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The total chromatic number of any multigraph with maximum degree five is at most seven

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Abstract

The result announced in the title is proved. A new proof of the total 6-colorability of any multigraph with maximum degree 4 is also given.

1 Introduction

A *total coloring of a multigraph G with k colors* is a mapping $f: E(G) \cup V(G) \to \{1, \dots, k\}$ such that for any two adjacent or incident elements a, b of $E(G) \cup V(G)$, $f(a) \neq f(b)$. The minimum k such that there exists a total coloring of a multigraph G with k colors is called *the total chromatic number of G* and is denoted by $\chi_2(G)$ (to distinguish it from the chromatic number $\chi(G)$ and from the edge chromatic number $\chi_1(G)$ of G).

Vizing [16] and Behzad [1] conjectured that for any positive integer Δ and for each simple graph G with maximum degree Δ,

$$\chi_2(G) \leqslant \Delta + 2. \qquad (1)$$

The validity of (1) is known to be true for graphs in several wide families (see [3, 4]). Hind [6] and then Chetwynd and Häggkvist [5] proved that it is 'almost true', i.e. that $\chi_2(G) \leqslant \Delta + o(\Delta)$ for each simple graph G with maximum degree Δ. But the exact bound (1) was published only for $\Delta \leqslant 3$ [14, 15] and for $\Delta = 4$ [7]. The inequality (1) is not true if we replace simple graphs by multigraphs, since for each $\Delta \geqslant 1$, there exists a multigraph S (so-called Shannon's triangle) with maximum degree Δ and $\chi_1(G) = \lfloor 1.5\Delta \rfloor$. In [10] (cf. also [8]) it was proved that for any integer $\Delta \geqslant 4$, $\Delta \neq 5$

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and for each multigraph G with maximum degree Δ,

$$\chi_2(G) \leqslant \lfloor 1.5\Delta \rfloor. \tag{2}$$

Thus, the main result of the present paper announced in its title completes the total coloring analogue (2) of Shannon Theorem on edge coloring and proves the validity of Vizing–Behzad conjecture for one more value of Δ. Note that the result of this paper was proved in [9] but was not published because of size of the proof. The present proof is sufficiently shorter (though not quite short).

The paper structure is as follows. After introducing in Section 2 definitions and notation, in Section 3 a new proof of the following known result is given.

Theorem 1. *For each multigraph G with maximum degree at most four,*

$$\chi_2(G) \leqslant 6.$$

Then the idea of the proof of Theorem 1 is developed in Sections 4 and 5 and gives

Theorem 2. *For each 5-regular multigraph G having a perfect matching,*

$$\chi_2(G) \leqslant 7.$$

Finally, in Section 6 the general case is reduced to that described by Theorem 2.

2 Notation

For any multigraph G, let $V(G)$ (respectively, $E(G)$) denote the set of vertices (respectively, edges) of G. All the elements of the set $V(G) \cup E(G)$ are called *elements of G*. For $W \subseteq V(G)$ (respectively, $F \subseteq E(G)$), by $G(W)$ (respectively, $G(F)$) is denoted the subgraph of G induced by W (respectively, spanned by F). As usual, for multigraphs G and H, the multigraph $R = G \cap H$ (respectively, $Q = G \cup H$) has $V(R) = V(G) \cap V(H)$ and $E(R) = E(G) \cap E(H)$ (respectively, $V(Q) = V(G) \cup V(H)$ and $E(Q) = E(G) \cup E(H)$).

Although, for $v, w \in V(G)$, there can be several edges of G connecting v with w we will still use expression (v, w). It will mean 'some edge connecting v with w'. In particular, the record $H = G \cup \{(v, w)\}$ means that H is obtained from G by adding an additional edge connecting v with w.

For $V_1, V_2 \subseteq V(G)$, $V_1 \cap V_2 = \emptyset$, $v \in V(G)$, let $E_G(V_1, V_2)$ denote the set of edges of G, connecting V_1 with V_2, $\deg_G(v, V_1) = |E_G(\{v\}, V_1 \setminus \{v\})|$ and $\deg_G(v) = \deg_G(v, V(G) \setminus \{v\})$. In clear cases the subscript will be omitted. *The maximum degree* $\Delta(G)$ of a multigraph G is $\max\{\deg_G(v) \mid v \in V(G)\}$. If $V_2 = V(G) \setminus V_1$ then we will call $E_G(V_1, V_2)$ *a cut* of G and denote it by $[V_1, V_2]$.

A coloring of a multigraph G with k colors is an arbitrary mapping $f: E(G) \cup V(G) \to \{1, \dots, k\}$. If $f(a)$ is not defined for several elements $a \in V(G) \cup E(G)$, we say that f is a *partial coloring of G*. *A proper (total) coloring of* *a multigraph G by k colors* is a coloring such that for any two adjacent or incident elements $a, b \in E(G) \cup V(G)$, $f(a) \neq f(b)$. Let $\chi_2(G)$ denote the minimum k such that there exists a proper coloring of G with k colors.

3 Coloring multigraphs with maximum degree four

Lemma 1. *Let G be a 4-regular connected multigraph, $G \neq K_5$. Then there exists a cut $[V_1, V_2]$ of G such that*

(1) *the induced subgraphs $G_i := G(V_i)$, $i \in \{1,2\}$ have no cycles of length at least three;*
(2) $\Delta(G_i) \leqslant 2$, $i \in \{1,2\}$;
(3) *the multigraph $\tilde{G} = G \backslash (E(G_1) \cup E(G_2))$ is connected.*

Proof. Borodin [2] and Kronk and Mitchem [11] independently proved that there is a cut of G satisfying statement (1). Among cuts with this property choose a cut $[V_1, V_2]$ containing maximal number of edges. This is the desired cut. Indeed, if $v \in V_1$ and $\deg_G(v, V_1) \geqslant 3$, then the cut $[V', V''] = [V_1 \backslash \{v\}, V_2 \cup \{v\}]$ has more edges than $[V_1, V_2]$ and v does not belong to any cycle of $G(V'')$. So, $[V_1, V_2]$ satisfies (2).

Suppose that a cut $[W_1, W_2]$ of $\tilde{G}$ has no edges. Denote $A_{ij} = V_i \cap W_j$. Then the cut $[V', V'']$ with $V' = A_{11} \cup A_{22}$, $V'' = A_{12} \cup A_{21}$ has more edges than $[V_1, V_2]$ and still $G(V')$ and $G(V'')$ have no cycles. □

Lemma 2. *Let G be a 4-regular connected multigraph, $G \neq K_5$ and $[V_1, V_2]$ be a cut of G satisfying Lemma 1. Then there exists a partition of $E(G)$ into two 2-factors F_1 and F_2 such that*

$$\Delta(G(F_i) \cap G(V_i)) \leqslant 1, \quad i \in \{1,2\}. \tag{3}$$

Proof. By assumption every component of $G_i = G(V_i)$, $i = 1,2$ is a path or a 2-cycle. Replace each such a path P_j by an edge p_j connecting its ends, and each such a cycle (x_k, y_k) by a loop l_k at the vertex x_k. Due to Lemma 1(3), the resulting multigraph G' is connected and therefore eulerian. Let $L' = (e'_1, e'_2, \ldots, e'_s)$ be an eulerian trail of G'. Replacing in L' every edge p_j not belonging to G back by the path P_j, and each loop l_k by the cycle (x_k, y_k), we obtain an eulerian trail $L = (e_1, e_2, \ldots, e_n)$ of G on which the edges of each P_j and each (x_k, y_k) lie in consecutive order. As Petersen [13] observed, the subgraphs F_1 and F_2 consisting of the edges having odd and even numbers in L, respectively, are 2-factors of G. Because of tricks with P_j and (x_k, y_k), F_1 and F_2 are what we need. □

Let $[W_1, W_2]$ be a cut of a multigraph G. Say that a path $P = (x_0, x_1, \ldots, x_r)$ in G is *a $[W_i, W_{3-i}]$-path* ($i \in \{1,2\}$) if it satisfies the following two conditions:

(1) $r \geqslant 4$ and r is even;
(2) $V(P) \cap W_{3-i} = \{x_2, x_4, \ldots, x_{r-2}\}$.

Let $[W_1, W_2]$ be a cut of a multigraph G and f be a proper partial coloring of W_i in G. We say that f is (G, W_i)-*good* if for each $[W_i, W_{3-i}]$-path $P = (x_0, x_1, \ldots, x_r)$ in G at least one of the following conditions is satisfied:
(a) there are uncolored vertices in $V(P) \cap W_i$;
(b) $f(x_0) \neq f(x_r)$;
(c) $f(x_0) \in \{f(x_3), f(x_5), \ldots, f(x_{r-3})\}$.

Lemma 3. *Let C be a cycle of length at least three or a path and $[W_1, W_2]$ be a cut of C such that*

$$\Delta(C(W_1)) \leqslant 1. \tag{4}$$

Suppose that f is a (C, W_1)-good coloring of W_1 with colors 1, 2 and 3 and W'_1 is the colored part of W_1. Then we can extend f to a proper coloring of $W'_1 \cup E(C)$ with the same colors 1, 2 and 3.

Proof. Suppose that C is a minimal counterexample with respect to the number of edges. If $e = (v, w) \in E(C)$ and $\{v, w\} \cap W'_1 = \emptyset$, then after coloring the edges of $C \setminus e$ (which is possible because of the minimality of C) we could color e with a color in $\{1, 2, 3\}$ not used for coloring the edges incident with e. We can also color e after coloring other edges if C is a path, $e = (v, w)$ is its first edge, and $|\{v, w\} \cap W'_1| \leqslant 1$. Thus,

$$\{v, w\} \cap W'_1 \neq \emptyset; \tag{5}$$

$$(C \text{ is a path and } (v, w) \text{ is its first edge}) \Rightarrow \{v, w\} \subset W'_1. \tag{6}$$

Case 1. There is an edge $e = (v, w) \in E(C)$ with $\{v, w\} \subset W'_1$. If $\{v, w\} = V(C)$, then since C is not a 2-cycle, we simply color e with $\alpha \in \{1, 2, 3\} \setminus \{f(v), f(w)\}$. Otherwise, because of (5), (6) and the symmetry of v and w we may suppose that there are vertices $x \in W_2, y \in W'_1, z \in V(C)$ with $(w, x) \in E(C)$, $(x, y) \in E(C)$, $(y, z) \in E(C)$. Let, for definiteness, $f(v) = 1$, $f(w) = 2$. We ought to put $f((v, w)) := 3$, $f((w, x)) := 1$. If $f(y) = 1$ then we could temporarily delete edge (x, y) and color it at the end with the color $\beta \in \{2, 3\} \setminus \{f((y, z))\}$. Let $f(y) \in \{2, 3\}$. Then we ought to put $f((x, y)) \in \{2, 3\} \setminus \{f(y)\}$, $f((y, z)) := 1$.

If $z \in W'_1$ then (v, w, x, y, z) is a $[W_1, W_2]$-path and, according to (b) above, $f(z) \neq f(v) = 1$ should hold. So, in this case we could temporarily delete edge (x, y) and color it at the end with color $f(z)$. If $z \in W_1 \setminus W'_1$ then due to (5) and (6) there exists $u \in W'_1 \setminus \{y\}$ incident with z. But this contradicts (4).

Thus, $z \in W_2$. Then again by (5) and (6) there are vertices $p \in W'_1, q \in V(C)$ with $(z, p) \in E(C)$, $(p, q) \in E(C)$. We deal with them like we did with y and z, since $f((y, z))$ has to be 1. We proceed in this manner until either we meet two adjacent vertices in W_1, and then use the conditions (b) and (4), or we meet a vertex $r \in W_1$ with $f(r) = 1$ and deal with r as we did with y.

Case 2. For each edge (v, w) of C, $|\{v, w\} \cap W'_1| = 1$. Then by (5) and (6), C is an even cycle, and by (7), $W'_1 = W_1$. Let $C = (x_1, x_2, \ldots, x_{2k})$,. and $W'_1 = \{x_1, x_3, \ldots, x_{2k-1}\}$. If $f(x_1) = f(x_3) = \cdots = f(x_{2k-1})$ then we color the edges of C with two remaining colors. Assume that $f(x_{2k-1}) \neq f(x_1)$. Put $f((x_{2k}, x_1)) = f(x_{2k-1})$ and then for $j = 1, 2, \ldots, k-1$, choose $f((x_{2j-1}, x_{2j})) \in \{1,2,3\} \setminus \{f((x_{2j-2}, x_{2j-1})), f(x_{2j-1})\}$, $f((x_{2j}, x_{2j+1})) \in \{1,2,3\} \setminus \{f((x_{2j-1}, x_{2j})), f(x_{2j+1})\}$, and at last $f((x_{2k-1}, x_{2k})) \in \{1,2,3\} \setminus \{f((x_{2k-2}, x_{2k-1})), f(x_{2k-1})\}$. □

Proof of Theorem 1. Let G be a connected multigraph with $\Delta(G) \leqslant 4$ such that adding any edge violates the condition $\Delta(G) \leqslant 4$. By this choice, there is a vertex $s \in V(G)$ such that all the other vertices have degree 4 in G.

Construct a 4-regular multigraph H as follows. If $\deg_G(s) = 4$ then put $H = G$. Otherwise take a copy G' of G and connect s with its copy s' in G' by $4 - \deg_G(s)$ edges. If $H = K_5$, then $\chi_2(H) = 5$.

Let $H \neq K_5$ and $[V_1, V_2]$ be a cut of H possessing the properties (1)–(3) of Lemma 1. Choose disjoint 2-factors F_1 and F_2 satisfying (4). We are going to color for $i = 1, 2$ the vertices in V_i and the edges in F_i with colors $3i - 2$, $3i - 1$ and $3i$.

In order to construct a $(H(F_1), V_1)$-good coloring of V_1, we construct a multigraph $\tilde{H}_1$ by adding some edges to $H_1 := H(V_1)$ in the following way. We add the edge (v, w) if and only if the following condition holds:

$$\text{one of the paths connecting in } F_1 \text{ vertices } v \text{ and } w \text{ is a } [V_1, V_2]\text{-path}. \tag{7}$$

Since for each $v \in V_1$ there is at most one vertex w such that v and w satisfy (7), we added to H_1 some matching. But H_1 is a forest (up to 2-cycles which do not affect vertex coloring). Hence, $\tilde{H}_1$ is strictly 3-degenerate (i.e. every subgraph has a vertex of degree less than 3) and so 3-colorable. Fix any 3-coloring f of $\tilde{H}_1$ with colors 1, 2 and 3. The edges added to H_1 provide for the cycles in F_1 of length at least three the conditions of Lemma 3 and hence a proper 3-coloring of their edges. But if (x, y) is a 2-cycle in F_1, then, by (4), $|\{x, y\} \cap V_1| \leqslant 1$ and we have two free colors in $\{1, 2, 3\}$ to color the edges of this 2-cycle. Thus, the edges in F_1 are colored.

We deal analogously with the vertices in V_2 and the edges in F_2. □

4 Analogues of Lemmas 1 and 2

Throughout this Section, G is a 5-regular multigraph having a perfect matching, and π is some such matching. If we find a cut $[V_1, V_2]$ of G possessing the properties (1)–(3) of Lemma 1 then we would color the edges in π by color 7, and the rest of G - as in the proof of Theorem 1. But we can only use the following weaker statement.

Lemma 4. *Let G be a 5-regular multigraph having a perfect matching. Then there exist a perfect matching π and a cut $[V_1, V_2]$ of G such that denoting $G_i := G(V_i)$ we have*

(1) $\Delta(G_i) \leqslant 2$, $i \in \{1, 2\}$;

(2) *the multigraph* $\tilde{G}_\pi := (G \setminus \pi) \setminus (E(G_1) \cup E(G_2))$ *has the same number of components as* $G \setminus \pi$;

(3) *if* $(v_1, v_2) \in E(G)$, $v_i \in V_i$, *and* $\deg_{G_i \setminus \pi}(v_i) = 2$ *for* $i \in \{1,2\}$ *then* $(v_1, v_2) \in \pi$.

Proof. Choose π and $[V_1, V_2]$ so as to maximize $|[V_1, V_2] \setminus \pi|$, and among the triples V_1, V_2, π with the maximum value of $|[V_1, V_2] \setminus \pi|$ choose a triple with the maximum possible $|[V_1, V_2]|$. We prove that this is a desired triple.

If $v \in V_1$ and $\deg_G(v, V_1) \geqslant 3$, then the cut $[V', V''] = [V_1 \setminus \{v\}, V_2 \cup \{v\}]$ has more edges than $[V_1, V_2]$ and $|[V', V''] \setminus \pi| \geqslant |[V_1, V_2] \setminus \pi|$, a contradiction. This proves (1).

Assume that for a component H of $G \setminus \pi$, the multigraph $H \cap \tilde{G}_\pi$ is not connected. Let a cut $[W_1, W_2]$ of $\tilde{G}_\pi$ have no edges and $V(H) \cap W_i \neq \emptyset$, $i = 1, 2$. Denote $A_{ij} = V_i \cap W_j$. Then the cut $[V', V'']$ with $V' = A_{11} \cup A_{22}$, $V'' = A_{12} \cup A_{21}$ contains more edges of $G \setminus \pi$ than $[V_1, V_2]$. A contradiction to the choice of π, V_1 and V_2 proves (2).

Now, if $(v_1, v_2) \in E(G) \setminus \pi$, $v_i \in V_i$, $\deg_{G_i \setminus \pi}(v_i) = 2$ for $i \in \{1,2\}$ then denoting $V_i' := (V_i \setminus \{v_i\}) \cup \{v_{3-i}\}$ we obtain

$$|[V', V''] \setminus \pi| = |[V_1, V_2] \setminus \pi| + 2,$$

a contradiction. □

The presence of cycles in G_1 and G_2 complicates the situation seriously and we need several tricks.

Let π and $[V_1, V_2]$ be chosen so that they satisfy (1)–(3) above. Say that a 4-cycle $C \subseteq G(V_1) \cup G(V_2)$ is *j-bad*, $j \in \{0, 1, 2\}$ if exactly j of its edges belong to π. The path of length 3 obtained from a 1-bad 4-cycle by deleting its edges belonging to π will be called *a bad path.* Also any odd cycle $C \subseteq (G(V_1) \cup G(V_2)) \setminus \pi$ is *a bad cycle.* Other paths and cycles are not so bad.

Let H be a component of $G \setminus \pi$. If the total number of bad paths and odd bad cycles in H is odd then we mark one of these paths or cycles. Thus,

(i) the total number of non-marked bad paths and odd bad cycles in each component of $G \setminus \pi$ is even;

(ii) the total number of marked paths and cycles in each component of $G \setminus \pi$ is at most one.

The analogue of Lemma 2 is the following statement.

Lemma 5. *Let G be a 5-regular multigraph and a perfect matching* π *and a cut* $[V_1, V_2]$ *of G satisfy statements of Lemma* 4. *Then there exists a partition* (F_1, F_2, F_3) *of* $E(G) \setminus \pi$ *such that denoting* $G_i := G(V_i)$ *we have*

(1) $\Delta(G(F_i)) \leqslant 2$, $i \in \{1, 2, 3\}$;

(2) $\Delta(G(F_i) \cap G_i) \leqslant 1$, $i \in \{1, 2\}$;

(3) *the edges in* F_3 *are exactly edges of non-marked bad paths, non-marked bad odd cycles and of those* 0-*bad* 4-*cycles* $C = (v_1, v_2, v_3, v_4) \subseteq G_i$, $i \in \{1, 2\}$ *for which there exist* $x, y \in V(G) \setminus V(C)$ *such that* $F_i \supseteq \{(x, v_1), (x, v_3), (y, v_2), (y, v_4)\}$;

(4) *for each vertex v of any cycle in* F_3,

$$\deg_{F_1}(v) = \deg_{F_2}(v) = 1\,;$$

(5) *for each bad path* $P = (v_1, v_2, v_3, v_4)$,

$$\deg_{F_i}(v_2) = \deg_{F_i}(v_3) = 1, \quad i \in \{1,2\}\,,$$

$$\deg_{F_i}(v_1) = \deg_{F_{3-i}}(v_4) \in \{1,2\}, \quad i \in \{1,2\}\,;$$

(6) *for each* 2-*bad* 4-*cycle* (v_1, v_2, v_3, v_4) *with* $(v_1, v_4), (v_2, v_3) \in \pi$ *such that* v_1 *and* v_3 *lie in the same component of* $G \setminus \pi$, *edges* (v_1, v_2) *and* (v_3, v_4) *belong to different* F_i, $i \in \{1,2\}$;

(7) *if* $P = (v_1, v_2, v_3, v_4) \subseteq G_i$, $i \in \{1,2\}$ *is a marked bad path, then* $E(P) \cap F_i = \{(v_2, v_3)\}$;

(8) *if* $C = (v_1, \ldots, v_{2t+1}) \subseteq G_i$, $i \in \{1,2\}$ *is a marked bad odd cycle, then* $E(C) \cap F_i = \{(v_{2j}, v_{2j+1}) \mid j = 1, \ldots, t\}$.

Proof. We will construct a partition (F_1, F_2, F_3) of edges in each component H of $G \setminus \pi$ and then just take the union of them. Probably, π will be changed, but the vertex-sets of components of $G \setminus \pi$ will not be changed and Lemma 4 (1)–(3) will hold all the time.

So, let H be a component of $G \setminus \pi$, $H^+ := H \cap (G_1 \cup G_2)$, $H^- := H \setminus E(H^+)$, and let $\Pi(H)_i$ denote the set of quadruples $(v_1, v_2, v_3, v_4) \subseteq V_i \cap V(H)$, $i \in \{1,2\}$ such that $(v_1, v_4), (v_2, v_3) \in \pi$ and $(v_1, v_2), (v_3, v_4) \in E(H)$.

Let $W_1, \ldots, W_s$ be all quadruples in $\Pi(H)_1 \cup \Pi(H)_2$, and $W_{s+1}, \ldots, W_m$ be the vertex-sets of components of $H^+ \setminus \bigcup_{j=1}^{s} W_j$. We start from constructing an auxiliary multigraph (probably with one loop) H_m as follows.

Set $H_0 := H^-$. Due to Lemma 4(2), H_0 is connected. Now, for $r = 1, \ldots, m$, apply the following procedure.

Procedure 1. (1) If $1 \leqslant r \leqslant s$, $W_r = \{v_1, v_2, v_3, v_4\}$, (v_1, v_4), $(v_2, v_3) \in \pi$ then put $V(H_r) := V(H_{r-1}) \cup \{x_{r,1}, x_{r,2}\}$, $E(H_r) := E(H_{r-1}) \cup \{(x_{r,j}, v_j), (x_{r,j}, v_{j+2}) \mid j = 1, 2\}$.

(2) If $|W_r| \in \{1,2\}$ then $H_r := H_{r-1}$.

(3) If $H(W_r)$ is the marked bad path $P = (v_1, v_2, v_3, v_4)$ then $H_r := H_{r-1} \cup \{(v_1, v_4)\}$.

(4) If $H(W_r)$ is a non-marked bad path $P = (v_1, v_2, v_3, v_4)$ then $V(H_r) := V(H_{r-1}) \cup \{x_r\}$, $E(H_r) := E(H_{r-1}) \cup \{(x_r, v_1), (x_r, v_4)\}$.

(5) If $H(W_r)$ is a not so bad path $P = (v_1, \ldots, v_t)$, $t \geqslant 3$, then put $V(H_r) := V(H_{r-1}) \cup \{x_{r,2}, \ldots, x_{r,t-1}\}$, $E(H_r) := E(H_{r-1}) \cup \{(x_{r,j}, x_{r,j+1}) \mid j = 2, \ldots, t-2\} \cup \{(x_{r,2}, v_1), (v_{r,t-1}, v_t)\}$.

(6) If $H(W_r)$ is the marked bad odd cycle $C = (v_1, \ldots, v_{2t+1})$ then H_r is obtained from H_{r-1} by adding a loop at v_1.

(7) If $H(W_r)$ is a non-marked cycle then $H_r := H_{r-1}$.

By Construction, the degree of each vertex in H_m is even. Let us see that

$$|E(H)| = |E(H_m)| \pmod 2.$$

Denote $e_1(W_r) := |E(H(W_r))|$, $e_2(W_r) := |E(H_r) \setminus E(H_{r-1})|$. Then

$$|E(H)| = |E(H_m)| = |E(H^+)| - \sum_{r=1}^{m} e_2(W_r) = \sum_{r=1}^{m} (e_1(W_r) - e_2(W_r)).$$

The difference $e_1(W_r) - e_2(W_r)$ is odd only if $H(W_r)$ is a non-marked bad odd cycle or a bad path, but due to (i) above, the number of such cases is even.

Since H is 4-regular, $|E(H)|$ is even, and whence $|E(H_m)|$ is even. Let Λ be an eulerian trail in H_m, and F_1' and F_2' be the sets of edges with odd and even numbers in Λ respectively. By construction, for each $v \in V(H_m)$,

$$\deg_{F_1'}(v) = \deg_{F_2'}(v). \tag{8}$$

Because of symmetry of F_1' and F_2', we can assume

(a) if $P = (v_1, v_2, v_3, v_4) \subseteq G_i \cap H$ is the marked bad path then edge $(v_1, v_4) \in E(H_m) \setminus E(H)$ belongs to F_{3-i};

(b) if $C = (v_1, \dots, v_{2t+1}) \subseteq G_i \cap H$ is the marked bad odd cycle then the loop at v_1 belongs to F_{3-i}.

Now we construct desired F_1, F_2 and F_3. For $r = 1, \dots, m$ we run the following Procedure 2, which starts with $F_3 := \emptyset$, $F_1 := F_1'$, $F_2 := F_2'$.

Procedure 2. (1) Let $1 \leqslant r \leqslant s$, $W_r = \{v_1, v_2, v_3, v_4\}$, $(x_{r,1}, v_1) \in F_1$, $(x_{r,1}, v_3) \in F_2$. If $(x_{r,2}, v_2) \in F_1$, then put $F_1 := (F_1 \setminus \{(x_{r,2}, v_2), (x_{r,1}, v_1)\}) \cup \{(v_1, v_2)\}$, $F_2 := (F_2 \setminus \{(x_{r,2}, v_4), (x_{r,1}, v_3)\}) \cup \{(v_3, v_4)\}$, $F_3 := F_3$.

If $(x_{r,2}, v_2) \in F_2$, then we change π: $\pi := (\pi \setminus \{(v_1, v_4), (v_2, v_3)\}) \cup \{(v_1, v_2), (v_3, v_4)\}$ and put $F_1 := (F_1 \setminus \{(x_{r,2}, v_2), (x_{r,1}, v_1)\}) \cup \{(v_1, v_4)\}$, $F_2 := (F_2 \setminus \{(x_{r,2}, v_4), (x_{r,1}, v_3)\}) \cup \{(v_2, v_3)\}$, $F_3 := F_3$. Note that because of Lemma 4(2) the change of π preserves the vertex-set of H and the properties (1)–(3) of Lemma 4.

By (1), above after step s, property (6) of Procedure 1 will be fulfilled.

(2) If $|W_r| \in \{1, 2\}$ then $F_i := F_i$, $i = 1, 2, 3$.

(3) Let $H(W_r)$ be the marked bad path $P = (v_1, v_2, v_3, v_4) \subseteq G_i$, $i \in \{1, 2\}$. Put $F_i := F_i \cup \{(v_2, v_3)\}$, $F_{3-i} := (F_{3-i} \setminus \{(v_1, v_4)\}) \cup \{(v_1, v_2), (v_3, v_4)\}$, $F_3 := F_3$.

Due to (8) and (a) above, after this step we will keep (8) and Lemma 5(7) will be fulfilled.

(4) If $H(W_r)$ is a non-marked bad path P then $F_1 := F_1$, $F_2 := F_2$, $F_3 := F_3 \cup E(P)$.

(5) If $H(W_r)$ is a not so bad path $P = (v_1, \dots, v_t)$, $t \geqslant 3$, and $(v_1, x_2) \in F_i$ then put $F_i := F_i \cup \{(v_{2j-1}, v_{2j}) \mid 1 \leqslant j \leqslant t/2\}$, $F_{3-i} := F_{3-i} \cup \{(v_{2j}, v_{2j+1}) \mid 1 \leqslant j \leqslant (t-1)/2\}$, $F_3 := F_3$.

(6) If $H(W_r)$ is the marked bad odd cycle $C = (v_1, \dots, v_{2t+1}) \subseteq G_i$, $i \in \{1, 2\}$ then put $F_i := F_i \cup \{(v_{2j}, v_{2j+1}) \mid 1 \leqslant j \leqslant t\}$, $F_{3-i} := F_{3-i} \cup \{(v_{2j-1}, v_{2j}) \mid 1 \leqslant j \leqslant t\} \cup \{(v_{2t+1}, v_1)\}$, $F_3 := F_3$.

Due to (8) and (b) above, after this step we will still keep (8) and Lemma 5(8) will be fulfilled.

(7) If $H(W_r)$ is a non-marked bad odd cycle C then $F_1 := F_1$, $F_2 := F_2$, $F_3 := F_3 \cup E(C)$.

(8) If $H(W_r)$ is a 4-cycle $C=(v_1,v_2,v_3,v_4)\subseteq G_i$, $i\in\{1,2\}$, and there exist $x,y\in V(G)\setminus V(C)$ such that $F_i\supseteq\{(x,v_1),(x,v_3),(y,v_2),(y,v_4)\}$ then $F_1:=F_1$, $F_2:=F_2$, $F_3:=F_3\cup E(C)$.

(9) If $H(W_r)$ is an even cycle $C=(v_1,\ldots,v_{2t})$ and C does not satisfy conditions (8) then put $F_1:=F_1\cup\{(v_{2j-1},v_{2j})\mid 1\leqslant j\leqslant t\}$, $F_2:=F_2\cup\{(v_{2j},v_{2j+1})\mid 1\leqslant j\leqslant t-1\}\cup\{(v_{2t},v_1)\}$, $F_3:=F_3$.

After all steps have been made, statement (1) of Lemma 5 follows from the fact that we kept equality (8) obtained in Procedure 1 on every step. Since any component of $(G(V_1)\cup G(V_2))\cap H$ is contained in some $H(W_r)$, it is easy to check Lemma 5(2) on each step. Statement (3) of Lemma 5 holds because F_3 changed only in cases (4), (8) and (9). Now Lemma 5(4) and (5) follow from (8) and Procedure 1(4). The validity of (6), (7) and (8) have been noted in the description of Procedure 2. □

5 Proof of Theorem 2

Let π, $[V_1,V_2]$ and F_1, F_2 and F_3 satisfy Lemma 5 and $V_3=\{v\in V(G)\mid \deg_{G(F_3)}(v)\geqslant 1\}$. For each $v\in V(G)$ with $\deg_{G(F_i)}(v)=1$, $i\in\{1,2\}$, let $e_i(v)$ denote the edge in F_i incident to v. Due to Lemma 5(5) we may choose the direction of each bad path $P=(v_1,v_2,v_3,v_4)\subseteq G_i\cap G(F_3)$ so that

$$\deg_{F_{3-i}}(v_1)=\deg_{F_i}(v_4)=1\,. \tag{9}$$

The coloring will be made in 6 stages. First we list them.

Stage 1. Color the edges in π with color 7.

Stage 2. Find a proper coloring f of $F_1\cup F_2$ and $V\setminus V_3$ such that for $i=1,2$,

(1) the elements of $F_i\cup(V_i\setminus V_3)$ are colored with colors $3i-2$, $3i-1$ and $3i$;

(2) for each odd cycle $C=(v_1,\ldots,v_{2t+1})\subseteq G(F_3)\cap G_{3-i}$,

$$f(e_i(v_1))\neq f(e_i(v_3));$$

(3) for each bad path $P=(v_1,v_2,v_3,v_4)\subseteq G(F_3)\cap G_{3-i}$,

$$f(e_i(v_1))\neq f(e_i(v_3));$$

(4) for each 4-cycle $C=(v_1,v_2,v_3,v_4)\subseteq G(F_3)\cap G_{3-i}$,

$$f(e_i(v_1))\neq f(e_i(v_3)),\quad f(e_i(v_2))\neq f(e_i(v_4)).$$

Stage 3. For each odd cycle $C=(v_1,\ldots,v_{2t+1})\subseteq G(F_3)\cap G_i$, color properly $V(C)$ and edges $(v_3,v_4),(v_5,v_6),\ldots,(v_{2t+1},v_1)$ with colors $3i-2$, $3i-1$ and $3i$ with recoloring the set $\{e_i(v_j)\mid j=1,\ldots,2t+1\}$.

Stage 4. For each bad path $P=(v_1,v_2,v_3,v_4)\subseteq G(F_3)\cap G_i$, color properly $V(P)$ and edge (v_3,v_4) with colors $3i-2$, $3i-1$ and $3i$ with recoloring the set $\{e_i(v_j)\mid j=1,2,3,4\}$.

Stage 5. For each odd cycle $C = (v_1, \ldots, v_{2t+1}) \subseteq G(F_3) \cap G_i$ and for each bad path $P = (v_1, v_2, v_3, v_4) \subseteq G(F_3) \cap G_i$, color properly uncolored edges with colors $3(3-i)-2$, $3(3-i)-1$ and $3(3-i)$.

Stage 6. Color the vertices and the edges of 4-cycles in $G(F_3)$.

Stage 1 is trivial. The work on other stages will be described for $i = 1$; for $i = 2$ it is symmetrical.

The most complicated stage is Stage 2. It needs two auxiliary multigraphs, R_1 and S_1. First put $R_1 := G(F_1)$. By Lemma 5(1) and (2),

$$\Delta(R_1) \leqslant 2, \qquad \Delta(R_1(V_1 \setminus V_3)) \leqslant 1. \tag{10}$$

For a multigraph L and $x, y \in V(L)$, denote by $\Psi(L, x, y)$ the multigraph obtained from L by shrinking x and y into one vertex.

Let $C = (v_1, \ldots, v_{2t+1})$ be an odd cycle in $G(F_3) \cap G_2$. Due to Lemma 5(4), after putting, $R_1 := \Psi(R_1, v_1, v_3)$ the multigraph R_1 still satisfies (10). Do this for each odd cycle $C \subseteq G(F_3) \cap G_2$. Now let $P = (v_1, v_2, v_3, v_4)$ be a bad path in $G(F_3)) \cap G_2$. According to (9), $R_1 := \Psi(R_1, v_1, v_3)$ still satisfies (10). Do this for each bad path $P \subseteq G(F_3) \cap G_2$. Then for each 4-cycle $C = (v_1, v_2, v_3, v_4) \subseteq G(F_3) \cap G_2$, we put $R_1 := \Psi(\Psi(R_1, v_1, v_3)), v_2, v_4)$ and again by Lemma 5(4) R_1 satisfies (10). Now R_1 is constructed.

Note that by construction of R_1, any proper coloring of its edges induces a proper coloring of edges in F_1 satisfying stage 2(2)–(4). Let $W_1 := V_1 \setminus V_3$, $W_2 := V(R_1) \setminus W_1$. In order to find an (R_1, W_1)-good 3-coloring of W_1, we construct multigraph S_1 by adding to $R_1(W_1)$ the set of edges (v, w) such that in R_1 there is a $[W_1, W_2]$-path connecting vertices v and w. Because of (10), any vertex $v \in W_1$ is connected by a $[W_1, W_2]$-path with at most one vertex. Hence, the set of added edges forms a matching, and $\Delta(S_1) \leqslant \Delta(R_1) + 1 \leqslant 3$. Let W_1' denote the union of the vertex-sets of components of S_1 isomorphic to K_4. By Brooks Theorem, there exists a proper coloring φ of vertices in $S_1 \setminus W_1'$.

Put for each $v \in W_1 \setminus W_1'$, $f(v) := \varphi(v)$. Then we need to color W_1'. Let

$$\{v_1, v_2, v_3, v_4\} \subseteq W_1' \subseteq V_1 \setminus V_3 \tag{11}$$

and $S_1(\{v_1, v_2, v_3, v_4\}) = K_4$. The only possibility (up to renumbering the vertices) for this situation to arise is that $C := R_1(\{v_1, v_2, v_3, v_4\})$ is a 4-cycle and (v_1, v_3), (v_2, v_4) are added while constructing S_1. In other words, there are paths $P_1 = (v_1, x_1, x_2, \ldots, x_{2k-1}, v_3)$ and $P_2 = (v_2, y_1, y_2, \ldots, y_{2l-1}, v_4)$ in R_1 such that

$$(V(P_1) \cup V(P_2)) \cap W_1' = \{x_2, x_4, \ldots, x_{2k-2}, y_2, y_4, \ldots, y_{2l-2}\}.$$

Since $V(C) \subseteq V_1$, C is a bad 4-cycle.

Case 1. C is a 2-bad 4-cycle, (v_1, v_4), $(v_2, v_3) \in \pi$. Since v_1 and v_3 are connected by P_1 in R_1, they lie in the same component of $G \setminus \pi$. By Lemma 5(6), one of the edges (v_1, v_2) and (v_3, v_4) (say, (v_1, v_2)) belongs to F_2. Since $(v_1, x_1) \in F_1$ we obtain $x_1 \neq v_2, v_4$ and $\deg_{G_1}(v_1) \geqslant 3$. But this contradicts Lemma 4(1).

Case 2. C is a 1-bad 4-cycle, $(v_1, v_4) \in \pi$. If (v_1, v_2, v_3, v_4) is marked bad path in H then by Lemma 5 (7), $(v_1, v_2), (v_3, v_4) \in F_2$ and we have $\deg_{G_1}(v_1) \geqslant 3$ for the same reasons as in Case 1. And if (v_1, v_2, v_3, v_4) is non-marked then, by definition, $V(C) \subseteq V_3$, a contradiction to (11).

Case 3. C is a 0-bad 4-cycle, $E(C) \cap F_i = \{(v_i, v_{i+1}), (v_{i+2}, v_{i+3})\}$, $i = 1, 2$. Then $x_1 = v_2, y_1 = v_1, x_{2k-1} = v_4, y_{2l-1} = v_3$. Suppose that $k \geqslant 3$. Then by definition, x_3 is not an end-vertex of a $[W_1, W_2]$-path in R_1. Hence, $x_3 \in W_1 \setminus W'_1$ and is already colored. Let for definiteness, $f(x_3) = 1$. We put $f(v_3) = f(v_1) = 1$, $f(v_2) = 2$, $f(v_4) = 3$. If $l \geqslant 3$ we do analogously.

Let $k = l = 2$, i.e. let $(v_1, v_2, x_2, v_4, v_3, y_2)$ be a cycle in R_1. Show that $x_2 \in V_2$. Suppose to the contrary that x_2 is the result of shrinking some z_1 and z_2 which lay before on some bad cycle or path in $G(F_3) \cap G_2$. Then at least one of them, say z_2, has degree 2 in $G_2 \setminus \pi$. By Lemma 4(3), z_2 should not be adjacent in $G \setminus \pi$ to vertices of C, but either v_1 or v_3 is adjacent to z_2 in F_1. Consequently, $x_2 \in V_2$. Analogously, $y_2 \in V_2$. Then $V(C) \subseteq V_3$, a contradiction to (11).

Thus, all vertices in $V_1 \setminus V_3$ are colored and, due to the construction of S_1, the resulting coloring is (R_1, W_1)-good. Applying Lemma 3 finishes this stage. After this stage has been completed for $i = 1, 2$, go to Stage 3.

Let $C = (v_1, \ldots, v_{2t+1})$ be an odd cycle in $G(F_3) \cap G_1$. For $j = 1, \ldots, 2t + 1$, denote by $e'_1(v_j)$ the edge in $E(R_1)$ adjacent to $e_1(v_j)$ (if such edge exists). Now, **Step** j, $j = 1, \ldots, t$ is as follows (assuming $v_{2t+2} = v_1$).

(1) If $e'_1(v_{2j+1}) \neq e_1(v_{2j+2})$ then choose $f(v_{2j+1}) \in \{1,2,3\} \setminus \{f(v_{2j}), f(e'_1(v_{2j+2}))\}$, $f(v_{2j+2}) \in \{1,2,3\} \setminus \{f(v_{2j+1}), f(e'_1(v_{2j+1}))\}$.

(2) If $e'_1(v_{2j+1}) = e_1(v_{2j+2})$ then choose $f(v_{2j+1}) \in \{1,2,3\} \setminus \{f(v_{2j})\}$, $f(v_{2j+2}) = f(v_{2j})$.

(3) In both cases put $f(e_1(v_{2j+1})) = f(v_{2j+2})$, $f(e_1(v_{2j+2})) = f(v_{2j+1})$, $f((v_{2j+1}, v_{2j+2})) \in \{1,2,3\} \setminus \{f(v_{2j+1}), f(v_{2j+2})\}$.

After Step t choose $f(v_2) \in \{1,2,3\} \setminus \{f(v_1), f(v_3)\}$. Note that after recoloring the edges $e_1(v_j)$ the inequalities (2)–(4) of Stage 2 still hold, since we obtain a proper edge coloring of R_1. We do this for each odd cycle in $G(F_3) \cap G_1$, then for each odd cycle in $G(F_3) \cap G_2$ and go to Stage 4.

Let $P = (v_1, v_2, v_3, v_4)$ be a bad path in $G(F_3) \cap G_1$. Choose $f(v_1)$ in $\{1,2,3\}$ distinct from the colors of the edges in F_1 incident with v_1. Then choose $f(v_4) \in \{1,2,3\} \setminus \{f(v_1), f(e'_1(v_3))\}$, $f(v_3) \in \{1,2,3\} \setminus \{f(v_4), f(e'_1(v_4))\}$, $f(e_1(v_4)) = f(v_3)$, $f(e_1(v_3)) = f(v_4)$, $f((v_3, v_4)) \in \{1,2,3\} \setminus \{f(v_3), f(v_4)\}$, $f(v_2) \in \{1,2,3\} \setminus \{f(v_1), f(v_3)\}$. After fulfillment of the described procedure for each bad path in $G(F_3)$ go to Stage 5.

Let $P = (v_1, v_2, v_3, v_4)$ be a bad path in $G(F_3) \cap G_1$. Because of (3) of Stage 2, it is possible to choose $f((v_1, v_2))$ and $f((v_2, v_3))$ from $\{4,5,6\} \setminus \{f(e_2(v_2))\}$ so that $f((v_1, v_2)) \neq f((v_2, v_3))$, $f((v_1, v_2)) \neq f(e_2(v_1))$, $f((v_2, v_3)) \neq f(e_2(v_3))$. Now, all vertices and edges of P are colored.

Let $C = (v_1, \ldots, v_{2t+1})$ be an odd cycle in $G(F_3) \cap G_1$. Because of (2) of Stage 2, it is possible to choose $f((v_1, v_2)), f((v_2, v_3))$ with the same properties as it was made for bad paths. Further, for $j = 2, 3, \dots, t$, choose $f((v_{2j}, v_{2j+1})) \in \{4,5,6\} \setminus \{f(e_2(v_{2j}), f(e_2(v_{2j+1}))\}$. Since there were no recolorings, (3) and (4) of Stage 2 still hold. We do this for each bad path and for each bad odd cycle in $G(F_3)$ and go to Stage 6.

Let $C = (v_1, v_2, v_3, v_4)$ be a 4-cycle in $G(F_3) \cap G_1$, and $x, y \in V(G) \setminus V(C)$ be such that $e_1(v_1) = (x, v_1)$, $e_1(v_3) = (x, v_3)$, $e_1(v_2) = (y, v_2)$, $e_1(v_4) = (y, v_4)$. If $x \notin V_3$ then colors 1, 2 and 3 were not used to color the edges incident with x distinct from $e_1(v_1)$ and $e_1(v_3)$. If $x \in V_3$ then due to Lemma 5(4) and (5), x is an end of a bad path in $G(F_3) \cap G_2$ and again colors 1, 2 and 3 were not used to color the edges incident with x distinct from $e_1(v_1)$ and $e_1(v_3)$. That means that recoloring $e_1(v_1)$ and $e_1(v_3)$ with colors 1, 2 and 3 would not affect coloring 'outside' our configuration. The same is true for y and e_2 and e_4.

Because of (4) Stage 2 there are two possible cases (up to renumbering vertices and/or colors):

(1) $f(e_2(v_1)) = f(e_2(v_2)) = 4$, $f(e_2(v_3)) = f(e_2(v_4)) = 5$;

(2) $f(e_2(v_1)) = f(e_2(v_2)) = 4$, $f(e_2(v_3)) = 5$, $f(e_2(v_4)) = 6$.

In both cases put $f((v_4, v_1)) = f((v_2, y)) = f(v_3, x)) = 1$, $f(v_1) = f(v_3) = f((v_4, y)) = 2$, $f(v_2) = f(v_4) = f((v_1, x)) = 3$, $f((v_3, v_4)) = 4$, $f((v_1, v_2)) = 5$, $f((v_2, v_3)) = 6$. □

6 General case

Assume that there exists a multigraph G with $\Delta(G) \leqslant 5$ and $\chi_2(G) \geqslant 8$. We can choose such G with the smallest possible number of vertices. Lemmas below list some properties of G.

Lemma 6. *G has no cut-vertices.*

Lemma 7. *Let $[V_1, V_2]$ be a cut of G and its edges be (v_1, w_1), (v_2, w_2), (v_3, w_3), where $\{v_1, v_2, v_3\} \subseteq V_1$, $\{w_1, w_2, w_3\} \subseteq V_2$ (some v_i-s and/or w_j-s can coincide). Suppose that there exists a 7-coloring f of G which is proper except that possibly $f(v_i) = f(w_i)$ for several $i \in \{1,2,3\}$. If*

$$f(v_j) = f(w_j) \Rightarrow f(v_j) \notin \{f((v_i, w_i)) \mid i \in \{1,2,3\}\}, \tag{12}$$

then there exists a proper 7-coloring of G.

Proof. Assume that there exists a 7-coloring f of G satisfying the conditions of the lemma.

Let j be the smallest index such that $f(v_j) = f(w_j)$. For $i \in \{1,2,3\}$, $i \neq j$, let

$$\varphi(i, j) = \begin{cases} f(v_i) & \text{if } f(v_i) \neq f(v_j), \\ f(w_i) & \text{if } f(v_i) = f(v_j). \end{cases}$$

Choose

$$\psi(j) \in \{1, \dots, 7\} \setminus (\{f(v_j), \varphi(j-1, j), \varphi(j+1, j)\} \cup \{f((v_k, w_k)) \mid 1 \leqslant k \leqslant 3\}), \tag{13}$$

where indices are taken modulo 3. Consider f' obtained from f by switching in $f(G(V_2))$ the colors $f(v_j)$ and $\psi(j)$. By the choice of $\psi(j)$ and by (12), f' also satisfies the conditions of our lemma and $f(v_i) \neq f(w_i)$ for $1 \leqslant i \leqslant j-1$. Moreover, because of (12) and (13) we have $f'(w_j) = \psi(j) \neq f(v_j) = f'(v_j)$.

Repeating this procedure at most three times, we obtain a proper coloring of G. □

Let f be a partial coloring of G with colors in $\{1, \dots, 7\}$. For $v \in V(G)$, denote by $O_f(v)$ (sometimes, simply by $O(v)$) the subset of $\{1, \dots, 7\}$ whose elements are not used in f to color v and the edges incident with v.

Lemma 8. *No cut $[V_1, V_2]$ of G with $|V_1| \geqslant 2$, $|V_2| \geqslant 2$ contains exactly three edges.*

Proof. Suppose that $[V_1, V_2] = \{e_1, e_2, e_3\}$, $e_i = (x_i, y_i)$, $i = 1, 2, 3$, $X = \{x_1, x_2, x_3\} \subseteq V_1$, $Y = \{y_1, y_2, y_3\} \subseteq V_2$ and $|V_1| \geqslant 2$, $|V_2| \geqslant 2$. By Lemma 6, $|X| \geqslant 2$, $|Y| \geqslant 2$. So we may assume

$$x_3 \neq x_2, \qquad y_3 \neq y_2. \tag{14}$$

Denote $G_1 = G(V_1) \cup \{(x_3, x_2)\}$, $G_2 = G(V_2) \cup \{(y_2, y_3)\}$, $G' = G_1 \cup G_2$. Due to the minimality of G, there exists a proper 7-coloring f' of G'. Then the restriction f of f' on $G(V_1) \cup G(V_2)$ is a partial 7-coloring of G with the following properties:

(a) the only non-colored elements of G are e_1, e_2 and e_3;

(b) $f(x_3) \neq f(x_2)$, $f(y_3) \neq f(y_2)$;

(c) $\exists \alpha_1 \in O(x_3) \cap O(x_2)$, $\exists \beta_1 \in O(y_3) \cap O(y_2)$;

(d) f is proper except that possibly $f(x_i) = f(y_i)$ for several $i \in \{1, 2, 3\}$.

Now we show that the colors used to color $G(V_2)$ can be renumbered, and edges e_1, e_2 and e_3 can be colored in such a way that the obtained coloring would satisfy conditions of Lemma 7.

Case 1. $f(x_1) \neq \alpha_1$ or $f(y_1) \neq \beta_1$.

Subcase 1.1. $\exists \alpha_2 \in O(x_1) \setminus \{\alpha_1, f(x_2), f(x_3)\}$. Choose $\beta_2 \in O(y_1) \setminus \{\beta_1\}$. Renumber the colors in $f(G(V_2))$ so that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, and put $f(e_2) = f(e_3) = \alpha_1$, $f(e_1) = \alpha_2$. Under the conditions of this case and by the choice of α_2, the resulting coloring satisfies the conditions of Lemma 7.

Subcase 1.2. $O(x_1) \subseteq \{\alpha_1, f(x_2), f(x_3)\}$, $O(y_1) \subseteq \{\beta_1, f(y_2), f(y_3)\}$. Because x_2 and x_3 are symmetrical for us, we may assume that $f(x_3) \in O(x_1)$.

If there exists $\beta_2 \in O(y_1) \setminus \{\beta_1, f(y_3)\}$ then renumbering the colors in $f(G(V_2))$ so that $\beta_1 = \alpha_1$, $\beta_2 = f(x_3)$, and putting $f(e_2) = f(e_3) = \alpha_1$, $f(e_1) = f(x_3)$, we obtain a coloring f such that $f(x_3) \neq f(y_3)$, $f(y_1) \notin \{\alpha_1, f(x_3)\}$. And by (b) above, $f(x_2) \notin \{\alpha_1, f(x_3)\}$. But such a coloring statisfies the conditions of Lemma 7. Thus, we may assume

$$O(y_1) = \{\beta_1, f(y_3)\}.$$

Now switching the roles of V_1 and V_2 and applying the symmetrical argument to that above we derive

$$O(x_1) = \{\alpha_1, f(x_3)\}.$$

Since $|O(x_1)| = |O(y_1)| = 2$, we have $x_1 \notin \{x_2, x_3\}$, $y_1 \notin \{y_2, y_3\}$. Hence, we can renumber colors in $f(G(V_2))$ so that $\beta_1 = \alpha_1$ and put $f(e_1) = f(e_2) = f(e_3) = \alpha_1$.

Since Case 1 is complete, we consider below only the possibility

$$f(x_1) = \alpha_1, \quad f(y_1) = \beta_1. \tag{15}$$

From (15), (14) and (c) above we obtain $|X| = |Y| = 3$. In view of (15) we may assume that

$$O(x_2) \supseteq \{\alpha_1, \alpha_2\}, \quad O(x_3) \supseteq \{\alpha_1, \alpha_3\}, \quad \alpha_2 \neq \alpha_3;$$

$$O(y_2) \supseteq \{\beta_1, \beta_2\}, \quad O(y_3) \supseteq \{\beta_1, \beta_3\}, \quad \beta_2 \neq \beta_3.$$

Case 2. There exists $\alpha_4 \in O(x_1) \setminus \{\alpha_2, \alpha_3\}$. Because of (b) above and in view of symmetry of x_2 and x_3, we may assume that

$$\alpha_4 \neq f(x_3). \tag{16}$$

If there exists $\beta_4 \in O(y_1) \setminus \{\beta_2, f(y_2)\}$ then renumber colors in $f(G(V_2))$ so that $\beta_1 = \alpha_3$, $\beta_2 = \alpha_1$ and $\beta_4 = \alpha_4$ and put $f(e_1) = \alpha_4$, $f(e_2) = \alpha_1$, $f(e_3) = \alpha_3$. By (15), (16) and the definition of β_4, the resulting coloring satisfies the conditions of Lemma 7.

Thus, below we assume $O(y_1) = \{\beta_2, f(y_2)\}$. Now suppose that $\alpha_4 \neq f(x_2)$. Then renumber colors in $f(G(V_2))$ so that $\beta_1 = \alpha_3$, $\beta_2 = \alpha_1$ and $f(y_2) = \alpha_4$ and again put $f(e_1) = \alpha_4$, $f(e_2) = \alpha_1$, $f(e_3) = \alpha_3$. By (15), (16) and $\alpha_4 \neq f(x_2)$, the resulting coloring satisfies the conditions of Lemma 7.

Let $\alpha_4 = f(x_2)$. Consider $\gamma \in O(x_1) \setminus \{\alpha_4\}$. If $\gamma = \alpha_3$ and $\beta = f(y_2)$ then we have Case 1 with x_2 replaced by x_1 and y_2 replaced by y_1. If $\gamma = \alpha_3$ and $\beta_3 \neq f(y_2)$ then renumber colors in $f(G(V_2))$ so that $\beta_1 = \alpha_2$, $\beta_3 = \alpha_1$ and $f(y_2) = \alpha_3$ and put $f(e_1) = \alpha_3$, $f(e_2) = \alpha_2$, $f(e_3) = \alpha_1$. If $\gamma \neq \alpha_3$ then renumber colors in $f(G(V_2))$ so that $\beta_1 = \alpha_3$, $\beta_2 = \alpha_1$ and $f(y_2) = \gamma$ and put $f(e_1) = \gamma$, $f(e_2) = \alpha_1$, $f(e_3) = \alpha_3$. In both of the last cases the resulting coloring satisfies the conditions of Lemma 7.

Thus, Case 2 is complete and we may assume $O(x_1) = \{\alpha_2, \alpha_3\}$. By symmetry of V_1 and V_2, we get $O(y_1) = \{\beta_2, \beta_3\}$. Renumber colors in $f(G(V_2))$ so that $\beta_1 = \alpha_2$, $\beta_2 = \alpha_3$ and $\beta_3 = \alpha_1$ and put $f(e_1) = \alpha_3$, $f(e_2) = \alpha_1$, $f(e_3) = \alpha_2$. We obtain a total 7-coloring of G. □

Theorem 3. *For each multigraph G with maximum degree at most five,*

$$\chi_2(G) \leqslant 7.$$

Proof. Suppose that the theorem is false. Choose among counterexamples to the theorem with the smallest possible number of vertices a multigraph G with the largest possible number of edges. By this choice, there is a vertex $s \in V(G)$ such that all the other vertices have degree 5 in G.

Construct a 5-regular multigraph H as follows. If $\deg_G(s) = 5$ then put $H = G$. Otherwise take a copy G' of G and connect s with its copy s' in G' by $5 - \deg_G(s)$ edges. If H has a perfect matching then by Theorem 2 it is totally 7-colorable and hence so is G, a contradiction. Thus, H has no perfect matching, and by Tutte's Theorem [12, Theorem 3.1.1], for some $X \subset V(H)$, there exist $|X| + 2$ components of $H - X$ each having odd number of vertices. Since each of these components is joined with X by an odd number of edges and $|[X, V(H)\setminus X]| \leqslant 5|X|$, there are three components C_1, C_2 and C_3 such that

$$|[V(C_i), V(H)\setminus V(C_i)]| \in \{1,3\}, \quad i = 1,2,3.$$

At most one of these three cuts separates s and s'. Let cuts $U_i = [V(C_i), V(H)\setminus V(C_i)]$ for $i = 1,2$ not to separate s and s'. By Lemma 6, U_i for $i = 1,2$ contains either 3 or 0 edges of G and either 3 or 0 edges of G'. Let, for definiteness, U_1 contain 3 edges of G. Then by Lemma 8, $U_1|_G = [\{s\}, V(G)\setminus\{s\}]$. For U_2 there remains only the possibility that $U_2 = [V(G)\cup\{s'\}, V(G')\setminus\{s'\}]$. For U_2 there remains only the possibility that $U_2 = [V(G)\cup\{s'\}, V(G')\setminus\{s'\}]$. But in this case $|[V(G), V(G')]| = 2$ and there is no room for cut $[V(C_3), V(H)\setminus V(C_3)]$. This is a contradiction. □

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