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Note

Spanning trees with pairwise nonadjacent endvertices

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Abstract

A spanning tree of a connected graph G is said to be an independency tree if all its endvertices are pairwise nonadjacent in G. We prove that a connected graph G has no independency tree if and only if G is a cycle, a complete graph or a complete bipartite graph the color classes of which have equal cardinality.

Keywords: Spanning tree with pairwise nonadjacent endvertices; Depth-first-search tree

1. Introduction

We use [1] for basic terminology and notation not defined here. In this paper, among other things, we are interested in the existence of spanning trees with pairwise non-adjacent endvertices in finite connected graphs.

The motivation of the first three authors is related to cycles in graphs and can be described as follows.

Suppose G is a finite connected graph on at least three vertices, and T is an arbitrary spanning tree of G. If two of the endvertices of T are joined by an edge e of G, then the graph $T+e$ obtained from T by adding e has a unique cycle C. If C is not a hamiltonian cycle of G, then at least one vertex v of C has degree at least three in $T+e$. Deleting one of the edges of C incident with v from $T+e$ results in a spanning tree T' of G which has one endvertex less than T. We can repeat this procedure to a

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newly found spanning tree as long as its endvertices are not pairwise nonadjacent in G, and as long as the tree is not a hamiltonian path of G with adjacent endvertices. In other words we either end up with a hamiltonian cycle or a spanning tree of G the endvertices of which form an independent set of G. We refer to such a spanning tree as an *independency tree*. So on one hand, independency trees are interesting 'blocking objects' in the context of finding hamiltonian cycles. On the other hand, the concept of an independency tree generalizes the concept of a hamiltonian path in a nonhamiltonian graph.

The motivation of the last two authors comes from graph colorings.

Stiebitz proved a theorem [5, Theorem 2] which implies the statement of Brooks' theorem ([2], and [1, Theorem 11.2]) for graphs that contain an independency tree. Using our characterization in the next section of those graphs that do not contain an independency tree, a new proof of Brooks' theorem can be obtained.

Our main result implies that a graph does not have an independency tree if and only if it is either a cycle, a complete graph or a complete bipartite graph the color classes of which have equal cardinality. Furthermore it will be shown that if a graph has an independency tree, then it has an independency tree which is a depth-first-search tree, as well. Such a tree is defined as follows. Let G be a graph on n vertices. A *numbering* η of G is a bijection from $V(G)$ onto $\{1,\dots,n\}$. If x is a vertex of G, then $\eta(x)$ is called the *number* of x. Now the *depth-first-search tree* (or *DFS tree*) $T(G,\eta)$ of G with respect to η is defined by the following algorithm. In every step of the algorithm a pair (T_i, v_i) consisting of a tree T_i contained in G and a vertex v_i of T_i is constructed. We start with the pair (T_0, v_0) where v_0 is the vertex with $\eta(V_0)=1$ and T_0 is the trivial tree consisting of v_0 only. Suppose that the pair (T_i, v_i) has already been constructed, then the pair (T_{i+1}, v_{i+1}) will be obtained as follows. If v_i has a neighbor outside T_i, then we choose v_{i+1} among the neighbors of v_i such that it is not contained in T_i and has minimal number under this condition, and define T_{i+1} to be the tree obtained from T_i by adding v_{i+1} and the edge $v_i v_{i+1}$. Otherwise, v_{i+1} is chosen to be the predecessor of v_i on the unique path connecting v_0 with v_i in T_i, and T_{i+1} is defined to be T_i. It is not hard to see that this algorithm terminates after a finite number of steps, and the tree obtained in the last step forms a spanning tree of G. Notice that if P: $x_1,\dots,x_k$ is a path in G and the numbering η has been chosen such that $\eta(x_i)=i$ for $i\in\{1,\dots,k\}$, then P is contained in $T(G,\eta)$. Finally, note that the endvertices of the DFS tree $T(G,\eta)$ which are different from v_0 form an independent set in G.

The procedure described in the motivation of the first three authors yields a polynomially bounded algorithm that either finds an independency tree or a hamiltonian cycle. In the latter case one easily extracts from the proof of our theorem in the next section a polynomially bounded algorithm that either finds an independency tree (in this case a hamiltonian path with nonadjacent endvertices) or shows that no such tree exists. The graphs that do not contain an independency tree turn out to be easily recognizable. The related decision problem, whether a given graph contains an independency tree with at most k endvertices, can be proved to be NP-complete for any fixed $k\geqslant 2$, as follows.

Let G be a graph, x a vertex of G and $k \geqslant 2$ an integer. Define G' to be the graph obtained from G by adding $k-1$ new vertices $x_1, \ldots, x_{k-1}$ each of which is adjacent to x and to no other vertex. Then every independency tree of G' contains $x_1, \ldots, x_{k-1}$ as endvertices. Consequently, G' has an independency tree with at most k endvertices if and only if G has a hamiltonian path starting at x. Since the hamiltonian path problem is NP-complete (see [4, p. 199]) the result follows.

2. The main theorem

Now we are ready to prove the following theorem. We denote by $K_{n/2,n/2}$ the complete bipartite graph both color classes of which have cardinality $n/2$.

Theorem. *Let G be a connected graph on $n \geqslant 3$ vertices. Then the following statements are equivalent.*

(i) *G is isomorphic to C_n, K_n, or n is even and G is isomorphic to $K_{n/2,n/2}$.*

(ii) *G has no independency tree.*

(iii) *G has no DFS tree in which all endvertices are pairwise nonadjacent in G.*

(iv) *G has a hamiltonian path and every hamiltonian path is contained in a hamiltonian cycle.*

Proof. Obviously (ii) implies (iii), thus it suffices to prove (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) This is obvious if G is isomorphic to C_n or K_n. Suppose T would be a spanning tree of $K_{n/2,n/2}$ in which all endvertices are pairwise nonadjacent. Then all the endvertices of T would belong to the same color class and the vertices in the other color class would all have degree at least two in T. Consequently, T would have at least n edges, a contradiction.

(iii) $\Rightarrow$ (iv) If G does not satisfy (iv) then it has a path P: $x_1, \ldots, x_k$ such that x_1 is not adjacent to x_k and all neighbors of x_1 and x_k are in $V(P)$. We consider a numbering η subject to

$$\eta(x_i) = i \quad \text{for } i \in \{1, \ldots, k\},$$

$$\eta(y) \text{ arbitrary for } y \notin V(P).$$

Then x_1 is not adjacent to any endvertex of $T(G,\eta)$ and, therefore, the endvertices of $T(G,\eta)$ are pairwise nonadjacent.

(iv) $\Rightarrow$ (i) Let C: $x_1, \ldots, x_n$ be a hamiltonian cycle of G, then an edge $e \in E(G) \backslash E(C)$ is called a *k-chord* if the shortest cycle formed by e and edges of C has length k. We distinguish two cases.

(1) G contains a hamiltonian cycle C: $x_1, \ldots, x_n$ with a 3-chord, say x_1x_3. Then $x_2, x_1, x_3, x_4, \ldots, x_n$ is a hamiltonian path of G and, consequently, $x_2x_n \in E(G)$. By similar arguments $x_ix_{i+2} \in E(G)$ for $1 \leqslant i \leqslant n-2$ and $x_{n-1}x_1 \in E(G)$, i.e., G contains all 3-chords of C. Now $x_4, x_5, \ldots, x_n, x_2, x_3, x_1$ is a hamiltonian path of G, hence $x_1x_4 \in E(G)$.

If $n \geqslant 5$, then $x_5, \ldots, x_n, x_2, x_3, x_4, x_1$ is a hamiltonian path and $x_1x_5 \in E(G)$. By repeating this argument we see that $x_1y \in E(G)$ for all $y \in V(G) \setminus \{x_1\}$. Similar arguments for the other vertices show that G is isomorphic to K_n.

(2) G contains no hamiltonian cycle with 3-chords. Let C: $x_1, \ldots, x_n$ be a hamiltonian cycle of G. If G is not a cycle, then C has a chord and without loss of generality we may assume $x_1x_i \in E(G)$ for some i with $4 \leqslant i \leqslant n-2$. Then $n \geqslant 6$, and both $x_2, x_3, \ldots, x_i, x_1, x_n, x_{n-1}, \ldots, x_{i+1}$ and $x_n, x_{n-1}, \ldots, x_i, x_1, x_2, \ldots, x_{i-1}$ are hamiltonian paths of G. Hence we obtain that $x_2x_{i+1}, x_nx_{i-1} \in E(G)$. Now $x_{i-2}, x_{i-3}, \ldots, x_2$, $x_{i+1}, x_{i+2}, \ldots, x_n, x_{i-1}, x_i, x_1$ is a hamiltonian path of G. This implies $x_1x_{i-2} \in E(G)$. Since C has no 3-chords, by symmetry arguments and by repeating the argument, we conclude that i is even and that $x_1x_j \in E(G)$ for all even numbers $j \in \{1, 2, \ldots, n\}$. Hence n is even. Similar arguments for the other vertices show that G is isomorphic to $K_{n/2,n/2}$. □

Chartrand and Kronk [3] showed that the graphs in (i) are exactly the so-called randomly traceable graphs. Since every randomly traceable graph satisfies (iv) (by [3, Lemma 1]), their theorem follows from ours. Similar questions have also been discussed by Thomassen [6].

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