |V_1| \leq |V_2|$. The quantity

$$i(G) = \min\left\{|E_G(V_1, V_2)|/|V_1|\colon (V_1, V_2) \text{ is a cut of } G\right\}, \tag{1.1}$$

is called the *isoperimetric number* of G. The isoperimetric number was introduced by Buser (Buser, 1978), who later showed (Buser, 1984) that for every even $n \geq 4$ there exists a simple cubic graph G on n vertices with isoperimetric number $i(G) \geq 1/128$.
A bisection of a graph G is a partition (V_1, V_2) of $V(G)$ into two sets of almost equal cardinality, i.e., a partition with $||V_1| - |V_2|| \leq 1$. The *bisection width* $\text{bw}(G)$ of a graph G is defined as the minimum number of edges between the vertex sets of bisections (V_1, V_2), shortly

$$\text{bw}(G) = \min\left\{|E_G(V_1, V_2)|\colon (V_1, V_2) \text{ is a bisection of } G\right\}. \tag{1.2}$$

From definitions (1.1) and (1.2) the following inequality for these characteristics follows

$$i(G) \leq \frac{2}{n}\text{bw}(G). \tag{1.3}$$

The attention to the isoperimetric number $i(G)$ and the bisection width $\text{bw}(G)$ may be explained by various interesting interpretations of these numbers (a measure of connectivity of graphs) and some practical applications, for example, in

VLSI designs (see (Goldberg and Gardner, 1984; Mohar, 1989) and the references therein). The problems of finding the isoperimetric number $i(G)$ and the bisection width $\mathrm{bw}(G)$ of a graph are known to be NP-hard (Mohar, 1989; Garey at al., 1976). Moreover, it was shown in (MacGregor, 1978; Bui et al., 1987) that the problem of finding the bisection width is NP-hard even within the class of graphs with maximum degree 3. Generalizations have been considered in (Barnes and Hoffman, 1984; Buser, 1978).

In (Bui et al., 1987) an algorithm was given for transforming a regular graph G with n vertices into a cubic graph G^* with $O(n^6)$ vertices such that any minimum bisection of G^* uses only edges of the graph G. Therefore we are interested in the examination of cubic graphs.

Clark and Entringer (Clark and Entringer, 1988) obtained that sup $\mathrm{bw}(G) \leq (n+138)/3$ for cubic graphs and also showed that for almost every r-regular simple graph G on n vertices $\mathrm{bw}(G) \geq c_r n$, where $c_r \to r/4$ as $r \to \infty$ (in particular, $c_3 \geq 1/11$).

The present paper is a continuation of (Kostochka and Melnikov, 1991), where it is proved that, for any integer q $(q \geq 2)$ and for any simple q-regular graph G on n vertices,

$$\mathrm{bw}(G) \leq (q-2)n/4 + O(q\sqrt{n}\log n). \tag{1.4}$$

As a corollary for any simple cubic graph G on n vertices,

$$\mathrm{bw}(G) \leq n/4 + O(\sqrt{n}\log n).$$

In (Kostochka and Melnikov, 1991) it was announced that for any sufficiently large n there exists a cubic graph G on n vertices such that $\mathrm{bw}(G) \geq n/9.9$ or a bit stronger that $i(G) \geq 1/4.95$. Bollobás (Bollobás, 1988) also estimated the isoperimetric number of regular graphs and in particular he proved that for almost every cubic simple graph $i(G) > 2/11$. The main goal of this paper is to present a complete proof of the following result: *For almost every cubic simple graph G on n vertices $i(G) \geq 1/4.95$.*

All graph theoretic notions not defined here, may be found in (Harary, 1969).

2. MINIMAL CUTS IN MARKED CUBIC GRAPHS

For a cut (V_1, V_2) of a graph G we denote

$$V_{i,j} = \left\{v \in V_i \,\middle|\, |E_G(v, V_{3-i})| = j\right\}, \quad V_{i,j,k} = \{v \in V_{i,j} \,|\, \deg_{G(V_{i,j})} v = k\}.$$

LEMMA 2.1. *Let G be a cubic graph with $i(G) \leq 2/9$ and (V_1, V_2) be a cut such that $i(G) = |E_G(V_1, V_2)|/|V_1|$ (recall that by definition $|V_1| \leq |V_2|$). Then*

$$V_{i,2} = V_{i,3} = \varnothing, \qquad i = 1, 2.$$

Proof. Suppose that $v \in V_{i,2} \cup V_{i,3}$. Since $i(G) \leq 2/9$, we have $|V_1| \geq 9$. Denoting $V_i' = V_i \setminus \{v\}$, $V_{3-i}' = V_{3-i} \cup \{v\}$ we obtain

$$\frac{|E_G(V_1', V_2')|}{\min\{|V_1'|, |V_2'|\}} \leq \frac{|E_G(V_1, V_2)|}{|V_1| - 1} \leq \frac{i(G)|V_1| - 1}{|V_1| - 1} < i(G).$$

LEMMA 2.2. *Under conditions of Lemma 2.1*

1) $V_{i,1,3} = V_{i,1,2} = \varnothing, \quad i = 1, 2;$

2) *if* $|V_2| \geq |V_1| + 3$, *then* $V_{2,1,1} = \varnothing$;

3) *if $G(V_{3-i,1,1})$ has more than five edges, then any vertex from V_i is adjacent to at most one vertex from $V_{i,1,1}$.*

Proof. Suppose that $v \in V_{i,1,2} \cup V_{i,1,3}$. Then $|E_G(V_1, V_2)| \geq 3$ and hence $|V_1| \geq \lceil 3 \cdot 9/2 \rceil = 14$. Let w_1 and w_2 be the vertices of $V_{i,1}$ adjacent to v. Putting $V_i' = V_i \setminus \{v, w_1, w_2\}$, $V_{3-i}' = V_{3-i} \cup \{v, w_1, w_2\}$ we see that $|E_G(V_1', V_2')| \leq |E_G(V_1, V_2)| - 1$ and

$$\frac{|E_G(V_1', V_2')|}{\min\{|V_i| - 3, |V_{3-i}| + 3\}} \leq \frac{|E_G(V_1, V_2)| - 1}{|V_1| - 3} \leq \frac{i(G)|V_1| - 1}{|V_1| - 3} < i(G).$$

Now, let $|V_2| \geq |V_1| + 3$, $v \in V_{2,1,1}$ and let $w \in V_{2,1}$ be adjacent to v. Denote $V_1' = V_1 \cup \{v, w\}$, $V_2' = V_2 \setminus \{v, w\}$. Then

$$|E_G(V_1', V_2')| = |E_G(V_1, V_2)|$$

and

$$\frac{|E_G(V_1', V_2')|}{\min\{|V_1'|, |V_2'|\}} = \frac{|E_G(V_1, V_2)|}{\min\{|V_1| + 2, |V_2| - 2\}} \leq \frac{i(G)|V_1|}{|V_1| + 1} < i(G).$$

At last, suppose that

$$\left|E\big(G(V_{3-i,1,1})\big)\right| > 5, \tag{2.1}$$

and vertices v_1, $v_2 \in V_{i,1,1}$ are adjacent to $v \in V_i$. It follows from above that $v \in V_{i,0}$. By (2.1), $|E_G(V_1, V_2)| > 10$, and $|V_1| > 10 \cdot 9/2 = 45$. If $(v_1, v_2) \in E(G)$, then transferring v_1, v_2, and v from V_i into V_{3-i}, we decrease the number of edges in the cut. Let

$$(v_1, v_2) \notin E(G), \quad (v_1, x_2) \in E(G), \quad (x_1, v_2) \in E(G), \quad \{x_1, x_2\} \subset V_{1,1}.$$

By (2.1) there is an edge $(z_1, z_2) \in E(G(V_{3-i,1,1}))$ whose ends are not adjacent to v_1, v_2, x_1, and x_2. Setting

$$V_i' = V_i \setminus \{v, v_1, v_2, x_1, x_2\} \cup \{z_1, z_2\}, \quad V_{3-i}' = V(G) \setminus V_i',$$

we see that

$$|E_G(V_1', V_2')| \leq |E_G(V_1, V_2)| - 1$$

and

$$\frac{|E_G(V_1', V_2')|}{\min\{|V_i| - 3, |V_{3-i}| + 3\}} \leq \frac{|E_G(V_1, V_2)| - 1}{|V_1| - 3} = \frac{i(G)|V_1| - 1}{|V_1| - 3} < i(G).$$

Now we need some definitions.

Let us call a cubic graph *marked* if the ends of any edge $e = (v, w)$ have marks $l(e, v)$ and $l(e, w)$ from $\{-1, 0, 1\}$ such that for any vertex u and the edges e_1, e_2, e_3 incident to u we have $\{l(e_1, u), l(e_2, u), l(e_3, u)\} = \{-1, 0, 1\}$.
We say that a cut (V_1, V_2) of a marked cubic graph G is *optimal* if

(i) $i(G) = |E_G(V_1, V_2)|/|V_1|$;

(ii) for this cut the expression

$$\delta(V_1, V_2) = \sum_{\{e=(v,w)\mid v\in V_1, w\in V_2\}} \big(l(e, v) - l(e, w)\big)$$

takes the minimum value over all the cuts for which (i) holds.
Let $e = (v, w)$ be an edge of a marked cubic graph G, $N_G(v) = \{w, w_1, w_2\}$, $N_G(w) = \{v, v_1, v_2\}$. The value

$$l(e, v) + l(e, w) + l\big((v, w_1), w_1\big) + l\big((v, w_2), w_2\big) + l\big((w, v_1), v_1\big) + l\big((w, v_2), v_2\big)$$

is called the *weight* $W(e)$ *of* e.
Suppose that (V_1, V_2) is a cut of a marked cubic graph G,

$$\{v, w, v_2, w_2\} \subset V_2, \quad \{v_1, w_1\} \subset V_1,$$
$$\big\{(v, w), (v, w_1), (v, w_2), (v_1, w), (v_2, w)\big\} \subset E(G),$$
$$V_1' = V_1 \cup \{v, w\}, \quad V_2' = V_2 \setminus \{v, w\}.$$

Then

$$\big|E_G(V_1', V_2')\big| = \big|E_G(V_1, V_2)\big|$$

and

$$\delta(V_1', V_2') = \delta(V_1, V_2) - \Big(l\big((v, w_1), w_1\big) - l\big((v, w_1), v\big)\Big) - \Big(l\big((w, v_1), v_1\big)$$
$$- l\big((w, v_1), w\big)\Big) + \Big(l\big((v, w_2), v\big) - l\big((v, w_2), w_2\big)\Big) + \Big(l\big((w, v_2), w\big) - l\big((w, v_2), v_2\big)\Big).$$

Since

$$l\big((v, w_1), v\big) + l\big((v, w_2), v\big) + l\big((v, w), v\big)$$
$$= l\big((w, v_1), w\big) + l\big((w, v_2), w\big) + l\big((w, v), w\big) = 0,$$

we have $\delta(V_1', V_2') = \delta(V_1, V_2) - W\big((v, w)\big)$. This yields the following assertion.

LEMMA 2.3. *Let* (V_1, V_2) *be the optimal cut of a marked cubic graph* G. *If* $G(V_{1,1})$ *has an edge* e_1 *with* $W(e_1) = t$, *then any edge* $e_2 \in E(G(V_{2,1}))$ *with* $W(e_2) > t$ *should be adjacent to* e_1. *In particular, at most two edges of* $G(V_{2,1})$ *can have a weight greater than* t.

3. A LOWER BOUND FOR THE ISOPERIMETRIC NUMBER OF CUBIC GRAPHS

We will show that for almost all cubic graphs G the inequality $i(G) \geq 1/4.95$ holds. To do this, Bollobás' configurations (Bollobás, 1985) will be used.

For an even n a $(3, n)$-configuration τ is a set X of $3n$ labelled vertices partitioned into n 3-element subsets with some perfect matching M on these $3n$ vertices. We may assume that $X = \{x_{ij} \mid 1 \leq i \leq n,\ j \in \{-1, 0, 1\}\}$.
There is a simple one-to-one correspondence between the set of $(3, n)$-configurations and the set of cubic marked graphs (c.m.g.) on n labelled vertices. We can simply merge every set $X_i = \{x_{i,-1}, x_{i,0}, x_{i,1}\}$ to the vertex v_i of degree 3. Bollobás showed that simply cubic marked graphs correspond to constant (nonzero) part of the set of $(3, n)$-configurations as $n \to \infty$.
We will identify cubic marked graphs on n labelled vertices with corresponding $(3, n)$-configurations.
A cut (V_1, V_2) $(|V_1| \leq |V_2|)$ of c.m.g. G is called *convenient* if (in the notions of the previous section)

(i) $V_{i,2} = V_{i,3} = \varnothing, \quad i = 1, 2$;

(ii) $V_{i,1,3} = V_{i,1,2} = \varnothing, \quad i = 1, 2$;

(iii) if $|V_2| \geq |V_1| + 3$, then $V_{2,1,1} = \varnothing$;

(iv) if $G(V_{3-i,1,1})$ has more than five edges, then any vertex from V_i is adjacent to at most one vertex from $V_{i,1,1}$;

(v) if $G(V_{1,1})$ has an edge e_1 with $W(e_1) = t$, then any edge $e_2 \in E(G(V_{2,1}))$ with $W(e_2) > t$ is adjacent to e_1;

(vi) if $G(V_{2,1})$ has an edge e_2 with $W(e_2) = t$, then any edge $e_1 \in E(G(V_{1,1}))$ with $W(e_1) < t$ is adjacent to e_2.

Lemmas 2.1–2.3 imply that any c.m.g. G with $i(G) \leq 1/4.95$ has some convenient cut such that

$$\big|E_G(V_1, V_2)\big|/|V_1| \leq 1/4.95. \tag{3.1}$$

We can introduce a probability distribution on the set of all $(3n-1)!!$ of $(3, n)$-configurations letting $\mathbf{P}\{\tau\} = 1/(3n-1)!!$ for every τ. Let us estimate the probability of existence of a convenient cut satisfying (3.1) in $(3, n)$-configurations. Theorem VII.32 from (Bollobás, 1985) implies that almost every (a.e.) random $(3, n)$-configuration is connected.

LEMMA 3.1. *The probability that there exists a cut* (V_1, V_2) *with* $|E_G(V_1, V_2)| = 1$; $4 \leq |V_1| \leq |V_2|$ *in a random* $(3, n)$*-configuration is* $O(1/n)$.

Proof. Since in this case $|V_1|$ should be odd, we estimate the required probability by the value

$$d_1 = \frac{1}{(3n-1)!!} \sum_{i=2}^{\lceil n/4\rceil - 1} \binom{n}{2i+1} (6i+3)(3n-6i-3)(6i+1)!!(3n-6i-5)!!$$
$$= \frac{n!}{(3n-1)!!} \sum_{i=2}^{\lceil n/4\rceil - 1} \frac{(6i+3)!!(3n-6i-3)!!}{(2i+1)!(n-2i-1)!}.$$

For $3 \leq i \leq \lceil n/4\rceil - 1$ the ratio of ith summand to $(i-1)$th summand is equal to

$$\frac{(6i+3)(6i+1)(6i-1)(n-2i+1)(n-2i)}{(2i+1)2i(3n-6i+3)(3n-6i+1)(3n-6i-1)}$$
$$= \frac{(6i+1)(6i-1)(n-2i)}{2i(3n-6i+1)(3n-6i-1)} \leq \frac{6i+1}{3n-6i+1} \leq 1.$$

Hence

$$d_1 = \frac{1}{(3n-1)!!}\frac{n}{4}\binom{n}{5}(15)!!(3n-15)!!$$
$$\le \left((1+o(1))\frac{n}{4}\frac{n^5}{5!}(15)!!/(3n)^{-7} = O(1/n)\right).$$

LEMMA 3.2. *The probability d_2 of existing a convenient cut (V_1, V_2) with $|E_G(V_1,V_2)| > 1$ and $V_{2,1,1} = \varnothing$ is $o(1)$.*

Proof. The following procedure gives us all the convenient cuts (V_1, V_2) with $|E_G(V_1,V_2)| = j$ and $|V_1| = i$:

1) select i vertices for V_1 ($\binom{n}{i}$ variants);
2) select end vertices for the j-matching between V_1 and V_2 ($\binom{i}{j}\binom{n-i}{j}j!$ variants);
3) mark by $-1, 0$, and 1 the ends of this matching (3^{2j} variants);
4) add marked edges in $G(V_1)$ to obtain the degrees of the vertices of V_1 to be equal to 3 ($(3i-j-1)!!$ variants);
5) add $2j$ marked edges connecting $V_{2,1}$ with $V_{2,0}$ ($\frac{(3n-3i-3j)!}{(3n-3i-5j)!}$ variants);
6) add marked edges to obtain a cubic marked graph ($(3n-3i-5j-1)!!$ variants).

So,

$$d_2 \le \frac{1}{(3n-1)!!}\sum_{j=2}^{\lceil n/9.9\rceil}\sum_{i=\lceil 4.95j\rceil}^{\lceil n/2\rceil}\binom{n}{i}\binom{i}{j}\binom{n-i}{j}3^{2j}(j!)(3i-j-1)!!$$
$$\times\frac{(3n-3i-3j)!(3n-3i-5j-1)!!}{(3n-3i-5j-1)!}$$
$$= \frac{n!}{(3n-1)!!}\sum_{j=2}^{\lceil n/9.9\rceil}\sum_{i=\lceil 4.95j\rceil}^{\lceil n/2\rceil}\frac{3^{2j}(3i-j-1)!!(3n-3i-3j)!}{j!(i-j)!(n-i-j)!(3n-3i-5j)!!} \tag{3.2}$$
$$\le \frac{\bar{c}n^{0.5}}{3^{1.5n}n^{0.5n}}\sum_{j=2}^{\lfloor n/9.9\rfloor}\sum_{i=\lfloor 4.95j\rfloor}^{\lfloor n/2\rfloor}3^{2j}\frac{(3i-j)^{1.5i-0.5j}}{j^j(i-j)^{i-j}}$$
$$\times\frac{3^{3n-3i-3j}(n-i-j)^{2(n-i-j)}}{(3n-3i-5j)^{0.5(3n-3i-5j)}}.$$

Putting

$$F(n,i,j) = \frac{(3i-j)^{1.5i-0.5j}(n-i-j)^{2(n-i-j)}}{3^{3i+j}j^j(i-j)^{i-j}(3n-3i-5j)^{0.5(3n-3i-5j)}},$$

we have

$$d_2 \le \frac{\bar{c}n^{0.5}3^{1.5n}}{n^{0.5n}}\sum_{j=2}^{\lfloor n/9.9\rfloor}\sum_{i=\lceil 4.95j\rceil}^{\lfloor n/2\rfloor}F(n,i,j). \tag{3.3}$$

Clearly,

$$\left(\log F(n,i,j)\right)'_j = -0.5\left(1+\log(3i-j)\right) - 2\left(1+\log(n-i-j)\right) - \log 3$$
$$-\left(1+\log j\right) + \left(1+\log(i-j)\right) + 2.5\left(1+\log(3n-3i-5j)\right)$$
$$= \log\frac{(i-j)\left(3(n-i-5j/3)\right)^{2.5}}{(3i-j)^{0.5}(n-i-j)^2 3j}.$$

Since for any fixed i and j such that $i \ge 4.9j$, $j \ge 2$ the derivative of the function $\psi(n) = (n-i-5j/3)^{2.5}(n-i-j)^{-2}$ is positive for $n \ge 2i$, we have

$$\frac{(i-j)\left(3(n-i-5j/3)\right)^{2.5}}{(3i-j)^{0.5}(n-i-j)^2 3j} \ge \frac{3(i-5j/3)^{2.5}}{(i-j/3)^{0.5}(i-j)j} = \frac{3(i/j-5/3)^{2.5}}{(i/j-1/3)^{0.5}(i/j-1)}.$$

The minimum of the last expression under the condition $i/j \ge 4.9$ is attained at $i/j = 4.9$ and is equal to

$$\frac{3(4.9-5/3)^{2.5}}{(4.9-1/3)^{0.5}(4.9-1)} > 3.$$

Consequently, $(\log F(n,i,j))'_j > \log 3 > 0$ and

$$\sum_{j=2}^{\lfloor n/9.9\rfloor}\sum_{i=\lfloor 4.95j\rfloor}^{\lfloor n/2\rfloor}F(n,i,j) \le \sum_{i=10}^{\lfloor n/2\rfloor}\frac{i}{4.9}F(n,i,i/4.9). \tag{3.4}$$

Put

$$\alpha = i/4.9n,\quad \Phi(\alpha,n) = \frac{3^{1.5n}}{n^{0.5(n-1)}}\alpha n F(n, 4.9\alpha n, \alpha n).$$

Then (3.3) and (3.4) imply

$$d_2 \le \sum_{i=10}^{\lfloor n/2\rfloor}\Phi\left(i/(4.9n), n\right).$$

Let us count

$$\Phi(\alpha,n) = \frac{3^{1.5n}\alpha(13.7\alpha n)^{0.5\cdot 13.7\alpha n}(n-5.9\alpha n)^{2(n-5.9\alpha n)}}{n^{0.5(n-3)}3^{15.7\alpha n}(3.9\alpha n)^{3.9\alpha n}(3n-19.7\alpha n)^{0.5(3n-19.7\alpha n)}(\alpha n)^{\alpha n}}$$
$$= \alpha n^{1.5}\left[\frac{3^{1.5-15.7\alpha-0.5(3-19.7\alpha)}(13.7\alpha n)^{6.85\alpha}\left(n(1-5.9\alpha)\right)^{2(1-5.9\alpha)}}{n^{0.5}(3.9\alpha n)^{3.9\alpha}\left(n(1-19.7\alpha/3)\right)^{1.5-9.85\alpha}(\alpha n)^{\alpha}}\right]^n$$
$$= \alpha n^{1.5}\left[\frac{3^{-5.85\alpha}13.7^{6.85\alpha}\alpha^{1.95\alpha}(1-5.9\alpha)^{2(1-5.9\alpha)}}{3.9^{3.9\alpha}(1-19.7\alpha/3)^{1.5-9.85\alpha}}\right]^n.$$

Denote by $\Phi_0(\alpha)$ the expression in square brackets and put $c = 3^{-5.85}13^{6.85}3.9^{-3.9}$. Then

$$\log\Phi_0(\alpha) = \alpha\log c + 1.95\alpha\log\alpha + 2(1-5.9\alpha)\log(1-5.9\alpha)$$
$$-1.5(1-19.7\alpha/3)\log(1-19.7\alpha/3),$$
$$\left(\log\Phi_0(\alpha)\right)' = \log c + 1.95 + 1.95\log\alpha - 11.8$$
$$-11.8\log(1-5.9\alpha) + 9.85 + 9.85\log(1-19.7\alpha/3)$$
$$= \log c + 1.95\log\alpha - 11.8\log(1-5.9\alpha) + 9.85\log(1-19.7\alpha/3),$$
$$\left(\log\Phi_0(\alpha)\right)'' = \frac{1.95}{\alpha} + \frac{11.8\cdot 5.9}{1-5.9\alpha} - \frac{9.85\cdot 19.7/3}{1-19.7\alpha/3}.$$

Hence, for $1/12 \le \alpha \le 1/9.8$,

$$\left(\log \Phi_0(\alpha)\right)' = \log c + 1.95 \log \frac{\alpha}{1-5.9\alpha} + 9.85 \log \frac{1-19.7\alpha/3}{1-5.9\alpha}$$

$$\ge \log c + 1.95 \log \frac{1}{12-5.9} + 9.85 \log \frac{9.8-19.7/3}{9.8-5.9} \ge 6.1 - 3.6 - 1.9 = 0.6.$$

For $0 < \alpha < 1/12$ we have $16.3\alpha/3 \le 1 - 19.7\alpha/3$ and

$$\left(\log \Phi_0(\alpha)\right)'' \ge \frac{1.95 \cdot 16.3/3 - 9.85 \cdot 19.7/3}{1-19.7\alpha/3} + \frac{11.8 \cdot 5.9}{1-5.9\alpha}$$
$$= \frac{-162.26}{3-19.7\alpha} + \frac{67.28}{1-5.8\alpha} = \frac{39.58 - 384.308\alpha}{(3-19.7\alpha)(1-5.8\alpha)} > 0.$$

Therefore, the function $\log \Phi_0(\alpha)$ is convex in the interval $0 < \alpha < 1/12$ and increases in $1/12 \le \alpha \le 1/9.8$. Thus

$$\max\left\{\Phi_0(\alpha) | 2/n \le \alpha \le 1/9.8\right\} = \max\left\{\Phi_0(2/n), \Phi_0(1/9.8)\right\}.$$

Let us estimate these values of $\Phi(\alpha)$:

$$\Phi_0(2/n) = \left[\frac{c^2(2/n)^{3.9}(1-5.9 \cdot 2/n)^{2n-23.6}}{(1-19.7 \cdot 2/3n)^{1.5n-19.7}}\right]^{1/n} = \left[O(n^{-3.9})\right]^{1/n},$$

$$9.8 \log \Phi_0(1/9.8) = \log c - 1.95 \log 9.8 + 2(9.8-5.9)\log(1-5.9/9.8)$$

$$-1.5(9.8-19.7/3)\log\left(1 - 19.7/(3 \cdot 9.8)\right) \le 6.2 - 4.45 - 7.186 + 5.379 \le -0.05.$$

Hence

$$\Phi(\alpha, n) \le \alpha n^{1.5} \max\left\{O(n^{-3.9}), \exp\{-0.05n\}\right\} = o(n^{-2}).$$

LEMMA 3.3. *The probability d_3 that there exists a convenient cut (V_1, V_2) with $0.5n - 1 \le |V_1| \le |V_2|$ such that $|E_G(V_1, V_2)|/|V_1| \le 1/4.95$ and either $G(V_{1,1})$ or $G(V_{2,1})$ contains less than 6 edges is $o(1)$.*

Proof. Obviously,

$$d_3 = \sum_{i=0.5n-1}^{n/2} \sum_{j=2t}^{i/4.95} \sum_{l=1}^{2} \sum_{t=0}^{5} p(i,j,l,t),$$

where $p(i,j,l,t)$ is the probability that there exists a convenient cut (V_1, V_2) with $|V_1| = i$, $|E_G(V_1, V_2)| = j$ and $|E_G(V_{l,1})| = t$.
First we estimate $p(n/2, j, 2, t)$. To do this, we repeat the first four steps of the procedure described in the proof of Lemma 3.2 and then make the following steps:

5') put t marked independent edges on $V_{2,1}$ and $2j - 2t$ marked edges connecting $V_{2,1}$ with $V_{2,0}$ ($\binom{2j}{2t}(2t-1)!!\frac{(3(n-i-j))!}{(3n-3i-5j+2t)!}$ ways);

6') add marked edges to obtain a cubic marked graph ($(3n - 3i - 5j + 2t - 1)!!$ variants).

The ratio $p(n/2, j, 2, t)$ to the summand with parameters $(n/2, j)$ in sum (3.2) is equal to

$$\binom{2j}{2t}(2t-1)!!\frac{(3(n-i-j))!}{(3n-3i-5j+2t)!}\frac{(3n-3i-5j+2t-1)!!}{(3n-3i-5j-1)!!}$$
$$\le \frac{(2j)^{2t}}{2^t t!(3n-3i-5j)^t} < j^t.$$

But in the course of the proof of Lemma 3.2 it was shown that this summand does not exceed $n^2 \exp\{-0.05\}$. Hence $p(n/2, j, 2, t) = o(n^{-3})$ for any j and t under consideration.
Evidently, the cases $l = 1$ and $i = 0.5n - 1$ are quite similar.

LEMMA 3.4. *The probability d_4 that $i(G) \le 1/5.2$ is $o(1)$.*

Proof. If $i(G) \le 1/5.2$ for some c.m.g. G, then there exists a convenient cut (V_1, V_2) with

$$|E_G(V_1, V_2)|/|V_1| \le 1/5.2.$$

Denote this event by $\mathcal{B}$. Obviously,

$$\mathbf{P}\{\mathcal{B}\} \le \mathbf{P}\{\mathcal{B}_1\} + \mathbf{P}\{\mathcal{B}_2\} + \mathbf{P}\{\mathcal{B}_3\},$$

where $\mathcal{B}_i$ means that $\mathcal{B}$ takes place and, in addition, $\mathcal{B}_1$ means that $|E_G(V_1, V_2)| \le 1$, $\mathcal{B}_2$ means that $E_G(V_{2,1}) = \emptyset$, and $\mathcal{B}_3 = \mathcal{B} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$.
By Lemmas 3.1 and 3.2, $\mathbf{P}\{\mathcal{B}_1\} + \mathbf{P}\{\mathcal{B}_2\} = o(1)$. Suppose that $\mathcal{B}_3$ takes place. By the definition of a convenient cut we have $|V_1| \in \{0.5n, 0.5n - 1\}$. We consider the case $|V_1| = 0.5n$. The other case is quite similar.
Let us estimate the probability p_j of the event that $\mathcal{B}_3$ takes place, $|V_1| = 0.5n$, and $|E_G(V_1, V_2)| = j$. To do this, we repeat first four steps of the procedure described in the proof of Lemma 3.2 and then make the following step:

5'') add marked edges to obtain a cubic marked graph ($(1.5n - j - 1)!!$ variants).

It is clear now that

$$p_j \le \frac{1}{(3n-1)!!}\binom{n}{n/2}\left(\binom{n/2}{j}\right)^2 3^{2j} j! \left((1.5n - j - 1)!!\right)^2$$
$$= \frac{n! 3^{2j}\left((1.5n-j-1)!!\right)^2}{(3n-1)!! j! \left((0.5n-j)!\right)^2}$$
$$\le \frac{(1+o(1))(2\pi n)^{0.5}(n/e)^n 3^{2j} 2\left((1.5n-j)/e\right)^{1.5n-j}}{(2\pi j)^{0.5}(j/e)^j 2^{0.5}(3n/e)^{1.5n} 2\pi(0.5n-j)\left((0.5n-j)/e\right)^{n-2j}}$$
$$\le \frac{(1+o(1))3^{2j-1.5n}(1.5-j/n)^{1.5n-j}}{n^{0.5}(j/n)^j(0.5-j/n)^{n-2j}}.$$

Putting $\alpha = j/n$, we have

$$p_j \le \frac{(1+o(1))}{n^{0.5}}\left(\frac{3^{2\alpha-1.5}(1.5-\alpha)^{1.5-\alpha}}{\alpha^{\alpha}(0.5-\alpha)^{1-2\alpha}}\right)^n.$$

Put

$$\varphi(\alpha) = \frac{3^{2\alpha-1.5}(1.5-\alpha)^{1.5-\alpha}}{\alpha^{\alpha}(0.5-\alpha)^{1-2\alpha}}.$$

It is clear that, for $0 < \alpha \le 1/10.4$,

$$\begin{aligned}(\log\varphi(\alpha))' &= -1-\log(1.5-\alpha)+2\log 3-1-\log\alpha+2+2\log(0.5-\alpha)\\ &= \log\frac{9(0.5-\alpha)}{(1.5-\alpha)\alpha} > \log\frac{9\cdot 0.4}{1.5\cdot 0.1} > 0.\end{aligned}$$

Hence $\max\{\varphi(\alpha)|\, 0<\alpha\le 1/10.4\} = \varphi(1/10.4)$ and

$$\mathbf{P}\{\mathcal{B}_3\} \le n\max\{p_j | 2 \le j \le n/10.4\} \le n\frac{(1+o(1))}{n^{0.5}}(\varphi(1/10.4))^n.$$

But $\varphi(1/10.4) < 0.999$.

LEMMA 3.5. *Let j be sufficiently large and $9.9j \le n \le 10.4j$. Then the probability d_5 of the event $\mathcal{D}_5$ that random c.m.g. G has a convenient cut (V_1, V_2) with $|V_1| = |V_2|$ and $|E_G(V_1,V_2)| = j$ does not exceed $jn^2\exp\{-0.005j\}$.*

Proof. Case 1. Let

$$\min\left\{W(e) | e \in E(G(V_{1,1}))\right\} \ge 0.$$

Let $p(1,n)$ be the probability of the event $\mathcal{D}_5$ provided Case 1 takes place, and $p(1,n,k,l)$ be the probability that, provided Case 1 takes place, the event $\mathcal{D}_5$ holds and, in addition, there are exactly k edges in $G(V_{1,1})$ and exactly l edges in $G(V_{2,1})$. Then

$$\begin{aligned}p(1,n) &\le \sum_{k=1}^{\lfloor j/2\rfloor}\sum_{l=1}^{\lfloor j/2\rfloor} p(1,n,k,l)\\ &\le j^2\max\{p(1,n,k,l) | 1\le k\le q\lfloor j/2\rfloor,\ 1\le l\le\lfloor j/2\rfloor,\ 9.9j\le n\le 10.4j\}.\end{aligned}$$

The following eleven steps give us all the required c.m.g.

1) select $n/2$ vertices for V_1 ($\binom{n}{n/2}$ variants);

2) select end vertices for the j-matching between V_1 and V_2 ($\binom{n/2}{j}j!$ variants);

3) select end vertices for the k-matching on $V_{1,1}$ ($\binom{j}{2k}(2k-1)!!$ variants);

4) select end vertices in $V_{1,0}$ for the edges whose other ends belong to $V_{1,1,1}$ ($\frac{(0.5n-j)!}{(0.5n-j-2k)!}$ variants);

5) select points for all the ends of all edges incident to $V_{1,1,1}$ in such a way that $W(e) \ge 0$ for every edge of the chosen k-matching ($(3^6 2^2 435/729)^2$ variants);

6) select points for the remaining edges of the j-matching (3^{2j-4k} variants);

7) select points in $V_{1,0}$ for the ends of edges whose other ends belong to $V_{1,1,0}$ ($\frac{(1.5n-3j-2k)!}{(1.5n-5j+2k)!}$ variants);

8) select points for the ends of edges whose both ends belong to $V_{1,0}$ ($(1.5n-5j+2k-1)!!$ variants);

9) select points for the l-matching in $V_{2,1}$ ($\binom{j}{2l}(2l-1)!!2^{2l}$ variants);

10) select points in $V_{2,0}$ for the ends of edges whose other ends belong to $V_{2,1}$ ($\frac{(1.5n-3j)!}{(1.5n-5j+2l)!}$ variants);

11) select points for the ends of edges whose both ends belong to $V_{2,0}$ ($(1.5n-5j+2l-1)!!$ variants).

Hence

$$\begin{aligned}p(1,n,k,l) \le{}& \frac{1}{(3n-1)!!}\binom{n}{n/2}\left(\binom{n/2}{j}\right)^2 j!\binom{j}{2k}(2k-1)!!\frac{(0.5n-j)!}{(0.5n-j-2k)!}\\ &\times\left(3^6 2^2\frac{435}{729}\right)^k 3^{2j-4k}\frac{(1.5n-3j-2k)!}{(1.5n-5j+2k)!}(1.5n-5j+2k-1)!!\binom{j}{2l}(2l-1)!!\\ &\times 2^{2l}\frac{(1.5n-3j)!}{(1.5n-5j+2l)!}(1.5n-5j+2l-1)!!.\end{aligned}$$

Since

$$\frac{3^{2k}(0.5n-j)!}{(0.5n-j-2k)!} \le \frac{(1.5n-3j)!}{(1.5n-3j-2k)!},$$

we have

$$\begin{aligned}p(1,n,k,l) \le h(n,k,l) ={}& \frac{n!3^{2j}j!2^{2k}(435/729)^k}{(3n-1)!!j!((0.5n-j)!)^2(j-2k)!2^k k!}\\ &\times\frac{(1.5n-3j)!(1.5n-5j+2k-1)!!j!2^{2l}(1.5n-3j)!}{(1.5n-5j+2k)!(j-2l)!2^l l!2^{0.75n-2.5j+l}(0.75n-2.5j+l)!}.\end{aligned}$$

Note that for even j

$$\frac{h(n,j/2,l)}{h(n,j/2-1,l)} = \frac{4(435/729)}{k(1.5n-4j)} < 1$$

and

$$\frac{h(n,k,j/2)}{h(n,k,j/2-1)} = \frac{4}{l(1.5n-4j)} < 1.$$

For $1 \le l \le 2j-1$, $1 \le k \le 2j-1$,

$$h(n,k,l) \le \frac{3(2\pi n)^{0.5}(n/e)^n 3^{2j} 2^{5j-1.5n}(435/729)^k}{2^{0.5}(3n/e)^{1.5n}\big(2\pi(0.5n-j)\big)\big((0.5n-j)/e\big)^{n-2j}(2\pi k)^{0.5}}$$
$$\times \frac{2\pi(1.5n-3j)\big((1.5n-3j)/e\big)^{3n-6j}(2\pi j)^{0.5}(j/e)^j}{(k/e)^k\big(2\pi(j-2k)\big)^{0.5}\big((j-2k)/e\big)^{j-2k}\big((0.75n-5j+2k/e)\big)^{0.75n-5j+2k}}$$
$$\times \frac{1}{\big(2\pi(0.75n-5j+2k)\big)^{0.5}\big(2\pi(j-2l)\big)^{0.5}\big((j-2l)/e\big)^{j-2l}(2\pi l)^{0.5}(l/e)^l}$$
$$\times \frac{1}{\big(2\pi(0.75n-5j+2l)\big)^{0.5}\big((0.75n-5j+2l)/e\big)^{0.75n-5j+2l}}$$
$$\le \frac{n^{-0.5n}3^{1.5n-4j}j^j 2^{5j-1.5n}(0.5n-j)^{2n-4j}(435/729)^k}{(j-2k)^{j-2k}k^k(0.75n-5j+2k)^{0.75n-5j+2k}l^l(j-2l)^{j-2l}}$$
$$\times \frac{1}{(0.75n-5j+2l)^{0.75n-5j+2l}}.$$

Putting $\alpha = k/j$, $\beta = n/j$, $\gamma = l/j$, we obtain that

$$h(n,k,l)$$
$$\le \frac{1}{j}\left[\frac{\beta^{-0.5\beta}3^{1.5\beta-4}1^1 2^{5-1.5\beta}(0.5\beta-1)^{2\beta-4}(435/729)^{\alpha}\alpha^{-\alpha}(1-2\gamma)^{2\gamma-1}}{(1-2\alpha)^{1-2\alpha}(0.75\beta-2.5+\alpha)^{0.75\beta-2.5+\alpha}(0.75\beta-2.5+\gamma)^{0.75\beta-2.5+\gamma}\gamma^{\gamma}}\right]^j.$$

Thus,

$$\log\Big(\max\{p(1,n,k,l) \,|\, 1 \le k < 0.5j,\ 1 \le l < 0.5j,\ 9.9j \le n \le 10.4j\}\Big)$$
$$\le -\log j + j\Big(\max\{\varphi(\alpha,\beta,\gamma) \,|\, 0<\alpha<0.5,\ 0<\gamma<0.5,\ 9.9 \le \beta \le 10.4\}\Big),$$

where

$$\varphi(\alpha,\beta,\gamma) = -0.5\beta(\log\beta) + (1.5\beta-4)(\log 1.5) + \log 2 + (2\beta-4)\log(0.5\beta-1)$$
$$+\alpha\log(435/729) - (1-2\alpha)\log(1-2\alpha) - \alpha\log\alpha - (0.75\beta-2.5+\alpha)\log(0.75\beta-2.5+\alpha)$$
$$-(1-2\gamma)\log(1-2\gamma) - \gamma\log\gamma + (0.75\beta-2.5+\gamma)\log(0.75\beta-2.5+\gamma).$$

For

$$0<\alpha<0.5,\quad 0<\gamma<0.5,\quad 9.9 \le \beta \le 10.4,$$

we have

$$\big(\varphi(\alpha,\beta,\gamma)\big)'_\beta = -0.5 - 0.5\log\beta + 1.5\log 1.5 + 2 + 2\log(0.5\beta-1)$$
$$-0.75 - 0.75\log(0.75\beta-2.5+\alpha) - 0.75 - 0.75\log(0.75\beta-2.5+\gamma)$$
$$\le -0.5\log\beta + 1.5\log 1.5 + 2\log(0.5\beta-1) - 1.5\log(0.75\beta-2.5)$$
$$\le 0.5\log\big((0.5\beta-1)/\beta\big) + 1.5\log\big(1.5(0.5\beta-1)/(0.75\beta-2.5)\big)$$
$$\le 0.5\log(0.5-1/10.4) + 1.5\log\big((0.5\cdot 9.9-1)/(0.5\cdot 9.9-5/3)\big)$$
$$< -0.45 + 1.5\cdot 0.2 < 0.$$

Hence it is sufficient to verify the assertion of the lemma for $\beta = 9.9$.
Since

$$\big(\varphi(\alpha,\beta,\gamma)\big)'_\alpha = 2+2\log(1-2\alpha)-1-\log\alpha-1-\log(0.75\beta-2.5+\alpha)+\log(435/729),$$

$$\big(\varphi(\alpha,\beta,\gamma)\big)''_\alpha = -\frac{4}{1-2\alpha} - \frac{1}{\alpha} - \frac{1}{0.75\beta-2.5+\alpha} < 0,$$

we can conclude that for fixed β and γ the function $\varphi(\alpha,\beta,\gamma)$ is concave in α and attains the maximum at $\alpha = x_1$, where x_1 is the root of the equation $(\varphi(\alpha,\beta,\gamma))'_\alpha = 0$. Here this equation has the form

$$\log\big(435(1-2x)^2/729\big) = \log x(0.75\beta-2.5+x),$$

and may be reduced to

$$(4-\delta)x^2 - (0.75\beta\delta - 2.5\delta + 4)x + 1 = 0, \tag{3.5}$$

where $\delta = 435/729$. The largest root of (3.5) is greater than 1. The smallest root of (3.5) is equal to

$$2\Big(0.75\beta\delta - 2.5\delta + 4 + \big((0.75\beta\delta - 2.5\delta+4)^2 - 4(4-\delta)\big)^{1/2}\Big)^{-1}. \tag{3.6}$$

For $\beta = 9.9$, $\delta = 435/729$, it follows from (3.6) that

$$0.0829 \le x_1 \le 0.083.$$

Analogously, under conditions of the lemma

$$\big(\varphi(\alpha,\beta,\gamma)\big)'_\gamma = 2\log(1-2\gamma) - \log\gamma - \log(0.75\beta-2.5+\gamma),$$
$$\big(\varphi(\alpha,\beta,\gamma)\big)''_\gamma = -\frac{4}{1-2\gamma} - \frac{1}{\gamma} - \frac{1}{0.75\beta-2.5+\gamma} < 0,$$

and for fixed β and α the function $\varphi(\alpha,\beta,\gamma)$ is concave in γ and attains the maximum at $\gamma = x_2$, where x_2 is the root of the equation (3.5), with $\delta = 1$. According to (3.6),

$$0.1166 \le x_2 \le 0.1167.$$

Since $(\varphi(\alpha,\beta,\gamma))'_\alpha$ does not depend on γ and $(\varphi(\alpha,\beta,\gamma))'_\gamma$ does not depend on α, we have

$$\varphi(x_1; 9.9; x_2) \le \varphi(0.083; 9.9; 0.1166)$$
$$-10^{-4}\big(\varphi(0.083;9.9;0.1166)\big)'_\alpha + 10^{-4}\big(\varphi(0.083;9.9;0.1166)\big)'_\gamma$$
$$\le -0.0051 - 10^{-4}(-0.01) + 10^{-4}\cdot 0.01 \le -0.005.$$

Thus, $p(1,n) \le j\exp\{-0.005j\}$.

Case 2. Let

$$\max\Big\{W(e)\Big| e \in E\big(G(V_{2,1})\big)\Big\} \le 0.$$

This case is symmetrical to Case 1.

Case 3. Let

$$\min\Big\{W(e)\Big| e \in E\big(G(V_{1,1})\big)\Big\} \le -1, \quad \max\Big\{W(e)\Big| e \in E(G(V_{2,1}))\Big\} \ge 1.$$

Then, by the definition of a convenient cut, at least $k-2$ edges $e \in E(G(V_{1,1}))$ have $W(e) \ge 1$ and at least $l-2$ edges $e \in E(G(V_{2,1}))$ have $W(e) \le -1$. Obviously, the probability of this event less than $p(1,n)$.

THEOREM. *For almost every cubic marked graph G on n vertices the inequality $i(G) \ge 1/4.95$ holds.*

Proof. Let G be a c.m.g. with $i(G) < 1/4.95$. Then there is a convenient cut (V_1, V_2) ($|V_1| \le |V_2|$) such that

$$|E_G(V_1, V_2)|/|V_1| \le 1/4.95.$$

With probability $1 - o(1)$ such a cut contains more than one edge. Lemma 3.2 and the definition of a convenient cut imply that with probability $1 - o(1)$ either $|V_1| = |V_2|$ or $|V_1| = |V_2| - 2$. Lemmas 2.4 and 2.5 give the required estimate in the former case. The latter case is quite analogous.
Bollobás (Bollobás, 1985) proved that simple cubic marked graphs correspond to constant (nonzero) part of the set of $(3,n)$-configurations as $n \to \infty$. Besides he showed that any simple cubic graph on n labelled vertices corresponds to exactly 6^n different $(3,n)$-configurations, and any nonsimple cubic graph on n labelled vertices corresponds to less that 6^n $(3,n)$-configurations. Hence the theorem implies the following assertion.

COROLLARY. *For almost every cubic simple graph G on n vertices $i(G) \ge 1/4.95$.*

The constant 1/4.95 could be slightly improved using the same ideas, but then the computations would be too complicate.

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