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Note

Covering and coloring polygon–circle graphs

Alexandr Kostochka[a], Jan Kratochvíl[b,*]

[a] *Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia*
[b] *Charles University, KAM MFF UK, 11800 Praha 1, Prague, Czech Republic*

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Abstract

Polygon–circle graphs are intersection graphs of polygons inscribed in a circle. This class of graphs includes circle graphs (intersection graphs of chords of a circle), circular arc graphs (intersection graphs of arcs on a circle), chordal graphs and outerplanar graphs. We investigate binding functions for chromatic number and clique covering number of polygon–circle graphs in terms of their clique and independence numbers. The bound $\alpha \log \alpha$ for the clique covering number is asymptotically best possible. For chromatic number, the upper bound we prove is of order 2^{ω}, which is better than previously known upper bounds for circle graphs.

1. Introduction

We will consider simple undirected graphs without loops or multiple edges. The vertex set and edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. The subgraph of a graph G induced by a set of vertices U will be denoted by $G|U$.

The independence number (the maximum size of a stable set), the clique number (the maximum size of a complete subgraph), the chromatic number (the minimum number of classes of a partition of the vertex set into stable sets) and the clique covering number (the minimum number of classes of a partition of the vertex set into complete subgraphs) of a graph G are denoted by $\alpha(G), \omega(G), \chi(G)$ and $\sigma(G)$, respectively.

1.1. Polygon–circle graphs

A well-known class of intersection graphs is the class of *circle graphs* which we denote by CIR. Circle graphs are defined as intersection graphs of chords of a circle, or, equivalently, as *overlap graphs* of intervals on a line (in the overlap graph, two vertices are adjacent if and only if the corresponding intervals are not disjoint and

* Corresponding author. E-mail: honza@kam.ms.mff.cuni.cz.

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none of them is a subinterval of the other one). They can be recognized in polynomial time [1], a nontrivial and elegant characterization by three obstructions is given in [2].

In 1989, M. Fellows (personal communication) suggested the following generalization of circle graphs. Call a graph a *polygon–circle graph* if it can be represented as the intersection graph of (convex) polygons inscribed in a circle. We denote this class of graphs by PC. Obviously, every circle graph is polygon–circle, and in fact, polygon–circle graphs are exactly the graphs which can be obtained from circle graphs by edge contractions. It is also clear that circular arc graphs (intersection graphs of arcs on a circle) form a subclass of PC, and one can also see that every chordal graph (i.e., graph with no induced cycle of length greater than three) is polygon–circle [7]. Under a different name ('spider graphs'), polygon–circle graphs were considered by Koebe, who gave a polynomial time recognition algorithm for them in [8].

Similarly to circle graphs being viewed on as overlap graphs, we can derive the following equivalent definition of polygon–circle graphs, using the fact that every PC graph has an intersection representation by polygons which have mutually distinct corners. We say that a graph G has an *alternating representation* if the vertices of G can be represented by pairs (I_v, M_v), where I_v is a closed interval with integral endpoints on the real line and M_v is a finite subset of $I_v \cap Z$ which contains the endpoints of I_v in such a way that (i) the sets M_v are mutually disjoint and (ii) for any two vertices u, v, uv is an edge of G if and only if there are integers $a < b < c < d$ such that $a, c \in M_u$ and $b, d \in M_v$ (or $a, c \in M_v$ and $b, d \in M_u$). A graph has an alternating representation if and only if it is a polygon–circle graph. We will exploit this definition of polygon–circle graphs in Section 3.

1.2. Binding functions

Obviously, $\omega(G) \leqslant \chi(G)$ for every graph G. It is well known that graphs without triangles (i.e., graphs that satisfy $\omega(G) = 2$) may have arbitrarily large chromatic number. This is, however, not the case for many special classes of graphs. The other extreme are perfect graphs which satisfy $\chi(G) = \omega(G)$. Gyarfás defined the notion of *binding function* in the following way. A function f is a binding function for χ and a class of graphs $\mathscr{A}$ if $\chi(G) \leqslant f(\omega(G))$ for any graph $G \in \mathscr{A}$ (in this case we simply write $\chi \leqslant f(\omega)$ for $\mathscr{A}$). Binding function for the clique covering number σ is defined in a similar way (in this case as a function of α). It is always interesting to know if a class of graphs under consideration admits a binding function for χ or σ. Many results in this direction can be found in [6]. In the sequel, we will denote by f_χ and f_σ the optimal binding functions, i.e., we set

$$f_\chi(\mathscr{A}, k) = \max\{\chi(G) \mid G \in \mathscr{A},\ \omega(G) \leqslant k\},$$

$$f_\sigma(\mathscr{A}, k) = \max\{\sigma(G) \mid G \in \mathscr{A},\ \alpha(G) \leqslant k\}.$$

For the class of circle graphs, it was known that $f_\sigma = \Theta(\alpha \log \alpha)$ and we will prove in Section 2 that the same holds true for the wider class of polygon–circle graphs. For chromatic number, so far the best published binding function was of order $2^\omega \omega^2$ [9]. We will improve this bound to $O(2^\omega)$ by proving this bound even for polygon–circle graphs in Section 3. However, in the case of chromatic number the best known lower bound for f_χ on circle graphs is $\Omega(\omega \log \omega)$ and we are not able to improve the lower bound for polygon–circle graphs, either. In fact, we propose the following problem:

Problem 1.1. Is it true that $f_\chi(\mathrm{PC}, k) = \Theta(f_\chi(\mathrm{CIR}, k))$?

2. Clique covering number

The goal of this section is to prove the following theorem:

Theorem 2.1. *For polygon–circle graphs, we have*

$$f_\sigma(\mathrm{PC}, \alpha) = (1 + o(1))\alpha \log \alpha.$$

Since the same result is proved for circle graphs in [9], we only have to prove the upper bound.

Let $G = (V, E)$ be a PC graph, denote $\sigma = \sigma(G)$ and $\alpha = \alpha(G)$. Fix a representation $P = \{P_v \mid v \in V\}$ of G by polygons inscribed in a circle C. First we define some technical notions.

A *v-arc* is any open arc on C determined by two consecutive vertices of P_v. If $U \subset V$ is a set of vertices of G, we say that a v-arc A is *U-empty* if no P_u, for $u \in U$, has all corners in A. An arc which is not U-empty is called *U-nonempty*. A polygon P_v (and the corresponding vertex v) is called *U-separating* if at least two of the v-arcs are nonempty. If v is not U-separating and determines one U-nonempty arc, we denote this arc by $A(v, U)$.

Next we define subsets $V^0, V^1, V_1, V^2, V_2, \ldots$ by means of recursion as follows:

$$V^0 = V,$$

$$V^i = \{v \in V^{i-1} \mid v \text{ is } V^{i-1}\text{-separating}\}, \quad i = 1, 2, \ldots,$$

$$V_i = V^{i-1} - V^i, \quad i = 1, 2, \ldots,$$

and we denote by G_i the subgraph of G induced by V_i, i.e., $G_i = G|V_i$.

Lemma 2.2. *For every $i = 1, 2, \ldots, \alpha$, we have $\alpha(G_i) \leqslant \alpha/i$.*

Proof. The statement is obvious if $i = 1$ or $\alpha(G_i) = 1$. Hence suppose $i \geqslant 2$ and let $\{v_1^1, \ldots, v_m^1\} \subset V_i$ be an independent set in G, $m \geqslant 2$. Since $v_1^1, \ldots, v_m^1$ are not

V^{i-1}-separating, for each $j = 1, 2, \ldots, m$, the arc $A(v_j^1, V^{i-1})$ contains all corners of the polygons representing the other v_l^1's. Since $v_1^1, \ldots, v_m^1$ are V^{i-2}-separating, for every $j = 1, \ldots, m$, there exist a v_j^1-arc $B(j) \neq A(v_j^1, V^{i-1})$ and a vertex $v_j^2 \in V^{i-2}$ such that all corners of $P_{v_j^2}$ lie in $B(j)$. Because v_j^1 is not V^{i-1}-separating, $v_j^2 \notin V^{i-1}$ and hence $v_j^2 \in V_{i-1}$. Observe also that $A(v_j^2, V^{i-2}) \supset A(v_j^1, V^{i-1})$. If $i-1 \geqslant 2$, we construct in a similar way a sequence $v_j^3, \ldots, v_j^i$ so that for every $l = 3, 4, \ldots, i$, $v_j^l \in V^{i-l}$ and $A(v_j^l, V^{i-l}) \supset A(v_j^{l-1}, V^{i-l+1})$. It follows that the polygons which represent mi vertices v_j^l $(l = 1, 2, \ldots, i,\ j = 1, 2, \ldots, m)$ are pairwise disjoint and the statement of the lemma follows. □

Lemma 2.3. *For* $i > \frac{1}{2}(\alpha + 1)$, $V_i = \emptyset$.

Proof. If $v \in V_i$, v is V^{i-1}-separating, and thus there are two V^{i-1}-nonempty v-arcs on C. Similar to the proof of the preceeding lemma, we can find $2i-1$ pairwise disjoint polygons in the representation. Hence $2i - 1 \leqslant \alpha$ and the statement follows. □

Lemma 2.4. *For every* $i \geqslant 1$, *we have* $\sigma(G_i) \leqslant \alpha(G_i) + 1$.

Proof. Set $R_i = \{v \in V_i \mid \text{all } v\text{-arcs are } V_i\text{-empty}\}$, $S_i = V_i - R_i$. It follows that for every $v \in R_i$, the polygon P_v intersects all polygons which represent the remaining vertices from V_i. Thus $\sigma(G_i) = \sigma(G|S_i)$ and $\alpha(G_i) = \alpha(G|S_i)$ if $S_i \neq \emptyset$ (and $\sigma(G_i) = \alpha(G_i) = 1$ otherwise). Now every polygon which represents a vertex v from S_i determines exactly one V_i-nonempty v-arc $A(v, V_i)$. Hence vertices $u, v \in S_i$ are adjacent in $G|S_i$ if and only if $A(u, V_i) \cup A(v, V_i) \neq C$ and $G|S_i$ is a circular arc graph. It is well known that for circular arc graphs $\sigma \leqslant \alpha + 1$ (cf. [6]). □

Proof of Theorem 2.1. Since $V = \bigcup_{i=1}^{\lfloor(\alpha+1)/2\rfloor} V_i$ (cf. Lemma 2.3), it follows from Lemmas 2.2 and 2.4 that

$$\sigma(G) \leqslant \sum_{i=1}^{\lfloor(\alpha+1)/2\rfloor} \sigma(G_i) \leqslant \sum_{i=1}^{\lfloor(\alpha+1)/2\rfloor} \left(\left\lfloor \frac{\alpha}{i} \right\rfloor + 1 \right) = (1 + o(1))\alpha \log \alpha. \qquad \square$$

3. Chromatic number

The following theorem will be proved in this section.

Theorem 3.1. *For polygon–circle graphs, we have*

$$f_\chi(\mathrm{PC}, \omega) < 2^{\omega+6}.$$

In the proof of this theorem, we will lean on the alternating representations of polygon–circle graphs, as they were described in Section 1.1. Suppose $G = (V, E)$ is a PC graph. Fix an alternating representation of G, say $\{(I_v, M_v) \mid v \in V\}$ for the rest of the proof. Let $x_0 \in V$ be the vertex such that $\min I_{x_0} < \min I_v$ for each $v \neq x_0$. Set

$$V_0 = \{x_0\},$$

$$V_i = \left\{ y \mid y \notin \bigcup_{j=0}^{i-1} V_j \wedge \exists z \in V_{i-1},\ zy \in E \right\}, \quad i = 1,2,\ldots$$

The sets V_i are called *levels* of G. It is clear that $\chi(G) \leqslant 2k$ provided every level can be colored by k colors. The following lemma is an analogue of a lemma of Gyarfás [4].

Lemma 3.2. *If $U \subset V_i$ and $z \in V_i$ $(i > 0)$ are such that $I_z \subset \bigcap_{u \in U} I_u$, then there is a $y \in V_{i-1}$ such that $yu \in E$ for every $u \in U \cup \{z\}$ and $I_y \not\subset I_u$ for any $u \in U$.*

Proof. We will prove a slightly stronger statement that obviously implies the lemma: If $x_0, x_1, \ldots, x_i = z$ is a shortest path from x_0 to z in G and $u \in V_j, j \geqslant i$ is such that $I_z \subset I_u$, then $x_{i-1}u \in E$ and $I_{x_{i-1}} \not\subset I_u$.

Indeed, $x_{i-1}z \in E$ implies that $M_{x_{i-1}} \cap I_z \neq \emptyset$, and hence $x_{i-1}u \notin E$ would yield $I_{x_{i-1}} \subset I_u$. Since $I_{x_0} \not\subset I_u$, we have $i > 0$ and we conclude by induction (applying the statement to $z' = x_{i-1}$ and u) that $x_{i-2}u \in E$, i.e., $u \in V_{i-1}$, a contradiction. □

Lemma 3.3. *Let $U \subset V$ be such that $\bigcap_{u \in U} I_u \neq \emptyset$. Then $\chi(G|U) = \omega(G|U)$.*

Proof. In this case $uv \notin E$ implies that either $I_u \subset I_v$ or $I_v \subset I_u$ and it follows that the complement of $G|U$ is transitively orientable, i.e., a comparability graph. Comparability graphs (and their complements) are perfect (cf. e.g. [3]). □

The following technical lemma will be used further on.

Lemma 3.4. *Let A and B be two families of closed real intervals such that any two intervals from B are disjoint while any two intervals from A have a nonempty intersection. Moreover, every interval from A contains at least two intervals from B. If $w : A \to Z^+$ is a weight function on A such that $\sum_{a \in A} w(a) \geqslant 2m - 1$ for some positive integer m, then there exist a subfamily $A' \subset A$ and an interval $b \in B$ such that $\sum_{a \in A'} w(a) \geqslant m$ and $b \subset \bigcap_{a \in A'} a$.*

Proof. We prove the statement by induction on the number of intervals in A. If $|A| = 1$, $A' = A$ suffices.

Let $|A| > 1$. If there are two intervals $a, a' \in A$ which are in inclusion, say $a \subset a'$, we set $A_1 = A - \{a'\}, w_1(x) = w(x)$ for $x \neq a$ and $w_1(a) = w(a) + w(a')$. By induction hypothesis, there is an $A'_1 \subset A_1$ such that $\bigcap_{x \in A'_1} x \supset b$, for some $b \in B$, and $\sum_{x \in A'_1} w_1(x) \geqslant m$. Set $A' = A'_1$ if $a \notin A'_1$ and $A' = A'_1 \cup \{a'\}$ otherwise.

If no two intervals of A are in inclusion, then the intervals can be numbered $a_1, a_2, \ldots, a_k$ so that, with $a_i = [l_i, r_i]$, we have $l_1 < l_2 < \cdots < l_k < r_1 < r_2 < \cdots < r_k$.

Let j be the first index such that $\sum_{i=1}^{j} w(a_i) \geqslant m$ (it follows that $\sum_{i=j}^{k} w(a_i) \geqslant m$ as well). The interval a_j contains two disjoint intervals $b_1, b_2 \in B$. If the right one, b_2, is contained neither in $\bigcap_{i=1}^{j} a_i$ nor in $\bigcap_{i=j}^{k} a_i$, then $l_k, r_1 \in b_2$ and $b_1 \in \bigcap_{i=1}^{j} a_i$. $\square$

Definition. Let m be a positive integer. We say that an alternative representation is *m-good* if for any clique C of size m, the intersection $\bigcap_{v \in C} I_v$ contains no other I_u, $u \in V(G)$. By $\mathscr{H}(m)$ we denote the subfamily of polygon–circle graphs G which have an m-good alternating representation and which satisfy $\omega(G) \leqslant 2m$. Let $h(m) = \max\{\chi(G) \mid G \in \mathscr{H}(m)\}$.

Lemma 3.5. *For each positive integer m,*

$$h(m) \leqslant 25 \cdot 2^m - 16m - 32.$$

Proof. We use induction on m. Any graph from $\mathscr{H}(1)$ is an interval graph, and hence $h(1) = 2$.

Suppose the inequality holds for every $m < k$. Consider a graph $G \in \mathscr{H}(k)$ and fix a k-good alternating representation $\{(I_v, M_v),\ v \in V\}$ of G. Partition G into levels and consider a level $H = G|V_h$ ($h \geqslant 1$). Let $U = \{v_1, v_2, \ldots, v_r\}$ be a set of vertices of H such that for any $i = 1, 2, \ldots, r-1$, $\max I_{v_i} < \min I_{v_{i+1}}$. We choose U of maximum possible size, and subject to this constraint, we pick the vertices v_i so that the right endpoints of the corresponding intervals are leftmost possible. Set $P_i = \max I_{v_{3i}}$, $i = 1, 2, \ldots, s = \lfloor \frac{1}{3} r \rfloor$ and choose P_0, P_{s+1} so that $P_0 < \min I_{v_1}$ and $P_{s+1} > \max I_{v_r}$. Denote $P = \{P_1, P_2, \ldots, P_s\}$, and partition the set of vertices of H into 3 subsets $U_1 = \{v \in V_h, |I_v \cap P| = 1\}$, $U_2 = \{v \in V_h, I_v \cap P = \emptyset\}$ and $U_3 = \{v \in V_h, |I_v \cap P| \geqslant 2\}$. We will show that $\chi(H|U_1) \leqslant 4k$, $\chi(H|U_2) \leqslant 4k$ and $H|U_3 \in \mathscr{H}(k-1)$.

(1) By Lemma 3.3, $\chi(H|U_{1,i}) = \omega(H|U_{1,i}) \leqslant 2k$ for $U_{1,i} = \{v \in U_1, I_v \cap P = \{P_i\}\}$. Since $I_u \cap I_v = \emptyset$ for $u \in U_{1,i}$ and $v \in U_{1,j}$ such that $|i-j| > 1$, we may use $2k$ colors to color the vertices from $\bigcup_{i=1}^{\lfloor s/2 \rfloor} U_{1,2i}$ and other $2k$ colors to color the vertices from $\bigcup_{i=0}^{\lfloor (s-1)/2 \rfloor} U_{1,2i+1}$.

(2) The intervals which lie within P_i and P_{i+1} do not intersect intervals which lie outside $[P_i, P_{i+1}]$ and we may use the same collection of colors for every set $U_{2,i} = \{v \in U_2, I_v \subset (P_i, P_{i+1})\}$, $i = 0, 1, \ldots, s$. Consider a particular i. By the choice of U and P, every interval which represents a vertex from $U_{2,i}$ contains the right endpoint of $I_{v_{3i+1}}$ or the right endpoint of $I_{v_{3i+2}}$. Thus, by Lemma 3.3, $4k$ colors suffice to color $H|U_{2,i}$.

(3) We show first that $\omega(H|U_3) \leqslant 2k-2$. To obtain a contradiction, suppose that there is a clique $C \subset U_3$ of size $2k-1$ in $H|U_3$. The families $A = \{I_v,\ v \in C\}$ and $B = \{I_{v_j},\ v_j \in U\}$ (together with a weight function $w(I_v) = 1$) satisfy the assumptions of Lemma 3.4. It follows that there are vertices $v^1, v^2, \ldots, v^k \in C$ and $v_j \in U$ such that $I_{v_j} \subset \bigcap_{i=1}^{k} I_{v^i}$, contradicting $G \in \mathscr{H}(k)$.

Now suppose that there is a clique C of size $k-1$ in $H|U_3$ such that $I_w \subset \bigcap_{v \in C} I_v$ for some $w \in U_3$. Since $I_w \supset [P_i, P_{i+1}]$ for some i, I_w contains 3 disjoint intervals $I_{v_{3i+1}}, I_{v_{3i+2}}$

and $I_{v_{3i+3}}$. Apply Lemma 3.2 to C and $z = v_{3i+2}$. For $y \in V(G)$, which we get by this lemma, $C' = C \cup \{y\}$ is a clique of size k in G and either $I_{v_{3i+1}}$ or $I_{v_{3i+3}}$ is contained in $\bigcap_{v \in C'} I_v$, contradicting $G \in \mathscr{H}(k)$.

It follows that $\chi(G|U_3) \leqslant h(k-1)$ and we have $\chi(G) \leqslant 2(4k+4k+h(k-1)) \leqslant 2(8k + 25 \cdot 2^{k-1} - 16(k-1) - 32) = 25 \cdot 2^k - 16k - 32$. $\square$

Proof of Theorem 3.1. Consider an alternating representation of G, partitioned into levels. By Lemma 3.2, each graph $H = G|V_i$ induced by a level V_i belongs to $\mathscr{H}(\omega)$. Hence $\chi(G) \leqslant 2h(\omega) < 50 \cdot 2^{\omega}$. $\square$

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