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List edge chromatic number of graphs with large girth

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Abstract

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It is shown that the list edge chromatic number of any graph with maximal degree Δ and girth at least $8\Delta(\ln \Delta + 1.1)$ is equal to $\Delta + 1$ or to Δ.

1. Introduction and results

Trying to obtain reasonable upper bound for total chromatic number $t\chi(G)$ of multigraphs G with given maximal degree $\Delta(G)$, Vizing [11–12] introduced the concept of list colouring (prescribed colouring in his terms). He also posed the following.

Conjecture 1. The list edge chromatic number $le\chi(G)$ of any graph G does not exceed $\Delta(G) + 1$.

Conjecture 2. The list edge chromatic number of any multigraph is equal to its edge chromatic number (i.e. chromatic index).

The notion of list colouring was also independently introduced by Erdős, Rubin and Taylor [4], and Conjecture 2—by Albertson and Tucker. Conjecture 1 implies that Total Colouring Conjecture is 'almost true'.

Conjectures 1 and 2 are proved only for very small classes of graphs: snarks, trees, cycles (cf. [3]), planar graphs with maximal degree at least 9 [2]. As far as we know the best general bounds for $le\chi(G)$ and $t\chi(G)$ are the following.

Bound 1 (Hind [5]). *$le\chi(G) \leq 1.8\,\Delta(G)$ for any multigraph G.*

Bound 2 ([7–8, 10]). *$t\chi(G) \leq 1.5\,\Delta(G)$ for any multigraph G with $\Delta(G) \geq 4$.*

Bound 3 (Bollobàs and Hind [1]). *$le\chi(G) \leq 1.75\,\Delta(G) + o(\Delta(G))$ for any graph G.*

Bound 4 (Hind [6]). *$t\chi(G) \leq \Delta(G) + 1 + 2\sqrt{\Delta(G)}$ for any graph G.*

The aim of the present paper is to prove that Conjecture 1 holds for the graphs with large girth.

Theorem 1. *Let G be a graph with maximal degree $\Delta(G) \leq \Delta$ and the girth $g(G) \geq 8\Delta(\ln \Delta + 1.1)$. Then $le\chi(G) \leq \Delta + 1$.*

This theorem immediately yields the following.

Corollary 2. *Let G be a graph with the maximal degree $\Delta(G) \leq \Delta$ and the girth $g(G) \geq 8\Delta(\ln \Delta + 1.1)$. Then $t\chi(G) \leq \Delta + 3$.*

The idea of the Theorem proof is that of [9] adapted for line graphs.

2. Notation

The girth $g(G)$ of the graph G is, as usual, the length of its shortest cycle. For any $v \in V(G)$ we let $N_G(v) := \{w \in V(G) \mid (v, w) \in E(G)\}$.

The line graph $L(G)$ of the graph G is that whose vertices are the edges of G and two vertices of $L(G)$ are adjacent iff the correspondent edges of G have an end vertex in common. The vertex of the graph $L(G)$ corresponding to the edge $(a, b) \in E(G)$ will sometimes be denoted by $\langle a, b\rangle$. The set $N_{L(G)}(\langle a, b\rangle)$ is splitted in a natural way into two cliques: $C(a, b)$ which contains all the vertices of $N_{L(G)}(\langle a, b\rangle)$ of the kind $\langle a, c\rangle$, and $C(b, a)$ which contains all the vertices of $N_{L(G)}(\langle a, b\rangle)$ of the kind $\langle b, d\rangle$.

The list chromatic number $l\chi(G)$ of the graph G is the smallest k such that if whenever each vertex $v \in V(G)$ is assigned a list $\Phi(v)$ of k admissible colours, then there exists such a proper colouring f of $V(G)$ that each vertex v is coloured by a colour $f(v) \in \Phi(v)$ (in this situation we will say that f is compatible with Φ).

It is known ([4, 12]) that there exist bipartite graphs with arbitrary large list chromatic number.

The list edge chromatic number $le\chi(G)$ is defined analogously. We can also say that $le\chi(G)$ is the list chromatic number of the line graph $L(G)$.

Call the walk $(v_1, v_2, \ldots, v_t)$ of the graph L *flexible* if $(v_i, v_{i+2}) \notin E(L)$ for any $1 \leq i \leq t-2$. We mean that no edge is used in a walk more than once.

3. Preliminaries

Lemma 1. *Let $L = L(G)$ be the line graph of a graph G and $(v_1, \ldots, v_t)$ be a flexible walk in L. If $t \geq 3$ and $(v_1, v_t) \in E(L)$, then $t \geq g(G)$.*

Proof. Let $(v_1, \ldots, v_t)$ be the shortest walk in L which satisfies the conditions of Lemma. Then the subgraph of L induced by the vertex set $(v_1, \ldots, v_t)$ should be a cycle. Since $(v_1, v_3) \notin E(L)$, we have $t \geq 4$. In this case the spanning subgraph of G generated by the edges corresponding to the vertices $v_1, \ldots, v_t$ also has to be a cycle. $\square$

Now, suppose that the Theorem 1 is not true. Then there exists a counter example G, minimal with respect to the number of edges. Let the maximal degree of G be equal to Δ and the set Φ of $(\Delta + 1)$-element lists for edges of G be such that there is no proper colouring of $E(G)$ compatible with Φ.

Consider $L = L(G)$ and some vertex $v_0 = \langle a_0, b_0 \rangle \in V(L)$. Due to the minimality of G, there is a colouring f of $V(L) \setminus \{v_0\} = E(G) \setminus \{(a_0, b_0)\}$ compatible with Φ.

In the next section we will construct a subset Γ of $V(L) \cup E(L)$. This set will play the main role in proving our Theorem. The edges of Γ will be directed, some of them in both directions. While constructing Γ, the arcs and vertices belonging to Γ will be called Γ-arcs and Γ-vertices, respectively. At the Step i of the construction we shall define Γ-arcs and Γ-vertices of level i. It will be shown that either we can colour $V(L)$ properly or Γ increases unboundedly, a contradiction to the finiteness of Γ. Roughly, the existence of a Γ-arc (v, u) will show the possibility to recolour v in the colour of u, if we could recolour u in another colour.

4. Construction of the set Γ

By the choice of G, $f(N_L(v_0)) \supset \Phi(v_0)$. Denote

$$O_1(v_0) := \{v \in N_L(v_0) \mid f(v) \in \Phi(v_0) \setminus f(N_L(v_0) \setminus \{v\})\}.$$

Step 0. The vertex v_0 is called the Γ-vertex of level 0.

Step 1. For every $v \in O_1(v_0)$ direct edge (v_0, v) towards v. Call these arcs the Γ-arcs (in particular, Γ_1-arcs) of level 1. The vertices of $O_1(v_0)$ will be called Γ-vertices of level 1.

Definition. Say that Γ-vertex v of level 1 is 1-free if

$$\Phi(v) \setminus f(\{v\} \cup N_L(v)) \neq \emptyset.$$

Definition. For any Γ-vertex $v \neq v_0$ we let $T_0(v) := \emptyset$,

$$T_1(v) := \{v_0\}, \qquad Q_2(v) := \{w \in N_L(v) \setminus \{v_0\} \mid f(w) \in \Phi(v) \setminus f(N_L(v) \setminus \{w\})\}.$$

If $v = \langle a, b\rangle$ is a Γ-vertex of level 1 then $|\{a, b\} \cap \{a_0, b_0\}| = 1$ by definition. In the case $a \in \{a_0, b_0\}$ we set $O_2(v) := Q_2(v) \cap C(b, a)$. Then go to Step 2.

Step k ($k > 1$).

(k 1). If there exists any $(k-1)$-free Γ-vertex, then the construction terminates. It also terminates if there are no Γ_1-arcs of level $k-1$. In all other cases go to (k 2).

(k 2). For every ordered pair (v, w) of vertices, where v $(\neq v_0)$ is a Γ-vertex and $w \in O_k(v)$, we direct the edge (v, w) towards w. Call all these arcs the Γ-arcs of level k. Go to (k 3).

Remark. It may happen that some edges generate Γ-arcs in both directions.

(k 3). Here we will determine for every Γ-arc of level k whether it is a Γ_1-arc or else a Γ_2-arc. Let

$$q = \lfloor 0.5g(G) \rfloor. \tag{1}$$

Consider an arbitrary Γ-arc $\overrightarrow{(v, w)}$ of level k. Let $f(v) = \alpha$, $f(w) = \beta$. Suppose, for some $s \geq 2$ there exists a directed path $\overrightarrow{(v_1, v_2)}, \ldots, \overrightarrow{(v_{s-1}, v_s)}$ consisting of Γ-arcs such that:

(a) $v_{s-1} = v$, $v_s = w$;

(b) $f(v_i) \in \{\alpha, \beta\}$, $1 \leq i \leq s$;

(c) $\overrightarrow{(v_i, v_{i+1})}$ is a Γ_1-arc for $1 \leq i \leq s-2$;

(d) there is a Γ-vertex u of level at most $k-q$ (see (1)) with $N_L(u) \cap \{v_1, \ldots, v_s\} \neq \emptyset$.

Then we call $\overrightarrow{(v, w)}$ a Γ_2-arc of level k and put $\mathcal{P}(v, w) = u$. If there are several such u's, then select any of them as $\mathcal{P}(v, w)$.

If there are no such paths as described above we call $\overrightarrow{(v, w)}$ a Γ_1-arc of level k. After having every Γ-arc of level k checked go to (k 4).

(k 4). A vertex $v \in V(L)$ will be called a Γ-vertex of level k if there is a Γ_1-arc entering v but there are no such Γ_1-arcs of level less than k. Go to (k 5).

(k 5). For a Γ-vertex w we denote $T_k(w) := \{v \in V(L) \mid \overrightarrow{(v, w)}$ is a Γ_1-arc of level at most $k\}$. Go to (k 6).

(k 6). Call a Γ-vertex v $(\neq v_0)$ k-free if

$$\Phi(v) \backslash f(\{v\} \cup (N_L(v) \backslash T_k(v)) \neq \emptyset.$$

Let us recall that, due to (k 1), if there is at least one k-free Γ-vertex then the construction terminates at the Step $k+1$.

Go to (k 7).

(k 7). For every Γ-vertex v $(\neq v_0)$ we set

$$Q_{k+1}(v) := \Big\{ w \in N_L(v) \Big\backslash \Big(\{v_0\} \cup T_k(v) \cup \bigcup_{j=2}^{k} O_j(v) \Big) \Big| \\ f(w) \in \Phi(v) \backslash f(N_L(v) \backslash (\{w\} \cup T_k(v))) \Big\}.$$

Go to (k 8).

(k 8). For every Γ-vertex $v = \langle a, b \rangle$ we define $O_{k+1}(v)$ as follows.

Case 1: $T_k(v) \setminus T_{k-1}(v) = \emptyset$.

Then $O_{k+1}(v) := \emptyset$.

Case 2: $C(a, b) \cap (T_k(v) \setminus T_{k-1}(v)) \neq \emptyset$ and $C(b, a) \cap (T_k(v) \setminus T_{k-1}(v)) \neq \emptyset$.

Then $O_{k+1}(v) := Q_{k+1}(v)$.

Case 3: $C(a, b) \cap (T_k(v) \setminus T_{k-1}(v)) = \emptyset$ and $C(b, a) \cap (T_k(v) \setminus T_{k-1}(v)) = \emptyset$.

Then $O_{k+1}(v) := Q_{k+1}(v) \cap C(b, a)$.

Go to Step $k + 1$.

By Γ we will denote the set consisting of all the Γ-arcs and Γ-vertexs.

5. Properties of Γ

At first we remark that the number of steps is finite. Indeed, by (k 7) and (k 8), for any Γ-vertex v ($\neq v_0$) and any $k \neq j$ we have $O_k(v) \cap O_j(v) = \emptyset$. So, any edge of the graph L can generate at most two Γ-arcs. Consequently, by (k 1), the construction lasts no more than $2\,|E(L)| + 1$ steps. We will assume below that the construction terminated on the Step $s + 1$ and hence the maximal level of Γ-arcs is s.

The levels of the Γ-vertex v and the Γ-arc $\overrightarrow{(v, w)}$ will be denoted by $Y(v)$ and $Y(v, w)$, respectively.

Claim 1. *If $\overrightarrow{(v, w)}$ is a Γ-arc, then $Y(v, w) \geq Y(v) + 1$. If $\overrightarrow{(v, w)}$ is a Γ_1-arc and $w \neq v_0$, then $Y(w) \leq Y(v, w)$.*

Proof. Due to (k 2), the initial vertex of any Γ-arc is a Γ-vertex of lower level. According to (k 4), the level of any Γ-vertex $w \neq v_0$ is equal to the lowest level of Γ_1-arcs entering w. □

Claim 2. *If $\overrightarrow{(v, w)}$ is a Γ-arc, $w \neq v_0$ and $\overrightarrow{(w, v)}$ is a Γ_1-arc, then $Y(w, v) = s$ and v is an s-free Γ-vertex.*

Proof. Let $Y(v, w) = k + 1$, $f(w) = \beta$, $Y(w, v) = r$. By (k 5), $w \in T_r(v)$. Hence, due to (k 7), $r \geq k + 1$ and $\beta \in \Phi(v) \setminus f(N_L(v) \setminus (\{w\} \cup T_k(v)))$. But then $\beta \in \Phi(v) \setminus f(N_L(v) \setminus T_r(v))$ and v is an r-free vertex. Thus, by (k 1), $r = s$. □

Claim 3. *If $\overrightarrow{(v, w)}$ and $\overrightarrow{(w, u)}$ are Γ-arcs, $f(v) = f(u)$ and $v \neq u$, then $\overrightarrow{(v, w)}$ is a Γ_1-arc and $Y(v, w) + 1 \leq Y(w, u)$.*

Proof. Let $Y(w, u) = k$. By (k 2), $u \in O_k(w) \subset Q_k(w)$. Then, because of ($k$ 7), $v \in T_{k-1}(w)$. It means that $\overrightarrow{(v, w)}$ is a Γ_1-arc and $Y(v, w) \leq k - 1$. □

From Claim 1 and (k 4) we obtain the following.

Claim 4. *For any Γ-vertex $w \neq v_0$ there exists a Γ_1-arc $\overrightarrow{(v, w)}$ such that $Y(w) = Y(v, w) \geqslant Y(v) + 1$.*

From (k 8) and (k 2) we obtain immediately the following.

Claim 5. *Let $\overrightarrow{(v, w)}$ be a Γ-arc of level k ($k \geqslant 2$), $v = \langle a, b\rangle \neq v_0$, $w \in C(b, a)$. Then there exists a Γ-vertex $u \in C(a, b)$ such that $\overrightarrow{(u, v)}$ is a Γ_1-arc of level $k - 1$.*

Claim 6. *Γ_1-arcs of level $k > 1$ do not enter vertices adjacent to v_0.*

Proof. Suppose, Claim 6 is false. Then among the Γ_1-arcs entering vertices adjacent to v_0 we choose a Γ_1-arc $\overrightarrow{(v, w)}$ of the smallest level $k \geqslant 2$.

Construct a sequence $v_k, v_{k-1}, \ldots, v_0$ as follows. Put $v_k := w$, $v_{k-1} := v$. Let now the sequence $v_k, \ldots, v_{k-t}$ ($t < k$) be constructed such that

(a) $\overrightarrow{(v_i, v_{i+1})}$ is a Γ_1-arc of level $i + 1$ ($k - t \leqslant i \leqslant k - 1$);

(b) if $v_i = \langle a_i, b_i\rangle$ ($k - t \leqslant i \leqslant k$), then $a_{i+1} = b_i$ ($k - t \leqslant i \leqslant k - 1$) and $a_i \neq a_{i+2}$ ($k - t \leqslant i \leqslant k - 2$).

Due to Claim 5, there exists $v_{k-t-1} \in C(a_{k-t}, b_{k-t})$ such that $\overrightarrow{(v_{k-t-1}, v_{k-t})}$ is a Γ_1-arc of level $k - t$. Since $v_{k-t-1} \in C(a_{k-t}, b_{k-t})$, we may assume $v_{k-t-1} = \langle a_{k-t-1}, b_{k-t-1}\rangle$, where $b_{k-t-1} = a_{k-t}$. Because $v_{k-t-1} \neq v_{k-t}$, we have $a_{k-t-1} \neq b_{k-t}$.

By the construction, $(v_{i+2}, v_i) \notin E(L)$ ($i = 0, 1, \ldots, k - 2$) if $g(G) \geqslant 4$. Hence, by Lemma 1, $k \geqslant g(G) - 1$. Then, according to (k 3), $\overrightarrow{(v_{k-1}, v_k)}$ should be a Γ_2-arc since $(v_0, v_k) \in E(L)$. □

Claim 7. *Let $(v_1, \ldots, v_{t+1})$ be a walk in L such that $f(v_1) = f(v_3) = f(v_5) = \cdots = \alpha$, $f(v_2) = f(v_4) = \cdots = \beta$ and $\overrightarrow{(v_2, v_1)}$ be a Γ-arc. Then:*

(a) *$\overrightarrow{(v_{i+1}, v_i)}$ is a Γ_1-arc of level $Y(v_2, v_1) + 1 - i$ ($1 \leqslant i \leqslant t$);*

(b) *$Y(v_2, v_1) \leqslant Y(v_{t+1}) + q - 1$;*

(c) *if $\overrightarrow{(v_2, v_1)}$ is a Γ_1-arc, then $Y(v_2, v_1) \leqslant Y(v_{t+1}) + q - 2$.*

Proof. Let $Y(v_2, v_1) = k$ and $t \geqslant 2$. According to (k 7), to have $v_1 \in Q_k(v_2)$ it is necessary that $v_3 \in T_{k-1}(v_2)$. If $v_3 \in T_{k-2}(v_2)$, then $v_1 \in Q_{k-1}(v_2)$. So, $v_3 \in T_{k-1}(v_2) \setminus T_{k-2}(v_2)$. It means that $\overrightarrow{(v_3, v_2)}$ is a Γ_1-arc of level $k - 1$. Analogously we obtain that (a) is true for any $2 \leqslant i \leqslant t$.

Suppose, $Y(v_{t+1}) \leqslant k - q$. Since $f(v_{t+1}) \in \{\alpha, \beta\}$, then $v_{t+1} \neq v_0$ and by Claim 4 there exists $w \in N_1(v_{t+1})$ with $Y(w) \leqslant k - q - 1$. But in this case $\overrightarrow{(v_3, v_2)}$ should be a Γ_2-arc because of (k 3). Analogously arguing, (c) is also true. □

Definition. We will denote by $L_{\alpha\beta}$ the subgraph of L induced by $\{v \in V(L) \mid f(v) \in \{\alpha, \beta\}$. By $L_{\alpha\beta}(v, w)$ will be denoted the connected component of $L_{\alpha\beta} \setminus \{(v, w)\}$ containing v with the added vertex w and edge (v, w).

Claim 8. *For any Γ_1-arc $\overrightarrow{(v, w)}$ with $f(\{v, w\}) = \{\alpha, \beta\}$ the graph $L_{\alpha\beta}(v, w)$ forms an oriented path of the length at most $q-2$ with the end vertex w. The subgraph of L induced by the vertex set of this path is also a path.*

Proof. Since L is a line graph, the connected component R of $L_{\alpha\beta}$ containing v has to be a path or a cycle. Due to Claim 7, it ought to generate in Γ an oriented path of the length at most $q-2$ consisting of Γ_1-arcs. Since L is $K_{1,3}$-free, then the induced subgraph $L(R)$ has to be a path, too. □

Remark. In view of Claim 8 we can consider $L_{\alpha\beta}(v, w)$ as a part of Γ, namely, as an oriented path.

Definition. For any Γ_1-arc $\overrightarrow{(v, w)}$ the digraph $F(v, w)$ will be defined by induction on $Y(v, w)$:

(1) If $v = v_0$, then $F(v, w)$ consists of vertices v, w, and Γ_1-arc $\overrightarrow{(v, w)}$.

(2) Let $F(x, y)$ be defined for all the Γ_1-arcs $\overrightarrow{(x, y)}$ with $Y(x, y) < k$. Consider a Γ_1-arc $\overrightarrow{(v, w)}$ of level k with $f(v) = \alpha$, $f(w) = \beta$. Let u_1 and u_2 be the first and the second vertices of the oriented path $L_{\alpha\beta}(v, w)$. We may suppose $u_1 = \langle a, b\rangle$, $u_2 \in C(b, a)$. By Claim 5 there exists a Γ_1-arc $\overrightarrow{(u, u_1)}$ such that $u \in C(a, b)$ and $Y(u, u_1) < Y(u_1, u_2)$. Select some u with this property and put $F(v, w) := L_{\alpha\beta}(v, w) \cup F(u, u_1)$.

Remark. Since $g(G) > 3$, $u \in C(a, b)$, $u_2 \in C(b, a)$, then $(u, u_2) \notin E(L)$.

Claim 9. *Let $\overrightarrow{(v, w)}$ be a Γ_1-arc. Then*:

(a) *$F(v, w)$ is an oriented path consisting of Γ_1-arcs; it leads from v_0 to w;*

(b) *if $\overrightarrow{(y_1, y_2)}$ and $\overrightarrow{(y_2, y_3)}$ belong to $F(v, w)$, $\{y_1, y_2\} \neq \{v, w\}$, then $Y(y_1, y_2) + 1 \leq Y(y_2, y_3)$;*

(c) *the subgraph of L induced by the vertex-set of $F(v, w)$ is a path which connects v_0 and w.*

Proof. We will use induction on the level of $\overrightarrow{(v, w)}$. If $Y(v, w) = 1$, the Claim 9 is obvious for $\overrightarrow{(v, w)}$.

Suppose, the Claim is true for all Γ_1-arcs of level at most $k-1$, and $\overrightarrow{(v, w)}$ is a Γ_1-arc of level k. Let $f(v) = \alpha$, $f(w) = \beta$. By the definition of $F(v, w)$, there exist Γ-vertices u_1, u_2, u such that:

(i) $\{u_1, u_2\} \subset V(L_{\alpha\beta}(v, w))$;

(ii) $\overrightarrow{(u, u_1)}$ and $\overrightarrow{(u_1, u_2)}$ are Γ_1-arcs;

(iii) $1 + Y(u, u_1) \leq Y(u_1, u_2) \leq Y(v, w) = k$;

(iv) $(u, u_2) \notin E(L)$;

(v) $F(v, w)$ is the union of $L_{\alpha\beta}(v, w)$ and $F(u, u_1)$.

By the induction assumption the Claim is true for $F(u, u_1)$. Due to Claim 8 and Claim 7, $L_{\alpha\beta}(v, w)$ is an oriented path from u_1 to w and the statement (b) of Claim 9 is valid for $L_{\alpha\beta}(v, w)$. Hence (b) is valid for $F(v, w)$.

Now, let $(v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_t)$ (where $v_m = u_1$, $v_t = w$) be the walk consisting of $F(u, u_1)$ and $L_{\alpha\beta}(v, w)$. Due to induction assumption and Claim 8 and since $(u, u_2) \notin E(L)$, this walk is flexible. Show that it is a simple path. Suppose, $i<j$ and $v_i = v_j$. By Lemma 1, $j - i \geq g(G)$. Therefore, according to (b),

$$Y(v_{j-1}, v_j) \geq g(G) - 1 + Y(v_i, v_{i+1}) \geq Y(v_i) + g(G),$$

and $\overrightarrow{(v_{j-1}, v_j)}$ should be a Γ_2-arc. The contradiction shows that (a) is true for $F(v, w)$.

At last suppose that $i + 1 < j$ and $(v_i, v_j) \in E(L)$. Since $(v_0, \ldots, v_t)$ is flexible, then $j - 1 \geq g(G) - 1$, $Y(v_{j-1}, v_j) \geq g(G) + Y(v_i) - 1$ and again $\overrightarrow{(v_{j-1}, v_j)}$ should be a Γ_2-arc. □

Definition. Let w be an s-free Γ-vertex, $f(w) = \beta$. By the definition of s-free Γ-vertex, there exists

$$\alpha \in \Phi(w) \setminus (\{\beta\} \cup f(N_L(w) \setminus T_s(w))).$$

For such α, construct (w, α)-trace $F(w, \alpha)$ by the following rules.

Case 1. If $\alpha \in \Phi(w) \setminus (\{\beta\} \cup f(N_L(w))$, then, by definition, $Y(w) = s$. Take any vertex $v \in T_s(w)$ and put $F(w, \alpha) := F(v, w)$.

Case 2. If there is exactly one vertex $v \in T_s(w)$ with $f(v) = \alpha$, then put $F(w, \alpha) := F(v, w)$.

Case 3. Let $\{y \in T_s(w) \mid f(y) = \alpha\} = \{v, x\}$ and $Y(v, w) \leq Y(x, w)$. Then $F(w, \alpha)$ is the union of $F(v, w)$ and $L_{\alpha\beta}(x, w)$.

Claim 10. *Let w be an s-free Γ-vertex, $f(w) = \beta$ and $\alpha = \Phi(w) \setminus (\{\beta\} \cup f(N_L(w) \setminus T_s(w)))$. Then*:

(a) *$F(w, \alpha)$ is a root tree with the root w every arc of which is a Γ_1-arc; this tree consists of one or two oriented paths, terminating in w; one of these paths starts from v_0;*

(b) *if $\overrightarrow{(y_1, y_2)}$ and $\overrightarrow{(y_2, y_3)}$ belong to $F(w, \alpha)$, $y_1 \neq y_3$, then $Y(y_1, y_2) + 1 \leq Y(y_2, y_3)$;*

(c) *the subgraph of L induced by $F(w, \alpha)$ is a path one end of which is v_0.*

Proof. If Cases 1 or 2 of the definition of $F(w, \alpha)$ take place, then Claim 10 follows from Claim 9. Let Case 3 hold. According to Claims 7, 8 and 9, $F(v, w)$ and $L_{\alpha\beta}(x, w)$ are oriented paths, consisting of Γ_1-arcs. They lead to w and $F(v, w)$ starts from v_0.

Let $(v_0, v_1, \ldots, v_t, \ldots, v_r)$ (where $v_{t-1} = v$, $v_t = w$, $v_{t+1} = x$) be the walk such that $(v_0, v_1, \ldots, v_t)$ is $F(v, w)$ and $(v_r, v_{r-1}, \ldots, v_t)$ is $L_{\alpha\beta}(x, w)$. Since $f(v) = \alpha = f(x)$, then $(v, x) \notin E(L)$ and $(v_0, v_1, \ldots, v_t, \ldots, v_r)$ is a flexible walk.

Suppose, $i<j$ and $v_i = v_j$. Due to Claims 8 and 9, $i<t<j$. As $(v_0, v_1, \ldots, v_r)$ is flexible, we have $j-i \geqslant g(G)$. By Claim 7,

$$Y(v_{t+1}, v_t) = Y(v_j, v_{j-1}) + j - 1 - t \leqslant Y(v_j, v_{j-1}) + q - 2.$$

So, $j-t \leqslant q-1$. By the definition of $F(w, \alpha)$, $Y(v_{t-1}, v_t) \leqslant Y(v_{t+1}, v_t)$. According to Claim 9,

$$Y(v_{t-1}, v_t) \geqslant t - 1 - i + Y(v_i, v_{i+1}) \geqslant t - i + Y(v_i).$$

Thus,

$$\begin{aligned} Y(v_{t+1}, v_t) &\geqslant t - i + Y(v_i) = Y(v_i) + (j-i) + (t-j) \\ &\geqslant Y(v_i) + g(G) - q + 1 > Y(v_i) + q, \end{aligned}$$

and $\overrightarrow{(v_{t+1}, v_t)}$ should be a Γ_2-arc. Hence all the vertices $v_0, v_1, \ldots, v_r$ are distinct and, taking Claims 8 and 9 into account, (a) and (b) are true for $F(w, \alpha)$.

At last suppose, $j>i+1$ and $(v_i, v_j) \in E(L)$. Again $i<t<j$. Since $(v_0, v_1, \ldots, v_r)$ is flexible, then $j-i \geqslant g(G)-1$. As above, we obtain $j-t \leqslant q-1$ and

$$\begin{aligned} Y(v_{t+1}, v_t) &\geqslant t - i + Y(v_i) = Y(v_i) + (j-i) + (t-j) \\ &\geqslant Y(v_i) + g(G) - 1 - q + 1 \geqslant Y(v_i) + q. \end{aligned}$$

Therefore, again $\overrightarrow{(v_{t+1}, v_t)}$ should be a Γ_2-arc. □

Claim 11. *There are no s-free Γ-vertices.*

Proof. Suppose, w is an s-free Γ-vertex, $f(w) = \beta$ and $\alpha \in \Phi(w) \setminus (\{\beta\} \cup f(N_L(w) \setminus T_s(w)))$. For every $v \in V(F(w, \alpha)) \setminus \{w\}$ there is exactly one vertex $\nu(v)$ such that the Γ_1-arc $\overrightarrow{(v, \nu(v))}$ belongs to $F(w, \alpha)$. Set

$$f'(y) = \begin{cases} f(y), & y \in V(L) \setminus F(w, \alpha); \\ \alpha, & y = w; \\ f(\nu(y)), & y \in (V(L) \cap F(w, \alpha)) \setminus \{w\}. \end{cases} \tag{2}$$

Due to (k 7) and the choice of α, we have

$$f'(y) \in \Phi(y) \quad \forall\, y \in V(L).$$

Assume that for some $(z, u) \in E(L)$

$$f'(z) = f'(u). \tag{3}$$

Since colours of vertices of $V(L) \setminus F(w, \alpha)$ do not change, we let $z \in F(w, \alpha)$.

Case 1. $z = w$. By the definition of s-free Γ-vertex,

$$\alpha \notin f(N_L(w) \setminus F(w, \alpha)).$$

According to Claim 10, there are at most two vertices of $F(w, \alpha)$ adjacent to w. And they are coloured by β in f'. Hence in this case (3) does not hold.

Case 2. $z = v_0$. By Claim 10, $\nu(v_0)$ is the only vertex of $F(w, \alpha)$ adjacent to v_0. Due to the definition of the Step 1,

$$f'(v_0) = f(\nu(v_0)) \neq f(u) = f'(u) \quad \text{if} \quad u \neq \nu(v_0).$$

And, by (2), $f'(\nu(v_0) \neq f(\nu(v_0)) = f'(v_0)$.

Case 3. $z \notin \{v_0, w\}$, $u \in V(L)\setminus F(w, \alpha)$. Since $\overrightarrow{(z, \nu(z))}$ is a Γ-arc, we have

$$f(\nu(z)) \in \Phi(z)\setminus f(\{z\} \cup (N_L(z)\setminus\{\nu(z), \nu^{-1}(z)\})).$$

Hence $f'(z) = f(\nu(z)) \neq f(u) = f'(u)$.

Case 4. $\{z, u\} \subset (V(L) \cap F(w, \alpha))\setminus\{v_0, w\}$. Due to Claim 10, either $z = \nu(u)$ or $u = \nu(z)$. Let $z = \nu(u)$. Then $f'(z) = f(\nu(z)) \neq f(z) = f'(u)$. □

From Claim 11 we obtain immediately the following.

Claim 12. *For any Γ-vertex $w \neq v_0$*

$$\Phi(w)\setminus\{f(w)\} \subset f(N_L(w)\setminus T_s(w)).$$

Taking Claim 2 into account, we have the following.

Claim 13. *For any Γ-vertex w and any $1 \leqslant j \leqslant s+1$*

$$O_j(w) \cap T_s(w) = \emptyset.$$

Claim 14. *For any Γ-vertex $w \neq v_0$ and any k $(Y(w) \leqslant k \leqslant s)$*

$$\left|\bigcup_{j=2}^{k+1} O_j(w)\right| \geqslant 1 + |T_k(w)|.$$

Proof. Let $w = \langle a, b\rangle$, $|T_k(w)| = t$. If $v_0 \in N_L(w)$, then by Claim 6 it belongs to $T_k(w)$.

Case 1. $T_k(w) \cap C(a, b) \neq \emptyset$ and $T_k(w) \cap C(b, a) \neq \emptyset$. Let $u \in N_L(w)\setminus T_k(w)$ and

$$f(u) \in \Phi(w)\setminus f(N_L(w)\setminus(\{u\} \cup T_k(w))). \tag{4}$$

Then, by (k 7), $u \in \bigcup_{j=2}^{k+1} Q_j(w)$, and in the case under consideration, by (k 8), u should belong to $\bigcup_{j=2}^{k+1} O_j(w)$. So, in view of Claim 13, the set $\bigcup_{j=2}^{k+1} O_j(w)$ consists of the vertices $u \in N_L(w)\setminus T_k(w)$ satisfying (4). Due to Claim 12, every colour of the Δ-element set $\Phi(w)\setminus\{f(w)\}$ belongs to $f(N_L(w)\setminus T_k(w))$. But $|N_L(w)\setminus T_k(w)| \leqslant 2(\Delta - 1) - t$. Hence at most $(2\Delta - 2 - t) - \Delta$ colours from $\Phi(w)$ occur in $N_L(w)\setminus T_k(w)$ twice. Thus, $|\bigcup_{j=2}^{k+1} O_j(w)| \geqslant \Delta - (\Delta - 2 - t) = t + 2$.

Case 2. $T_k(w) \cap C(a, b) \neq \emptyset$, $T_k(w) \cap C(b, a) = \emptyset$. Then $\bigcup_{j=2}^{k+1} O_j(w)$ consists of vertices $u \in C(b, a)\setminus T_k(w)$ such that (4) holds. Again at most $\Delta - 2 - t$ colours from $\Phi(w)$ occur twice in colouring of $N_L(w)\setminus T_k(w)$. Suppose, there are m such colours. Every one of them is used for colouring some vertex of $C(a, b)$.

Therefore, at most $(\Delta - 1) - t - m$ colours, which occur once in $N_L(w) \setminus T_k(w)$, are used for colouring $C(a, b)$. Consequently,

$$\left| \bigcup_{j=2}^{k+1} O_j(w) \right| \geq \Delta - m - (\Delta - 1 - t - m) = t + 1. \qquad \square$$

According to Claims 13 and 14, for any Γ-vertex w

$$|T_s(w)| \leq \Delta - 2.$$

Then Claim 14 implies the following.

Claim 15. *For any Γ-vertex $w \neq v_0$ and any $Y(w) \leq k \leq s$*

$$\frac{|\bigcup_{j=2}^{k+1} O_j(w)|}{|T_k(w)|} \geq \frac{1 + |T_k(w)|}{|T_k(w)|} \geq \frac{\Delta - 1}{\Delta - 2}.$$

6. Completion of the proof of Theorem 1

It will be proved that the construction of Γ does not stop if there are no s-free vertices.

Let $\mathcal{P}$ be the mapping defined in (k 3), u be some Γ-vertex and $(u, y) \in E(L)$. Denote by $\mathcal{P}_y^{-1}(u)$ the set of Γ_2-arcs $\overrightarrow{(v, w)}$ such that $\mathcal{P}(v, w) = u$ and the arc $\overrightarrow{(v, w)}$ is the last arc of some bicoloured oriented path, which contains y.

The number of Γ_2-arcs entering y does not exceed $2(\Delta - 1))$. At most Δ bicoloured oriented paths use Γ-arcs going out of y, since $|\Phi(y)| = \Delta + 1$. Besides, if some Γ-arc goes out of y, then less than $2(\Delta - 1)$ Γ_2-arcs enter y. So, $|\mathcal{P}_y^{-1}(u)| \leq 3(\Delta - 1)$. Hence for any Γ-vertex u

$$|\mathcal{P}^{-1}(u)| \leq 3(\Delta - 1)(2\Delta - 2) = 6(\Delta - 1)^2. \tag{5}$$

Denote by e_i, e_i' and e_i'' the numbers of Γ-arcs, Γ_1-arcs and Γ_2-arcs of level i, respectively. Then from (5) and (k 3) we obtain

$$\sum_{i=1}^{k} e_i'' \leq 6(\Delta - 1)^2 \sum_{i=0}^{k-q} |V_i| \quad \forall\, k \geq 1. \tag{6}$$

where V_i denotes the set of Γ-vertices of level i. Because of (k 4),

$$|V_i| \leq e_i' \quad \forall\, i \geq 1. \tag{7}$$

We have

$$\sum_{i=1}^{k+1} e_i' = \sum_{i=1}^{k+1} e_i - \sum_{i=1}^{k+1} e_i'' = \sum_{v \in \bigcup_{j=0}^{k} V_j} \sum_{i=1}^{k+1} |O_i(v)| - \sum_{i=1}^{k+1} e_i''.$$

Due to (6), (7) and Claim 15, the last expression is not less than

$$\sum_{v \in \bigcup_{j=0}^{k} V_j} \frac{\Delta - 1}{\Delta - 2} |T_k(v)| - 6(\Delta - 1)^2 \sum_{i=0}^{k-q+1} |V_i| \geq \frac{\Delta - 1}{\Delta - 2} \sum_{i=0}^{k} e_i' - 6(\Delta - 1)^2 \sum_{i=1}^{k-q+1} e_i'. \tag{8}$$

Denote $m_i := \sum_{j=1}^{i} e_i'$. By definition, $m_1 \geqslant 2(\Delta+1) - 2(\Delta-1) = 4$. We will show that

$$m_{k+1} \geqslant \frac{(q-1)(\Delta-1)}{q(\Delta-2)} m_k \quad \forall\, k \geqslant 1. \tag{9}$$

Set $m_i := 0$ for $i \leqslant 0$. Suppose that k is the smallest positive integer for which (9) is not proved. Due to (8),

$$m_{k+1} \geqslant \frac{\Delta-1}{\Delta-2} m_k - 6(\Delta-1)^2 m_{k-q+1}.$$

By the minimality of k,

$$m_{k-q+1} \leqslant m_k \left(\frac{(q-1)(\Delta-1)}{q(\Delta-2)}\right)^{1-q}.$$

Consequently,

$$m_{k+1} \geqslant \frac{(q-1)(\Delta-1)}{q(\Delta-2)} m_k + \left(\frac{\Delta-1}{\Delta-2} - \frac{(q-1)(\Delta-1)}{q(\Delta-2)} - 6(\Delta-1)^2\left(\frac{(q-1)(\Delta-1)}{q(\Delta-2)}\right)^{1-q}\right) m_k.$$

But

$$\frac{\Delta-1}{\Delta-2} - \frac{(q-1)(\Delta-1)}{q(\Delta-2)} - 6(\Delta-1)^2\left(\frac{(q-1)(\Delta-1)}{q(\Delta-2)}\right)^{1-q}$$
$$= \frac{\Delta-1}{\Delta-2}\left(\frac{1}{q} - 6(\Delta-1)^2\left(\frac{\Delta-2}{\Delta-1}\right)^q \left(1+\frac{1}{q-1}\right)^{q-1}\right)$$
$$\geqslant \frac{\Delta-1}{q(\Delta-2)}\left(1 - 17(\Delta-1)^2 q\left(\frac{\Delta-2}{\Delta-1}\right)^q\right).$$

Remark that the derivative of the function

$$\varphi(q) := q\left(\frac{\Delta-2}{\Delta-1}\right)^q$$

is negative while $q \geqslant \Delta - 1$. Hence for any $\Delta \geqslant 3$, $q \geqslant (\Delta-1)4(\ln(\Delta-1)+1.1)$ we have

$$1 - 17(\Delta-1)^2 q\left(\frac{\Delta-2}{\Delta-1}\right)^q$$
$$\geqslant 1 - 17(\Delta-1)^2(\Delta-1)4(\ln(\Delta-1)+1.1)\left(1-\frac{1}{\Delta-1}\right)^{(\Delta-1)4(\ln(\Delta-1)+1.1)}$$
$$\geqslant 1 - 17(\Delta-1)^3 4(\ln(\Delta-1)+1.1)\mathrm{e}^{-4(\ln(\Delta-1)+1.1)}$$
$$= 1 - \frac{17\cdot 4}{\Delta-1}(\ln(\Delta-1)+1.1)\mathrm{e}^{-4.4} > 1 - 68\mathrm{e}^{-4.4} > 0.$$

Since $q = \lfloor 0.5g(G) \rfloor \geqslant 4(\Delta - 1)(\ln(\Delta - 1) + 1.1)$, the last chain of inequalities yields (9). This implies endless construction of Γ, a contradiction. □

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