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Note

The colour theorems of Brooks and Gallai extended

A.V. Kostochka[a,1], M. Stiebitz[b,*], B. Wirth[c]

[a] *Novosibirsk State University, 630090 Novosibirsk, Russia*
[b] *Institute of Math., Technische Universität Ilmenau, D-98684 Ilmenau, Germany*
[c] *Institute of Math., Universität Köln, Weyertal 86, D-50931 Köln, Germany*

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Abstract

One of the basic results in graph colouring is Brooks' theorem [4] which asserts that the chromatic number of every connected graph, that is not a complete graph or an odd cycle, does not exceed its maximum degree. As an extension of this result, Gallai [6] characterized the subgraphs of k-colour-critical graphs induced by the set of all vertices of degree $k-1$. The choosability version of Brooks' theorem was proved, independently, by Vizing [9] and by Erdös et al. [5]. As Thomassen pointed out in his talk at the Graph Theory Conference held at Oberwolfach, July 1994, one can also prove a choosability version of Gallai's result.

All these theorems can be easily derived from a result of Borodin [2, 3] and Erdös et al. [5] which enables a characterization of connected graphs G admitting a color scheme L such that $|L(x)| \geqslant d_G(x)$ for all $x \in V(G)$ and there is no L-colouring of G. In this note, we use a reduction idea in order to give a new short proof of this result and to extend it to hypergraphs.

A *hypergraph* $G=(V,E)$ consists of a finite set $V=V(G)$ of *vertices* and a set $E=E(G)$ of subsets of V, called *edges*, each having cardinality at least two. An edge e with $|e| \geqslant 3$ is called a *hyperedge* and an edge e with $|e|=2$ is called an *ordinary edge*. The *degree* $d_G(x)$ of a vertex x in G is the number of the edges in G containing x. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. If $\Delta(G)=\delta(G)=r$, then G is called *r-regular*. Let $X \subseteq V(G)$. The subhypergraph $G[X]$ of G induced by X is defined as follows: $V(G[X])=X$ and $E(G[X])=\{e \in E(G) | e \subseteq X\}$; further, $G-X=G[V(G)-X]$. For $x \in V(G)$, let $G \backslash x$ denote the subhypergraph of G with $V(G\backslash x)=V(G)-\{x\}$ and $E(G\backslash x)=E(G-\{x\}) \cup \{e-\{x\} | x \in e \in E(G) \;\&\; |e| \geqslant 3\}$. For a hyperedge e, let $\langle e \rangle$ denote the hypergraph $(e, \{e\})$.

* Corresponding author.

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Let G be a connected hypergraph. A vertex x of G is called a *separating vertex* of G iff $G \backslash x$ is disconnected. An edge e of G is called a *bridge* of G iff $G - \{e\} = (V(G), E(G) - \{e\})$ has precisely $|e|$ components. By a *block* of G we mean a maximal connected subhypergraph B of G such that no vertex of B is a separating vertex of B. Any two blocks of G have at most one vertex in common and, obviously, a vertex of G is a separating vertex of G iff it is contained in more than one block of G. An *end-block* of G is a block that contains at most one separating vertex of G.

By a *brick* we mean a hypergraph of the form $\langle e \rangle$ for some hyperedge e, or an odd cycle consisting only of ordinary edges, or a complete graph. A connected hypergraph all of whose blocks are bricks is called a *Gallai tree*; a *Gallai forest* is a hypergraph whose components are Gallai trees.

Consider a hypergraph G and assign to each vertex x of G a set $L(x)$ of colours (positive integers). Such an assignment L of sets to vertices in G is referred to as a *colour scheme* (or briefly, a *list*) for G. An *L-colouring* of G is a mapping f of $V(G)$ into the set of colours such that $f(x) \in L(x)$ for all $x \in V(G)$ and $|\{f(x) | x \in e\}| \geqslant 2$ for each $e \in E(G)$. If G admits an L-colouring, then G is said to be *L-colourable*. In case of $L(x) = \{1, \dots, k\}$ for all $x \in V(G)$, we also use the terms *k-colouring* and *k-colourable*, respectively. A hypergraph G is called *k-choosable* iff G is L-colourable for every list L of G satisfying $|L(x)| = k$ for all $x \in V(G)$. The *chromatic number* $\chi(G)$ (*choice number* $\chi_l(G)$) of G is the least integer k such that G is k-colourable (k-choosable).

The choosability concept was introduced, independently, by Vizing [9] and Erdös et al. [5].

We call a pair (G, L) consisting of a connected hypergraph G and a list L for G a *bad pair of order* n iff $\sum_{x \in V(G)} d_G(x) = n$, $|L(x)| \geqslant d_G(x)$ for all $x \in V(G)$ and G is not L-colourable.

Reduction. Let (G, L) be a bad pair of order $n \geqslant 1$, y a non-separating vertex of G, and $\alpha \in L(y)$. For the connected hypergraph $G' = G \backslash y$ define a list $L' = L_y^\alpha$ by setting $L'(x) = L(x) - \{\alpha\}$ if $\{x, y\} \in E(G)$ and $L'(x) = L(x)$ otherwise.

Then, clearly, $|L'(x)| \geqslant d_{G'}(x)$ for all $x \in V(G')$ and (G', L') is a bad pair of some order k with $k < n$.

Lemma 1. Let (G, L) *be a bad pair of order* $n \geqslant 0$. *Then the following statements hold.*

(1) $|L(x)| = d_G(x)$ *for all* $x \in V(G)$.

(2) *Every hyperedge* e *of* G *is a bridge of* G *and, therefore,* $\langle e \rangle$ *is a block of* G.

(3) *If* G *has no separating vertex, then* $L(x)$ *is the same for all* $x \in V(G)$ *and* G *is regular.*

(4) G *is a Gallai tree.*

Proof (*By induction on n*). If (G, L) is a bad pair of order $n = 0$, then $L(x) = \emptyset$ where x is the only vertex of G and the statements (1)–(4) are obviously true. In what follows, let (G, L) be a bad pair of order $n \geqslant 1$.

For the proof of (1) consider an arbitrary vertex x of G. Since G is connected, there is a non-separating vertex $y \neq x$ in G. Now, consider the bad pair (G', L') where $G' = G \setminus y$ and $L' = L_y^\alpha$ for some $\alpha \in L(y)$. Note that $L(y) \neq \emptyset$. Then, by the induction hypothesis, $|L'(x)| = d_{G'}(x)$ which implies immediately that $|L(x)| = d_G(x)$. This proves (1).

To prove (2) suppose that some hyperedge e of G is not a bridge of G. Then, for some vertex $x \in e$, the hypergraph $G' = (V(G), E(G) - \{e\} \cup \{e - \{x\}\})$ is connected and, therefore, (G', L) is a bad pair with $|L(x)| \geqslant d_G(x) > d_{G'}(x)$, a contradiction to (1). This proves (2).

For the proof of (3) assume that G has no separating vertex. If G contains a hyperedge e, then, because of (2), $G = \langle e \rangle$ and (3) is obviously true. Otherwise G is a graph, i.e. G has only ordinary edges, and, for proving (3), it suffices to show that $L(x) = L(y)$ for all $x, y \in V(G)$. Suppose that this is not true. Then there are two adjacent vertices x, y in G such that $L(x) \neq L(y)$. W.l.o.g. assume that $\alpha \in L(y) - L(x)$. Set $G' = G \setminus y = G - \{y\}$ and $L' = L_y^\alpha$. Then, clearly, (G', L') is a bad pair with $|L'(x)| > d_{G'}(x)$, a contradiction to (1). This proves (3).

For the proof of (4), we consider two possible cases.

Case 1: G has a separating vertex. Then G has at least two end-blocks, say B_1 and B_2. For $i = 1, 2$, there is a non-separating vertex y_i of G contained in B_i. Then, by induction (using the reduction operation), $G \setminus y_i$ is a Gallai tree and, clearly, every block $B \neq B_i$ of G is a block of $G \setminus y_i$, too. This implies that G is a Gallai tree.

Case 2: G has no separating vertex, i.e. G is a block. Because of (2) and (3), we may assume that G is a graph which is regular of some degree $r \geqslant 1$ and $L(x) = C$ for all $x \in V(G)$ where C is a set of r colours. Let y be a vertex of G and set $G' = G \setminus y = G - \{y\}$. Then, by induction, G' is a Gallai tree and every block of the graph G' is a complete graph or an odd cycle. If G' consists of a single block, then, clearly, both G' and G are complete graphs. So, let G' have at least two blocks. Since every end-block of G' must be $(r-1)$-regular, the degree of y in G is at least $2(r-1)$ which yields $r = 2$, and, therefore, the graph G is a cycle. Since G is not L-colourable and the same two colours are available at each vertex of the cycle G, we conclude that G is an odd cycle.

This proves (4). $\square$

Lemma 1 is not quite new. In particular, its graph version was proved independently by Borodin [2, 3] and Erdös et al. [5]. However, their proofs use different ideas and are longer. Proofs of statements (1) and (3) in the graph version based on a sequential colouring argument were given by Vizing [9] and Lovász [7].

The next result is a simple consequence of Lemma 1. A different proof of its graph version has recently been given by Thomassen [8].

Theorem 2. *Let L be an arbitrary list for a given hypergraph G. Furthermore, let X be a subset of $V(G)$ such that $G[X]$ is connected and $|L(x)| \geqslant d_G(x)$ for all $x \in X$. Assume that $G - X$ is L^*-colourable where L^* is the restriction of L to $V(G) - X$. If G is not L-colourable, then $G[X]$ is a Gallai tree and $|L(x)| = d_G(x)$ for each $x \in X$.*

Proof. Consider an arbitrary L^*-colouring f of $G - X$ and choose for each edge $e \in E(G)$ with $e - X \neq \emptyset$, a vertex $v(e) \in e - X$. For the connected hypergraph $G' = G[X]$, define a list L' by

$$L'(x) = L(x) - \{f(v(e)) | x \in e \in E(G) \;\&\; e - X \neq \emptyset\}$$

for each $x \in X$. Then $|L'(x)| \geqslant d_{G'}(x)$ for every vertex x of G' and, clearly, every L'-colouring of G' yields an L-colouring of G. Therefore, if G is not L-colourable, then (G', L') is a bad pair and, by Lemma 1, G' is a Gallai tree and $|L'(x)| = d_{G'}(x)$ for all $x \in X$ implying that $|L(x)| = d_G(x)$ for all $x \in X$. This proves Theorem 2. □

Let G be a connected hypergraph. Clearly, $\chi(G) \leqslant \chi_l(G)$ and Theorem 2 implies that

$$\chi(G) \leqslant \chi_l(G) \leqslant \Delta(G) + 1.$$

If G is a brick, i.e., if $|E(G)| = 1$ or G is a complete graph or an odd cycle, then

$$\chi(G) = \chi_l(G) = \Delta(G) + 1.$$

Brooks [4] proved that the complete graphs and the odd cycles are the only connected graphs whose chromatic number is larger than their maximum degree. This famous result has a number of different proofs. Some of them are listed in [1]. That a similar result is true also for the choice number, was proved, independently, by Vizing [9] and Erdös et al. [5].

Theorem 3. *If G is a connected hypergraph that is not a brick, then $\chi(G) \leqslant \chi_l(G) \leqslant \Delta(G)$.*

Proof. Suppose that G is not L-colourable for some list L for G where $|L(x)| = \Delta(G)$ for each $x \in V(G)$. Then, by Theorem 2, G is a Gallai tree which is regular of degree $\Delta(G)$. This implies that G is a brick, i.e., a connected hypergraph consisting of exactly one hyperedge, or an odd cycle consisting only of ordinary edges, or a complete graph. This proves Theorem 3. □

Another consequence of Theorem 2 is the following result stated for the chromatic number as well as for the choice number.

Theorem 4. *Let ξ stand for χ or χ_l. Let G be a hypergraph with $\xi(G) \geqslant k$ for some $k \geqslant 2$. Furthermore, let X be a subset of $V(G)$ such that $G[X]$ is connected and $d_G(x) \leqslant k - 1$ for each $x \in X$. If $\xi(G - X) \leqslant k - 1$, then the following statements hold.*

(1) $\xi(G) = k$,

(2) $d_G(x) = k - 1$ for each $x \in X$, and

(3) $G[X]$ is a Gallai tree.

Proof. We prove Theorem 4 only for the choice number. Since $\chi_l(G) \geqslant k$, G is not L-colourable for some list L for G where $|L(y)| = k - 1$ for each $y \in V(G)$. Then $|L(x)| \geqslant d_G(x)$ for all $x \in X$. Let $L^* = L\,|_{G-X}$. The hypergraph $G - X$ being L^*-colourable, (2) and (3) are consequences of Theorem 2. Furthermore, Theorem 2 implies that G is L-colourable for every list L satisfying $|L(y)| = k$ for each $y \in V(G)$, i.e. $\chi_l(G) = k$. □

A hypergraph G is called *k-colour-critical* (*k-choice-critical*) iff $\chi(H) < \chi(G) = k$ ($\chi_l(H) < \chi_l(G) = k$) for every proper subhypergraph H of G.

Clearly, if G is k-colour-critical (k-choice-critical) for some $k \geqslant 2$, then $\delta(G) \geqslant k - 1$. The vertices of G whose degrees are equal to $k - 1$ are called the *low vertices* of G and the subhypergraph of G induced by the set of low vertices is called the *low-vertex subhypergraph* of G.

Theorem 4 immediately implies the following result.

Theorem 5. *If a hypergraph G is k-colour-critical or k-choice-critical for some $k \geqslant 2$, then the low-vertex subhypergraph of G is a Gallai forest (possibly empty).*

For colour-critical graphs the above Theorem was proved by Gallai [6]. Moreover, using his result, Gallai established a lower bound for the number of edges of a k-colour-critical graph. Since Gallai's proof can be applied also to k-choice-critical graphs, we obtain the following.

Theorem 6. *Let G be a k-colour-critical or k-choice-critical graph for some $k \geqslant 4$. If G is not a complete graph, then $2 \cdot |E(G)| \geqslant ((k-1) + ((k-3)/(k^2-3)))|V(G)|$.*

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