On $K_{s,t}$-minors in graphs with given average degree, II

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Abstract

Let $K^*_{s,t}$ denote the graph obtained from $K_{s,t}$ by adding all edges between the s vertices of degree t in it. We show how to adapt the argument of an our previous paper (Discrete Math. **308** (2008), 4435–4445) to prove that if $t/\log_2 t \geq 1000s$, then every graph G with average degree at least $t + 8s\log_2 s$ has a $K^*_{s,t}$ minor. This refines a corresponding result by Kühn and Osthus.

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1 Introduction

Graphs in this paper are undirected simple graphs. For a graph G, $V(G)$ is the set of its vertices, $E(G)$ is the set of its edges, $e(G) = |E(G)|$, and $v(G) = |V(G)|$. By $G[X]$ we denote the subgraph of G induced by the vertex set X. We let $N_G(v)$ denote the set of neighbors of v in G and $N_G[v] = N_G(v) \cup \{v\}$. Similarly, for $X \subseteq V(G)$, we define $N(X) := \bigcup_{x\in X} N(x)$. A *minor* of a graph G is a graph H that can be obtained from G by a sequence of vertex and edge deletions and edge contractions. For a graph H, let $D(H)$ denote the minimum number t such that every graph G with average degree at least t has an H-*minor*, i.e., a minor isomorphic to H.

Mader [8] proved that $D(K_r) \leq 8r\ln r$. Later, Kostochka [2, 3] and Thomason [14] found the order of magnitude of $D(K_r)$, and then Thomason [15] found the asymptotics of $D(K_r)$ as $r \to \infty$. Myers and Thomason [12, 9] determined $D(H)$ for almost every H, showing, in particular, that for almost all H, the extremal graphs not containing H are quasi-random (built deterministically from randomly generated subcomponents). Their methods work better for dense and balanced graphs.

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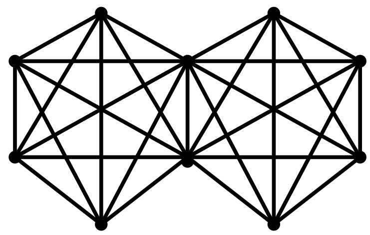

Figure 1: Graph $M(2,3,4)$ has no $K_{3,4}$-minor.

An example of a sparse and unbalanced H is the complete bipartite graph $K_{s,t}$, where s is fixed and t is large with respect to s. For this reason, Myers [10, 11] studied $D(K_{s,t})$ when s is fixed and t is large. Let $M(r,s,t)$ be the graph obtained by taking r copies of K_{s+t-1} arranged so that each two copies share the same fixed $s-1$ vertices (Fig. 1 shows $M(2,3,4)$). Myers [11] observed that $M(r,s,t)$ has no $K_{s,t}$-minor and that

$$e(M(r,s,t)) = \frac{1}{2}(t+2s-3)(n-s+1) + \binom{s-1}{2}, \tag{1}$$

where $n = |V(M(r,s,t))| = rt+s-1$. He proved that for $t > 10^{29}$ and $n \geq 3$, each n-vertex graph G with more than $\frac{1}{2}(t+1)(n-1)$ edges has a $K_{2,t}$ minor. The graphs $M(r,2,t)$ witness that this bound is sharp when $|V(G)| \equiv 1 \pmod{t}$. In connection with graph coloring, Chudnovsky, Reed, and Seymour [1] proved that Myers' bound is true for all t.

Myers conjectured that a similar, more general statement holds for $K_{s,t}$-minors.

Conjecture 1 *Let s be a positive integer. Then there exists a constant $C(s)$ such that, for all positive integers t, if G has average degree at least $C(s)\cdot t$, then G has a $K_{s,t}$-minor.*

Let $K^*_{s,t} = K_{s+t} - E(K_t)$. In other words, $K^*_{s,t}$ is the graph obtained from $K_{s,t}$ by adding all $\binom{s}{2}$ possible edges into the s-vertex partite set. Myers noted that the average degree that forces G to contain a $K_{s,t}$-minor also likely forces a $K^*_{s,t}$-minor, that is, $D(K_{s,t}) = D(K^*_{s,t})$ when s is fixed and t is large.

Myers' Conjecture was proved independently in [5] and [7] using different methods. Kühn and Osthus [7] showed the following.

Theorem 1 ([7]) *For every $0 < \epsilon < 10^{-16}$, there exists a number $t_0 = t_0(\epsilon)$ such that for all integers $t \geq t_0$ and $s \leq \epsilon^6 t/\log t$, every graph of average degree at least $(1+\epsilon)t$ contains $K_{s,t}$ as a minor.*

They also showed that $K_{s,t}$ can be replaced with $K^*_{s,t}$ if the restriction $s \leq \epsilon^6 t/\log t$ is replaced with $s \leq \epsilon^7 t/\log t$.

In [5], the following fact was proved.

Theorem 2 *Let n, s and t be positive integers with*

$$t > (240s\log_2 s)^{8s\log_2 s+1}.$$

Let G be an n-vertex graph such that $e(G) \geq \frac{t+3s}{2}(n-s+1)$. Then G has a $K^*_{s,t}$-minor. Furthermore, for infinitely many n, there exists a graph G_n of order n and size at least $\frac{t+3s-5\sqrt{s}}{2}(n-s+1)$ that has no $K_{s,t}$-minor.

From Theorem 2 we have that for huge t,

$$t + 3s - 5\sqrt{s} \leq D(K_{s,t}) \leq D(K^*_{s,t}) \leq t + 3s.$$

Hence, Myers' insight that $D(K_{s,t})$ is the same as $D(K^*_{s,t})$ is true asymptotically in s.

Observe that while Theorem 1 provides a weaker bound on the second term of $D(K_{s,t})$ (essentially, the second term in their bound is $(s\ln t/t)^{1/6}t$ while in Theorem 2 it is the asymptotically (in s) exact $3s$), it applies for a much wider (essentially best possible) range of t for a given s than Theorem 2, namely for $t \geq C \cdot s\log t$. Kühn and Osthus [7] also proved the following fact showing that the statement of their theorem would be incorrect if $s \geq 18t/\ln t$.

Proposition 3 ([7, Proposition 10]) *There exists n_0 such that for each integer $n \geq n_0$ and each $\alpha > 0$, there is an n-vertex graph G with average degree at least $n/2$ that does not have a $K_{s,t}$ minor with $s = \lceil 2n/\alpha \ln n \rceil$ and $t = \lceil \alpha n \rceil$.*

In particular, it implies that the statement of Theorem 2 is not correct when $t = s\ln s$ and s is large.

The goal of the present note is to show how to adapt the proof of Theorem 2 to prove the following.

Theorem 4 *Let $1 \leq s \leq t \leq n$ be integers such that $n \geq 2s$ and*

$$s \leq t/1000\log_2 t. \tag{2}$$

Let G be an n-vertex graph with $e(G) \geq \frac{t+8s\log_2 s}{2}(n-s+1)$. Then G has a $K^*_{s,t}$-minor.

This theorem applies to the same range of t in terms of s as Theorem 1 (and even a bit better, since the range does not depend on ϵ), but gives the better estimate of the second term. The first author plans to use Theorem 4 to improve the result of [4], where Theorem 2 was used.

The idea of this note is that in the proof of Theorem 2, we needed t that is much larger than s only in the case when n is small, essentially when $n < t + C \cdot s\ln t$. This note shows that in this range we can prove the bound of Theorem 4 (which is weaker) for $n \leq t + C \cdot s\ln s$. The setup and this case are handled in Section 2. In Section 3 we list useful lemmas from [5] and in Section 4 we present the proof of the main case.

2 Setup and graphs of small order

In [6] we proved the following.

Theorem 5 *Let $t \geq 6300$. Let G be a graph of order $n \geq 3$ with $e(G) > \frac{1}{2}(t+3)(n-2)+1$. Then G has a $K^*_{3,t}$-minor.*

So, it is enough to prove Theorem 4 for $s \geq 4$. Similarly to the proof in [5], we say that a graph G is (s,t)-*irreducible* if
(i) $v(G) \geq 2s$;
(ii) $e(G) \geq 0.5(t + 8s\log_2 s)(v(G) - s + 1)$;
(iii) G has no proper minor G' possessing (i) and (ii).

For an edge e of a graph G, $t_G(e)$ denotes the number of triangles in G containing e. Similarly to Lemma 3 in [5], the following lemma holds.

Lemma 6 *If G is an n-vertex (s,t)-irreducible graph and $s < t/1000\log_2 t$, then*
(a) $n \geq t + 8s\log_2 s - 1.5s + 1$;
(b) $t_G(e) \geq 0.5(t + 8s\log_2 s) - 1$ for every $e \in E(G)$;
(c) if $W \subset V(G)$ and $v(G) - |W| \geq 2s$, then W is incident with at least $0.5(t+8s\log_2 s)|W|$ edges; in particular, $\delta(G) \geq 0.5(t + 8s\log_2 s)$;
(d) G is s-connected;
(e) $e(G) < 0.5(t + 8s\log_2 s)n$.

Proof. The proofs of (b)–(d) are almost exactly the same as in the proof of Lemma 3 in [5]. So we present only the proof of (a) which slightly differs from that of Lemma 3 in [5]. Since G has at most $\binom{n}{2}$ edges, for the quadratic function $f(n) := n^2 - n - (t+8s\log_2 s)(v(G)-s+1)$ we have $f(n) \geq 0$. The roots of $f(n)$ are

$$n_{1,2} = \frac{1}{2}\left(t + 8s\log_2 s + 1 \pm \sqrt{(t + 8s\log_2 s + 1)^2 - 4(t + 8s\log_2 s)(s-1)}\right).$$

Since $(t+8s\log_2 s+1)^2 - 4(t+8s\log_2 s)(s-1) > (t+8s\log_2 s - 3s+1)^2$ for $t \geq 1000s\log_2 t$, either $n < 1.5s$ or $n > t + 8s\log_2 s - 1.5s + 1$. This together with (i) proves (a). □

Since we aim for a weaker bound than in [5], instead of Lemmas 4, 5, and 6 there, we prove just one.

Lemma 7 *Each (s,t)-irreducible graph with no $K^*_{s,t}$-minor has at least $10t/9$ vertices.*

Proof. Suppose that an (s,t)-irreducible graph G has $n = t + d$ vertices, where $d \leq t/9$. By Lemma 6(a), $d \geq 8s\log_2 s - 1.5s + 1$. If at most $s-1$ vertices of G have degree greater than t, then $2e(G) \leq (s-1)n + t(n-s+1) = tn + d(s-1) \leq tn + (s-1)t/9$. Since $n > t > 1000s\log t$, this is less than $(t + 8s\log_2 s)(n-s+1)$, a contradiction to (ii). So, we may assume that vertices $v_1, \ldots, v_s$ have degree at least t in G.

Let $k = \lceil \log_{3/2} d \rceil$. We will find s disjoint dominating sets S_i with $|S_i| \leq k+1$ for each $1 \leq i \leq s$.

Initialize $S_i^0 = \{v_i\}$ for $1 \leq i \leq s$. For $1 \leq i \leq s$, consecutively run the following procedure. Define U_i^j be the set of vertices not dominated by S_i^{j-1}. By the choice of $v_1, \dots, v_s$, $|U_i^1| \leq d-1$ for $1 \leq i \leq s$. Step j for constructing S_i is as follows. If U_i^j is empty or $j = k+1$, then set $S_i = S_i^j$ and stop. Otherwise, define $W_i^j = V(G) - U_i^j - \bigcup_{q=1}^{i-1} S_q - S_i^{j-1} - \{v_{i+1}, \dots, v_s\}$ and let S_i^{j+1} be obtained from S_i^j by adding a vertex $v_i^j \in W_i^j$ that has the most neighbors in U_i^j.

Since for every i, we do at most k steps and in each Step j add at most one vertex to S_i^j, we have $|S_i| \leq k+1$ for every i. It follows that for all i and j,

$$|V(G) - W_i^j| = |U_i^j \cup S_i^{j-1} \cup \{v_{i+1}, \dots, v_s\} \cup \bigcup_{q=1}^{i-1} S_q| \leq |U_i^1| + s(k+1) < d + s(k+1). \quad (3)$$

By Lemma 6(c), $\delta(G) \geq 0.5(t + 8s\log_2 s)$. So by (3) for all i and j and each $u \in U_i^j$,

$$\frac{|N_G(u) \cap W_i^j|}{|W_i^j|} \geq \frac{\delta(G) - |V(G) - W_i^j|}{n - |V(G) - W_i^j|} \geq \frac{0.5t - (d + s(k+1))}{n - (d + s(k+1))} \geq \frac{\frac{7t}{18} - s(k+1)}{t - s(k+1)}.$$

Since $k + 1 \leq 2 + \log_{3/2} d \leq \log_{3/2} \left(\frac{3}{2}\right)^2 \frac{t}{9} \leq 1.71 \log_2 \frac{t}{4}$, we conclude from (2) that for all i and j and each $u \in U_i^j$,

$$\frac{|N_G(u) \cap W_i^j|}{|W_i^j|} \geq \frac{7}{18} - \frac{s(k+1)}{t} \geq \frac{7}{18} - \frac{s(1.71\log_2 t)}{t} > \frac{1}{3}. \quad (4)$$

Hence by the choice of v_i^j, it has at least $\frac{|U_i^j|}{3}$ neighbors in U_i^j. Thus for all i and j, $|U_i^j| < (2/3)^j d$. In particular,

$$|U_i^{k+1}| < d\left(\frac{2}{3}\right)^{\lceil \log_{3/2} d \rceil} \leq d \cdot \frac{1}{d} = 1.$$

It follows that U_i^{k+1} is empty for $1 \leq i \leq s$. This means S_i is a dominating set of G with size at most $k+1$. Also, $G[S_i]$ is connected because each v_i^j was chosen among the neighbors of S_i^{j-1}. Since $s \geq 4$ and $d \geq 8s\log_2 s - 1.5s + 1 \geq 59$,

$$\frac{s(k+1)}{d} \leq \frac{s(1.71\log_2(\frac{9}{4}d))}{d} \leq \frac{1.71s\log_2(18s\log_2 s)}{7s\log_2 s} \leq \frac{1.71(\log_2 s + \log_2\log_2 s + 4.18)}{7\log_2 s}. \quad (5)$$

The derivative of the function $\phi(x) = \frac{x + \log_2 x + 4.18}{x}$ is negative for $x \geq 2$. It follows that for $s \geq 4$ by (5) we have

$$\frac{s(k+1)}{d} \leq \frac{1.71(\log_2 4 + \log_2\log_2 4 + 4.18)}{7\log_2 4} = \frac{1.71 \cdot 7.18}{14} < 1.$$

So, contracting each S_i to a single vertex, we do not touch at least $n - d = t$ vertices. Thus the resulting graph contains $K_{s,t}^*$. $\square$

3 Lemmas

The statements and proofs of Lemmas 7, 8, 9, and 10 in [5] do not need any change, since no relation between s and t is involved there. We will refer to the following two of them.

Lemma 8 ([5], Lemma 9) *Let s, k, and n be positive integers and $\alpha \geq 2$. Suppose that $n \leq \alpha(k+1)$. Let G be a $(3s \log_{\alpha/(\alpha-1)} n)$-connected graph with n vertices and $\delta(G) \geq k + 3(s-1)\log_{\alpha/(\alpha-1)} n$. Then $V(G)$ contains s disjoint subsets $A_1, \ldots, A_s$ such that for every $i = 1, \ldots, s$,*
(i) $G[A_i]$ is connected;
(ii) $|A_i| \leq 3 \log_{\alpha/(\alpha-1)} n$;
(iii) A_i dominates $G - A_1 - \ldots - A_{i-1}$.

Lemma 9 ([5], Lemma 10) *Let H be a graph and k be a positive integer. If C is an inclusion minimal k-separable set in H and $S = N(C) - C$, then the subgraph of H induced by $C \cup S$ is $(1 + \lceil \frac{k}{2} \rceil)$-connected.*

The statement of Lemma 11 in [5] also is correct in our setting, and the proof smoothly goes through when $s \geq 4$ and $t/\log_2 t > 1000s$. It will be our main tool:

Lemma 10 ([5], Lemma 11) *Let G be a $100s \log_2 t$-connected graph. Suppose that G contains a vertex subset U with $t + 100s\log_2 t \leq |U| \leq 3t$ such that $\delta(G[U]) \geq 0.4t + 100s \log_2 t$. Then G has a $K^*_{s,t}$-minor.*

4 Handling large graphs

The proof in the last section of [5] also works with small changes (we need some changes, since the range of t is different), but for convenience of the reader, instead of pointing out and commenting the differences we present below an updated version of this proof.

If Theorem 4 does not hold, then there exists an (s,t)-irreducible graph G with no $K^*_{s,t}$-minor. Let $n = v(G)$. By Lemma 7, $n \geq 10t/9$.

CASE 1. G is $200s\log_2 t$-connected. If G has a vertex v with $t + 100s\log_2 t \leq \deg(v) \leq 3t - 1$, then G satisfies Lemma 10 with $U = N[v]$ and we are done. Thus, we can assume that every vertex in G has either 'small' ($< t + 100s\log_2 t$) or 'large' ($\geq 3t$) degree. Let V_0 be the set of vertices of 'small' degree. If $|V_0| > t + 100s\log_2 t$, then there is some $V_0' \subseteq V_0$ such that

$$t + 100s\log_2 t \leq |\bigcup_{v \in V_0'} N[v]| \leq 3t - 1.$$

In this case, we can apply Lemma 10 with $U = \bigcup_{v \in V_0'} N[v]$.

Now, let $|V_0| \leq t + 100s\log_2 t$. By Lemma 6(e), the average degree of G is less than $t + 8s\log_2 s$. Since every vertex outside of V_0 has degree at least $3t$, we get

$$0.5t|V_0| + 3t(n - |V_0|) < (t + 8s\log_2 s)n$$

and hence by (2), $n < \frac{2.5|V_0|}{2-8s\log_2 s/t} < 3t$. Since (again by (2)) $n \geq 10t/9 > t + 100s\log_2 t$, we can apply Lemma 10 with $U = V(G)$ to find a needed minor.

CASE 2. G is not $200s\log_2 t$-connected. Let S be a separating set with at most $k = \lceil 200s\log_2 t\rceil - 1$ vertices and let $V(G) - S = V_1 \cup V_2$ where vertices in V_1 are not adjacent to vertices in V_2. Then each of V_1 and V_2 is a k-separable set. For $j = 1, 2$, let W_j be an inclusion minimal k-separable set contained in V_j and $S_j = N(W_j) - W_j$. By Lemma lem24, the graph $G_j = G[W_j \cup S_j]$ is $100s\log_2 t$-connected.

CASE 2.1. $|W_j \cup S_j| \geq t + 100s\log_2 t$ for some $j \in \{1, 2\}$. Then $|W_j| \geq t - 100s\log_2 t$. Let $G_j = G[W_j \cup S_j]$. By Lemma 6(b), $\delta(G_j) \geq 0.5(t + 8s\log_2 s)$. If $|W_j \cup S_j| \leq 3t$, then we apply Lemma 10 with $U = W_j \cup S_j$. So suppose

$$|W_j \cup S_j| > 3t. \tag{6}$$

As in Case 1, we may suppose that the degree of each $w \in W_j$ is either 'small' ($< t + 100s\log_2 t$) or 'large' ($\geq 3t$). Let W_j' be the set of vertices $w \in W_j$ of 'small' degree. As in Case 1, we conclude that $|W_j'| \leq t + 100s\log_2 t$. Since every vertex in $W_j - W_j'$ has degree at least $3t$, we get

$$0.5t|W_j'| + 3t|W_j - W_j'| < (t + 8s\log_2 s)|W_j \cup S_j|.$$

Since $|S_j| \leq k$, by (6), $3t|W_j - W_j'| \geq 3t(|W_j \cup S_j| - k - |W_j'|) \leq (3t - k)|W_j \cup S_j| - 3t|W_j'|$. So again by (2),

$$|W_j \cup S_j| \leq \frac{2.5|W_j'|}{2 - (k + 8s\log_2 s)/t} \leq \frac{2.5(1.1t)}{2 - 0.208} < 3t,$$

a contradiction to (6).

CASE 2.2. $|W_j \cup S_j| < t + 100s\log_2 t$ for both $j \in \{1, 2\}$. Let $H_j = G[W_j]$. By Lemma 6(c) and the fact that $|S_j| \leq k$,

$$\delta(H_j) \geq 0.5t - |S_j| \geq 0.5t - k. \tag{7}$$

Suppose that S_0 is a separating set in H_j with $|S_0| < 100s\log_2 t$. Let $W_j - S = W_{j,1} \cup W_{j,2}$ where vertices in $W_{j,1}$ are not adjacent to vertices in $W_{j,2}$. For $\ell = 1, 2$, let $e'(W_{j,\ell})$ denote the number of edges incident to $W_{j,\ell}$. By Lemma 6(c), $e'(W_{j,\ell}) \geq 0.5(t + 8s\log_2 s)|W_{j,\ell}|$. Since $e_G(S_j \cup S_0, W_{j,\ell}) \leq |S_j \cup S_0||W_{j,\ell}|$, we have

$$\sum_{w \in W_{j,\ell}} d_G(w) = 2e'(W_{j,\ell}) - e_G(S_j \cup S_0, W_j) > (t - 1.5k)|W_j|.$$

It follows that some $w_\ell \in W_{j,\ell}$ has degree greater than $t - 1.5k$. Thus,

$$2(t - 1.5k) \leq d_G(w_1) + d_G(w_2) < (|W_{j,1}| + |S_j \cup S_0|) + (|W_{j,1}| + |S_j \cup S_0|) \leq$$

$$|W_j \cup S_j| + |S_j| + |S_0| \leq (t + 100s\log_2 t) + 100s\log_2 t + k.$$

So, $t < 200s\log_2 t + 4k < 1000s\log_2 t$, a contradiction to (2). Therefore, H_j is $100s\log_2 t$-connected. By this, (7), and Lemma 8 (for $k = 0.3t$ and $\alpha = 4$), $V(H_j)$ contains s disjoint subsets $A_1^j, \dots, A_s^j$ such that for every $i = 1, \dots, s$,
(i) $G[A_i^j]$ is connected;
(ii) $|A_i^j| \le 3\log_{4/3}|W_j| < 7.23\log_2|W_j| \le 7.23\log_2(1.1t)$;
(iii) A_i^j dominates $W_j - A_1^j - \dots - A_{i-1}^j$.

Since G is s-connected, $|S_j| \ge s$, $j = 1, 2$, and there are s pairwise vertex disjoint S_1, S_2-paths $P_1, \dots, P_s$. We may assume that the only common vertex of P_i with S_j is p_{ij}. By Lemma 6(b), each p_{ij} has at least $0.5t - 200s\log_2 t$ neighbors in W_j. Thus, we can choose $2s$ distinct vertices q_{ij} such that $q_{ij} \in W_j - \bigcup_{k=1}^s A_k^j$ and $p_{ij}q_{ij} \in E(G)$.

Define $F_i = A_i^1 \cup A_i^2 \cup V(P_i) + q_{i1} + q_{i2}$, $i = 1, \dots, s$. Then for every $i = 1, \dots, s$,
(i) $G[F_i]$ is connected;
(ii) F_i-s are pairwise disjoint;
(iii) F_i dominates $\bigcup_{j=1}^2 W_j - F_1 \dots - F_{i-1}$.

Since by (2),

$$|\bigcup_{j=1}^{2} W_j - F_1 \dots - F_{i-1}| \ge 2(t - 400s\log_2 t) - 14.46s\log_2 1.1t - 2s > t,$$

G has a $K^*_{s,t}$-minor, a contradiction. ■

Comments. 1. Lemma 8 was reproved in [6] in a slightly stronger form.

2. The factor 1000 in (2) and maybe the factor 8 in front of $s\log_2 s$ in Theorem 4 can be improved with more work, but Proposition 3 shows that the theorem will not hold if we replace both 1000 and 8 with 1/18. Still, as Deryk Osthus observed, it could be that the statement holds for all $s \le t$ if we do not change 8.

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