Graphs with chromatic number close to maximum degree

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Abstract

Let G be a color-critical graph with $\chi(G) \geq \Delta(G) = 2t+1 \geq 5$ such that the subgraph of G induced by the vertices of degree $2t+1$ has clique number at most $t-1$. We prove that then either $t \geq 3$ and $G = K_{2t+2}$ or $t = 2$ and $G \in \{K_6, O_5\}$, where O_5 is a special graph with $\chi(O_5) = 5$ and $|O_5| = 9$. This result for $t \geq 3$ improves a case of a theorem by Rabern [9] and for $t = 2$ answers a question raised by Kierstead and Kostochka in [6].

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1 Introduction

In this paper we consider finite, simple, undirected graphs. Given a graph G, we write $V(G)$ for its vertex set and $E(G)$ for its edge set. Furthermore, we write $\Delta(G)$ for its maximum degree, $\delta(G)$ for its minimum degree, $\omega(G)$ for its clique number and $\chi(G)$ for its chromatic number.

A graph G is called *critical*, or *color-critical* if $\chi(H) < \chi(G)$ whenever H is a proper subgraph of G. Let ρ be a *monotone graph parameter*, that is a mapping that assigns to each graph G a real number such that $\rho(H) \leq \rho(G)$ whenever H is a isomorphic to a subgraph of G. If we want to show that every graph G satisfies $\chi(G) \leq \rho(G)$, then it suffices to establish this inequality for all critical graphs. This follows from the simple fact that each graph contains a critical graph with the same chromatic number.

The only critical graph with chromatic number $k \in \{1,2\}$ is the complete graph K_k on k vertices and the only critical graphs with chromatic number 3 are the odd cycles C_{2p+1}.

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However, for any given integer $k \geq 4$, a characterization of all critical graphs with chromatic number k seems to be unlikely.

Critical graphs were first defined and investigated by Dirac [3] in 1951. He observed that $\delta(G) \geq \chi(G) - 1$ for every critical graph G with $\chi(G) \geq 1$ and proved as a generalization the following result.

Theorem 1.1 (Dirac [4] 1953) *If G is a critical graph with $\chi(G) = k$ for an integer $k \geq 1$, then G is $(k-1)$-edge connected.*

In view of this, the vertices of a critical graph G with degree $\chi(G) - 1$ are called the *low vertices* and the others are called the *high vertices*. For a critical graph G, we denote by $\mathcal{L}(G)$ the subgraph induced by the low vertices of G, and by $\mathcal{H}(G)$ the subgraph induced by the high vertices of G.

Dirac's simple observation concerning the minimum degree in critical graphs implies, in particular, that every graph G satisfies

$$\chi(G) \leq \Delta(G) + 1. \tag{1}$$

Brooks's fundamental result from 1941 characterizes the graphs for which (1) holds with equality.

Theorem 1.2 (Brooks [1] 1941) *If a graph G satisfies $\chi(G) = \Delta(G) + 1$, then either G contains $K_{\Delta(G)+1}$ or $\Delta(G) = 2$ and G contains an odd cycle.*

Observe that Brooks's theorem is equivalent to the statement that the only critical graphs G with $\chi(G) = \Delta(G) + 1$ or, equivalently with $\mathcal{H}(G) = \emptyset$, are the complete graphs and the odd cycles. This result was generalized by Gallai.

Theorem 1.3 (Gallai [5] 1963) *If G is a critical graph with $\chi(G) \geq 1$, then each block of $\mathcal{L}(G)$ is a complete graph or an odd cycle.*

The *Ore-degree* of an edge xy in a graph G is the sum $\theta_G(xy) = d_G(x) + d_G(y)$ of the degrees of its ends. The *Ore-degree* of a graph G is defined as $\theta(G) = \max_{xy \in E(G)} \theta_G(xy)$. The counterpart of (1) for the Ore-degree of a graph G is

$$\chi(G) \leq \lfloor \theta(G)/2 \rfloor + 1. \tag{2}$$

This also follows easily from Dirac's observation that $\delta(G) \geq \chi(G) - 1$ for every critical graph G. Clearly, equality in (2) holds for complete graphs and odd cycles. However, for small odd θ there are other critical graphs for which (2) holds with equality.

Example 1 Let $\mathcal{O}_4$ be a family of graphs defined recursively as follows. The graph obtained from disjoint graphs $K_4 - xy$ and $K_4 - x'y'$ by identifying x and x' and joining y and y' belongs to the family. If G belongs to the family, then the graph G' obtained as follows also belongs to the family: Choose a vertex z with $d_G(z) = 4$ and split z into two vertices z_1

and z_2 of degree two. Add two new vertices u and v that are adjacent to z_1, z_2 and to each other. The resulting graph is G', see Figure 1. It is easy to show that each graph $G \in \mathcal{O}_4$ is a critical graph such that $\theta(G) = 7$ and $\chi(G) = 4$, see also [7].

Example 2 Let O_5 be the graph obtained from $K_5 - xy$ and K_4 by joining two vertices of K_4 to x and the two other vertices of K_4 to y, see Figure 2. Then O_5 is a critical graph satisfying $|V(O_5)| = 9$, $\theta(O_5) = 9$ and $\chi(O_5) = 5$.

The Ore-degree of a graph is closely related to Ore's famous theorem about the existence of Hamilton cycles (for the complement). Ore-type bounds for the chromatic number of a graph were first investigated by Kierstead and Kostochka [6, 7].

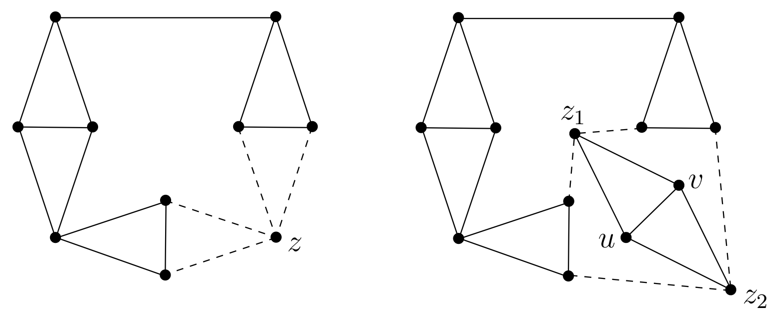

Figure 1: Two graphs in $\mathcal{O}_4$.

Theorem 1.4 (Kierstead and Kostochka [6] 2009) *If $7 \leq \chi(G) = \lfloor \theta(G)/2 \rfloor + 1$, then G contains the complete graph $K_{\chi(G)}$.*

Theorem 1.4 is equivalent to the following statement about critical graphs.

Theorem 1.5 (Kierstead and Kostochka [6] 2009) *The complete graph $K_{\Delta(G)+1}$ is the only critical graph G with $\chi(G) \geq \Delta(G) \geq 7$ such that $\mathcal{H}(G)$ is edgeless.*

The proof of Theorem 1.5 given in [6] uses a list coloring argument for an auxiliary bipartite graph and is based on Gallai's characterization of the low vertex subgraph $\mathcal{L}(G)$ of a critical graph G and a result from [11] saying that if G is a critical graph, then $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$ has. The proof only works if $\Delta(G) \geq 7$. Very recently, Rabern [9] found a simpler argument that also works for critical graphs with $\chi(G) \geq \Delta(G) \geq 6$. He proved a stronger result:

Theorem 1.6 (Rabern [9] 2010) *The complete graph $K_{\Delta(G)+1}$ is the only critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\omega(\mathcal{H}(G)) \leq \lfloor \Delta(G)/2 \rfloor - 2$.*

In this paper, using the ideas of the proof of Theorem 1.6, we improve its bound for odd $\Delta(G) \geq 7$. Furthermore, we also consider the case $\Delta(G) = 5$, see Theorem 1.8.

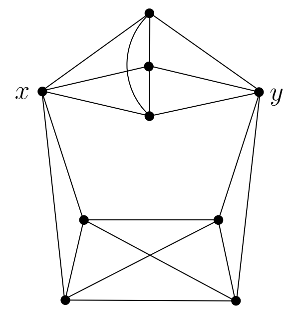

Figure 2: The graph O_5.

Theorem 1.7 *If G is a critical graph and $t \geq 3$ is an integer such that $\chi(G) \geq \Delta(G) = 2t+1$ and $\omega(\mathcal{H}(G)) \leq t-1$, then $G = K_{2t+2}$.*

Theorem 1.7 is a partial case of a more general result which will be stated and proved in Section 3.

Theorem 1.8 *If G is a critical graph satisfying $\chi(G) \geq \Delta(G) = 5$ and $\omega(\mathcal{H}(G)) \leq 1$, then $G \in \{K_6, O_5\}$.*

Corollary 1.9 *If $\chi(G) = \lfloor \theta(G)/2 \rfloor + 1 = 5$, then $G \in \{K_5, O_5\}$.*

Theorem 1.8 answers a question raised by Kostochka and Kierstead [6] in the affirmative. Description of critical graphs with $\chi(G) = \Delta(G) = 4$ and $\omega(\mathcal{H}(G)) \leq 1$ remains unknown. Observe that there is no critical graph with $\chi(G) = \Delta(G) \leq 3$.

2 Proper partitions and Mozhan's lemma

Our main tool to prove both theorems is an useful observation made by Mozhan [8] developing an idea by Catlin [2]. Let us start with some terminology.

In the sequel, let G denote a critical graph with $\chi(G) \geq 4$. For a vertex set $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G induced by X and let $E_G(X) = E(G[X])$. Further, let $G - X = G[V(G) \setminus X]$. If $x \in V(G)$ we write $G - x$ rather than $G - \{x\}$. For a vertex $v \in V(G)$ and a vertex set $X \subseteq V(G)$, let $N_G(v : X) = \{u \in X \mid uv \in E(G)\}$ and let $d_G(v : X) = |N_G(v : X)|$. We put $N_G(x) = N_G(x : V(G))$ and $d_G(x) = d_G(x : V(G))$.

A sequence $(x, X_1, \ldots, X_p)$ is called a (n ordered) *partition* of G if $x \in V(G)$ and $X_1, \ldots, X_p$ are pairwise disjoint subsets of $V(G - x)$ whose union is $V(G - x)$. We call such a partition a $(t_1, \ldots, t_p)$-*partition* of G if $\chi(G - x) = t_1 + \ldots + t_p$ and $\chi(G[X_i]) = t_i$ for $i = 1, \ldots, p$. By an *optimal* $(t_1, \ldots, t_p)$-*partition* of G we mean a $(t_1, \ldots, t_p)$-partition $(x, X_1, \ldots, X_p)$ of G such that the weight

$$w(X_1, \ldots, X_p) = |E_G(X_1)| + \cdots + |E_G(X_p)|$$

is minimum over all $(t_1, \dots, t_p)$-partitions of G. Observe that for any sequence $(t_1, \dots, t_p)$ such that $\chi(G) = 1 + t_1 + \cdots + t_p$, there exists a $(t_1, \dots, t_p)$-partition of G. If $\mathcal{P} = (x, X_1, \dots, X_p)$ is a $(t_1, \dots, t_p)$-partition, we will denote the component of $G[X_i \cup \{x\}]$ containing the vertex x by $K^i(\mathcal{P})$. Note that $G[X_i \cup \{x\}]$ is not necessarily connected.

The next result is a simple consequence of Brooks's theorem and the fact that the chromatic number is subadditive in the sense that every partition $(Y_1, \dots, Y_s)$ of the vertex set of a graph H satisfies $\chi(H) \leq \chi(H[Y_1]) + \cdots + \chi(H[Y_s])$.

Lemma 2.1 (Mozhan [8] 1983) *Let G be a critical graph and let $\mathcal{P} = (x, X_1, \dots, X_p)$ be an optimal $(t_1, \dots, t_p)$-partition of G for integers $t_1, \dots, t_p \geq 1$. Then the following statements hold:*

(a) $\chi(G[X_i \cup \{x\}]) = t_i + 1$ *and* $d_G(x : X_i) \geq t_i$ *for all* $i \in \{1, \dots, p\}$.

(b) *If x is a low vertex of G, then $d_G(x : X_i) = t_i$ for all $i \in \{1, \dots, p\}$.*

(c) *If $d_G(x : X_i) = t_i$ for some $i \in \{1, \dots, p\}$, then either $K^i(\mathcal{P}) = K_{t_i+1}$ or $t_i = 2$ and $K^i(P)$ is an odd cycle.*

Proof. By definition, $\chi(G) = 1 + t_1 + \cdots + t_p$ and $\chi(G[X_i]) = t_i$ for all $i \in \{1, \dots, p\}$. Since $(X_1, \dots, X_p)$ is a partition of $V(G - x)$, we have $\chi(G[X_i \cup \{x\}] = t_i + 1$ and hence $d_G(x : X_i) \geq t_i$ for all $i \in \{1, \dots, p\}$. This proves (a). If x is a low vertex of G, then $d_G(x) = \chi(G) - 1 = t_1 + \cdots + t_p$. By (a), this implies that $d_G(x : X_i) = t_i$ for all $i \in \{1, \dots, p\}$. This proves (b).

For the proof of (c), assume that $d_G(x : X_i) = t_i$ for some $i \in \{1, \dots, p\}$. By (a) it then follows that $\chi(K^i(\mathcal{P})) = t_i + 1$. We claim that $\Delta(K^i(\mathcal{P})) \leq t_i$. Suppose this is false. Then choose a vertex y in $K^i(\mathcal{P})$ with $d_{K^i(\mathcal{P})}(y) > t_i$ closest to x and let $P = (x_1 = x, x_2, \dots, x_s = y)$ be a shortest path in $K^i(\mathcal{P})$ joining x and y. Now, let φ be a (proper) coloring of $G[X_i]$ with t_i colors. Then φ induces a coloring of $K^i(\mathcal{P}) - x$ with t_i colors. Clearly, $x \neq y$ and $d_G(x_k : X_i) = t_i$ for $1 \leq k < s$. Since $\chi(K^i(\mathcal{P})) = t_i + 1$, this implies that among the t_i neighbors of $x = x_1$ in $K^i(\mathcal{P})$ all t_i colors occur. Hence if we recolor x_1 with $\varphi(x_2)$ and uncolor x_2, we obtain a coloring of $G[(X_i \cup \{x_1\})] - x_2$ with t_i colors such that $d_G(x_2 : (X_i \cup \{x_1\}) \setminus \{x_2\}) = t_i$. Now we can repeat the argument. Hence if we recolor x_k by $\varphi(x_{k+1})$ for $1 \leq k < s$ and uncolor $x_s = y$, we obtain a t_i-coloring of $G[X_i']$ with $X_i' = (X_i \cup \{x\}) \setminus \{y\}$. Thus $(y, X_i, \dots, X_{i-1}, X_i', X_{i+1}, \dots, X_p)$ is a $(t_1, \dots, t_p)$-partition of G with a smaller weight. This however contradicts the choice of our partition $(x, X_1, \dots, X_p)$. This proves the claim that $\Delta(K^i(\mathcal{P})) \leq t_i$. Since $\chi(K^i(\mathcal{P})) = t_i + 1$, by Brooks's theorem either $K^i(\mathcal{P}) = K_{t_i+1}$ or $t_i = 2$ and $K^i(\mathcal{P})$ is an odd cycle. This proves (c) and hence the lemma. $\square$

Let $\mathcal{P} = (x, X_1, \dots, X_p)$ be an optimal $(t_1, \dots, t_p)$-partition of G and let $y \in X_i$ be a vertex for some $i \in \{1, \dots, p\}$. For $j \in \{1, \dots, p\}$ we let

$$Y_j = \begin{cases} X_j & \text{if } j \neq i, \\ (X_i \cup \{x\}) \setminus \{y\} & \text{if } j = i. \end{cases}$$

We then say that $\mathcal{P}' = (y, Y_1, \ldots, Y_p)$ is obtained from $\mathcal{P}$ by *swapping* x with y and write $\mathcal{P}' = \mathcal{P}/(x, y)$. Clearly, $\mathcal{P}'$ is a $(t_1, \ldots, t_p)$-partition of G if and only if $\chi(G[Y_i]) = t_i$. If $d_G(x : X_i) = t_i$ and $y \in V(K^i(\mathcal{P}))$, then it follows from Lemma 2.1(c) that $\chi(G[Y_i]) = t_i$ and $\mathcal{P}' = (y, Y_1, \ldots, Y_p)$ is a $(t_1, \ldots, t_p)$-partition of G with the same weight as $\mathcal{P}$. So, we obtain the following statement.

Lemma 2.2 *Let G be a critical graph and let $\mathcal{P} = (x, X_1, \ldots, X_p)$ be an optimal $(t_1, \ldots, t_p)$-partition of G for integers $t_1, \ldots, t_p \geq 1$. If $d_G(x : X_i) = t_i$ for some $i \in \{1, \ldots, p\}$ and $y \in V(K^i(\mathcal{P}))$, then $\mathcal{P}/(x, y)$ is an optimal $(t_1, \ldots, t_p)$-partition of G.*

We call $\mathcal{P} = (x, X_1, \ldots, X_p)$ a *proper $(t_1, \ldots, t_p)$-partition* of G if $\mathcal{P}$ is an optimal $(t_1, \ldots, t_p)$-partition of G and x is a low vertex of G. As a simple consequence of Lemma 2.1(b)(c) and Lemma 2.2 we obtain the following result.

Lemma 2.3 *Let G be a critical graph and let $\mathcal{P} = (x, X_1, \ldots, X_p)$ be a proper $(t_1, \ldots, t_p)$-partition of G for integers $t_1, \ldots, t_p \geq 1$. Then the following statements hold:*

(a) *For all $i \in \{1, \ldots, p\}$, we have $d_G(x : X_i) = t_i$ and, moreover, either $K^i(\mathcal{P}) = K_{t_i+1}$ or $t_i = 2$ and $K^i(\mathcal{P})$ is an odd cycle.*

(b) *If $y \in \bigcup_{i=1}^{p} V(K^i(\mathcal{P}))$ is a low vertex of G, then $\mathcal{P}/(x, y)$ is a proper $(t_1, \ldots, t_p)$-partition of G.*

Lemma 2.4 *Let G be a critical graph and let $\mathcal{P} = (x, X_1, \ldots, X_p)$ be a proper $(t_1, \ldots, t_p)$-partition of G for integers $t_1, \ldots, t_p \geq 1$. Furthermore, let $y \in V(K^i(\mathcal{P}) - x)$ be a low vertex of G with $i \in \{1, \ldots, p\}$. If y has a neighbor in G belonging to $K^j(\mathcal{P}) - x$ for $j \in \{1, \ldots, p\} \setminus \{i\}$, then $N_G(x : X_j) = N_G(y : X_j)$.*

Proof. Since y is a low vertex of G belonging to $K^i(\mathcal{P}) - x$, Lemma 2.3(b) implies that $\mathcal{P}' = \mathcal{P}/(x, y)$ is a proper $(t_1, \ldots, t_p)$-partition of G. Since y has a neighbor in G belonging to $K^j(\mathcal{P}) - x$, we conclude that $K^j(\mathcal{P}) - x = K^j(\mathcal{P}') - y$. Hence, by Lemma 2.3(a), $N_G(x : X_j) = N_G(y : X_j)$. $\square$

Let $\mathcal{P} = (x, X_1, \ldots, X_p)$ be a proper $(t_1, \ldots, t_p)$-partition of G for integers $t_1, \ldots, t_p \geq 1$. Then we denote the special vertex x by $v(\mathcal{P})$ and the i^{th} set X_i of $\mathcal{P}$ by $V_i(\mathcal{P})$. For $i \in \{1, \ldots, p\}$, we put $\tilde{K}^i(\mathcal{P}) = K^i(\mathcal{P}) - x$. Lemma 2.3 implies that $\tilde{K}^i(\mathcal{P})$ is either a complete graph with t_i vertices or $t_i = 2$ and $\tilde{K}^i(\mathcal{P})$ is an odd path (i.e., a path with an odd number of edges). Observe that $\tilde{K}^i(\mathcal{P})$ is a component of the graph $G[X_i]$. By $\mathcal{K}^i(\mathcal{P})$ we denote the set of all components K of $G[X_i]$ such that either K is a complete graph with t_i vertices or $t_i = 2$ and K is an odd path.

If H is a subgraph of G and $x \in V(G) \setminus V(H)$ is a vertex, then we denote by $H + x$ the graph obtained from H by adding the vertex x and joining x to each vertex of H by an edge. If $H + x$ is a subgraph of G, then we say that x is *completely joined* to H (and to $V(H)$).

Lemma 2.5 *Let G be a critical graph such that $\chi(G) = 1+t_1+\cdots+t_p$ for integers $t_1,\ldots,t_p \geq 2$. If $\Delta(G) \leq \chi(G)+p-2$ and $\omega(\mathcal{H}(G)) \leq \min\{t_1,\ldots,t_p\}-1$, then there exists a proper $(t_1,\ldots,t_p)$-partition of G.*

Proof. Since $\chi(G) = 1+t_1+\cdots+t_p$ and G is critical, there is an optimal $(t_1,\ldots,t_p)$-partition $\mathcal{P}$ of G. Let $v = v(\mathcal{P})$. If v is a low vertex of G, then $\mathcal{P}$ is a proper $(t_1,\ldots,t_p)$-partition of G and we are done. Otherwise v is a high vertex of G. Then, there is an $i \in \{1,\ldots p\}$ such that $d_G(v : V_i(\mathcal{P})) = t_i$. Otherwise, we conclude from Lemma 2.1(a) that $\Delta(G) \geq d_G(v) \geq (t_1+1)+\cdots+(t_p+1) = \chi(G)+p-1$, a contradiction to $\Delta(G) \leq \chi(G)+p-2$. Then Lemma 2.1(c) implies that $t_i \geq 3$ and $K^i(\mathcal{P}) = K_{t_i+1}$ or $t_i = 2$ and $K^i(\mathcal{P})$ is an odd cycle. Since $\omega(\mathcal{H}(G)) \leq t_i - 1$, it then follows that there is a low vertex y of G belonging to $\tilde{K}^i(P) = K^i(\mathcal{P}) - v(\mathcal{P})$. From Lemma 2.2 it then follows that $\mathcal{P}' = \mathcal{P}/(v(\mathcal{P}), y)$ is an optimal $(t_1,\ldots,t_{p+1})$-partition of G. Since $v(\mathcal{P}') = y$ is a low vertex of G, $\mathcal{P}'$ is a proper $(t_1,\ldots,t_{p+1})$-partition of G. This proves the lemma. □

Lemma 2.6 (Main Lemma) *Let G be a critical graph such that $\chi(G) = 1+t_1+\cdots+t_p$ for integers $t_1,\ldots,t_p \geq 1$, let $\mathcal{P}$ be a proper $(t_1,\ldots,t_p)$-partition of G, and let $i,j \in \{1,\ldots,p\}$ be two different integers such that $t_i,t_j \geq 3$ and $\omega(\mathcal{H}(G)) \leq \min\{t_i,t_j\}-1$. Then the following statements hold:*

(a) *Let $K \in \mathcal{K}^h(\mathcal{P})$ for $h \in \{i,j\}$. Then K is a complete graph K_{t_h} containing at least one low vertex. Furthermore, for $u \in V(K)$, either $N_G(u) = V(K-u) \cup N_G(u : \bigcup_{k\neq h} V_k(\mathcal{P}))$ or $K = \tilde{K}^h(\mathcal{P})$ and $N_G(u) = V(K-u) \cup \{v(\mathcal{P})\} \cup N_G(u : \bigcup_{k\neq h} V_k(\mathcal{P}))$.*

(b) *$\tilde{K}^i(\mathcal{P})$ contains a low vertex u of G such that u has a neighbor in G belonging to $\tilde{K}^j(\mathcal{P})$.*

(c) *Each low vertex x of G belonging to $U = V(\tilde{K}^i) \cup V(\tilde{K}^j) \cup \{v(\mathcal{P})\}$ is in G completely joined to $U \setminus \{x\}$.*

Proof. Statement (a) follows immediately from Lemma 2.3 and from the assumption that $\omega(\mathcal{H}(G)) \leq t_h - 1$ and the fact that K is a component of $G[V_h(\mathcal{P})]$.

For the proof of (b), suppose this is false. Then every low vertex u of G belonging to $\tilde{K}^i(\mathcal{P})$ satisfies $N_G(u) \cap V(\tilde{K}^j(\mathcal{P})) = \emptyset$. To arrive at a contradiction, we shall construct an infinite sequence $(L_0, L_1, \ldots)$ of distinct graphs all belonging to $\mathcal{K} = \mathcal{K}^i(\mathcal{P}) \cup \mathcal{K}^j(\mathcal{P})$. First, we put $\mathcal{P}_0 = \mathcal{P}$, $x_0 = v(\mathcal{P})$, $L_0 = \tilde{K}^j(\mathcal{P})$, and $L_1 = \tilde{K}^i(\mathcal{P})$. By (a), L_1 contains a low vertex x_1. For $k \geq 0$, we now construct recursively the partition $\mathcal{P}_{k+1}$ and the graph L_{k+2} by defining $\mathcal{P}_{k+1} = \mathcal{P}_k/(x_k, x_{k+1})$ and

$$L_{k+2} = \begin{cases} \tilde{K}^i(\mathcal{P}_{k+1}) & \text{if } k \text{ is odd,} \\ \tilde{K}^j(\mathcal{P}_{k+1}) & \text{if } k \text{ is even.} \end{cases}$$

By (a), L_{k+2} contains a low vertex x_{k+2}. This completes our construction.

Let $W = V_i(\mathcal{P}) \cup V_j(\mathcal{P}) \cup \{v(\mathcal{P})\}$. We now claim that for each integer $k \geq 0$ the sequence $(L_0, L_1, \ldots, L_k, L_{k+1})$ satisfies the following properties:

(1) $L_0, L_1, \ldots, L_k, L_{k+1}$ are pairwise distinct graphs from $\mathcal{K}$, where L_h belongs to $\mathcal{K}^i(\mathcal{P})$ if h is odd and to $\mathcal{K}^j(\mathcal{P})$ if h is even.

(2) For $h \in \{1, \ldots, k\}$, $x_h \in V(L_h)$ is a low vertex of G such that $N_G(x_h : W) = V(L_h - x_h) \cup \{x_{h-1}\} \cup V(L_{h+1})$. Moreover, $N_G(x_0 : W) = V(L_1) \cup V(L_0)$.

The proof of the claim is by induction on k. If $k = 0$, the claim is evidently true. Now, assume that the claim holds for the sequence $\mathcal{S} = (L_0, L_1, \ldots, L_k, L_{k+1})$ with $k \geq 0$. Furthermore, let $x_{k+1} \in V(L_{k+1})$ be a low vertex. By Lemma 2.3(b), we conclude from (1) and (2) that $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_1 = \mathcal{P}_0/(x_0, x_1), \ldots, \mathcal{P}_k = \mathcal{P}_{k-1}/(x_{k-1}, x_k), \mathcal{P}_{k+1} = \mathcal{P}_k/(x_k, x_{k+1})$ is a sequence of proper partitions of G. Now, consider the proper partition $\mathcal{P}' = \mathcal{P}_{k+1}$ of G. For $h \in \{1, \ldots, k+1\}$, let $L'_h = (L_h - x_h) + x_{h-1}$. Then we have

$$\mathcal{K}^i(\mathcal{P}') = \mathcal{K}^i(\mathcal{P}) \setminus \{L_h \mid 1 \leq h \leq k+1, h \text{ odd}\} \cup \{L'_h \mid 1 \leq h \leq k+1, h \text{ odd}\}$$

and

$$\mathcal{K}^j(\mathcal{P}') = \mathcal{K}^j(\mathcal{P}) \setminus \{L_h \mid 1 \leq h \leq k+1, h \text{ even}\} \cup \{L'_h \mid 1 \leq h \leq k+1, h \text{ even}\}.$$

Observe that $L_0 \in \mathcal{K}^j(\mathcal{P}')$. Furthermore, we have $v(\mathcal{P}') = x_{k+1}$. To show that the sequence $\mathcal{S}' = (L_0, L_1, \ldots, L_k, L_{k+1}, L_{k+2})$ satisfies (1) and (2), we distinguish two cases.

Case 1: *k is odd.* Then $L_{k+2} = \tilde{K}^i(\mathcal{P}')$ belongs to $\mathcal{K}^i(\mathcal{P}')$. Since (1) and (2) hold for the sequence $\mathcal{S}$, we have $N_G(x_{k+1} : W) \cap \{x_h \mid 0 \leq h \leq k-1, h \text{ even}\} = \emptyset$. Since $\mathcal{K}^i(\mathcal{P}')$ is a complete graph containing x_{k+1}, it follows that $L_{k+1} \notin \{L'_h \mid 1 \leq h \leq k+1, h \text{ odd}\}$. This implies that $L_{k+2} \in \mathcal{K}^i(\mathcal{P}) \setminus \{L_h \mid 1 \leq h \leq k+1, h \text{ odd}\}$. Hence, the sequence $\mathcal{S}'$ satisfies (1). Furthermore, we have $\tilde{K}^j(\mathcal{P}') = L'_{k+1}$ and, therefore, $x_{k+1} \in V(L_{k+1})$ is a low vertex of G such that

$$N_G(x_{k+1} : W) = V(L'_{k+1}) \cup V(L_{k+2}) = V(L_{k+1} - x_{k+1}) \cup \{x_k\} \cup V(L_{k+2}).$$

Hence, $\mathcal{S}'$ also satisfies (2).

Case 2: *k is even.* Then $L_{k+2} = \tilde{K}^j(\mathcal{P}')$ belongs to $\mathcal{K}^j(\mathcal{P}')$. Since (1) and (2) hold for the sequence $\mathcal{S}$, it follows that $N_G(x_{k+1} : W) \cap \{x_h \mid 1 \leq h \leq k-1, h \text{ odd}\} = \emptyset$. Consequently, $L_{k+1} \notin \{L'_h \mid 1 \leq h \leq k+1, h \text{ even}\}$. This implies that L_{k+2} belongs to $\mathcal{K}^j(\mathcal{P}) \setminus \{L_h \mid 1 \leq h \leq k+1, h \text{ even}\}$.

Next, we claim that $L_{k+2} \neq L_0$. Suppose this is false. Then x_{k+1} is completely joined to L_0. Since the low vertex $x_1 \in V(L_1)$ has no neighbor in $L_0 = \tilde{K}^j(\mathcal{P})$, we have $k \geq 2$. Now, we can choose a low vertex $x \in L_0$. Observe that x is adjacent to x_0 and, therefore, x has a neighbor in L'_1. By Lemma 2.3(b), $\mathcal{P}^* = \mathcal{P}'/(x_{k+1}, x)$ is a proper partition with $v(\mathcal{P}^*) = x$. Since $L'_1 \in \mathcal{K}^i(\mathcal{P}^*)$ and x has a neighbor in L'_1, this implies that $L'_1 = \tilde{K}^i(\mathcal{P}^*)$. Consequently, x is completely joined to L'_1. But then x has a neighbor in $L_1 = \tilde{K}^i(\mathcal{P})$. So by Lemma 2.4, x is adjacent to x_1, a contradiction. Hence $L_{k+2} \neq L_0$.

Consequently, the sequence $\mathcal{S}'$ satisfies (1). Furthermore, we have $\tilde{K}^i(\mathcal{P}') = L'_{k+1}$ and, therefore, $x_{k+1} \in V(L_{k+1})$ is a low vertex of G such that

$$N_G(x_{k+1} : W) = V(L'_{k+1}) \cup V(L_{k+2}) = V(L_{k+1} - x_{k+1}) \cup \{x_k\} \cup V(L_{k+2}).$$

Hence, $\mathcal{S}'$ also satisfies (2).

This shows that for each $k \geq 0$, the sequence $(L_0, L_1, \dots, L_k, L_{k+1})$ satisfies (1) and (2). Since our graph G is finite, this gives a contradiction. This completes the proof of (b).

Finally, we prove (c). By (b), $\tilde{K}^i(\mathcal{P})$ contains a low vertex u of G such that u has a neighbor in G belonging to $\tilde{K}^j(\mathcal{P})$. Then Lemma 2.4 implies that u is completely joined to $\tilde{K}^j(\mathcal{P})$. Now again by Lemma 2.4, every low vertex $x \in U$ is completely joined to $U \setminus \{x\}$ in G. □

3 A generalization of Theorem 1.7

Theorem 1.7 is the case $p = 1$ of the following more general statement.

Theorem 3.1 *Let G be a critical graph with $\Delta(G) \leq \Delta$ and $\chi(G) \geq \Delta - p + 1$ for integers Δ, p satisfying $\Delta \geq 4p + 3$ and $p \geq 1$. Let ℓ, r be integers satisfying $\Delta - p = \ell(p+1) + r$ and $0 \leq r \leq p$, further put*

$$b = \left\lceil \frac{\ell(p-1) + 2p - r}{2p} \right\rceil.$$

If $\omega(\mathcal{H}(G)) \leq \ell - b$, then $G = K_{\chi(G)}$.

Proof. Assume that Theorem 3.1 is false. Then there exists a critical graph $G \neq \mathcal{K}_{\chi(G)}$ such that $\Delta(G) \leq \Delta$, $\chi(G) \geq \Delta - p + 1$ and $\omega(\mathcal{H}(G)) \leq \ell - b$, where Δ, p, ℓ, r, b are integers satisfying the hypothesis of the theorem. We choose such G with the minimum $|V(G)|$. To arrive at a contradiction, we shall show that this leads to a coloring of G using $k = \chi(G) - 1$ colors. Based on Lemma 2.6, we shall first exhibit a set U of $k + 1$ vertices such that U contains at least $p + 2$ low vertices and each low vertex contained in U has no neighbor outside U. Then we show that a certain coloring of $G - U$ with k colors can be extended to a coloring of G with k colors.

From the assumption we easily conclude that $\ell \geq 3$ and $1 \leq b \leq \ell - 1$. Next, we define a sequence $(t_1, \dots, t_p)$ of integers as follows. For $i \in \{1, \dots, p\}$, let $t_i = \ell + 1$ if $1 \leq i \leq r$ and $t_i = \ell$ otherwise. Finally, let $t_{p+1} = k - t_1 - \cdots - t_p$. Since $t_1 + \dots t_p = \ell p + r$ and $k = \chi(G) - 1 \geq \Delta - p = \ell(p+1) + r$, we have $t_{p+1} \geq \ell$. Consequently, we have $\chi(G) = k+1 = 1+t_1+\cdots t_{p+1}$ and $\omega(\mathcal{H}(G)) \leq \ell - b \leq \ell - 1 \leq \min\{t_1, \dots, t_{p+1}\} - 1$. Then by Lemma 2.5, there exists a proper $(t_1, \dots, t_{p+1})$-partition $\mathcal{P}$ of G. Let $U = \{v(\mathcal{P})\} \cup \bigcup_{i=1}^{p+1} V(\tilde{K}^i(\mathcal{P}))$, let X denote the set of all low vertices of G belonging to U, and let $Y = U \setminus X$. For a set $M \subseteq V(G)$, let $M^c = V(G) \setminus M$.

By construction, $|U| = 1+t_1+\cdots+t_{p+1} = k+1 = \chi(G)$ and $t_i \geq \ell \geq 3$ for $i = 1, \dots, p+1$. So, by Lemma 2.6, $\tilde{K}^i(\mathcal{P}) = K_{t_i}$ for $i = 1, \dots, p+1$, and every vertex $x \in X$ is completely joined to $U \setminus \{x\}$ in G. This implies, in particular, that $d_G(x : U) = k$ and $d_G(x : U^c) = 0$ for every $x \in X$. Furthermore, since $\omega(\mathcal{H}(G)) \leq \ell - b$ and $v(\mathcal{P}) \in X$, we conclude that $|X \cap V(\tilde{K}^j(\mathcal{P}))| \geq (t_j - (\ell - b)) \geq b$ for $j = 1, \ldots, p+1$ and, therefore, $|X| \geq 1 + (p+1)b \geq p+2$. Since $\Delta(G) \leq \Delta \leq k + p$, this implies that

(1) $d_G(y : X^c) \leq k - 2$ for all $y \in Y$.

We also claim that

(2) $d_G(y : U) \geq k - p(\ell - b)$ for all $y \in Y$.

This follows from the fact that a vertex $y \in Y$ belongs to $K^i(\mathcal{P}) = K_{t_i+1}$ with $1 \leq i \leq p+1$ and therefore,

$$
\begin{aligned}
d_G(y : U) &= t_i + \sum_{j \neq i} d_G(y : V(\tilde{K}^j(\mathcal{P})) \\
&\geq t_i + \sum_{j \neq i} |X \cap V(\tilde{K}^j(\mathcal{P}))| \\
&\geq t_i + \sum_{j \neq i} (t_j - (\ell - b)) \\
&= k - p(\ell - b).
\end{aligned}
$$

In fact, the above proof yields the following strengthening of (2).

(3) If $d_G(y : U) = k - p(\ell - b)$ for a vertex $y \in Y \cap V(\tilde{K}^i(\mathcal{P}))$ and $i \in I = \{1, \ldots, p+1\}$, then $d_G(y : Y \cap V(\tilde{K}^j(\mathcal{P})) = 0$ and $|Y \cap V(\tilde{K}^j(\mathcal{P}))| = \ell - b$ for all $j \in I \setminus \{i\}$.

As the graph G is critical with $\chi(G) = k+1$ and $G \neq K_{k+1}$, we conclude that $\omega(G) \leq k$. Since $|U| = k+1$, this implies that $G[U]$ is not a complete graph. Since $G[X]$ is complete, it then follows that $G[Y]$ is not a complete graph. Therefore, we can choose a pair (u, v) of two distinct vertices in Y with $uv \notin E(G)$. Then $u \in V(\tilde{K}^i(\mathcal{P}))$ and $v \in V(\tilde{K}^j(\mathcal{P}))$ where $i \neq j$. Now, let $H = G[U^c \cup \{u, v\}]$ and let H' be the graph obtained from H by identifying u and v, that is, we replace u, v by a new vertex $w = w(u, v)$ and join w to each vertex in $N_H(u) \cup N_H(v)$ by an edge. Since G is critical and $\chi(G) = k+1$, we have $\chi(H) \leq k$ and, therefore, $\chi(H') \leq k+1$.

We claim that $\chi(H') = k+1$. Otherwise, there is a coloring φ of H with a set C of k colors such that $\varphi(u) = \varphi(v)$. Then φ can be extended to a coloring φ' of $G' = G - X$ using the same k colors from C. To see this, observe that, by (1), each vertex $y \in Y$ satisfies $d_G(y : V(G')) \leq d_G(y : X^c) \leq k - 2$. Eventually, we can extend φ' to a coloring of G using the colors from C. To see this, we associate to each vertex $x \in X$ a list $L(x) = C \setminus \{\varphi'(u) \mid u \in N_G(x : X^c)\}$ of colors available for x. Then $|X| = r \geq 3$ and each vertex $x \in X$ is adjacent to u and v in G and satisfies $d_G(x) = k$ and $d_G(x : X) = r - 1$. Hence, we have $|L(x)| \geq k - (k - (r-1) - 1) = r$ for all $x \in X$. Consequently, there is a coloring φ'' of $G[X]$ such that $\varphi''(x) \in L(x)$ for all $x \in X$. Then $\varphi' \cup \varphi''$ is a coloring of G with k colors, contradicting $\chi(G) = k+1$. This proves that $\chi(H') = k+1$.

Consequently, there is a critical subgraph G' of H' with $\chi(G') = k+1$. Since $\chi(H) \le k$, we have $w = w(u,v) \in V(G')$ and, therefore, $d_{G'}(w) \ge k$. Recall that $\Delta(G) \le \Delta$, $k \ge \Delta - p = \ell(p+1) + r$ and $b = \left\lceil \frac{\ell(p-1)+2p-r}{2p} \right\rceil$. By (2) this implies that

$$\begin{aligned} k \le d_{G'}(w) &\le d_G(u:U^c) + d_G(v:U^c) \le 2\Delta(G) - d_G(u:U) - d_G(v:U) \\ &\le 2\Delta - 2k + 2p(\ell - b) \le 2p + 2p(\ell - b) \le \ell(p+1) + r \\ &= \Delta - p \le k. \end{aligned}$$

Then we conclude that w is a low vertex of G', $\Delta(G) = \Delta$, $\chi(G) - 1 = k = \Delta - p$, $b = \frac{\ell(p-1)+2p-r}{2p}$ and, moreover, $d_G(u:U) = d_G(v:U) = k - p(\ell - b)$ and $d_G(z:U^c) = \Delta - d_G(z:U) = k/2$ for $z \in \{u,v\}$. We also conclude that $N_G(u:U^c)$ and $N_G(v:U^c)$ are disjoint sets, each with $k/2$ elements.

The vertex w being a low vertex of the critical graph G', we have $\Delta(G') \le \Delta$ and $\omega(\mathcal{H}(G')) \le \omega(\mathcal{H}(G)) \le \ell - b$. Since $\chi(G') = k+1 = \Delta - p + 1$ and $|G'| < |G|$, it then follows that $G' = K_{k+1}$. Consequently, for the vertex pair (u,v), consisting of two distinct vertices of Y with $uv \notin E(G)$, we obtain the following result:

(4) There is a set $W = W(u,v) \subseteq U^c$ of k vertices such that $G[W] = K_k$ and the pair $(N_G(u:U^c), N_G(v:U^c))$ is a partition of W with $|N_G(u:U^c)| = |N_G(v:U^c)| = k/2$.

Since $u \in V(\tilde{K}^i(\mathcal{P}))$ and $v \in V(\tilde{K}^j(\mathcal{P}))$ with $i \ne j$ and since $d_G(z:U) = k - p(\ell - b)$ for $z \in \{u,v\}$, it follows from (3) that $d_G(u : Y \cap V(\tilde{K}^h(\mathcal{P}))) = 0$ for all $h \in I \setminus \{i\}$, $d_G(v : Y \cap V(\tilde{K}^h(\mathcal{P}))) = 0$ for all $h \in I \setminus \{j\}$ and, moreover, $|Y \cap V(\tilde{K}^h(\mathcal{P}))| = \ell - b \ge 1$ for all $h \in I$, where $I = \{1, \dots, p+1\}$. Since this holds for any pair (u,v) of distinct vertices in Y with $uv \notin E(G)$, we conclude that two vertices of Y are adjacent in G if and only if they belong to the same complete graph $\tilde{K}^i(\mathcal{P})$ for some $i \in I$.

If $p \ge 2$, then there exists a set $\{u_1, u_2, u_3\} \subseteq Y$ of three vertices that are independent in G. Then it follows from (4) that, for $1 \le i < j \le 3$, $G[W(u_i,u_j)] = K_k$ and the pair $(N_G(u_i:U^c), N_G(u_j:U^c))$ is a partition of $W(u_i,u_j)$ with $|N_G(u_i:U^c)| = |N_G(u_j:U^c)| = k/2$. This implies that the sets $N_G(u_1:U^c), N_G(u_2:U^c), N_G(u_3:U^c)$ are pairwise disjoint and $G[N_G(u_1:U^c) \cup N_G(u_2:U^c) \cup N_G(u_3:U^c)] = K_{3k/2}$. Therefore, $\omega(G) \ge 3k/2 > k$, a contradiction.

If $p = 1$, then we have $b = \frac{\ell(p-1)+2p-r}{2p} = \frac{2-r}{2}$. Since $0 \le r \le 1$ and b is an integer, this implies that $r = 0$ and $b = 1$. Hence we obtain that $\chi(G) = \Delta(G) = \Delta = 2\ell + 1$, $k = 2\ell$, and $|Y \cap V(\tilde{K}^h(\mathcal{P}))| = \ell - b = \ell - 1 \ge 2$ for $h \in \{1,2\}$. Then we can choose three vertices $u, v_1, v_2 \in Y$ such that $uv_1, uv_2 \notin E(G)$. Then it follows from (4) that, for $i = 1,2$, $G[W(u,v_i)] = K_\ell$ and the pair $(N_G(u:U^c), N_G(v_i:U^c))$ is a partition of $W(u,v_i)$ with $|N_G(u:U^c)| = |N_G(v_i:U^c)| = \ell$. First, assume that $N_G(v_1:U^c) \ne N_G(v_2:U^c)$. Then $G_u = G[N_G(u:U^c)] = K_\ell$ and each vertex in G_u has degree at least $2\ell + 1 = \Delta$. Hence $\omega(\mathcal{H}(G)) \ge \ell$, a contradiction. Now, assume that $N_G(v_1:U^c) = N_G(v_2:U^c)$. Then $G_1 = G[N_G(v_1:U^c)] = K_\ell$ and each vertex in G_1 has degree at least $2\ell + 1 = \Delta$. Hence $\omega(\mathcal{H}(G)) \ge \ell$, a contradiction, too. This completes the proof of Theorem 3.1 $\square$

4 Proof of Theorem 1.8

Assume that Theorem 1.8 is false. Then there is a critical graph $G \notin \{K_6, O_5\}$ such that $\chi(G) \geq \Delta(G) \geq 5$ and $\omega(\mathcal{H}(G)) \leq 1$. We choose such G with the minimum $|V(G)|$.

By Brooks's theorem, $\chi(G) = \Delta(G) = 5$. Since G is critical, we have $\delta(G) \geq 4$. To come to a contradiction, we shall prove that there exists a coloring of G with 4 colors. For a set $M \subseteq V(G)$, let $M^c = V(G) \setminus M$.

Claim 4.1 *G contains no K_5^--subgraph.*

Proof. Suppose that G contains a subgraph $L = K_5^- = K_5 - xy$. Since G is a critical graph with $\chi(G) = \Delta(G) = 5$, we conclude that $xy \notin E(G)$. Clearly, $K = L - x - y$ is a K_3.

Let $H = G - V(K)$ and let H' be the graph obtained from H by identifying x and y to a new vertex $v = v(x, y)$. Since G is critical and $\chi(G) = 5$, we have $\chi(H) = 4$ and, therefore, $\chi(H') \leq 5$.

We claim that $\chi(H') = 5$. Indeed, otherwise, there is a colouring φ of H with a set C of 4 colors such that $\varphi(x) = \varphi(y)$. Since $\omega(\mathcal{H}(G)) \leq 1$, K contains at most one high vertex of G. Then using a simple greedy strategy, φ can be extended to a coloring φ' of G using the same 4 colors from C (the last vertex to color is chosen to be of degree 4 and has two neighbors, y and x, of the same colour), contradicting $\chi(G) = 5$. This proves the claim.

Consequently, H' contains a critical subgraph G' with $\chi(G') = 5$. Since $\chi(H) = 4$, we have $v = v(y, x) \in V(G')$ and, therefore, $d_{G'}(v) \geq 4$. Since

$$d_{G'}(v) \leq d_G(x : V(L)^c) + d_G(y : V(L)^c) = (d_G(x) - 3) + (d_G(y) - 3),$$

we need $d_G(x) = d_G(y) = 5$, i.e., both x and y are high vertices of G. So, all vertices of $N_G(x) \cup N_G(y)$ are low vertices. Hence, $d_{G'}(u) = 4$ for each vertex $u \in N_{G'}(v)$. We conclude that $\Delta(G') \leq 5$, $\omega(\mathcal{H}(G')) \leq 1$, and $G' \neq O_5$ (note that each low vertex of O_5 is adjacent to at least one high vertex). Since G' satisfies the conditions of the theorem, has fewer vertices than G and is not O_5, we have $G' = K_5$. Since $d_G(x : V(L)^c) = d_G(y : V(L)^c) = 2$, this implies that $G'' = G[V(L) \cup V(G' - v)]$ is isomorphic to O_5. Since G'' is a critical subgraph of G and $\chi(G'') = 5$, we obtain that $G = G'' = O_5$, a contradiction. $\square$

Claim 4.2 *Let $K = K_4$ be a subgraph of G. Then for each $v \in V(G) - V(K)$, at most one neighbor of v in K is a low vertex.*

Proof. Suppose this is false. Then G contains a subgraph $K = K_4$ such that there are two low vertices $x, y \in V(K)$ and a vertex $u \in V(K)^c$ with $ux, uy \in E(G)$. Let x', y' denote the two vertices of $K - x - y$. Since G is 5-critical and $G \neq K_5$, it does not contain a K_5-subgraph. By Claim 4.1, G does not contain K_5^--subgraph. This implies that $ux', uy' \notin E(G)$. Since $x'y' \in E(G)$, by symmetry, we may assume that x' is a low vertex of G. If u is a low vertex of G, then $G[\{x', x, y, u\}]$ is a K_4^- contained in $\mathcal{L}(G)$ as an induced subgraph. This, however, is a contradiction to Theorem 1.3, saying that each block of $\mathcal{L}(G)$ is either a complete graph or an odd cycle. Hence u is a high vertex of G. Since $ux' \notin E(G)$, there is a vertex $z \in V(G) \setminus \{x', y', x, y, u\}$ such that $zx' \in E(G)$.

Let $H = G - V(K) + zu$. Since G is critical, we have $\chi(G - V(K)) \leq 4$ and $\chi(H) \leq 5$. We claim that $\chi(H) = 5$. Indeed, otherwise, there is a coloring φ of $G - V(K)$ with a set C of 4 colors such that $\varphi(u) \neq \varphi(z)$. Then we let $\varphi(x') = \varphi(u)$ and color y', x, y greedily from C in this order. Since y has two neighbors of the same color, we will succeed, contradicting $\chi(G) = 5$.

So, $\chi(H) = 5$. Consequently, we have $zu \notin E(G)$ and there is a critical subgraph G' of H with $\chi(G') = 5$. Since $\chi(G - V(K)) = 4$, we have $uz \in E(G')$. Note that $d_H(u) = 4$ and $d_H(z) \leq d_G(z)$. Hence $\Delta(G') \leq 5$ and $\omega(\mathcal{H}(G')) \leq 1$. Since G is a smallest counterexample, this implies that $G' \in \{K_5, O_5\}$.

First, assume that $G' = O_5$. Observe that u is a high vertex of G, but a low vertex of G'. Since $\omega(\mathcal{H}(G)) \leq 1$, this implies that each neighbor of u in G is a low vertex of G. However, in $G' = O_5$ the vertex v is adjacent to some high vertex of G'. This implies that z is a high vertex of G'. Consequently, $zy' \notin E(G)$ and $d_H(z) = d_G(z)$, that is, z is a high vertex of G. Since G is critical and K is a complete subgraph of G, graph $G - V(K)$ is connected. Next, we claim that $G' = H$. Suppose this is false. Then there is an edge $vw \in E(H) - E(G')$ with $v \in V(G')$. If v is a high vertex of G' or $v = u$, we conclude that $d_G(v) \geq 6 > \Delta(G)$, a contradiction. If v is a low vertex of G' with $v \neq u$, then v is a high vertex of G. Since $G' = O_5$, the low vertex v is adjacent to some high vertex v' of G'. Then v' is a high vertex of G and and $vv' \in E(G)$, implying that $\omega(\mathcal{H}(G)) \geq 2$, a contradiction. This proves the claim that $G' = H = V(G) - V(K) + uz$. Clearly, y' is adjacent to some vertex $w \in V(G') - \{u, z\}$. Since Since $wy' \notin E(G')$, w is low in G', but a high of G. Hence, in $G' = O_5$ vertex w has a neighbor w' that is a high vertex in G'. Then w' is a high vertex of G. So, edge ww' in G joins two high vertices of G, contradicting $\omega(\mathcal{H}(G)) \leq 1$.

Now, assume that $G' = K_5$. Then $G'' = G[V(K) \cup V(G')]$ is isomorphic to O_5^-, where the missing edge is zy'. If the edge zy' belongs to G, then G contains O_5 and, as before, we conclude that $G = O_5$, a contradiction. If $zy' \notin E(G)$, then $d_{G''}(y') = 3$ implies that there is edge $y'w$ in G such that $w \notin V(K)$. Since $\omega(\mathcal{H}(G)) \leq 1$, we then conclude that $w \in V(G'')^c$. Furthermore, we conclude that there are at most three edges joining a vertex of $V(G'')$ (namely y' or z) with a vertex of $V(G'')^c$. This contradicts Theorem 1.1, saying that G is 4-edge connected.

Hence, in both cases we arrived at a contradiction. Thus the claim is proved. $\square$

To complete the proof of the theorem, we shall investigate the structure of proper $(2,2)$-partitions of G. In the sequel, such a partition is briefly called a *proper partition* of G. An edge uv of $\mathcal{L}(G)$ is called a *low edge* of G.

Claim 4.3 *Let $\mathcal{P}$ be a proper partition of G, let $C = K^i(\mathcal{P})$ with $i \in \{1,2\}$ and let $j \in \{1,2\} \setminus \{i\}$. Then C is an odd cycle containing a low edge of G. If $u \in V(C)$ is a low vertex of G, then $\mathcal{P}' = \mathcal{P}/(v(\mathcal{P}), u)$ is a proper partition and every graph $P \in \mathcal{K}^j(\mathcal{P})$ is an odd path such that either $N(u : V_j(\mathcal{P})) \cap V(P) = \emptyset$ or $N(u : V_j(\mathcal{P})) \cap V(P)$ consists of the two endvertices of P.*

Proof. By Lemma 2.3 and the assumption that $\omega(\mathcal{H}(G)) \leq 1$, C is an odd cycle and contains a low edge. Let $u \in V(C)$ be a low vertex of G. By Lemma 2.3, $\mathcal{P}' = \mathcal{P}/(v(\mathcal{P}), u)$ is a proper partition, where $\mathcal{K}^j(\mathcal{P}') = \mathcal{K}^j(\mathcal{P})$ consists of odd paths. Since $|N_G(u : V(\mathcal{P}))| = 2$ and $K^j(\mathcal{P}')$ is an odd cycle containing u, the two vertices in $N_G(u; V_j(\mathcal{P}))$ are the two endvertices of exactly one path in $\mathcal{K}^j(\mathcal{P})$. □

Since G is a critical graph with $\chi(G) = \Delta(G) = 5$ and $\omega(\mathcal{H}(G)) \leq 1$, by Lemma 2.5, there is a proper partition $\mathcal{P}$ of G. Starting with $\mathcal{P}$, we construct recursively a sequence $C_1, \dots, C_{k+1}$ of odd cycles, a sequence $\mathcal{P}_1, \dots, \mathcal{P}_{k+1}$ of proper partitions and sequence $u_1v_1, \dots, u_kv_k$ of edges as follows. Put $\mathcal{P}_1 = \mathcal{P}$ and $C_1 = K^1(\mathcal{P})$. Furthermore, choose a low edge u_1v_1 of C_1 such that $u_1 = v(\mathcal{P}_1)$ if possible. Now, let $h \geq 1$. Let $\mathcal{P}_{h+1} = \mathcal{P}_h/(v(\mathcal{P}_h), v_h)$ and, let

$$C_{h+1} = \begin{cases} K^2(\mathcal{P}_{h+1}) & \text{if } h \text{ is odd,} \\ K^1(\mathcal{P}_{h+1}) & \text{if } h \text{ is even.} \end{cases}$$

If C_{h+1} contains a vertex from the set $\{u_1, \dots, u_h\}$, then $k = h$ and we stop. Otherwise, we choose a low edge $u_{h+1}v_{h+1} \in E(C_{h+1})$, such that $u_{h+1} = v(\mathcal{P}_{h+1})$ if possible. Then we continue the construction with $h + 1$. Since G is a finite graph, k is well defined.

Now, we choose such a proper partition $\mathcal{P}$ with the minimum k. We use the same notation as above. Clearly, $k \geq 2$ and, for $1 \leq i \leq k+1$, C_i is an odd cycle containing the edge u_iv_i. Furthermore, if $1 \leq h \leq k$, then $v(\mathcal{P}_{h+1}) = v_h \in V(C_{h+1})$, and $C_1 - v_1, \dots, C_h - v_h$ are pairwise vertex-disjoint odd paths satisfying

$$\{C_i - v_i \mid 1 \leq i \leq h \text{ odd}\} \subseteq \mathcal{K}^1(\mathcal{P}_{h+1}),$$

and

$$\{C_i - v_i \mid 1 \leq i \leq h \text{ even}\} \subseteq \mathcal{K}^2(\mathcal{P}_{h+1}).$$

The odd cycle $C_{k+1} = K^p(\mathcal{P}_{k+1})$ with $p = 1, 2$ and $k + 1 \equiv p \bmod 2$ belongs to $G[V_p(\mathcal{P}_{k+1}) \cup \{v_k\}]$, contains the vertex v_k and, moreover, a vertex from the set $U = \{u_1, \dots, u_k\}$. This implies that C_{k+1} contains exactly one vertex from the set U, say u_j. Then $C_{k+1} - v_k = C_j - v_j$ and $k + 1 \equiv j \bmod 2$. We claim that $j = 1$. Otherwise, $P_2' = (v(P_2), V_2(P_2), V_1(P_2))$ is a proper partition and $C_2, \dots, C_{k+1}$ is the corresponding sequence of odd cycles, contradicting the choice of $\mathcal{P} = \mathcal{P}_1$. This shows that $j = 1$ and, therefore, k is even, $C_{k+1} = K^1(\mathcal{P}_{k+1})$, $v_k = v(\mathcal{P}_{k+1})$, and $C_{k+1} - v_k = C_1 - v_1$.

By definition, u_1v_1 is a low edge of G belonging to the odd cycle C_1 and, moreover, there is a vertex w_1 such that $N_G(v_1 : V(C_1)) = \{u_1, w_1\}$. Note that $N_G(v_k : V_1(\mathcal{P}_{k+1})) = \{u_1, w_1\}$. Furthermore, C_2 is an odd cycle containing the low edge u_2v_2 and $C_2 - v_2$ is a path in $\mathcal{K}^2(\mathcal{P}_{k+1})$ containing the vertex v_1. Hence u_2 is an endvertex of the path $C_2 - v_2$. Let w_2 denote the other endvertex of $C_2 - v_2$. Since u_1 is a low vertex of G contained in $C_{k+1} = K^1(\mathcal{P}_{k+1})$, we conclude from Claim 4.3, that $v_1 \in N_G(u_1 : V_2(\mathcal{P}_{k+1})) = \{u_2, w_2\}$. Since $v_1 = v(\mathcal{P}_2)$, our construction rule implies that $u_2 = v_1$.

Clearly, $P = C_2 - v_1$ is an odd path and v_2 is an endvertex of P. By our construction, we conclude that $P \in \mathcal{K}^2(\mathcal{P})$. Since u_1 is a low vertex of G contained in $C_1 = K^1(\mathcal{P})$ and

u_1 is adjacent to $w_2 \in V(P)$, Claim 4.3 implies that $N_G(u_1 : V_2(\mathcal{P})) = \{v_2, w_2\}$ and v_2, w_2 are the two endvertices of P. This implies that C_2 is a K_3 with $V(C_2) = \{v_1 = u_2, w_2, v_2\}$. Since $d_G(u_1) = 4$ and $u_1 v_k \in E(G)$, we conclude that $v_k = v_2$ and $k = 2$. Consequently, $G[\{u_1, v_1, w_2, v_2\}] = K_4$ and $N_G(w_1)$ contains the vertices $v_k = v_2$ and v_1. Since v_1, v_2 are low vertices of G, this gives a contradiction to Claim 4.2. This contradiction completes the proof of Theorem 1.8.

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