A Stability Theorem on Fractional Covering of Triangles by Edges*

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Abstract

Let $\nu(G)$ denote the maximum number of edge-disjoint triangles in a graph G and $\tau^*(G)$ denote the minimum total weight of a fractional covering of its triangles by edges. Krivelevich proved that $\tau^*(G) \leq 2\nu(G)$ for every graph G. This is sharp, since for the complete graph K_4 we have $\nu(K_4) = 1$ and $\tau^*(K_4) = 2$. We refine this result by showing that if a graph G has $\tau^*(G) \geq 2\nu(G) - x$, then G contains $\nu(G) - \lfloor 10x \rfloor$ edge-disjoint K_4-subgraphs plus an additional $\lfloor 10x \rfloor$ edge-disjoint triangles. Note that just these K_4's and triangles witness that $\tau^*(G) \geq 2\nu(G) - \lfloor 10x \rfloor$. Our proof also yields that $\tau^*(G) \leq 1.8\nu(G)$ for each K_4-free graph G. In contrast, we show that for each $\epsilon > 0$, there exists a K_4-free graph G_ϵ such that $\tau(G_\epsilon) > (2-\epsilon)\nu(G_\epsilon)$.

1 Introduction

The main motivation for this paper is an old conjecture of Tuza about packing and covering of triangles by edges. A *triangle packing* in a graph G is a set of pairwise edge-disjoint triangles. A *triangle edge cover* in G is a set of edges meeting all triangles. We denote by $\nu(G)$ the maximum cardinality of a triangle packing in G, and by $\tau(G)$ the minimum cardinality of a triangle edge cover for G. It is clear that for every graph G we have $\nu(G) \leq \tau(G) \leq 3\nu(G)$.

In [6], Tuza proposed the following conjecture.

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Conjecture 1. *For every graph G, $\tau(G) \leq 2\nu(G)$.*

The complete graphs K_4 and K_5 show that this bound is tight. The conjecture is known to be true for certain special classes of graphs, for example K_5-free chordal graphs and planar graphs (Tuza [7]), more generally graphs without a subdivision of $K_{3,3}$ (Krivelevich [5]), and tripartite graphs (Haxell and Kohayakawa [3]). The only general bound known [2] shows that $\tau(G) \leq \frac{66}{23}\nu(G)$ for every graph G.

In [5], Krivelevich proved two weaker versions of Tuza's conjecture, involving *fractional* parameters. A *fractional triangle edge cover* in G is a function $\phi : E(G) \to [0, 1]$ such that $\sum_{e\in T} \phi(e) \geq 1$ for every triangle T in G. Thus a triangle edge cover of G can be viewed as a fractional triangle edge cover that takes only the values 0 or 1. The parameter $\tau^*(G)$ is defined to be the minimum of $\sum_{e\in E(G)} \phi(e)$ over all fractional triangle edge covers ϕ of G. Then $\tau^*(G) \leq \tau(G)$ for every graph G. Using a result of Füredi [1], Krivelevich [5] proved that $\tau^*(G) \leq 2\nu(G)$ for every graph G. Again this bound is tight, for example for K_4. (For the fractional triangle packing parameter ν^*, Krivelevich also proved that $\tau(G) \leq 2\nu^*(G)$ for every graph G.)

Our aim in this paper is to prove the following stability version of the theorem of Krivelevich.

Theorem 1. *Let G be a graph and suppose $\tau^*(G) \geq 2\nu(G) - x$. Then G contains $\nu(G) - \lfloor 10x \rfloor$ K_4-subgraphs and an additional $\lfloor 10x \rfloor$ triangles, all of which are pairwise edge-disjoint.*

In particular, equality holds in the theorem of Krivelevich if and only if G is an edge-disjoint union of copies of K_4 (plus possibly edges that are not in triangles). It also follows from the proof of Theorem 1 that $\tau^*(G) \leq 1.8\nu(G)$ for the class of K_4-free graphs. This is in contrast with ordinary triangle packing, where for each $\epsilon > 0$, there is a K_4-free graph G_ϵ such that $\tau(G_\epsilon) > (2-\epsilon)\nu(G_\epsilon)$. A series of such examples is as follows. For large n, let G'_n be an n-vertex triangle-free graph with independence number $\alpha(G'_n) < n^{2/3}$ (it is known that for large n there are many such graphs, see e.g. [4]). Let G_n be obtained from G'_n by adding a vertex v_0 adjacent to all other vertices. Each triangle in G_n contains v_0, and there is a one-to-one correspondence between the triangles in G_n and the edges of G'_n. In particular, $\nu(G_n)$ equals the size of a maximum matching in G'_n and hence $\nu(G_n) \leq n/2$. On the other hand, among the smallest coverings of the triangles in G_n by edges there always exists a covering that uses only edges incident with v_0. It follows that $\tau(G_n) = n - \alpha(G'_n) > n - n^{2/3}$. Now for each $\epsilon > 0$, we can find an n such that $n - n^{2/3} > (2 - \epsilon)n/2$.

Our proof of Theorem 1 is a structural argument based on certain special subgraphs of G detailed in Section 2. In Section 3 we show how to define a fractional triangle cover of G, and in the last section we prove an upper bound on the total weight of this cover.

2 Patterns and their properties

In this section we will define a special set of edge-disjoint subgraphs in G called a $\mathcal{T}$-*pattern*, where $\mathcal{T}$ is a set of $\nu(G)$ edge-disjoint triangles in G. Each subgraph of a $\mathcal{T}$-pattern will be a copy of one of the following: K_5, K_3, the 5-wheel W_5, K_5^-, K_4^- (here K_r^- denotes the graph obtained from the complete graph K_r by deleting an edge), or a graph formed by gluing together copies of K_4 which we call a K_4-*structure*. When referring to W_5, which is the graph formed by adding a new vertex x to a 5-cycle C_5 and joining x to all vertices of C_5, we sometimes call x the *hub* and C_5 the *rim*, and the edges incident to x the *spokes*.

A K_4-*structure* is defined inductively and algorithmically as follows. A K_4-subgraph Q of G is a K_4-structure, and Q itself is the only *block* of this structure. Note that $\nu(Q) = 1$. Let Q_1 be a K_4-structure, and let Q_2 be a K_4-subgraph of G with $V(Q_2) = \{v_1, v_2, v_3, v_4\}$ such that $v_1v_2 \in E(Q_1)$ and all other edges of Q_2 are not in $E(Q_1)$. Then $Q = Q_1 \cup Q_2$ is a K_4-structure and its blocks are Q_2 and all blocks of Q_1. In this case, v_1v_2 is called the *attachment edge for* Q_2 of the new K_4-structure. Note that $\nu(Q) \geq \nu(Q_1) + 1$.

We define the *central edge* in a K_4^- to be the edge connecting the two vertices of degree 3. The following three statements are easy to check.

Lemma 2. *A graph obtained from K_4^- by deleting an edge that is not central contains a triangle.*

Lemma 3. *The graph obtained by deleting the edges of a 4-cycle from K_5 has two edge-disjoint triangles. In particular, any graph obtained from K_5 by deleting two arbitrary edges, or three edges forming a path, contains two edge-disjoint triangles.*

Lemma 4. *Any graph obtained from W_5 by deleting an edge contains two edge-disjoint triangles. The same holds if we delete two edges such that both of them are on the rim or one of them is on the rim and the other is a spoke sharing a vertex with it.*

To state a fact analogous to Lemmas 2-4 for K_4-structures, we introduce the notion of heavy edges. Two edges e and e' in a K_4-structure Q are *parallel* if there exists a sequence $e = e_1, e_2, \ldots, e_k = e'$ of edges of Q such that for every $i = 1, \ldots, k-1$, the edges e_i and e_{i+1} are in the same block of Q and are vertex-disjoint. (Note that it is possible for two parallel edges to share a vertex, if $k \geq 4$, see Fig. 1.) If Q consists of only one block, then it has no heavy edges. Suppose now that Q consists of $k > 1$ blocks. Order them $B_1, \ldots, B_k$ in the order of construction of Q. We define heavy edges in $k-1$ steps. In Step 1 we call *heavy* the attachment edge for B_2 and the edges in B_1 and B_2 parallel to it. In Step $i, 2 \leq i \leq k-1$, consider the attachment edge e_{i+1} for B_{i+1}. We declare that e_{i+1} and the edge parallel to it in B_{i+1} are now heavy. Moreover, if e_{i+1} already was heavy, but some edge e' parallel to it is not heavy, then we make e' also heavy. Note that there could be at most one such edge e'. After $k-1$ steps, all edges that are not yet heavy are *light*. Note

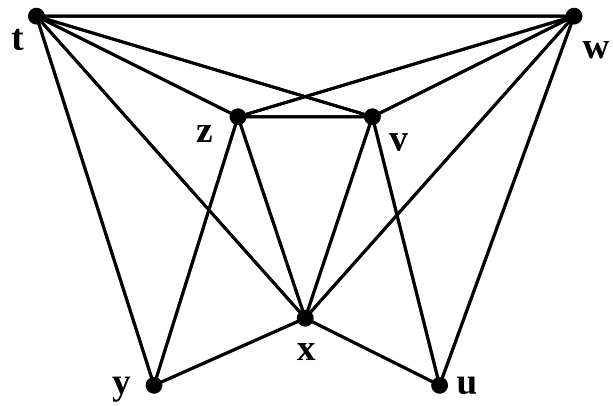

Figure 1: The edges xy, zt, vw and ux of the 3-block K_4-structure in the figure are parallel to each other, and edges xy and xu share a vertex.

also that each block B_i of Q contains a pair of parallel heavy edges, and hence each triangle in B_i contains a heavy edge.

Lemma 5. *Any graph obtained from a K_4-structure Q with k blocks by deleting any edge contains k edge-disjoint triangles. Moreover, after deleting from Q at most one light edge from each block, the remaining subgraph still contains k edge-disjoint triangles.*

Proof. For the first statement, suppose $B_1, \ldots, B_k$ are the blocks of Q in order of construction. If $k = 1$ then the statement is clearly true. Since each B_i contains a triangle edge-disjoint from $\bigcup_{j=1}^{i-1} E(B_j)$, we may assume that the deleted edge e is in B_k and not in $\bigcup_{j=1}^{k-1} E(B_j)$. By induction we may assume that $\bigcup_{j=1}^{k-1} B_j$ contains $k-1$ edge-disjoint triangles that are also disjoint from the attachment edge e' of B_k. But then these together with a triangle in B_k that avoids e gives the required triangle packing of size k.

To prove the second statement, suppose that the blocks of Q are $B_1, \ldots, B_k$, in the order of construction of Q. For $i = 1, \ldots, k$, let e_i be any light edge in B_i (if B_i has a light edge at all). Choose in B_1 any triangle T_1 not containing e_1. Now for $i = 2, \ldots, k$, choose in B_i the triangle T_i not containing e_i and the attachment edge for B_i, if e_i exists, and any triangle T_i not containing the attachment edge for B_i, if e_i does not exist. By construction, all triangles $T_1, \ldots, T_k$ are edge-disjoint. $\square$

Lemma 6. *Let Q be a K_4-structure Q with k blocks. Let e_1 and e_2 be two edges of Q.*
(a) If e_1 and e_2 are not parallel to each other, then $\nu(Q - e_1 - e_2) \geq k$;
(b) if e_1 and e_2 share a vertex and at least one of them is light, then $\nu(Q - e_1 - e_2) \geq k$.

Proof. We use induction on k. Both statements are trivial for $k = 1$. Let $k > 1$ and $B_1, \ldots, B_k$ be the blocks of Q in the order of construction. For $s = 1, 2$, let j_s be the minimum j such that $e_s \in E(B_j)$. Since every B_i contains a triangle edge-disjoint from $\bigcup_{j=1}^{i-1} E(B_j)$, we may assume that $k = j_2 \geq j_1$. Let $Q' = \bigcup_{j=1}^{k-1} B_j$ and e_2' be the attachment edge for B_k.

Case 1: $j_1 = k$. Since e_1 and e_2 are not parallel to each other, B_k contains a triangle T disjoint from both e_1 and e_2. Now by Lemma 5, $Q' - e_2'$ contains $k-1$ edge-disjoint triangles which together with T satisfy the lemma.

Case 2: $j_1 < k$ and e_2 is light. Then B_k contains a triangle T not containing e_2 and e_2'. In this case, by Lemma 5, $Q' - e_1$ contains $k-1$ edge-disjoint triangles which together with T satisfy the lemma.

Case 3: $j_1 < k$ and e_2 is heavy. Suppose that the conditions for (a) hold. Since e_2' is parallel to e_2, it is not parallel to e_1. By the induction assumption, $Q' - e_1 - e_2'$ has $k-1$ edge-disjoint triangles which together with a triangle in B_k containing e_2' satisfy the lemma. Thus, e_1 and e_2 are parallel, but e_1 is light. By the definition of the heavy edges, this might happen only if e_1 lies in a block $B_i \neq B_k$ that contains e_2'. But then e_1 cannot share a vertex with e_2. $\square$

For a family $\mathcal{T}$ of $\nu(G)$ edge-disjoint triangles in G, and for each of the subgraph types $P = K_5, W_5$, K_5^- and K_4^-, we define a $\mathcal{T}$-P to be a P-subgraph of G that contains $\nu(P)$ triangles of $\mathcal{T}$ and is otherwise edge-disjoint from $\mathcal{T}$. We say that a K_4-structure Q with k blocks is a $\mathcal{T}$-K_4-structure if Q contains k triangles in $\mathcal{T}$, and Q is otherwise edge-disjoint from $\mathcal{T}$. This implies in particular that $\nu(Q) = k$. (Note then that the K_4-structure Q depicted in Figure 1 is not a $\mathcal{T}$-K_4-structure for any $\mathcal{T}$, since it has three blocks but $\nu(Q) = 4$.) Observe that if Q' is a K_4 that shares exactly one edge e with Q, and Q' contains a triangle of $\mathcal{T}$ and its other edges (except possibly e) are not in any triangle of $\mathcal{T}$, then $Q \cup Q'$ is also a $\mathcal{T}$-K_4-structure.

A $\mathcal{T}$*-pattern* $\mathcal{P}$ is a collection of edge-disjoint $\mathcal{T}$-K_4-structures, $\mathcal{T}$-K_5's, $\mathcal{T}$-K_5^-'s, $\mathcal{T}$-W_5's, $\mathcal{T}$-K_4^-'s and simply of the members of $\mathcal{T}$ in G that together contain all the members of $\mathcal{T}$. In particular, $\mathcal{T}$ itself is a $\mathcal{T}$-pattern. The members of $\mathcal{P}$ will be called *pieces of* $\mathcal{P}$. The members of $\mathcal{T}$ that are pieces of $\mathcal{P}$ will be called $\mathcal{P}$*-lonely*. The *type* of a pattern $\mathcal{P}$ is the 5-tuple $(x_1, \ldots, x_5)$, where x_1 is the total number of blocks in all $\mathcal{T}$-K_4-structures in $\mathcal{P}$, x_2 is the number of $\mathcal{T}$-K_5's in $\mathcal{P}$, x_3 is the number of $\mathcal{T}$-K_5^-'s in $\mathcal{P}$, x_4 is the number of $\mathcal{T}$-W_5's in $\mathcal{P}$, and x_5 is the number of $\mathcal{T}$-K_4^-'s in $\mathcal{P}$. We say that a pattern $\mathcal{P}$ is *better than* a pattern $\mathcal{P}'$ if the type of $\mathcal{P}$ is lexicographically greater than that of $\mathcal{P}'$.

3 A fractional covering

Let $\mathcal{T}$ be a family of ν edge-disjoint triangles in G and let $\mathcal{P}$ be a $\mathcal{T}$-pattern so that $\mathcal{P}$ is the best among all patterns of all families of ν edge-disjoint triangles in G. An edge of G is *unused*, if it does not belong to any piece of $\mathcal{P}$. We define a function $\phi : E(G) \to [0,1]$ according to the rules below.

(R0) Initially, $\phi(e) = 1/2$ for every $e \in E(G)$ that belongs to a $\mathcal{P}$-lonely triangle and

$\phi(e) = 0$ for every other $e \in E(G)$. The weights of unused edges will not change.

Now we start increasing the values of $\phi(e)$ for some used e considering the pieces of $\mathcal{P}$ one by one. Let $P \in \mathcal{P}$.

(R1) If $P = K_5$, then let $\phi(e) = 1/3$ for each edge e.

(R2) Let P be a $\mathcal{P}$-K_4-structure. If $\nu(P) = 1$, then we let $\phi(e) = 1/3$ for each $e \in E(P)$. If P has at least 2 blocks, then let $\phi(e) = 1/2$ if e is heavy, and $\phi(e) = 3/10$, otherwise.

(R3) If $P = W_5$, then let $\phi(e) = 2/5$ if e is incident to the hub, and $\phi(e) = 3/10$, otherwise.

(R4) Let $P = K_4^-$ and xy be its central edge. We say that P is *extendable* if there exists a vertex v such that vx and vy both are unused edges. Otherwise, we say that P is *fixed*. If P is extendable, then let $\phi(xy) = 1$ and $\phi(e) = 1/5$ for the other 4 edges. If P is fixed, then let $\phi(xy) = 4/5$ and $\phi(e) = 1/4$ for the other 4 edges.

(R5) Let $P = K_5^-$. For each of the 6 edges e incident with a vertex of degree 3 in P, let $\phi(e) = 2/5$, for each of the remaining 3 edges, let $\phi(e) = 1/3$.

(R6) Let e be an edge of a $\mathcal{P}$-lonely triangle. Recall that by (R0), the current value of $\phi(e)$ is 1/2. Among all triangles of G containing e, choose a triangle T' with the minimum value of $\phi(T') := \sum_{e' \in E(T')} \phi(e')$. If $\phi(T') = \beta < 1$, then we redefine $\phi(e) := 3/2 - \beta$ so that for the new ϕ we have $\phi(T') = \beta - 1/2 + (3/2 - \beta) = 1$. Do this for every edge of every $\mathcal{P}$-lonely triangle.

Lemma 7. *If $\mathcal{P}$ is a best $\mathcal{T}$-pattern, then ϕ defined above is a fractional covering of triangles in G.*

Proof. Consider an arbitrary triangle $T = (v_1, v_2, v_3)$ in G. Suppose that $\phi(T) < 1$. By (R6), T does not contain any edge of any $\mathcal{P}$-lonely triangle. By (R4), T does not contain the central edge of any extendable $\mathcal{T}$-K_4^- in $\mathcal{P}$. Furthermore, if T contains the central edge of a fixed $\mathcal{T}$-K_4^- in $\mathcal{P}$, then to have $\phi(T) < 1$, the two other edges of T are unused, a contradiction to the definition of a fixed $\mathcal{T}$-K_4^-. Thus, by Lemmas 2–5, if T does not contain two edges from the same piece of $\mathcal{P}$, then we can find ν edge-disjoint triangles in G that do not contain any edge of T, a contradiction to the definition of ν. So, we may assume that v_1v_2 and v_2v_3 both belong to some $P \in \mathcal{P}$.

Case 1: $P = K_4^-$. Since v_1v_3 is not the central edge of P, we may assume that $V(P) = \{v_1, v_2, v_3, v_4\}$ and the central edge of P is v_2v_4. Since $v_1v_3 \in E(T)$, $G[\{v_1, v_2, v_3, v_4\}] = K_4$. In this case, $\mathcal{P}$ is not the best. Indeed, if v_1v_3 belongs to a $\mathcal{T}$-K_4-structure Q, then we delete P from the pattern and increase Q by adding the block $G[\{v_1, v_2, v_3, v_4\}]$. If v_1v_3 is in another piece P' of $\mathcal{P}$, then we (possibly) alter $\mathcal{T}$ by destroying P' into $\nu(P')$ edge-disjoint triangles that are disjoint from v_1v_3 and the rest of $\mathcal{P}$, which will become new lonely triangles, and we replace P with the $\mathcal{T}$-K_4-structure $G[\{v_1, v_2, v_3, v_4\}]$. Otherwise we simply replace P with $G[\{v_1, v_2, v_3, v_4\}]$. In all cases, the first coordinate of the type of the new pattern is greater than that of $\mathcal{P}$.

Case 2: $P = W_5$. If $E(T) \subset E(P)$ then by (R3), $\phi(T) \geq 1$. So, $v_1v_3 \notin E(P)$. Since $\phi(v_1v_3) = \phi(T) - \phi(v_1v_2) - \phi(v_2v_3) < 1 - 3/10 - 3/10 < 1/2$, by Lemmas 2–5, for the piece P' of $\mathcal{P}$ containing v_1v_3 (if it exists) we have $\nu(P' - v_1v_3) = \nu(P')$. On the other hand, adding any edge to a W_5 increases its packing number, and so $\nu(P + v_1v_3) = 3$, a contradiction to the maximality of ν.

Case 3: $P \in \{K_5, K_5^-\}$. If $v_1v_3 \in E(P)$, then by (R1) or (R5), $\phi(T) \geq 1/3 + 1/3 + 1/3 = 1$. So, $v_1v_3 \notin E(P)$. Then $P = K_5^-$ and we may assume that $V(P) = \{v_1, \ldots, v_5\}$. By (R5), in this case $\phi(v_1v_2) = \phi(v_2v_3) = 2/5$ and so we have a problem only if $\phi(v_1v_3) < 1/5$, which means that v_1v_3 is not in any piece of $\mathcal{P}$. Then we simply add v_1v_3 to P and get a better pattern.

Case 4: P is a $\mathcal{T}$-K_4-structure. Let P have k blocks. Suppose first that v_1v_3 also is in $E(P)$. Then in order to have $\phi(T) < 1$, we need that $k \geq 2$ and $\phi(v_1v_2) = \phi(v_1v_3) = \phi(v_2v_3) = 3/10$. Since every triangle inside a block of P contains a heavy edge, all edges of T belong to different blocks. In particular, $k \geq 3$. Then by Lemma 5, $\nu(P - E(T)) = \nu(P)$. This contradicts the maximality of ν. So, $v_1v_3 \notin E(P)$ and $\phi(v_1v_3) < 1 - 6/10 = 2/5$. Then by Lemmas 2–5, for the piece P' containing v_1v_3 we have $\nu(P' - v_1v_3) = \nu(P')$. On the other hand by Lemma 6 $\nu(P - v_1v_2 - v_2v_3) = k$, and hence $\nu(G - E(T)) \geq \nu$, a contradiction. □

4 The weight of the covering

Lemma 8. *Let $\mathcal{P}$ be a best $\mathcal{T}$-pattern. Let $k \geq 1$. For each $\mathcal{T}$-K_4-structure Q of $\mathcal{P}$ with k blocks, $\phi(Q) \leq 1.9k + 0.1$.*

Proof. For $k = 1$, the statement is trivial. Let $k = 2$. Then Q has 3 heavy edges and 8 light ones, and so

$$\phi(Q) = 3(1/2) + 8(3/10) = 3.9 = 1.9(2) + 0.1.$$

Suppose that the statement is proved for all $k' < k$. Let B_k be the last block in Q and Q' be the union of all other blocks of Q. By our assumption, $\phi(Q') \leq 1.9(k-1)+0.1$. When we add B_k to Q', we add 4 light edges and one heavy edge plus at most one edge of Q' turns from light to heavy. Thus, $\phi(Q) - \phi(Q') \leq 4(3/10) + 1/2 + 2/10 = 1.9$. □

Lemma 9. *Let $\mathcal{P}$ be a best $\mathcal{T}$-pattern. For each $\mathcal{P}$-lonely triangle T, $\phi(T) \leq 1.9$. Moreover, if G does not contain a K_4, then $\phi(T) \leq 1.8$.*

Proof. Suppose that $\phi(T) > 1.8$ and $V(T) = \{v_1, v_2, v_3\}$. We may assume that for $i = 1, 2, 3$, $\phi(v_iv_{i+1}) = \alpha_i$ (taking indices modulo 3). If $\alpha_i > 1/2$, then by definition,

$$\text{there is a vertex } w_i \text{ such that} \quad \phi(v_iw_i) + \phi(v_{i+1}w_i) = 1 - \alpha_i. \tag{1}$$

Recall that if e is not in a $\mathcal{P}$-lonely triangle, then

$$\phi(e) \in \{0, 1/5, 1/4, 3/10, 1/3, 2/5, 1/2, 4/5, 1\}. \tag{2}$$

Thus since v_iw_i and $v_{i+1}w_i$ cannot be in $\mathcal{P}$-lonely triangles,

$$\text{if } \alpha_i > 1/2, \text{ then } \alpha_i \in \{1, 4/5, 3/4, 7/10, 2/3, 3/5, 11/20\}. \tag{3}$$

If for some i, $\phi(v_iw_i) = \phi(v_{i+1}w_i) = 0$, then we may replace T in $\mathcal{P}$ with the K_4^- obtained by adding to T the edges v_iw_i and $v_{i+1}w_i$. The new pattern is better than $\mathcal{P}$, a contradiction to the choice of $\mathcal{P}$. So,

$$\text{for } i = 1, 2, 3, \ \max\{\phi(v_iw_i), \phi(v_{i+1}w_i)\} > 0. \tag{4}$$

We may assume that $\alpha_1 = \max\{\alpha_i : 1 \le i \le 3\}$. Then

$$\alpha_1 \ge \frac{1}{3}\phi(T) > \frac{1}{3}(1.8) = \frac{3}{5} \quad \text{and by (2), } \min\{\phi(v_1w_1), \phi(v_2w_1)\} = 0. \tag{5}$$

Case 1: For some $j \in \{2, 3\}$, $\alpha_j > 1/2$ and $w_1 \ne w_j$. We may assume that $j = 2$.

Case 1.1: No two edges in $F := \{v_1w_1, v_2w_1, v_2w_2, v_3w_2\}$ belong to the same piece of $\mathcal{P}$. Since $\phi(e) < 1/2$ for every $e \in F$, none of them belongs to a lonely triangle or is the central edge of a $\mathcal{T}$-K_4^-. So, by Lemmas 2–5, $\nu(G - F) = \nu(G)$. This contradicts the fact that we can replace T in $\mathcal{P}$ with (v_1, w_1, v_2) and (v_2, w_2, v_3).

Case 1.2: v_1w_1 and v_3w_2 are in the same piece P of $\mathcal{P}$. Then by (5), $\phi(v_2w_1) = 0$. Suppose first that $v_2w_2 \in E(P)$. Since $1/2 < \alpha_2 = 1 - \phi(v_2w_2) - \phi(v_3w_2)$, we have that one of $\phi(v_3w_2)$ and $\phi(v_2w_2)$ is less than $1/4$, and so by (2) it is $1/5$. It follows that $P = K_4^-$, but we have already 5 vertices in P, a contradiction. So, $v_2w_2 \notin E(P)$.

Case 1.2.1: $P = K_4^-$. Since $v_1v_3 \notin E(P)$, the only possibility is that $E(P) = \{v_1w_1, w_1w_2, w_1v_3, v_1w_2, v_3w_2\}$. Then replacing P and T in $\mathcal{P}$ with the $K_5 - v_2w_2$ on the vertex set $\{v_1, w_1, v_2, w_2, v_3\}$ would create a pattern better than $\mathcal{P}$, a contradiction.

Case 1.2.2: $P \in \{K_5, K_5^-\}$. Since $v_1v_3 \notin E(P)$, $P = K_5^-$. But then by (R5) $\alpha_1 = 3/5$, a contradiction to (5).

Case 1.2.3: $P = W_5$. Since $\alpha_1 > 0.6$, v_1w_1 is on the rim of P. If $v_2w_2 \in P' \in \mathcal{P}$, then by Lemmas 2–5, $\nu(P' - v_2w_2) = \nu(P')$. Thus if $\nu(P - v_1w_1 - v_3w_2) = 2$, then as above, replacing T in $\mathcal{P}$ with the triangles $v_1w_1v_2$ and $v_2w_2v_3$ and rearranging triangles within P and P', we find $\nu + 1$ edge-disjoint triangles in G, a contradiction. So, by Lemma 4, v_3w_2 is incident to the hub of P. In particular, $\phi(v_3w_2) = 2/5$ and hence $\phi(v_2w_2) = 0$. Since $v_1v_3 \notin E(P)$, w_2 is the hub of P. In particular, w_1w_2 is an edge of P. Then $\nu(P + w_1v_2 + v_2w_2) = 3$, a contradiction to the maximality of ν.

Case 1.2.4: P is a K_4-structure. Then $\phi(v_3w_2) \ge 3/10$. Since $\alpha_2 > 1/2$, by (2), $\phi(v_2w_2) = 0$. Since $v_1v_3 \notin E(P)$, edges v_1w_1 and v_3w_2 are in distinct blocks of P, in particular, $\nu(P) \ge 2$. By (5), since $\alpha_1 > 1/2$ we know that v_1w_1 is light. Similarly since $\alpha_2 > 1/2$ and $\phi(v_2w_2) = 0$ we see that v_3w_2 is light. Thus by Lemma 5, $\nu(P - v_1w_1 - v_3w_2) = \nu(P)$, and replacing $v_1v_2v_3$ by $v_1v_2w_1$ and $v_2v_3w_2$ we find $\nu + 1$ edge-disjoint triangles in G.

Case 1.3: v_1w_1 and v_2w_2 are in the same piece P of $\mathcal{P}$. Since Case 1.2 does not hold, $v_3w_2 \notin E(P)$. Since $v_2w_1, v_2v_1 \notin E(P)$, $P \notin \{K_4^-, K_5, K_5^-\}$.

Case 1.3.1: $P = W_5$. Since $\alpha_1 > 3/5$, v_1w_1 is on the rim of P. Adding to $P = W_5$ the edge w_1v_2 creates a K_4 in P, and there is a triangle in P edge-disjoint from this K_4. Then replacing P with this K_4 and triangle gives a pattern better than $\mathcal{P}$.

Case 1.3.2: P is a K_4-structure. Repeating the proof of Case 1.2.4 with the roles of v_2w_2 and v_3w_2 switched and using the fact that $w_1v_2 \notin P$, we come to a contradiction again.

Case 1.4: v_2w_1 and v_3w_2 are in the same piece P of $\mathcal{P}$. Then by (5), $\phi(v_1w_1) = 0$. Suppose first that $v_2w_2 \in E(P)$. Since $1/2 < \alpha_2 = 1 - \phi(v_2w_2) - \phi(v_3w_2)$, we have that one of $\phi(v_3w_2)$ and $\phi(v_2w_2)$ is less than $1/4$, and so by (2), it is $1/5$. Therefore $P = K_4^-$. Since $v_2v_3 \notin E(P)$, we have $w_1w_2, w_1v_3 \in E(P)$. Then we replace P and T in $\mathcal{P}$ with $G[\{v_1, v_2, v_3, w_1, w_2\}] - v_1w_2 = K_5^-$, a contradiction to the choice of $\mathcal{P}$. So, $v_2w_2 \notin E(P)$.

Since $v_2w_2, v_2v_3 \notin E(P)$, $P \notin \{K_4^-, K_5, K_5^-\}$.

Case 1.4.1: $P = W_5$. Since $\alpha_2 > 1/2$, $\phi(v_2w_2) < 1/2 - 3/10 = 1/5$ and hence $\phi(v_2w_2) = 0$. As in Case 1.3.1, adding to $P = W_5$ the edge w_2v_2 creates a K_4 in it, and there is a triangle in P edge-disjoint from this K_4.

Case 1.4.2: P is a K_4-structure. As in Case 1.3.2, essentially repeating the proof of Case 1.2.4, we come to a contradiction again.

Case 1.5: v_2w_1 and v_2w_2 are in the same piece P of $\mathcal{P}$. Since Case 1.4 does not hold, $v_3w_2 \notin E(P)$.

Case 1.5.1: $P = K_4^-$. Since $v_2v_1, v_2v_3 \notin E(P)$, there exists $v_0 \neq v_1, v_3$ such that $G[\{v_2, v_0, w_1, w_2\}]$ contains P. Since edges v_2w_1 and v_2w_2 are not central in P and share a vertex, the central edge of P is either w_1w_2 or v_2v_0. If the central edge of P is w_1w_2, then we destroy P and T and remove w_2v_3 from the piece P' containing it (if it exists), but add triangles $v_1v_2w_1$, $w_1w_2w_0$, and $v_2v_3w_2$, a contradiction to the maximality of ν. Otherwise, the central edge is v_0v_2. If $\phi(v_3w_2) = 0$, then we replace P and T in $\mathcal{P}$ by the copy Q of W_5 with edge set $E(P) \cup E(T) \cup \{v_1w_1, v_3w_2\}$. This creates a pattern better than $\mathcal{P}$, a contradiction. Suppose that $\phi(v_3w_2) > 0$. Since $\alpha_2 > 1/2$, $\phi(v_3w_2) < 1 - 1/2 - 1/5 = 3/10$, and hence $\phi(v_3w_2) \leq 1/4$. It follows that the piece P' containing v_3w_2 is a K_4^-. So we downgrade P' to a triangle and again replace P and T with Q.

Case 1.5.2: $P = W_5$. Since $\phi(v_2w_2) \geq 3/10$ and $\alpha_2 > 1/2$, we have $\phi(v_3w_2) = 0$. By (5), v_2w_1 is on the rim of P. So, by Lemma 4, $\nu(P - v_2w_1 - v_2w_2) = 2$. Thus we can replace $P \cup T$ with triangles $v_1w_1v_2$, $v_2w_2v_3$, and the two triangles in $P - v_2w_1 - v_2w_2$, a contradiction to the maximality of ν.

Case 1.5.3: P is a K_4-structure. As in Case 1.5.2 we know that $\phi(v_3w_2) = 0$. We claim that $\nu(P - v_2w_1 - v_2w_2) = \nu(P)$. If $\nu(P) = 1$, then this is clear. If $\nu(P) \geq 2$, then by (5), v_2w_1 is a light edge in P, and the claim follows from Lemma 6(b). Thus again we can use triangles $v_1w_1v_2$ and $v_2w_2v_3$ to find $\nu + 1$ edge-disjoint triangles in G, a contradiction.

Case 1.5.4: $P \in \{K_5, K_5^-\}$. Again, since $\phi(v_2w_2) \geq 1/3$ and $\alpha_2 > 1/2$, we have $\phi(v_3w_2) = 0$. If $P = K_5$, then by Lemma 3, $\nu(P - v_2w_1 - v_2w_2) = 2$, and we find $\nu + 1$ edge-disjoint triangles replacing T with triangles $v_1w_1v_2$ and $v_2w_2v_3$, a contradiction. So, $P = K_5^-$. By (5) and (R5), $\phi(v_2w_1) = 1/3$. This means that v_2w_1 is disjoint from the non-edge in P. If the non-edge in P contains w_2, then by Lemma 3, $\nu(P - v_2w_1 - v_2w_2) = 2$, and we get a contradiction in the same way. So, we may assume that $V(P) = \{w_1, v_2, w_2, w_4, w_5\}$ and $w_4w_5 \notin E(P)$. In particular, $w_1w_2 \in E(G)$. If $\alpha_3 \leq 1/2$, then $\phi(T) \leq 2/3 + 2/3 + 1/2 = 11/6 < 1.9$ (note that in this case G is not K_4-free). So we may assume $\alpha_3 > 1/2$. Now the situation is symmetric between w_1 and w_2 in the sense that $\alpha_1 = \alpha_2$ and $\phi(v_1w_1) = \phi(v_2w_2) = 0$. Since $\alpha_3 > 1/2$ and $w_1 \neq w_2$, we may therefore assume that $w_3 \neq w_1$, and Case 1 holds with $j = 3$. Since $\phi(v_1w_1) = 0$, now the only possible cases are 1.1, 1.2, 1.3 and 1.6 below. Thus the proof will be complete once we have finished Case 1.6.

Case 1.6: v_2w_2 and v_3w_2 are in the same piece P of $\mathcal{P}$. Since $\alpha_2 > 1/2$, $\phi(v_2w_2) = \phi(v_3w_2) = 1/5$, P is an extendable K_4^-, and $\alpha_2 = 3/5$. Since $v_2v_3 \notin E(P)$, there exists w_4 such that $V(P) = \{v_2, w_2, v_3, w_4\}$ and w_2w_4 is the central edge. Note that $w_4 \neq v_1$ since $v_1v_2 \notin P$, and $w_4 \neq w_1$ since we are not in Case 1.4. By the definition of an extendable K_4^-, there exists a vertex w_5 such that w_2w_5 and w_4w_5 are unused edges. Let e' be the edge in $\{v_1w_1, v_2w_1\}$ with $\phi(e') > 0$ and P' be the piece of $\mathcal{P}$ containing e'. Since $\phi(e') < 1/2$, by Lemmas 2–5 we know $P' - e'$ contains $\nu(P')$ edge disjoint triangles. We replace $P \cup T \cup P'$ with these triangles and triangles $w_1v_1v_2$, $w_2v_3v_2$, and $w_2w_4w_5$. This contradicts the maximality of ν.

Case 2: For $i = 2, 3$, either $\alpha_i = 1/2$ or $w_i = w_1$. By (4) we know $\alpha_1 \leq 1 - 1/5 = 0.8$, and so $\alpha_2 + \alpha_3 = \phi(T) - \alpha_1 > 1$. So, we may assume that $\alpha_2 > 1/2$ and hence $w_2 = w_1$. Then T is contained in K_4 induced by $\{v_1, v_2, v_3, w_1\}$, and we want to prove now that $\phi(T) \leq 1.9$. So, suppose $\phi(T) > 1.9$.

Let $F = \{w_1v_1, w_1v_2, w_1v_3\}$. By (5), $\phi(e) = 0$ for some $e \in F$ incident to w_1. As in the proof of (4), we see that there is only one such e. By (5), $e \neq w_1v_3$.

Case 2.1: $e = w_1v_1$. Since $\alpha_2 > 1/2$, $\phi(F - e) < 1/2$. It follows that $\phi(w_1v_2), \phi(w_1v_3) \in \{1/5, 1/4\}$. If w_1v_2 and w_1v_3 are in distinct K_4^--pieces of $\mathcal{P}$, then we downgrade these pieces to triangles, but upgrade T to the $K_4 = G[\{v_1, v_2, v_3, w_1\}]$. The new pattern is better than $\mathcal{P}$. So, suppose that w_1v_2 and w_1v_3 are in the same K_4^--piece P'. Since $v_2v_3 \notin E(P')$, there is w_0 such that w_0w_1 is the central edge in P'. Then we replace the pieces T and P' in $\mathcal{P}$ by the K_5^--subgraph $G[\{v_1, v_2, v_3, w_1, w_0\}] - w_0v_1$ of G. The new pattern is better than $\mathcal{P}$.

Case 2.2: $e = w_1v_2$. If $\phi(w_1v_1), \phi(w_1v_3) \in \{1/5, 1/4\}$, then we argue as in Case 2.1 with the roles of w_1v_1 and w_1v_2 switched. So since $\alpha_1 \geq \alpha_2$, we may assume that $\phi(w_1v_3) \geq 3/10$. If $\phi(w_1v_1) + \phi(w_1v_3) \geq 6/10$, then

$$\alpha_3 = \phi(T) - (1 - \phi(w_1v_1)) - (1 - \phi(w_1v_3)) > 1.9 - 2 + 6/10 = 1/2,$$

so since we are in Case 2 we must have $w_3 = w_1$, which contradicts (1). Thus $\phi(w_1v_1) \leq 1/4$. For $i = 1, 3$, let P_i be the piece of $\mathcal{P}$ containing w_1v_i. Then P_1 is a K_4^-, and since $\phi(w_1v_3) \geq 3/10$ we know P_3 is not a K_4^-. Thus in particular $P_1 \neq P_3$. If P_3 is not a K_4-structure, then we downgrade P_1 and P_3 but upgrade T to the $K_4 = G[\{v_1, v_2, v_3, w_1\}]$. If P_3 is a K_4-structure, then we downgrade P_1, but increase

P_3 by adding to it the block $G[\{v_1, v_2, v_3, w_1\}]$. In either case we get a better pattern, contradicting the choice of $\mathcal{P}$. $\square$

Theorem 10. *Let $\mathcal{P}$ be a best $\mathcal{T}$-pattern. If $\mathcal{P}$ has exactly y pieces that are K_4-structures, then $\sum_{e\in E(G)} \phi(e) \leq 1.9\nu + 0.1y$. Furthermore, if G is K_4-free, then $\sum_{e\in E(G)} \phi(e) \leq 1.8\nu$.*

Proof. Note that $\phi(e) = 0$ for every e that is not in $\bigcup_{P\in\mathcal{P}} E(P)$. Let P be a piece of $\mathcal{P}$. By definition, if $P = K_4^-$, then $\phi(P) = 9/5$; if $P = K_5$, then $\phi(P) = 10/3 = \frac{5}{3}\nu(P)$; if $P = K_5^-$, then $\phi(P) = 3(1/3) + 6(2/5) = 17/5 = 1.7\nu(P)$; if $P = W_5$, then $\phi(P) = 5(3/10) + 5(2/5) = 3.5 = \frac{7}{4}\nu(P)$. Using Lemmas 8 and 9, we obtain the first statement.

If G does not contain K_4, then the pieces of $\mathcal{P}$ are only W_5, K_4^- and lonely triangles. Using the calculations of the previous paragraph and the second statement of Lemma 9, we derive the second statement of our theorem. $\square$

The proof of Theorem 1 now follows immediately. Suppose G is such that $\tau^*(G) \geq 2\nu(G) - x$. Then by Theorem 10 we find $2\nu - x \leq 1.9\nu + 0.1y$, implying that $y \geq \nu - 10x$. Therefore G has at least $\nu - 10x$ K_4-structures and hence at least $\nu - 10x$ edge-disjoint K_4's, together with an additional $10x$ edge-disjoint triangles. $\square$

We end with the remark that we are still quite far from understanding the behaviour of $\tau^*(G)$ for K_4-free graphs G. With some extra work we are able to show that our bound of $\tau^*(G) \leq 1.8\nu(G)$ can be improved to $\tau^*(G) \leq 1.75\nu(G)$, but in terms of lower bounds we know only that the ratio $\tau^*(G)/\nu(G)$ can be as large as 1.25, which is attained by the 5-wheel W_5. It would be interesting to close this gap for the class of K_4-free graphs.

References

[1] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1** (1981) 155–162.

[2] P. E. Haxell, Packing and covering triangles in graphs, *Discrete Math.* **195** (1999) 251–254.

[3] P. E. Haxell and Y. Kohayakawa, Packing and covering triangles in tripartite graphs, *Graphs Combin.* **14** (1998) 1–10.

[4] J. H. Kim, The Ramsey number $R(3, t)$ has order of magnitude $t^2/\log t$, *Random Structures Algorithms* **7** (1995) 173–207.

[5] M. Krivelevich, On a conjecture of Tuza about packing and covering of triangles, *Discrete Math.* **142** (1995) 281–286.

[6] Zs. Tuza, Conjecture, Finite and Infinite Sets, Eger, Hungary 1981, A. Hajnal, L. Lovász, V. T. Sós (Eds.), Proc. Colloq. Math. Soc. J. Bolyai, Vol. 37, North-Holland, Amsterdam, (1984), 888.

[7] Zs. Tuza, A conjecture on triangles of graphs, *Graphs Combin.* **6** (1990) 373–380.