COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
52. COMBINATORICS, EGER (HUNGARY), 1987

The Nonexistence of Certain Generalized Friendship Graphs

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A graph G is called a p_k-graph, if any two vertices in G are connected by a unique path of length k, where $k \geq 1$. Obviously, p_1-graphs are the complete graphs. The p_2-graphs were described in [3]. Kotzig [4] has conjectured that there exists no p_k-graphs for all $k \geq 3$. In the same paper he noticed certain properties of p_k-graphs and the nonexistence of p_k-graphs for $3 \leq k \leq 8$. In survey [1] it was proved that for $k \geq 3$ any p_k-graph G (if it exists) must have the following properties:

(i) G is an edge-disjoint union of $(k+1)$-cycles, and contains no other $(k+1)$-cycles.

(ii) Any two $(k+1)$-cycles of G have at least three common vertices. In particular, G is 2-connected.

(iii) Any two $(k+1)$-cycles of G have at most $k-1$ vertices in common.

(iv) For any $v \in V(G)$ the parity of a number of $(k+2)$-cycles, containing v, is equal to the parity of $\deg_G v/2$.

(v) $(k+1) \cdot c_{k+1} \equiv (k+2) \cdot c_{k+2} (\mathrm{mod}\ 4)$, where c_i denotes the number of i-cycles in G.

In this paper we will prove that besides (i)-(v) any p_k-graph G has the following properties:

(vi) $c_{k+1} \geq 3$, and, by (v), for even k $c_{k+1} \geq 4$.

(vii) Any two $(k+1)$-cycles of G have at most $k-2$ common vertices.

(viii) Let C be the longest cycle of G. Then $k+5 \leq |C| \leq (4k-6)/3$. In particular, $k \geq 21$.

1. Definition and notation

For any path P (cycle C) the notation $|P|$ ($|C|$) denotes the number of edges of P (of C). If v and w are the vertices belonging to the path P, then the term vPw denotes the segment of P connecting v and w. Similarly, the record $vP_1wP_2yP_3v$ denotes the cycle which contains the paths vP_1w, wP_2y, yP_3v.

Following W. T. Tutte [6], for a cycle C of a connected graph G and a vertex v not belonging to C we define the C-bridge containing v as the spanning subgraph of G generated by the edges (x, y) for which in $G - V(C)$ there is a path between v and $\{x, y\} \cap V(G - C)$.

2. Some properties of p_k-graphs

Theorem 1. *For any p_k-graph G we have $|E(G)| > 2(k+1)$.*

Proof. Suppose $E(G) = E(F_1) \cup E(F_2)$, where F_1, F_2 are the $(k+1)$-cycles. Due to (iii) there exists $v \in V(F_1) - V(F_2)$. According to (iv) there exists a $(k+2)$-cycle C_1, containing v. Let w be a vertex of C_1 with $\deg_G w = 4$. By (iv), there exists a $(k+2)$-cycle $C_2 \neq C_1$, which contains w. Denote $n_i = |\{v \in \in V(G);\ \deg_G v = 2i\}|$, $v_{ij} = \{v \in C_j | \deg_G v = 2i\}$, $i = 1, 2$; $j = 1, 2$. If $x \in V_{11} \cap V_{12}$, and x_1, x_2 are the neighbours of x, then there exist two k-paths $C_1 - \{x\}$ and $C_2 - \{x\}$, connecting x_1 and x_2. Hence $V_{11} \cap V_{12} = \emptyset$, $|V_{11}| + |V_{12}| \leq n_1$ and $2k + 4 = |C_1| + |C_2| = |V_{11}| + |V_{21}| + |V_{12}| + |V_{22}| \leq n_1 + 2n_2$. On the other hand, $2k + 2 = |E(G)| = n_1 + 2n_2$, a contradiction. ■

We shall use below the following

Theorem. (of Thomason [5]) *Let H be a multigraph. One can divide $E(H)$ into two cycles. For any edges $e_1, e_2 \in E(H)$ the number of decompositions of $E(H)$ into two cycles such that e_1 and e_2 belong to different cycles, is even; the number of decompositions of $E(H)$ into two cycles such that e_1 and e_2 belong to the same cycle, is even, too. In particular, the number of partitions of $E(H)$ into two cycles is at least four.* ■

Proposition 2. *Let C_1 and C_2 be edge-disjoint cycles of p_k-graph G, and $|C_1| + |C_2| \geq 2k + 1$. Then there is at most one bipartition $\{\tilde{C}_1, \tilde{C}_2\}$ of the graph $H = C_1 \cup C_2$ such that $|\tilde{C}_1| = k$ or $|\tilde{C}_2| = k$.*

Proof. Suppose that $H = C_1 \cup C_2 = C_3 \cup C_4$, $|C_1| = |C_3| = k, |E(H)| \geq 2k + 1$. Then C_2 (as well as C_4) contains more than half of the number of vertices of degree 2 in H. Hence, there exists a vertex $v \in V(C_2) \cap V(C_4)$ such that $\deg_H v = 2$. One can choose v so that a neighbour w of v is adjacent to four vertices: v, w_1, w_2, w_3. At least one of the edges (w, w_i) (let it be (w, w_1)) belongs to $E(C_1) \cap E(C_3)$. Then

$C_1 - \{(w, w_1)\}$ and $C_2 - \{(w, w_1)\}$ are the $(k-1)$-paths between w and w_1, not containing v. Consequently, there exist two paths of length k, which connect v and w_1. ■

Proposition 3. *Any two $(k+1)$-cycles of the p_k-graph G have at most $k-2$ common vertices.*

Proof. Suppose that F_1 and F_2 are $(k+1)$-cycles, $|V(F_1) \cap V(F_2)| \geq k-1$. Then, by (iii), $|V(F_1) \cap V(F_2)| = k-1$. Let $H = F_1 \cup F_2$, $\{v_1, v_2\} = V(F_1) - V(F_2)$, $\{v_3, v_4\} = V(F_2) - V(F_1)$. Due to the Theorem of Thomason there exists a bipartition $\{C_1, C_2\} \neq \{F_1, F_2\}$ of $E(H)$ such that $v_1 \in C_1, v_3 \in C_2$. By construction, $|C_i| = k-1+|V(C_i) \cap \{v_1, v_2, v_3, v_4\}| \in \{k, k+1, k+2\}$, $i = 1, 2$. Since $|C_1| \neq k+1 \neq |C_2|$, $\{|C_1|, |C_2|\} = \{k, k+2\}$. Let, for definiteness, $\{v_1, v_2, v_4\} \subset V(C_1)$. Then there is a bipartition $\{C_3, C_4\} \neq \{F_1, F_2\}$ of $E(H)$ such that $v_1 \in C_3$, $v_4 \in C_4$. As above, $\{|C_3|, |C_4|\} = \{k+2, k\}$; but this contradicts Proposition 2. ■

Proposition 4. *Any p_k-graph G contains a cycle C with $|C| \geq k+5$.*

Proof. Suppose the contrary. First assume that for some edge-disjoint cycles C_1 and C_2 we have

$$|C_1| + |C_2| = 2k + 4. \tag{1}$$

Due to the theorem of Thomason there is at least four edge-decompositions of $H = C_1 \cup C_2$ into two cycles. The possible pairs of length of these cycles are: a) $\{k+4, k\}$, b) $\{k+3, k+1\}$, c) $\{k+2, k+2\}$. If all the vertices of H have degree four, then only the case c) is possible. So every vertex $v \in V(H)$ lies in at least $8 \cdot (k+2)$-cycles, and, therefore, a certain pair $\{(v, v_1), (v, v_2)\}$ of edges lies in at least two $(k+2)$-cycles. I.e., there is at least two k-paths between v_1 and v_2. Thus, H contains a vertex u of degree two. Since G is p_k-graph, the case b) takes place at most once. By Proposition 2, the case a) takes place at most once, too. Hence the vertex u lies in at least two $(k+2)$-cycles. But this is impossible, and (1) is not true.

Let $F_1, \ldots, F_t$ be all $(k+1)$-cycles of G. By Theorem 1, $t \geq 3$. We may assume that if there are vertices of degree two in the graph $F_1 \cup F_2 \cup F_3$, then those are in F_3.

The possibilities for lengths of cycles of 2-decompositions of $H_1 = F_1 \cup F_2$ are: a) $\{k+1, k+1\}$, b) $\{k+2, k\}$, c) $\{k+3, k-1\}$, d) $\{k+4, k-2\}$. Due to Proposition 2, the possibility b) takes place at most once. The same is true for possibility a). If we come across the possibility c), then the $(k+3)$-cycle together with F_3 satisfy (1), a contradiction. So, there is a 2-decomposition $H_1 = C_1 \cup C_2$, where C_1 is a $(k+4)$-cycle.

Case 1. $|V(H_1)| > k+4$. Then there is a vertex $v \in V(H_1) - V(C_1)$. By the construction of H_1, $\deg_{H_1} v = 2$. Consider $H_2 = C_1 \cup F_3$. Due to the choice of F_3

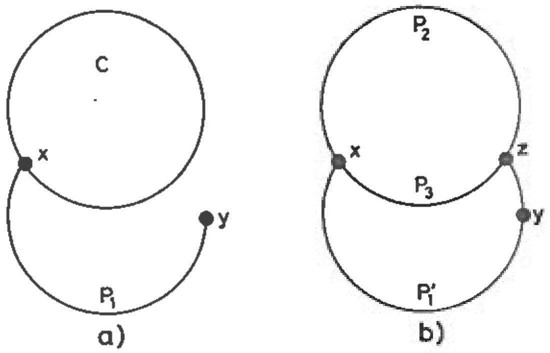

Figure 1.

and v, $|V(C_1) \cap V(F_3)| \leq k$, and the number n_1 of vertices of degree two in H_2 is more or equal to $((k+4)-k)+((k+1)-k)=5$. Since G has no ℓ-cycles for $\ell \geq k+5$, there are at least three decompositions of H_2 into cycles of length $k+3$ and $k+2$. For every such decomposition the number of vertices of degree two in H_2, belonging to the $(k+2)$-cycle of this decomposition is equal to $(n_1-1)/2$. As $n_1 \geq 5$, some vertex of degree two in H_2 lies in at least two $(k+2)$-cycles. But this is impossible.

Case 2. $|V(H_1)| = k+4$. Let $\{v_1, v_2, v_3\} = V(F_1) - V(F_2)$, $\{v_4, v_5, v_6\} = V(F_2) - V(F_1)$. Due to the theorem of Thomason there is a decomposition of H_1 into cycles C_3 and C_4 such that $v_1 \in C_3$, $v_4 \in C_4$, $\{C_3, C_4\} \neq \{F_1, F_2\}$. Since (1) is impossible, we have $\{|C_3|, |C_4|\} = \{k+2, k\}$. Let $|C_3| = k+2$. Therefore, either $v_5 \in C_3$ or $v_6 \in C_3$. Let $v_5 \in C_3$. Then, due to the theorem of Thomason there is a decomposition of H_1 into cycles C_5 and C_6 such that $v_1 \in C_5$, $v_5 \in C_6$, $\{C_5, C_6\} \neq \{F_1, F_2\}$. We have $\{|C_5|, |C_6|\} = \{k+2, k\}$ again, which contradicts Proposition 2. ■

3. Forbidden configurations

Below by G we denote a p_k-graph. In accordance with [2] the configuration in Fig. 1a) is said to be a *monocle*, if the cycle C is even. We call the configuration in Fig. 1b) a $^+$-monocle, if the cycle xP_2yP_3x is even. The configurations in Fig. 2a) and 2b), where $y \neq z$, $u \neq v$, and the cycle yP_2zP_3y, uP_5vP_6u are odd, are called a *binocle* and a $^+$*binocle*, respectively.

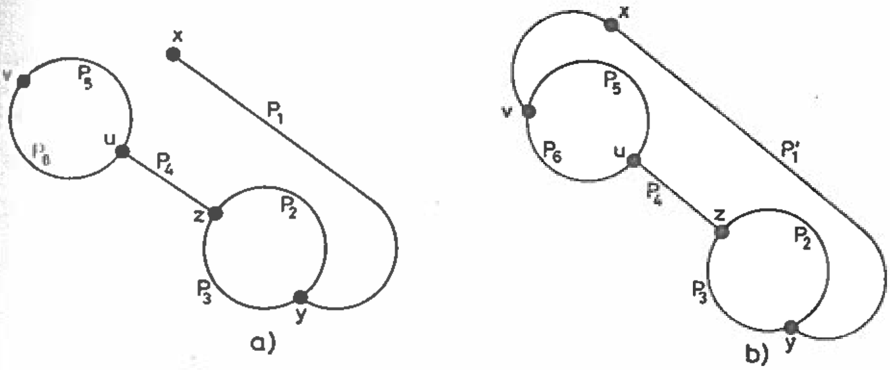

Figure 2.

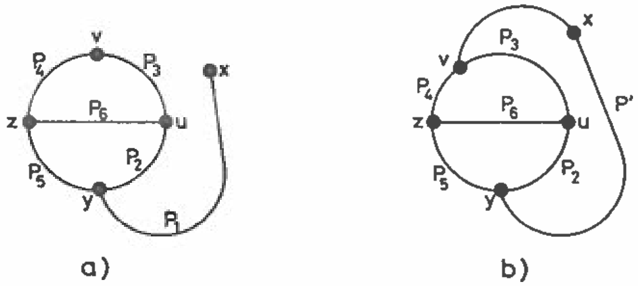

Figure 3.

Similarly, the configurations in Figs. 3a) and 3b), where the cycle $yP_2uP_3vP_4zP_5y$ is even, are called θ-monocle and θ^+-monocle,resp. The configuration in Fig. 4, where the cycle $V_1P_4v_2P_5v_3P_6v_1$ is even, is said to be a *wheel* with *rim* $v_1P_4v_2P_5v_3P_6v_1$ and *centre* x.

Lemma 1. [1,2] *If G contains a monocle (see Fig. 1a) and*

$$|C| \leq 2k, \tag{M1}$$

then $|C| + 2|P_1| \leq 2k - 2$.

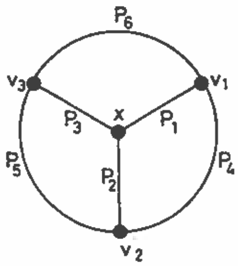

Figure 4.

Lemma 1 implies:

Lemma 2. *If G contains a $^+$-monocle (see Fig. 1b)) and*

$$(M^+1) \qquad |P_2| + |P_3| \le 2k,$$

then $|P_2| + |P_3| + 2|P_1'| \le 2k$.

Lemma 3. *Let G contain a binocle (see Fig. 2a)), which satisfies the following conditions:*

$$(B1) \qquad |P_2| + |P_3| + 2|P_4| + |P_5| + |P_6| \le 2k;$$

$$(B2) \qquad |P_3| < |P_2| + |P_5| + |P_6|, \quad |P_2| < |P_3| + |P_5| + |P_6|.$$

Then $2|P_1| + 2|P_4| + |P_2| + |P_3| + |P_5| + |P_6| \le 2k - 2$.

Proof. By definition of a binocle, $|P_2| + |P_3| + |P_5| + |P_6|$ is even. If (B2) is true, then one can choose the vertex v so that the path $P_7 = vP_5uP_4zP_2y$ and $P_8 = vP_6uP_4zP_3y$ have the same lengths. According to (B1), $|P_7| = |P_8| \le k$. Since in this situation $|P_1| + |P_7|$ must be less than k, lemma is valid. ■

Since for the validity of (B2) it suffices that $|P_5| + |P_6| \ge |P_3| + |P_2|$, Lemma 3 implies

Lemma 4. *Let G contain a $^+$-binocle (see Fig. 2b)) and*

$$(B^+1) \qquad |P_2| + |P_3| + 2|P_4| + |P_5| + |P_6| \le 2k.$$

Then $2|P_1'| + |P_2| + |P_3| + 2|P_4| + |P_5| + |P_6| \le 2k$.

Lemma 5. *Let G contain a θ-monocle (see Fig. 3a)), which satisfies the following conditions:*

$$(\theta 1) \qquad |P_2| + |P_3| + |P_4| + |P_5| \le 2k,$$

$$|P_2| < |P_3| + |P_4| + |P_5|, |P_5| < |P_2| + |P_3| + |P_4|. \quad (\theta 2)$$

Then $2|P_1| + |P_2| + |P_3| + |P_4| + |P_5| + 2|P_6| \le 2k - 2$.

Proof. If (θ2) is true, then one can choose the vertex v so that the paths $P_7 = vP_4zP_6uP_2y$ and $P_8 = vP_3uP_6zP_5y$ have the same lengths. By (θ1) and Lemma 2, we have $|P_2|+|P_3|+|P_4|+|P_5|+2|P_6| \le 2k$, and hence $|P_7| = |P_8| \le k$. The condition that G is a p_k-graph implies $|P_7| + |P_1| \le k - 1$. ■

Since for the validity of (θ2) it suffices that $|P_3| + |P_4| \ge |P_2| + |P_5|$, the following lemma is true.

Lemma 6. *Let G contain a θ^+-monocle (see Fig. 3b)) and*

$$|P_2| + |P_3| + |P_4| + |P_5| \le 2k. \quad (\theta^+ 1)$$

Then $2|P_1'| + |P_2| + |P_3| + |P_4| + |P_5| + 2|P_6| \le 2k$.

Lemma 7. *Let G contain a wheel (see Fig. 4) and*

$$|P_4| + |P_5| + |P_6| \le 2k. \quad (W1)$$

Then $2|P_1| + 2|P_2| + 2|P_3| + P_4| + |P_5| + |P_6| \le 2k$.

Proof. Let $|P_5| = \max\{|P_4|, |P_5|, |P_6|\}$. Then by deleting the edge of P_1, incident with x, we obtain a θ-monocle with the lens $v_1P_4v_2P_5v_3P_6v_1$, which satisfies the conditions of Lemma 5. ■

4. The nonexistence of very long cycles in G

We assume below that $E(G) = \cup_{i=1}^{t} E(F_i)$, where F_i's are $(k+1)$-cycles.

Lemma 8. *G contains no cycles with length exceeding 2k.*

Proof. Let C be the shortest of even cycles with the length $> 2k$. First suppose that

> the ends x and z of the path P_1 with $|P_1| \le k$ divide C into two segments P_2 and P_3, whose lengths have the same parity as $|P_1|$ (see Fig. 5.) (2)

One can assume that $|P_2| \le |P_3|$. If $|P_1| + |P_2| \le 2k$, then, by Lemma 2, $2|P_3| + |P_1| + |P_2| \le 2k$. But $|C| \ge 2k + 2$. Thus, $|P_1| + |P_2| > 2k$ and, due to minimality of $|C|$, we have $|P_1| \ge |P_3|$. However, $|P_1| \le k$ and $|P_3| \ge |C|/2 \ge k+1$, a contradiction.

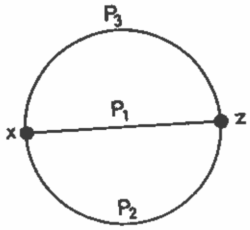

Figure 5.

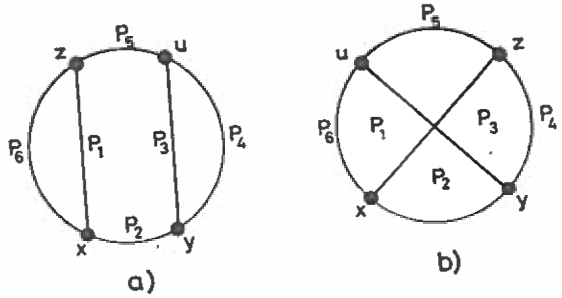

Figure 6.

Now suppose that

(3) the graph $F_i \cap C$ is not connected for some i.

Then one of the situations represented in Figs. 6a), 6b) takes place, where $C = xP_2P_4P_5P_6x$ and the path $zP_1xP_2yP_3u$ is contained in F_i.

Case 1. The situation represented in Fig. 6a) takes place (it is possible that $x = y$ and/or $z = u$). Since situation (2) is not the case, the cycles xP_1zP_6x and yP_3uP_4y are odd. If $|P_1| + |P_3| < |P_4| + |P_6|$, then, due to the choice of C, the length of the cycle $C_1 = zP_1xP_2yP_3uP_5z$ does not exceed $2k$. But then, by lemma 2, $|C_1| + 2\max\{|P_4|, |P_6|\} \leq 2k$, violating the inequality $|C| > 2k$. Therefore,

(4) $$|P_1| + |P_3| \geq |P_4| + |P_6|.$$

Since $|C| > 2k$, by Lemma 4, $2k < 2|P_2| + |P_1| + |P_6| + |P_3| + |P_4|$. Considering parity we deduce that

$$2k+2 \leq 2|P_2|+|P_1|+|P_6|+|P_3|+|P_4| = 2(|P_1|+|P_2|+|P_3|)+(|P_4|+|P_6|-|P_1|-|P_3|)$$

By (4), we obtain $k+1 \leq |P_1| + |P_2| + |P_3|$. Hence $k+1 = |P_1| + |P_2| + |P_3|$, $x = u$ and $|C| = |P_4| + |P_2| + |P_6| \leq |P_1| + |P_2| + |P_3| = k+1$, a contradiction.

Case 2. The situation of the Fig. 6b) takes place. Similarly to Case 1, the cycles $C_2 = xP_1zP_4yP_2x$ and $C_3 = yP_3uP_5zP_4y$ are odd. Thus, the cycles $C_4 = xP_2yP_3uP_5zP_1x$ and $C_5 = zP_1xP_6uP_3yP_4z$ are even. By Lemma 6, $|C_4| > 2k$, $|C_5| > 2k$. Hence, due to the choice of C, $|C| \leq |C_4|$, $|C| \leq |C_5|$. Consequently,

$$|P_1| + |P_3| \geq |P_4| + |P_6|, \quad |P_1| + |P_3| \geq |P_2| + |P_5|,$$

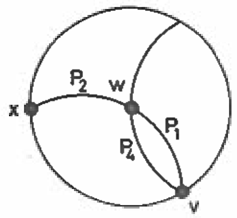

Figure 7.

and $|C| \leq 2(|P_1| + |P_3|) \leq 2k$, a contradiction. Thus, if $E(F_i \cap C) \neq \emptyset$, then $F_i \cap C$ is a path.

There is a vertex $v \in V(C)$ such that $(v, v_1) \in E(C) \cap E(F_i), (v, v_2) \in E(C) \cap E(F_j)$ for some $j \neq i$. Let w be the first vertex of $F_i \cap F_j$, met when moving on $F_j - E(C)$ from v (see Fig. 7.). By the Property (ii) such a vertex does exist. In Fig. 7. $F_i = vP_1wP_2xP_3v, P_4 \subset F_j$. At least one of three cycles of $F_i \cup P_4$ is even. This even cycle together with the path $C - (P_3 - \{v\})$ contradicts Lemma 1. So, in G there are no even cycles of the length $\ell > 2k$.

Let C be an odd cycle in G with $|C| > 2k$, and P_1 be a path, internally-disjoint from C whose endvertices x and z lie in C (such a path exists). In Fig. 5, $C = xP_2zP_3x$. One of the cycles $C_1 = xP_1zP_2x, C_2 = xP_1zP_3x$ (let it be C_1) is even. As noted above, $|C_1| \leq 2k$, violating Lemma 2.

Corollary 9. *The conditions (M1), (M^+1), (θ 1), (θ^+1), (W1) of Lemmas 1, 2, 5, 6, 7 are always true in G.*

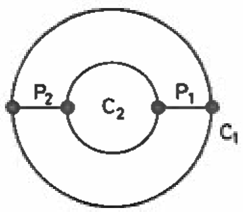

Figure 8.

If the cycles C_1 and C_2 in G are vertex-disjoint, then due to biconnectedness of G, there are vertex-disjoint paths P_1 and P_2 from C_1 to C_2 (see Fig. 8.). Using this fact and Lemma 8, we can prove, similarly to Property (ii) (see [1], Proposition 6) the following lemma.

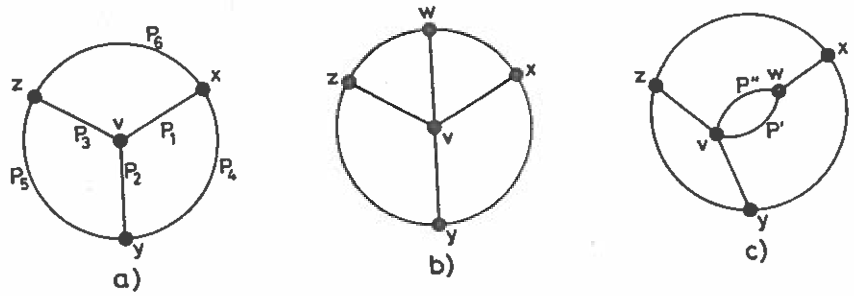

Figure 9.

Lemma 10. *Let G contain cycles C_1 and C_2 with $|V(C_1) \cap V(C_2)| \leq 1$. Then $|C_1| + |C_2| \leq 2k - 2$. Moreover, if $|C_2|$ is even, then $|C_1| \leq k + 1 - |C_2|/2$.*

Lemma 11. *Let C be a cycle of G and $|C| \geq (4k-5)/3$. Then the degree of every vertex $v \in V(G) - V(C)$ is equal to 2. (or equivalently, every C-bridge is a path).*

Proof. First suppose that there is a C-bridge, which has more than two vertices common with C. Then G contains a subgraph represented in Fig. 9a). As far as G is Eulerian, $\deg_G v \geq 4$. Therefore, at least one of the situations represented in Fig. 9b),9c) takes place (in Fig. 9c) $w = x$ is possible). By Lemma 10, the cycle $vP'wP''v$ in Fig. 9c) is odd. Thus, in either case the configuration of Fig. 9a) takes place, where at least one of the cycles

$$C_1 = vP_1xP_4yP_2v, \quad C_2 = vP_2yP_6zP_3v, \quad C_3 = vP_3zP_5xP_1v$$

is even and contains no more than half of $E(C)$. Let us assume for definiteness, that we have obtained that $|C_1|$ is even and $|P_4| \leq |C|/2$. Consider the wheel with the rim C_1 and the centre z. By Lemma 7,

$$2k \geq 2|P_6| + 2|P_5| + 2|P_3| + |P_1| + |P_2| + |P_4| \geq (2|C| - |P_4|) + 4 \geq 3|C|/2 + 4.$$

Therefore, $|C| \leq (4k-8)/3$, a contradiction to the Lemma's condition.

Thus, every C-bridge has exactly two vertices with C in common. Let $v \in V(G) - C, \deg_G v \geq 4, v \in F_i \cap F_j$. By Lemma 10, $|F_i \cap C| \geq 2$. Let the path $x_1P_1x_2$ be a segment of F_i (see Fig. 10a)), and let y_1, y_2 be the first vertices of $C \cup P_1$, which we meet, going from v on F_j in two different directions. Then we have one of the configurations of Fig. 10b), 10c) (the configuration of Fig. 9a) does not occur). In the former case at least one of the cycles

$$vP_2y_1P_1v, \quad vP_3y_2P_1v, \quad vP_2y_1P_1y_2P_3v$$

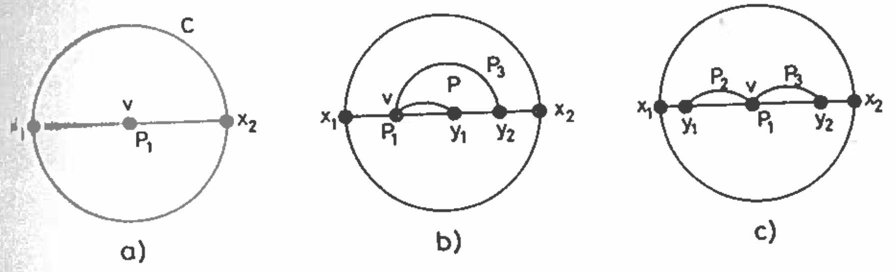

Figure 10.

is an even cycle, contradicting together with C to Lemma 10. In the latter case either at least one of the cycles

$$C_4 = vP_2y_1P_1v, \quad C_5 = vP_3y_2P_1v$$

is even (this contradicts Lemma 10), or, by Lemma 3, $|C_4|+|C_5| \geq 2k+1$. However, by construction, $|C_4| + |C_5| \leq (|F_i| - 1) + (|F_j| - 1) \leq 2k$. ■

Lemma 12. *Let C be a cycle of the maximal length in G, and let the paths P_1, P_2 be C-bridges, arranged as in Fig. 11b) (possibly, $z = u$ and/or $x = y$). Let $|C| = k + \delta$, $\delta \geq 2$ and let the cycle $C_1 = xP_3yP_2zP_5uP_1x$ be even. Then, if*

$$\max\{|P_3|, |P_5|\} \leq |P_1| + |P_2| + \min\{|P_3|, |P_5|\} \tag{5}$$

then $|P_3| + |P_5| \geq |P_1| + |P_2| + 2\delta$.

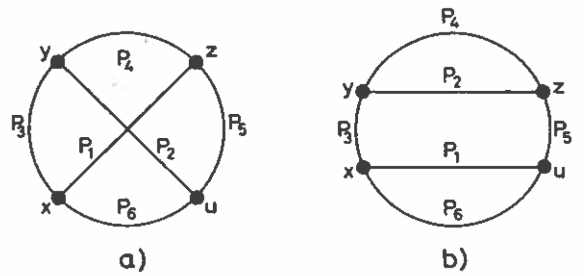

Figure 11.

Proof. We may assume that $|P_1| \geq |P_2|, |P_3| \geq |P_5|, P_2 \subset F_1$. If $x = y, z = u$, then, since P_1 and P_2 are C-bridges, we have $P_1 \cap F_1 = \{x, u\}$. But in this case

$P_1 \cup F_1$ is a $^+$-monocle with the lens xP_1uP_2x. This contradicts Lemma 2, since $|P_1| \ge |P_2|$. Therefore, $|P_3| \ge 1$, $x \ne y$. Due to (5), the vertex v opposite to y on C_1 belongs to P_1. If $v = x$, then the paths $xP_1uP_5zP_2y$ and xP_3y have the same length. Hence, by adding edges of $P_6 - \{u\}$ and $P_4 - \{z\}$ one can make from these paths two k-paths, which connect the same pair of vertices. We argue similarly if $v = u$. If $v \in P_1 - \{u, x\}$, then we have a θ-monocle with the lens C_1, the crack P_6 and the handle $P_4 - \{z\}$. But then, by Lemma 5,

$$\begin{aligned} 2k - 2 \ge & 2|P_6| + 2|P_4| - 2 + |P_1| + |P_2| + |P_3| + |P_5| = \\ = & 2|C| - 2 + (|P_1| + |P_2| - |P_3| - |P_5|) = \\ = & 2k - 2 + 2\delta + |P_1| + |P_2| - |P_3| - |P_5|. \quad \blacksquare \end{aligned}$$

Lemma 13. *Let C be a cycle of the maximal length in G, $C = k + \delta$, and the paths P_1 and P_2 be C-bridges. Denote by $\xi(P_i)$ one of the segments into which P_i divides the cycle C, $i = 1, 2$. Then $|\xi(P_1)| + |\xi(P_2)| \ge 2\delta + |P_1| + |P_2|$.*

Proof. Due to the maximality of $|C|$, we may assume that

(6) $$|P_i| \le |\xi(P_i)| \le |C|/2, \quad i = 1, 2.$$

If $|P_1| + |\xi(P_1)|$ is even, then by Lemma 2, $|\xi(P_1)| + |P_1| + 2(|C| - |\xi(P_1)|) \le 2k$, whence $|\xi(P_1)| \ge 2\delta + |P_1|$, and, by (6), the Lemma is true. Thus, we may suppose that $|P_i| + |\xi(P_i)|$ is odd for $i = 1, 2$.

Case 1. The paths P_1 and P_2 are arranged as in Fig. 11a), $\xi(P_1) = xP_3yP_4z$, $\xi(P_2) = yP_4zP_5u$. Then the cycle $xP_3yP_2uP_5zP_1x$ is even, and, by Lemma 6,

$$2k \ge 2|P_6| + 2|P_4| + |P_3| + |P_5| + |P_1| + |P_2| = 2(k + \delta) + |P_1| + |P_2| - |P_3| - |P_5|.$$

Case 2. The paths P_1 and P_2 are arranged as in Fig. 11b), $\xi(P_1) = P_6$, $\xi(P_2) = P_4$. Since at least one of the three cycles of $C \cup P_i$ is even, then, due to (6) and Lemma 2, we have $2|P_i| + |C| \le 2k$, $i = 1, 2$. Therefore, if $|P_3| \le |P_5|$, then $2k \ge |P_1| + |P_2| + |C| \ge |P_1| + |P_6| + 2|P_3| + |P_2| + |P_4|$. Thus, by Lemma 4,

$$2k \ge 2|C| + |P_1| + |P_2| - |P_4| - |P_6| = 2k + 2\delta + |P_1| + |P_2| - |P_4| - |P_6|.$$

Due to (6), only the following case is not considered.

Case 3. The path P_1 and P_2 are arranged as in Fig. 11b), $\xi(P_1) = P_6$, $\xi(P_2) = yP_3xP_6uP_5z$. Then the cycle $C_1 = xP_1uP_5zP_2yP_3x$ is even. By (6), $|P_4| \ge |C|/2 = (k + \delta)/2$. Therefore, the $^+$-monocle $C_1 \cup P_4$ contradicts to Lemma 2.

Let C be a cycle of the maximal length in G, and let the path P be a C-bridge, $\xi(P)$ be a segment into which the set $\{x, y\} = P \cap C$ divide C. If the cycle $C_1 = xPy\xi(P)x$ is odd, then for every $e \in E(P)$ we mark the vertex opposite to e on C by the mark e. Since $|\xi(P)| \ge |P|$, only the vertices of $C - P$ will be marked. We do so for every pair of paths $\{P, \xi(P)\}$, where P is a C-bridge and $|P| + |\xi(P)|$ is odd.

Lemma 14. *Let C be a cycle of the maximal length in G, $|C| = k+\delta, \delta \geq (k-5)/3$. Let a vertex $v \in V(C)$ be marked by the marks $e_1(e_1 \in P_1)$ and $e_2(e_2 \in P_2)$, where P_1, P_2 are C-bridges. Then the configuration in Fig. 11a) is impossible and in configuration of Fig. 11b) we have $v \notin P_3 \cup P_5$ and $|P_3| + |P_5| \geq 2\delta + |P_1| + |P_2|$.*

Proof. Suppose that the configuration of Fig. 11a) takes place and $v \in P_4$. By the rules of marking, the cycles $C_1 = xP_1zP_4yP_3x$ and $C_2 = uP_2yP_4zP_5u$ are odd. Therefore, the cycle $xP_1zP_5uP_2yP_3x$ is even and, by Lemma 6,

$$2k \geq 2|P_6| + 2|P_4| + |P_1| + |P_2| + |P_3| + |P_5|. \tag{7}$$

Since v is opposite to e_1 on C_1, then $|P_1| + |zP_4v| > |P_3| + |yP_4v|$. Similarly, in C_2 we have $|P_2| + |yP_4v| > |P_5| + |zP_4v|$. Summing up these inequalities with (7), we deduce $2k > 2|C|$, a contradiction.

Now let the configuration of Fig. 11b) take place.

Case 1. $v \in P_3$. Since v is marked by e_1 and e_2,

$$|P_2| + |yP_3v| > |P_5| + |P_6| + |xP_3v|, \quad |P_1| + |xP_3v| > |P_4| + |P_5| + |yP_3v|.$$

Thus, we have the inequality $|P_1| + |P_2| > 2|P_5| + |P_4| + |P_6|$, violating the maximality of $|C|$.

Case 2. $v \in P_4$ Let $|P_3| \geq |P_5|$. As above,

$$|P_2| + |yP_4v| > |zP_4v|, \quad |P_1| + |P_5| + |zP_4v| > |P_3| + |yP_4v|.$$

Summing up these inequalities, we obtain $|P_1| + |P_2| + |P_5| > |P_3|$ and, by Lemma 12, $|P_3| + |P_5| \geq 2\delta + |P_1| + |P_2|$. ■

Theorem 5. *The length of any cycle of G is not greater than $(4k-6)/3$.*

Proof. Suppose that C is a cycle of the maximal length in G, $|C| = k + \delta$, $\delta \geq (k-5)/3$. By Lemma 11, every C-bridge is a path.

Case 1. $|C|$ is odd. Every C-bridge P divides C into two segments $\varphi(P)$ and $\psi(P)$, where $|\varphi(P)| + |P|$ is odd, $|\psi(P)| + |P|$ is even. Since G is Eulerian, there are at least two C-bridges. Therefore, by Lemma 13, there is a C-bridge P such that $|\varphi(P)| \geq \delta + |P|$. Due to Lemma 2, $\psi(P)| + |P| + 2|\varphi(P)| \leq 2k$, whence $|\psi(P)| \geq |P| + 2\delta$. So, $|C| = |\varphi(P)| + |\psi(P) \geq 3\delta + 2|P|$, and $\delta \leq k/2 - 1$. By Proposition 4, $\delta \geq 5$. Thus, by Lemma 13, there is at most one C-bridge P_0 such that

$$|P_0| = 1, \quad |\varphi(P_0)| \leq 3. \tag{8}$$

If such P_0 exists, denote $G_1 = G - E(P_0)$, otherwise let $G_1 = G$. By Theorem 1, $|E(G_1)| - |C| \geq 3k + 3 - 1 - (k + k/2 - 1) > 3k/2 > |C|$. Hence there is a vertex $v \in V(C)$, which received at least two marks of edges of $E(G_1) - E(C)$.

According to Lemma 14 we may assume that the configuration of Fig. 11b) takes place, where $v \in P_4$, $\varphi(P_2) = P_4$, $\varphi(P_1) = xP_3yP_4zP_5u$. Then the cycle xP_1uP_6x is even and, by Lemma 2, $|P_6| \geq 2\delta + |P_1|$. Due to Lemma 14, $|P_3| + |P_5| \geq 2\delta + |P_1| + |P_2|$, whence $\delta + k = |C| \geq |P_4| + (2\delta + |P_1| + |P_2|) + (2\delta + |P_1|)$ and $\delta \leq (k - |P_4| - 2|P_1| - |P_2|)/3$. Due to the maximality of $|C|$, $|P_4| = |\varphi(P_2)| \geq$ $\geq |P_2| + 1$. Since $P_2 \neq P_0$, and (8) is not true for P_2, $|P_2| + |P_4| \geq 5$. Therefore, $\delta \leq (k - 5 - 2|P_1|)/3 \leq (k-7)/3$.

Case 2. $|C|$ is even. Let $\mathcal{P}_2$ be a set of C-bridges P such that P divides C into segments of the same parity as P. Let $\mathcal{P}_1$ be a set of other C-bridges. We show that

$$\sum_{P \in \mathcal{P}_2} |p| \leq 2.$$

First suppose that the C-bridges P_1 and P_2 from $\mathcal{P}_2$ are arranged as in Fig. 11a) and $|P_3| + |P_4| \leq |P_5| + |P_6|$. Lemma 7 implies

$$2(|P_2| + |P_5| + |P_6|) + |P_1| + |P_3| + |P_4| \leq 2k,$$

$$3|C|/2 \leq 2k - 2|P_1| - |P_2|, |C| \leq (4k-6)/3.$$

Let now the paths P_1 and P_2 from $\mathcal{P}_2$ be arranged as in Fig. 11b), and $|P_3| \geq |P_5|$. Then, by Lemma 12, either $|P_3| \geq |P_1| + |P_2| + |P_5| + 1$ and, since the cycle $xP_1uP_5zP_2yP_3x$ is even, we have $|P_3| \geq 4$, or $|P_3| + |P_5| \geq |P_1| + |P_2| + 2\delta \geq 4$. In each case $|P_3| + |P_5| \geq 4$. Due to Lemma 2, $|P_4| \geq |P_2| + 2\delta$, $|P_6| \geq |P_1| + 2\delta$. Therefore, $k + \delta = C \geq |P_2| + 2\delta + |P_1| + 2\delta + 4 \geq 4\delta + 6$, and $\delta \leq (k-6)/3$. Thus, if (9) is not true, $\mathcal{P}_2 = |P_1|$, where $|P_1| \geq 3$. Denoting by $\xi(P_1)$ the shortest of the segments into which P_1 divides C, we have (according to Lemma 2)

$$|P_1| + |\xi(P_1)| + 2(|C| - |\xi(P_1)|) \leq 2k,$$

$$|P_1| + 3|C|/2 \leq 2k, |C| \leq 2(2k - |P_1|)/3 \leq (4k-6)/3.$$

So, (9) is true. Hence, by Lemma 8,

$$\sum_{P \in \mathcal{P}_1} |P| \geq 3(k+1) - |C| - \sum_{P' \in \mathcal{P}_2} |P'| \geq 3k + 3 - 2k - 2 \geq k + 1.$$

Since every edge lying on some C-bridge from $\mathcal{P}_1$ marks two vertices of C, there is a vertex $v \in V(C)$, marked by at least two edges. Let v be marked by $e_1 \in E(P_1)$ and $e_2 \in E(P_2)$. Due to Lemma 14, we may suppose that the configuration of Fig. 11b) takes place, where $v \in P_4$ and $|P_3| + |P_5| \geq |P_1| + |P_2| + 2\delta$. By Lemma 13, $|P_6| + |P_4| \geq 2\delta + |P_1| + |P_2|$. Thus, $\delta + k = |C| \geq 4\delta + 4, \delta \leq (k-4)/3$. But then $\sum_{P \in \mathcal{P}_1} |P| \geq 3(k+1) - |C| - 2 > |C|$, and there exists a vertex $wV(C)$

...h that w is marked by at least three marks. Due to Lemma 14, the situation of Fig. 12 takes place, and

$$|P_5| + |P_9| \geq 2\delta + |P_1| + |P_2|, |P_6| + |P_8| \geq |P_2| + |P_3| + 2\delta.$$

By Lemma 13, $P|_4| + |P_7| \geq 2\delta + |P_1| + |P_3|$. Thus,

$$k + \delta = |C| \geq 6\delta + 2(|P_1| + |P_2| + |P_3|) \geq 6\delta + 6$$

and $\delta \leq (k-6)/5$.

The Theorem is proved. ■

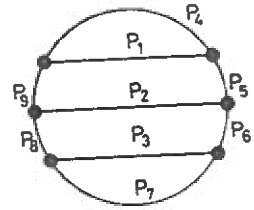

Figure 12.

Corollary 6. *There are no p_k-graphs for $3 \leq k \leq 20$.*

Proof. Suppose that there exists a p_k-graph G, and C is a cycle of the maximal length in G, $|C| = k + \delta$. By Proposition 4, $\delta \geq 5$. Due to Theorem 5, $\delta \leq (k-6)/3$. Hence $k \geq 21$. ■

Remark. Developing the ideas of [3] it can be shown that there are no p_k-graphs for $3 \leq k \leq 30$.

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